Ambiguities and Divergences in RG Functions

Anders Eller Thomsen

Based on F. Herren, AET [2104.07037]

Asymptotic Safety meets Particle Physics & Friends,
16 December 2021

UNIVERSITÄT BERN

AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Introduction
Flavorful trouble in the RG

Renormalization group flow

Callan–Symanzik equation for renormalized n-point functions:

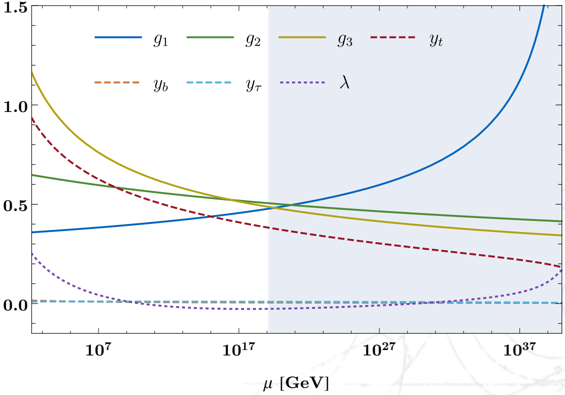
$$\left(\frac{\partial}{\partial t} + \beta_g \frac{\partial}{\partial g} + n\gamma\right) G^{(n)}(\{p\}) = 0, \qquad \beta_I(g_I) = \frac{\mathrm{d}}{\mathrm{d}t} g_I \equiv \frac{\mathrm{d}}{\mathrm{d}\ln\mu} g_I$$

Renormalization group flow

Callan–Symanzik equation for renormalized n-point functions:

$$\left(\frac{\partial}{\partial t} + \beta_g \frac{\partial}{\partial g} + n\gamma\right) G^{(n)}(\{p\}) = 0, \qquad \beta_I(g_I) = \frac{\mathrm{d}}{\mathrm{d}t} g_I \equiv \frac{\mathrm{d}}{\mathrm{d}\ln\mu} g_I$$

SM RG flow with 3rd generation Yukawa couplings:

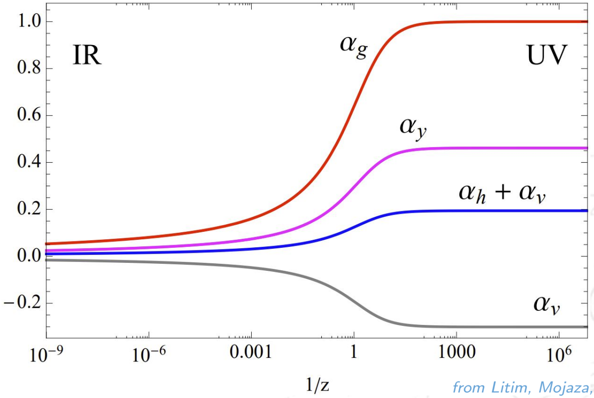


Renormalization group flow

Callan–Symanzik equation for renormalized n-point functions:

$$\left(\frac{\partial}{\partial t} + \beta_g \frac{\partial}{\partial g} + n\gamma\right) G^{(n)}(\{p\}) = 0, \qquad \beta_I(g_I) = \frac{\mathrm{d}}{\mathrm{d}t} g_I \equiv \frac{\mathrm{d}}{\mathrm{d}\ln\mu} g_I$$

Asymptotic safety in the Litim–Sannino model:



from Litim, Mojaza, Sannino [1501.03061]

Renormalization group flow

Callan–Symanzik equation for renormalized n-point functions:

$$\left(\frac{\partial}{\partial t} + \beta_g \frac{\partial}{\partial g} + n\gamma\right) G^{(n)}(\{p\}) = 0, \qquad \beta_I(g_I) = \frac{\mathrm{d}}{\mathrm{d}t} g_I \equiv \frac{\mathrm{d}}{\mathrm{d}\ln\mu} g_I$$

- The SM (and it extensions) has nontrivial flavor structure (matrix couplings)
- Improved precision necessitates their inclusion in the RG function
- Their inclusion causes new conceptual problems starting at 3-loop order:
  - The RG flow can generate spurious limit cycles
  - The $\overline{\text{MS}}$ counterterms are no longer uniquely defined
  - *RG functions can seemingly be divergent!*

Flavor symmetry in the SM

The quark sector of the SM,

$$\mathcal{L} = i\bar{q}\not{D}q + i\bar{u}\not{D}u + i\bar{d}\not{D}d + |D_\mu H|^2 - \left(\bar{q}\, y_u\, u\tilde{H} + \bar{q}\, y_d\, dH \ + \ \text{H.c.}\ \right),$$

has flavor symmetry (maximal symmetry of the kinetic terms)

$$G_F = \mathrm{SU}(3)_q \times \mathrm{SU}(3)_u \times \mathrm{SU}(3)_d \times \mathrm{U}(1)^3 \supset \mathrm{U}(1)_B$$

Flavor symmetry in the SM

The quark sector of the SM,

$$\mathcal{L} = i\bar{q}\not{D}q + i\bar{u}\not{D}u + i\bar{d}\not{D}d + |D_\mu H|^2 - \left(\bar{q}\, y_u\, u\tilde{H} + \bar{q}\, y_d\, dH \;+\; \text{H.c.}\,\right),$$

has flavor symmetry (maximal symmetry of the kinetic terms)

$$G_F = \mathrm{SU}(3)_q \times \mathrm{SU}(3)_u \times \mathrm{SU}(3)_d \times \mathrm{U}(1)^3 \supset \mathrm{U}(1)_B$$

