



Local Unitarity

First applications at NNLO and N3LO

Zeno Capatti

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A talk for ACAT 2022

Bari, October 25th, 2022

Photo: posing in front of Monte Carlo's casino, 2021

“It was at that time that I suggested an obvious name for the statistical method—a suggestion not unrelated to the fact that Stan had an uncle who would borrow money from relatives because he “just had to go to Monte Carlo.” The name seems to have endured.”

Nicholas Metropolis, “The beginning of the Monte Carlo method”,
Los Alamos Science Special Issue 1987

Local Unitarity: framing the problem

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A cross-section admits a perturbative expansion when $\alpha < 1$

$$\sigma = \sum_{L=1}^{\infty} \alpha^L \sigma^{(L)}$$

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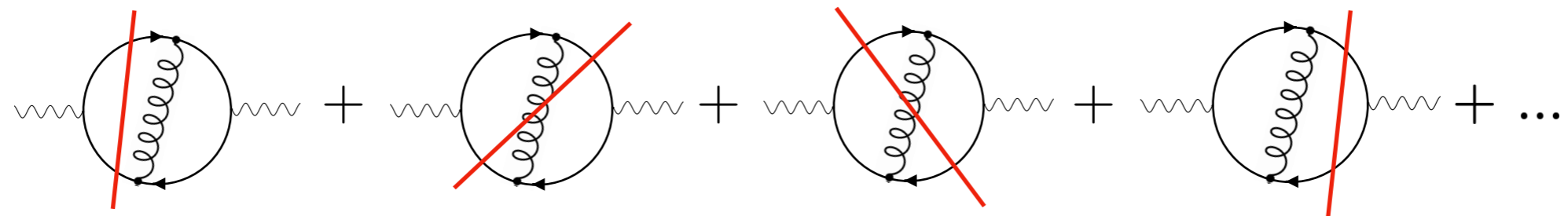
The coefficients can be represented as a sum of interference diagrams

Local Unitarity: framing the problem

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The coefficients can be represented as a sum of interference diagrams

$$\sigma^{(2)} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \dots$$


The diagram shows the second-order cross-section $\sigma^{(2)}$ as a sum of four interference diagrams. Each diagram consists of a circular loop with a wavy line entering from the left and a wavy line exiting to the right. Inside the loop, there are two fermion lines with arrows indicating their direction. A red diagonal line is drawn across each loop, representing an interference term. The first diagram has a red line from the bottom-left to the top-right. The second diagram has a red line from the top-left to the bottom-right. The third diagram has a red line from the top-left to the bottom-right. The fourth diagram has a red line from the bottom-left to the top-right. The diagrams are separated by plus signs, and the sequence ends with an ellipsis.

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The diagrams are interference diagrams for $\sigma^{(2)}$, each consisting of a circular loop with a wavy line entering from the left and a wavy line exiting to the right. The loop contains a fermion line with arrows indicating a clockwise direction. A vertical red line is drawn through each loop, representing a Cutkosky cut. The first diagram has the red line on the left side. The second and third diagrams have the red line on the right side. The fourth diagram has the red line on the left side. A red arrow points from the text "Cutkosky cut" to the red line in the fourth diagram.

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- Diagram 1: Red line is vertical, labeled "Cutkosky cut".
- Diagram 2: Red line is diagonal from top-left to bottom-right.
- Diagram 3: Red line is diagonal from top-right to bottom-left.
- Diagram 4: Red line is vertical, labeled "Cutkosky cut".

Interference diagrams themselves can be represented as integrals of amplitudes

Local Unitarity: framing the problem

A cross-section admits a perturbative expansion when $\alpha < 1$

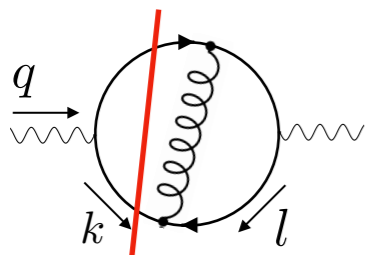
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Interference diagrams themselves can be represented as integrals of amplitudes

$$\text{diagram} = \int d^4k \delta^+(k^2) \delta^+((q-k)^2)$$

The diagram on the left is a circular loop with a wavy line entering from the left with momentum q and a wavy line exiting to the right. The loop contains a fermion line with arrows. A vertical red line is drawn through the loop. The momentum k is labeled on the left side of the loop, and the momentum l is labeled on the right side of the loop.

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Phase space integral

The diagram on the left shows a circular loop with a wavy line entering from the left with momentum q and a wavy line exiting to the right. A vertical red line is drawn through the loop. The momentum k is labeled on the bottom-left part of the loop, and the momentum l is labeled on the bottom-right part of the loop.

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Cutkosky cut

Interference diagrams themselves can be represented as integrals of amplitudes

$$\text{diagram} = \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

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Phase space integral

Loop integral

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Phase space integral

Loop integral

Problem: both types of integrals have **infrared** (collinear, soft) **divergences** and thresholds

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Phase space integral

Loop integral

Problem: both types of integrals have **infrared** (collinear, soft) **divergences** and thresholds

Many good methods around to deal with this that work **either** for loop or for phase-space integrals

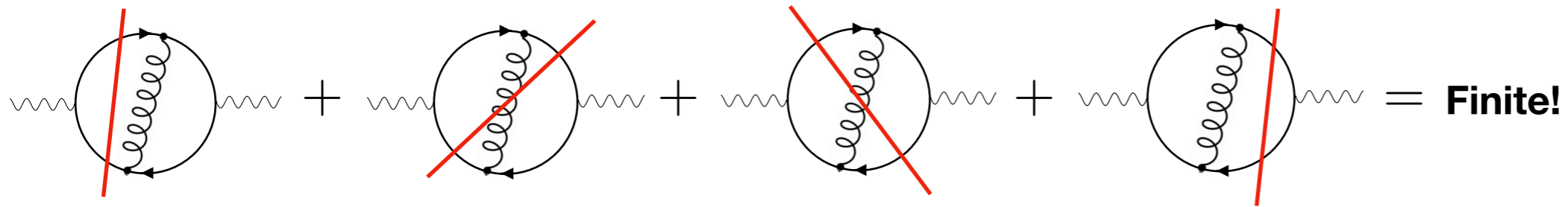
A local KLN cancellation mechanism

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Our approach instead **combines** singularities of loop and phase-space integrals at the local level through KLN

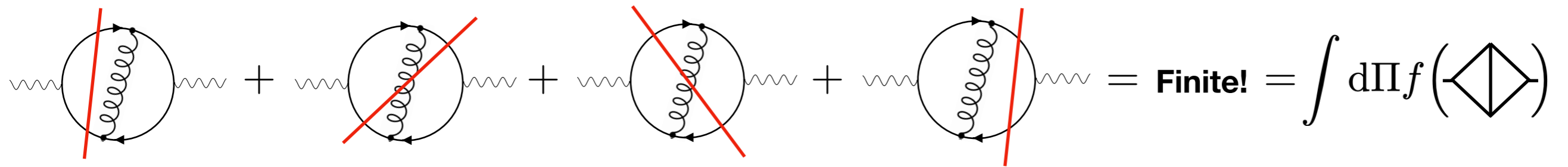
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