Physics is invariant under transformations

$$\left.\begin{aligned} y_u &\longrightarrow U_q\, y_u\, U_u^\dagger \\ y_d &\longrightarrow U_q\, y_d\, U_d^\dagger \end{aligned}\right\} \quad \text{e.g.,} \quad (y_u\,, y_d) \longrightarrow (V_{\mathrm{CKM}}^\dagger \hat{y}_u,\, \hat{y}_d)$$

Flavor symmetry in the SM

The quark sector of the SM,

$$\mathcal{L} = i\bar{q}\not{D}q + i\bar{u}\not{D}u + i\bar{d}\not{D}d + |D_\mu H|^2 - \big(\bar{q}\, y_u\, u\tilde{H} + \bar{q}\, y_d\, dH \;+\; \text{H.c.}\,\big),$$

has flavor symmetry (maximal symmetry of the kinetic terms)

$$G_F = \mathrm{SU}(3)_q \times \mathrm{SU}(3)_u \times \mathrm{SU}(3)_d \times \mathrm{U}(1)^3 \supset \mathrm{U}(1)_B$$

Physics is invariant under transformations

$$\left.\begin{aligned} y_u &\longrightarrow U_q\, y_u\, U_u^\dagger \\ y_d &\longrightarrow U_q\, y_d\, U_d^\dagger \end{aligned}\right\} \quad \text{e.g.,} \quad (y_u\,, y_d) \longrightarrow (V_{\mathrm{CKM}}^\dagger \hat{y}_u,\, \hat{y}_d)$$

If flavor transformations are unphysical, one can perform arbitrary flavor rotations along the RG flow...

γ-pole at 3-loop order

Renormalization condition for 2-point functions: ($\overline{\text{MS}}$, $d = 4 - 2\epsilon$)

$$Z^\dagger \text{—(1PI)—} Z \;+\; Z^\dagger \text{———} Z \;=\; \text{finite}, \qquad Z = \mathbb{1} + \sum_{n=1}^{\infty} \frac{z^{(n)}}{\epsilon^n}$$

with field anomalous dimension

loop-counting operator

$$\gamma = Z^{-1} \frac{\mathrm{d}}{\mathrm{d}t} Z = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}}{\epsilon^n} \quad \Longrightarrow \quad \gamma^{(0)} = -\zeta z^{(1)}, \qquad \zeta = k_I g_I \partial^I$$

γ-pole at 3-loop order

Renormalization condition for 2-point functions: ($\overline{\text{MS}}$, $d = 4 - 2\epsilon$)

$$Z^\dagger \text{—(1PI)—} Z \;+\; Z^\dagger \text{———} Z \;=\; \text{finite}, \qquad Z = \mathbb{1} + \sum_{n=1}^{\infty} \frac{z^{(n)}}{\epsilon^n}$$

with field anomalous dimension

loop-counting operator

$$\gamma = Z^{-1} \frac{\mathrm{d}}{\mathrm{d}t} Z = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}}{\epsilon^n} \quad \Longrightarrow \quad \gamma^{(0)} = -\zeta z^{(1)}, \qquad \zeta = k_I g_I \partial^I$$

In SM $\gamma^{(1)} \neq 0$ at the 3-loop order for $Z^\dagger = Z$:

Bednyakov, Pikelner, Velizhanin [1406.7171] Herren, Mihaila, Steinhauser [1712.06614]

$$(4\pi)^6 \gamma_q^{(1)} = \frac{g_1^2}{96} \left[y_u y_u^\dagger,\, y_d y_d^\dagger \right] + \frac{1}{32} \left[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger \right] + \frac{1}{32} \left[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger \right]$$

$$(4\pi)^6 \gamma_u^{(1)} = \frac{1}{16} y_u^\dagger \left[y_d y_d^\dagger,\, y_u y_u^\dagger \right] y_u$$

Renormalization condition for 2-point functions: ($\overline{\text{MS}}$, $d = 4 - 2\epsilon$)

$$Z^\dagger \text{—}(\text{1PI})\text{—} Z \;+\; Z^\dagger \text{———} Z \;=\; \text{finite}, \qquad Z = \mathbb{1} + \sum_{n=1}^{\infty} \frac{z^{(n)}}{\epsilon^n}$$

with field anomalous dimension

loop-counting operator

$$\gamma = Z^{-1} \frac{\mathrm{d}}{\mathrm{d}t} Z = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}}{\epsilon^n} \quad \Longrightarrow \quad \gamma^{(0)} = -\zeta z^{(1)}, \qquad \zeta = k_I g_I \partial^I$$

In SM $\gamma^{(1)} \neq 0$ at the 3-loop order for $Z^\dagger = Z$:

Bednyakov, Pikelner, Velizhanin [1406.7171] Herren, Mihaila, Steinhauser [1712.06614]

$$\gamma^{(1)} - \gamma^{(1)\dagger} = \left[z^{(1)}, \zeta z^{(1)} \right]$$

$\gamma^{(1,2)}$ can be made to vanish with $Z' = UZ$ for some divergent rotation U.

The Local RG

An excellent tool for RG analysis

Four-dimensional QFT

Most general renormalizable theory in 4D (ignoring relevant couplings):

$$\mathcal{L} = +\tfrac{1}{2}(D_\mu\phi)_a(D^\mu\phi)_a + i\psi_i^\dagger\bar{\sigma}^\mu(D_\mu\psi)^i + \mathcal{L}_{\mathrm{gh}} + \mathcal{L}_{\mathrm{gf}}$$
$$-\tfrac{1}{4}a^{-1}_{AB}F^A_{\mu\nu}F^{B\mu\nu} - \tfrac{1}{2}\left(Y_{aij}\psi^i\psi^j + \mathrm{H.c.}\right)\phi_a - \tfrac{1}{24}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d$$

Four-dimensional QFT

Most general renormalizable theory in 4D (ignoring relevant couplings):

$$\mathcal{L} = +\tfrac{1}{2}(D_\mu\phi)_a(D^\mu\phi)_a + i\psi_i^\dagger\bar{\sigma}^\mu(D_\mu\psi)^i + \mathcal{L}_{\text{gh}} + \mathcal{L}_{\text{gf}}$$
$$-\tfrac{1}{4}a^{-1}_{AB}F^A_{\mu\nu}F^{B\mu\nu} - \tfrac{1}{2}\left(Y_{aij}\psi^i\psi^j + \text{H.c.}\right)\phi_a - \tfrac{1}{24}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d$$

Compactly, the action is

all field sources

$$S = S_{\text{kin}}[\Phi] + \int \mathrm{d}^d x\left(g_I\mathcal{O}^I(x) + \mathcal{J}_\alpha\Phi^\alpha\right)$$

set of all marginal couplings

The vacuum functional

$$e^{i\mathcal{W}[\mathcal{J}]} = \int[\mathcal{D}\Phi]\, e^{iS[\Phi,\mathcal{J}]}$$

generates all the connected n-point functions.

Everything is a source

LRG was developed to probe the trace anomaly by introducing new sources:

Shore '87; Jack, Osborn '90; Osborn '91; Jack, Osborn [1312.0428]; Baume, Keren-Zur, Rattazzi, Vitale [1401.5983]

$$\underbrace{[T^{\mu}{}_{\mu}]}_{\text{stress-energy tensor}} = \beta_I [\mathcal{O}^I] + \upsilon \cdot \partial_\mu \underbrace{[J^\mu_F]}_{\text{flavor current; } J^\mu_F \in \mathfrak{g}_F} \qquad \begin{cases} T_{\mu\nu}: & \eta_{\mu\nu} \to \gamma_{\mu\nu}(x) \\ \mathcal{O}^I: & g_I \to g_I(x) \\ J^\mu_F: & D_\mu \to D_\mu - a_\mu(x) \end{cases}$$

Everything is a source

LRG was developed to probe the trace anomaly by introducing new sources:

Shore '87; Jack, Osborn '90; Osborn '91; Jack, Osborn [1312.0428]; Baume, Keren-Zur, Rattazzi, Vitale [1401.5983]

$$\underbrace{[T^{\mu}{}_{\mu}]}_{\text{stress-energy tensor}} = \beta_I [\mathcal{O}^I] + \upsilon \cdot \partial_\mu \underbrace{[J_F^\mu]}_{\text{flavor current; } J_F^\mu \in \mathfrak{g}_F} \qquad \begin{cases} T_{\mu\nu} : & \eta_{\mu\nu} \to \gamma_{\mu\nu}(x) \\ \mathcal{O}^I : & g_I \to g_I(x) \\ J_F^\mu : & D_\mu \to D_\mu - a_\mu(x) \end{cases}$$

All renormalization is source renormalization,

$$S = S_{\mathrm{kin}}[\Phi,\, \gamma,\, a_0] + \int \mathrm{d}^d x \sqrt{\gamma} \left(g_{0,I} \mathcal{O}^I + \mathcal{J}_{0,\alpha} \Phi^\alpha \right) + S_{\mathrm{ct}}[\gamma,\, g_0,\, a_0],$$

so the renormalized vacuum functional is

$$\mathcal{W}[\gamma,\, g,\, a,\, \mathcal{J}] = \mathcal{W}_0\big[\gamma,\, g_0(g),\, a_0(a, g),\, \mathcal{J}_0(\mathcal{J}, g)\big]$$

Weyl transformation

S is symmetric under the Weyl symmetry, with infinitesimal generator

$$\Delta_\sigma^W = \int \mathrm{d}^d x \bigg(2\sigma\gamma^{\mu\nu} \frac{\delta}{\delta\gamma^{\mu\nu}} - \sigma\beta_I \frac{\delta}{\delta g_I} + \sigma\mathcal{J}_\beta \big[(d - \Delta_\alpha)\delta^\beta{}_\alpha - \gamma^\beta{}_\alpha \big] \frac{\delta}{\delta\mathcal{J}_\alpha} + \big[\partial_\mu\sigma\, \upsilon - \sigma\, D_\mu g_I\, \rho^I \big] \cdot \frac{\delta}{\delta a_\mu} \bigg)$$

field anomalous dimension ($\gamma^\beta{}_\alpha$)

β-function (β_I)

RGs of the G_F current; $\upsilon, \rho^I \in \mathfrak{g}_F$

Weyl transformation

S is symmetric under the Weyl symmetry, with infinitesimal generator

field anomalous dimension

$$\Delta^W_\sigma = \int \mathrm{d}^d x \Big(2\sigma\gamma^{\mu\nu} \frac{\delta}{\delta\gamma^{\mu\nu}} - \sigma\beta_I \frac{\delta}{\delta g_I} + \sigma\mathcal{J}_\beta \big[(d-\Delta_\alpha)\delta^\beta{}_\alpha - \gamma^\beta{}_\alpha\big] \frac{\delta}{\delta\mathcal{J}_\alpha}$$

β-function

$$+ \big[\partial_\mu\sigma\, \upsilon - \sigma\, D_\mu g_I\, \rho^I\big] \cdot \frac{\delta}{\delta a_\mu} \Big)$$

RGs of the G_F current; $\upsilon, \rho^I \in \mathfrak{g}_F$

The symmetry is anomalous ($\Delta^W_\sigma S_{\mathrm{ct}} \neq 0$)

$$\Delta^W_\sigma \mathcal{W} = \int \mathrm{d}^d x\, \mathcal{A}^W_\sigma(\gamma,\, g,\, a)$$

Δ^W_σ contains the trace anomaly equation

$$[T^\mu{}_\mu] = \beta_I[\mathcal{O}^I] + \upsilon\cdot\partial_\mu[J^\mu_F] - \eta_a\partial^2[\mathcal{O}^a_M] \qquad \text{(FSCC)}$$

Flat-space constant-coupling limit:
$\gamma_{\mu\nu}(x) = \eta_{\mu\nu},\; g_I(x) = g_I,\; a_\mu = 0$

RG transformation

Accounting identity for mass dimension:

$$\Delta^\mu \mathcal{W} = 0, \qquad \Delta^\mu = \mu \frac{\partial}{\partial \mu} + \int \mathrm{d}^d x \left(2\gamma^{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} + (d - \Delta_\alpha) \mathcal{J}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right)$$

RG transformation

Accounting identity for mass dimension:

$$\Delta^{\mu}\mathcal{W} = 0, \qquad \Delta^{\mu} = \mu\frac{\partial}{\partial\mu} + \int \mathrm{d}^d x \left(2\gamma^{\mu\nu}\frac{\delta}{\delta\gamma_{\mu\nu}} + (d - \Delta_\alpha)\mathcal{J}_\alpha\frac{\delta}{\delta\mathcal{J}_\alpha}\right)$$

The generator of the RG is $\Delta^{\mathrm{RG}} = \Delta^{\mu} - \Delta^{W}_{\sigma=1}$, from which we recover the *CS equation*

$$0 = \Delta^{\mathrm{RG}}\mathcal{W} = \left(\frac{\partial}{\partial t} + \beta_I\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta\gamma^\beta{}_\alpha\frac{\delta}{\delta\mathcal{J}_\alpha}\right)\mathcal{W} \qquad \text{(FSCC)}$$

Exactly what we would get from $\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}t} = 0$:

$$\left(\frac{\partial}{\partial t} + \beta_g\frac{\partial}{\partial g} + n\gamma\right) G^{(n)}(\{p\}) = 0$$

Flavor transformations

G_F is a symmetry of S with generator

$$\Delta^F_\omega = \int \mathrm{d}^d x \left(D_\mu \omega \cdot \frac{\delta}{\delta a_\mu} - (\omega\, g)_I \frac{\delta}{\delta g_I} - (\omega\, \mathcal{J})_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right), \qquad \omega \in \mathfrak{g}_F,$$

but it is typically anomalous: Keren-Zur [1406.0869]

$$\Delta^F_\omega \mathcal{W} = \int \mathrm{d}^d x \, \mathcal{A}^F_\omega(\gamma,\, g,\, a)$$

Flavor transformations

G_F is a symmetry of S with generator

$$\Delta^F_\omega = \int \mathrm{d}^d x \left(D_\mu \omega \cdot \frac{\delta}{\delta a_\mu} - (\omega\, g)_I \frac{\delta}{\delta g_I} - (\omega\, \mathcal{J})_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right), \qquad \omega \in \mathfrak{g}_F,$$

but it is typically anomalous: Keren-Zur [1406.0869]

$$\Delta^F_\omega \mathcal{W} = \int \mathrm{d}^d x\, \mathcal{A}^F_\omega(\gamma,\, g,\, a)$$

The Weyl generator can be combined with a flavor rotation to generate a class of Weyl symmetries:

$$\Delta^{W'}_\sigma = \Delta^W_\sigma + \Delta^F_{\sigma\alpha}, \qquad \alpha(g) \in \mathfrak{g}_F,$$

$$\left[\Delta^F_\omega,\, \Delta^{W'}_\sigma\right] = \left[\Delta^{W'}_\sigma,\, \Delta^{W'}_{\sigma'}\right] = 0, \qquad \Delta^{W'}_\sigma \mathcal{W} = \int \mathrm{d}^d x\, \mathcal{A}^{W'}_\sigma$$

An ambiguity in the RG

Ambiguity in RG functions defined by the Weyl transformation:

$$\beta'_I = \beta_I + (\alpha\, g)_I, \quad \upsilon' = \upsilon + \alpha, \quad \rho'^I = \rho^I - \partial^I \alpha, \quad \gamma'^\alpha{}_\beta = \gamma^\alpha{}_\beta - \alpha^\alpha{}_\beta.$$

$\Longrightarrow$ *The RG flow has a flavor rotation ambiguity*

An ambiguity in the RG

Ambiguity in RG functions defined by the Weyl transformation:

$$\beta'_I = \beta_I + (\alpha\, g)_I, \quad \upsilon' = \upsilon + \alpha, \quad \rho'^I = \rho^I - \partial^I \alpha, \quad \gamma'^{\alpha}{}_{\beta} = \gamma^{\alpha}{}_{\beta} - \alpha^{\alpha}{}_{\beta}.$$

$\Longrightarrow$ *The RG flow has a flavor rotation ambiguity*

Flavor-improved RG functions are invariant:

$$B_I = \beta_I - (\upsilon\, g)_I, \qquad P^I = \rho^I + \partial^I \upsilon, \qquad \Gamma^{\alpha}{}_{\beta} = \gamma^{\alpha}{}_{\beta} + \upsilon^{\alpha}{}_{\beta}.$$

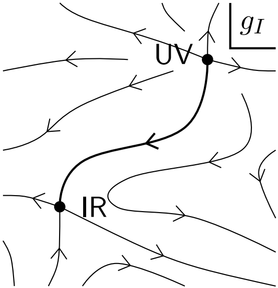
We can choose a "gauge" where $\upsilon = 0$:

$$\widehat{\Delta}^W_\sigma = \Delta^W_\sigma + \Delta^F_{-\sigma\,\upsilon} = \int \mathrm{d}^d x \bigg(2\sigma \gamma^{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} - \sigma B_I \frac{\delta}{\delta g_I} + \sigma \mathcal{J}_\beta \big[(d - \Delta_\alpha) \delta^{\beta}{}_{\alpha} - \Gamma^{\beta}{}_{\alpha} \big] \frac{\delta}{\delta \mathcal{J}_\alpha} - \sigma\, D_\mu g_I\, P^I \cdot \frac{\delta}{\delta a_\mu} \bigg),$$

But generally $B_I \neq \frac{\mathrm{d} g_I}{\mathrm{d} t}$ ...

How to recognize a CFT

Fixed Points

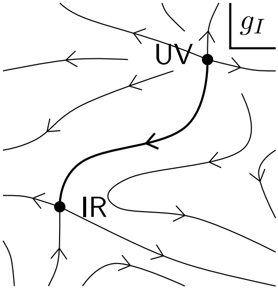


Traditionally CFTs were understood to be FPs:

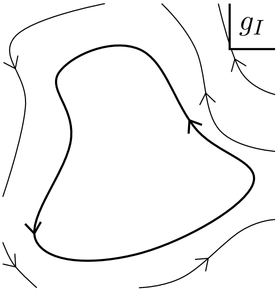
$$[T^\mu{}_\mu] = \beta_I [\mathcal{O}^I] = 0$$

How to recognize a CFT

Fixed Points



Limit Cycles



Traditionally CFTs were understood to be FPs:

ignores J_F^μ

~~$[T^\mu{}_\mu] = \beta_I[\mathcal{O}^I] = 0$~~

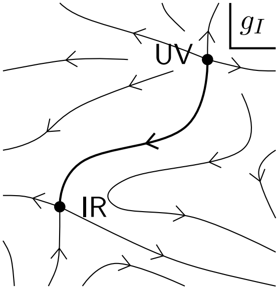
Limit cycles are actually CFTs

Fortin, Grinstein, Stergiou [1206.2921, 1208.3674]

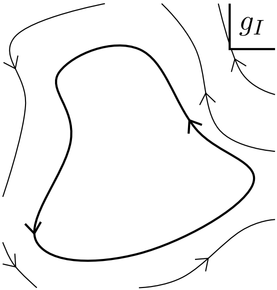
$$[T^\mu{}_\mu] = B_I[\mathcal{O}^I] = 0$$

How to recognize a CFT

Fixed Points



Limit Cycles



Traditionally CFTs were understood to be FPs:

ignores J_F^μ

~~$[T^\mu{}_\mu] = \beta_I [\mathcal{O}^I] = 0$~~

Limit cycles are actually CFTs

Fortin, Grinstein, Stergiou [1206.2921, 1208.3674]

$$[T^\mu{}_\mu] = B_I [\mathcal{O}^I] = 0$$

B_I is a more physical β-function

Divergences and Ambiguities

Does $\overline{\text{MS}}$ break down at higher loop order?

How to compute RG functions

In $\overline{\text{MS}}$ ($d = 4 - 2\epsilon$) the counterterms are arranged by poles

$$\delta g_I = \sum_{n=1}^{\infty} \frac{\delta g_I^{(n)}}{\epsilon^n}, \qquad Z = 1 + \sum_{n=1}^{\infty} \frac{z^{(n)}}{\epsilon^n}$$

The RG functions are determined recursively from the poles

$$\beta_I^{(-1)} = -k_I g_I, \qquad \beta_I^{(n)} = (\zeta - k_I)\delta g_I^{(n+1)} - \sum_{k=0}^{n-1} \beta_J^{(k)} \partial^J \delta g_I^{(n-k)}, \qquad n \geq 0$$

$$\gamma^{(n)} = -\zeta z^{(n+1)} + \sum_{k=0}^{n-1} \left[\beta_I^{(k)} \partial^I z^{(n-k)} - z^{(n-k)} \gamma^{(k)} \right], \qquad n \geq 0$$

**Similar formulas hold for υ involving a counterterm of the flavor current.*

Callan–Symanzik equation revisited

The evolution of *renormalized* amplitudes is governed by the CS Eq.:

$$0 = \Delta^{\mathrm{RG}}\mathcal{W} = \left(\frac{\partial}{\partial t} + \beta_I \partial^I + \int \mathrm{d}^d x \, \mathcal{J}_\beta \gamma^\beta{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W} \qquad \text{(FSCC)}$$

Callan–Symanzik equation revisited

The evolution of *renormalized* amplitudes is governed by the CS Eq.:

$$\left(\frac{\partial}{\partial t} + \left(\epsilon\beta_I^{(-1)} + \beta_I^{(0)}\right)\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{(0)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{\epsilon^n}\left(\beta_I^{(n)}\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{(n)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}$$

Callan–Symanzik equation revisited

The evolution of *renormalized* amplitudes is governed by the CS Eq.:

$$\left(\frac{\partial}{\partial t} + \left(\epsilon\beta_I^{(-1)} + \beta_I^{(0)}\right)\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{(0)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W} = -\sum_{n=1}^{\infty} \frac{1}{\epsilon^n}\left(\beta_I^{(n)}\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{(n)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}$$

Recall the G_F Ward identity (FSCC):

$$0 = \Delta_\omega^F \mathcal{W} = \left((\omega\, g)_I \partial^I - \int \mathrm{d}^d x\, \mathcal{J}_\beta \omega^\beta{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}, \qquad \omega \in \mathfrak{g}_F$$

Callan–Symanzik equation revisited

The evolution of *renormalized* amplitudes is governed by the CS Eq.:

$$\left(\frac{\partial}{\partial t} + \left(\epsilon \beta_I^{(-1)} + \beta_I^{(0)}\right)\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{(0)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}$$
$$= -\sum_{n=1}^{\infty} \frac{1}{\epsilon^n}\left(\beta_I^{(n)}\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{(n)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}$$

Recall the G_F Ward identity (FSCC):

$$0 = \Delta_\omega^F \mathcal{W} = \left((\omega\, g)_I \partial^I - \int \mathrm{d}^d x\, \mathcal{J}_\beta \omega^\beta{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}, \qquad \omega \in \mathfrak{g}_F$$

The RG flow is finite due to

RG Finiteness *(theorem)*

$$\gamma^{(n)} \in \mathfrak{g}_F \quad \text{and} \quad \beta_I^{(n)} = -\left(\gamma^{(n)}\, g\right)_I, \qquad n \geq 1$$

RG finiteness in the SM

3-loop RG divergences in the SM: using counterterms from Herren, Mihaila, Steinhauser [1712.06614]

$$(4\pi)^6 \gamma_q^{(1)} = \frac{g_1^2}{96}\left[y_u y_u^\dagger,\, y_d y_d^\dagger\right] + \frac{1}{32}\left[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\right] + \frac{1}{32}\left[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\right]$$

$$(4\pi)^6 \gamma_u^{(1)} = \frac{1}{16} y_u^\dagger \left[y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u$$

$$(4\pi)^6 \beta_{y_u}^{(1)} = -\frac{g_1^2}{96}\left[y_u y_u^\dagger,\, y_d y_d^\dagger\right] y_u - \frac{1}{32}\left[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\right] y_u - \frac{1}{32}\left[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u + \frac{1}{16} y_u y_u^\dagger \left[y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u$$

RG finiteness in the SM

3-loop RG divergences in the SM: using counterterms from Herren, Mihaila, Steinhauser [1712.06614]

$$(4\pi)^6 \gamma_q^{(1)} = \frac{g_1^2}{96}\left[y_u y_u^\dagger,\, y_d y_d^\dagger\right] + \frac{1}{32}\left[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\right] + \frac{1}{32}\left[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\right]$$

$$(4\pi)^6 \gamma_u^{(1)} = \frac{1}{16} y_u^\dagger \left[y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u$$

$$(4\pi)^6 \beta_{y_u}^{(1)} = -\frac{g_1^2}{96}\left[y_u y_u^\dagger,\, y_d y_d^\dagger\right] y_u - \frac{1}{32}\left[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\right] y_u - \frac{1}{32}\left[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u + \frac{1}{16} y_u y_u^\dagger \left[y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u$$

RG finiteness in the SM

3-loop RG divergences in the SM: using counterterms from Herren, Mihaila, Steinhauser [1712.06614]

$$(4\pi)^6\gamma_q^{(1)} = \underbrace{\frac{g_1^2}{96}\left[y_u y_u^\dagger,\, y_d y_d^\dagger\right]} + \underbrace{\frac{1}{32}\left[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\right]} + \underbrace{\frac{1}{32}\left[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\right]}$$

$$(4\pi)^6\gamma_u^{(1)} = \underbrace{\frac{1}{16} y_u^\dagger \left[y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u}$$

$$(4\pi)^6\beta_{y_u}^{(1)} = -\underbrace{\frac{g_1^2}{96}\left[y_u y_u^\dagger,\, y_d y_d^\dagger\right] y_u} - \underbrace{\frac{1}{32}\left[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\right] y_u}$$
$$\qquad - \underbrace{\frac{1}{32}\left[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u} + \underbrace{\frac{1}{16} y_u y_u^\dagger \left[y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u}$$

$\beta_{y_u}^{(1)} = -(\gamma^{(1)}\, y_u)$, $\beta_{y_u}^{(2)} = -(\gamma^{(2)}\, y_u)$, *etc.* in the SM

$$(\omega\, y_u)^i{}_j = \omega_q{}^i{}_k y_u{}^k{}_j - y_u{}^i{}_k \omega_u{}^k{}_j + \omega_h y_u{}^i{}_j$$

SM RG functions are RG finite at 3-loop order

Renormalization ambiguity

$\mathcal{W}$ is invariant under flavor rotations, $R \in G_F$: e.g., $y_u \longrightarrow R_q y_u R_u^\dagger$ in the SM

$$\mathcal{W}[\gamma,\, g,\, \mathcal{J},\, a] = \mathcal{W}[\gamma,\, Rg,\, R\mathcal{J},\, a^R] =$$
$$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, Rg_0,\, R\mathcal{J}_0,\, a_0^R], \quad (R\,g_0)_I = g_{0,I}(R\,g)$$

Renormalization ambiguity

$\mathcal{W}$ is invariant under flavor rotations, $R \in G_F$: e.g., $y_u \longrightarrow R_q y_u R_u^\dagger$ in the SM

$$\mathcal{W}[\gamma,\, g,\, \mathcal{J},\, a] = \mathcal{W}[\gamma,\, Rg,\, R\mathcal{J},\, a^R] =$$
$$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, Rg_0,\, R\mathcal{J}_0,\, a_0^R], \quad (R\,g_0)_I = g_{0,I}(R\,g)$$

Take a divergent rotation instead:

$$U = \exp\left[-\sum_{n=1}^{\infty} \frac{1}{\epsilon^n} u^{(n)}(g)\right], \qquad u^{(n)} \in \mathfrak{g}_F$$

$$\mathcal{W}[\gamma,\, g,\, \mathcal{J},\, a] = \mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, U g_0,\, U\mathcal{J}_0,\, a_0^U]$$

It results in a change of counterterms, e.g., Ambiguity in taking $\sqrt{Z^\dagger Z}$

$$(U\mathcal{J}_0)_\alpha = \mathcal{J}_{0,\beta} U^{\dagger\beta}{}_\alpha = \mathcal{J}_\beta (Z^{-1}U^\dagger)^\beta{}_\alpha \implies \tilde{Z}^\alpha{}_\beta = U^\alpha{}_\gamma Z^\gamma{}_\beta.$$

Ambiguity in RG the functions

$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, U g_0,\, U\mathcal{J}_0,\, a_0^U]$ *but produce different RG functions!*

$$\Delta\gamma \equiv \gamma^U - \gamma = -\beta_I U \partial^I U^\dagger$$
$$\Delta\beta_I \equiv \beta_I^U - \beta_I = -(\Delta\gamma\, g)_I,$$
$$\Delta\upsilon \equiv \upsilon^U - \upsilon = -\Delta\gamma,$$

Ambiguity in RG the functions

$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, U g_0,\, U\mathcal{J}_0,\, a_0^U]$ *but produce different RG functions!*

$$\Delta\gamma \equiv \gamma^U - \gamma = -\beta_I U \partial^I U^\dagger$$
$$\Delta\beta_I \equiv \beta_I^U - \beta_I = -(\Delta\gamma\, g)_I,$$
$$\Delta\upsilon \equiv \upsilon^U - \upsilon = -\Delta\gamma,$$

i) By choosing U, one can engineer any $\Delta\gamma = \alpha(g) \in \mathfrak{g}_F$
   - We can match any RG functions in $\Delta_\sigma^W + \Delta_{\sigma\alpha}^F$, all of which provide valid descriptions of the flow
   - The ambiguity in the Weyl symmetry is of the same form as the renormalization ambiguity in the RG functions

Ambiguity in RG the functions

$\mathcal{W}_0[\gamma, g_0, \mathcal{J}_0, a_0] = \mathcal{W}_0[\gamma, U g_0, U\mathcal{J}_0, a_0^U]$ *but produce different RG functions!*

$$\Delta\gamma \equiv \gamma^U - \gamma = -\beta_I U \partial^I U^\dagger$$
$$\Delta\beta_I \equiv \beta_I^U - \beta_I = -(\Delta\gamma\, g)_I,$$
$$\Delta\upsilon \equiv \upsilon^U - \upsilon = -\Delta\gamma,$$

i) By choosing U, one can engineer any $\Delta\gamma = \alpha(g) \in \mathfrak{g}_F$
   - We can match any RG functions in $\Delta_\sigma^W + \Delta_{\sigma\alpha}^F$, all of which provide valid descriptions of the flow
   - The ambiguity in the Weyl symmetry is of the same form as the renormalization ambiguity in the RG functions

ii) The flavor-improved RG functions are invariant:

$$B_I = \beta_I - (\upsilon\, g)_I, \qquad \Gamma = \gamma + \upsilon$$

Ambiguity in RG the functions

$\mathcal{W}_0[\gamma, g_0, \mathcal{J}_0, a_0] = \mathcal{W}_0[\gamma, U g_0, U \mathcal{J}_0, a_0^U]$ *but produce different RG functions!*

$$\Delta\gamma \equiv \gamma^U - \gamma = -\beta_I U \partial^I U^\dagger$$
$$\Delta\beta_I \equiv \beta_I^U - \beta_I = -(\Delta\gamma\, g)_I,$$
$$\Delta\upsilon \equiv \upsilon^U - \upsilon = -\Delta\gamma,$$

i) By choosing U, one can engineer any $\Delta\gamma = \alpha(g) \in \mathfrak{g}_F$
- We can match any RG functions in $\Delta_\sigma^W + \Delta_{\sigma\alpha}^F$, all of which provide valid descriptions of the flow
- The ambiguity in the Weyl symmetry is of the same form as the renormalization ambiguity in the RG functions

ii) The flavor-improved RG functions are invariant:

$$B_I = \beta_I - (\upsilon\, g)_I, \qquad \Gamma = \gamma + \upsilon$$

iii) RG-finiteness is conserved $\qquad \beta_I^{(n)} = -\big(\gamma^{(n)}\, g\big)_I$
- (β_I, γ) that are not RG finite cannot be made so by a shift

The preferred set of RG functions

We can choose counterterms to realize the flavor-improved RG functions:

$$\upsilon^U = \upsilon - \Delta\gamma = 0 \implies \left(\beta_I^U, \gamma^U\right) = \left(B_I, \Gamma\right)$$

The preferred set of RG functions

We can choose counterterms to realize the flavor-improved RG functions:

$$\upsilon^U = \upsilon - \Delta\gamma = 0 \implies \big(\beta_I^U, \gamma^U\big) = \big(B_I, \Gamma\big)$$

One additional consideration is

$$[T^\mu{}_\mu] = B_I[\mathcal{O}^I] - \eta_a \partial^2 [\mathcal{O}_M^a] \implies B_I \text{ is finite}$$

The preferred set of RG functions

We can choose counterterms to realize the flavor-improved RG functions:

$$\upsilon^U = \upsilon - \Delta\gamma = 0 \implies \left(\beta_I^U, \gamma^U\right) = \left(B_I, \Gamma\right)$$

One additional consideration is

$$[T^\mu{}_\mu] = B_I[\mathcal{O}^I] - \eta_a \partial^2 [\mathcal{O}_M^a] \implies B_I \text{ is finite}$$

The preferred set of RG functions

We can choose counterterms to realize the flavor-improved RG functions:

$$\upsilon^U = \upsilon - \Delta\gamma = 0 \implies \left(\beta_I^U, \gamma^U\right) = \left(B_I, \Gamma\right)$$

One additional consideration is

$$[T^\mu{}_\mu] = B_I[\mathcal{O}^I] - \eta_a \partial^2 [\mathcal{O}_M^a] \implies B_I \text{ is finite}$$

Flavor-improved RG functions

The set $\left(B_I, \Gamma\right)$ is unambiguous and (presumably) finite

The preferred set of RG functions

We can choose counterterms to realize the flavor-improved RG functions:

$$\upsilon^U = \upsilon - \Delta\gamma = 0 \implies \left(\beta_I^U, \gamma^U\right) = \left(B_I, \Gamma\right)$$

One additional consideration is

$$[T^\mu{}_\mu] = B_I[\mathcal{O}^I] - \eta_a \partial^2 [\mathcal{O}_M^a] \implies B_I \text{ is finite}$$

Flavor-improved RG functions

The set $\left(B_I, \Gamma\right)$ is unambiguous and (presumably) finite

In principle we know only that at least one element in

$$\left\{ \left(B_I, \Gamma + \alpha\right) \,:\, \alpha(g) \in \mathfrak{g}_F,\ (\alpha\, g)_I = 0 \right\}$$

is finite $\implies$ *RG-finiteness.* *Is there more than one element?*

Summary

i) The occurrence of a certain class of ϵ poles in the RG functions is consistent with the Callan–Symanzik equation and not a sign of the theory or renormalization scheme breaking down.

ii) There is an ambiguity in choosing renormalization constants due to the flavor symmetry.

iii) Using the ambiguity, it is always possible to remove all the poles simultaneously from γ and β_I to recover finite RG functions.

iv) The flavor-improved RG functions (Γ, B_I) are unambiguous and (presumably) finite and therefore a preferred choice.

Summary

i) The occurrence of a certain class of ϵ poles in the RG functions is consistent with the Callan–Symanzik equation and not a sign of the theory or renormalization scheme breaking down.

ii) There is an ambiguity in choosing renormalization constants due to the flavor symmetry.

iii) Using the ambiguity, it is always possible to remove all the poles simultaneously from γ and β_I to recover finite RG functions.

iv) The flavor-improved RG functions (Γ, B_I) are unambiguous and (presumably) finite and therefore a preferred choice.

No need to panic if you encounter an RG pole!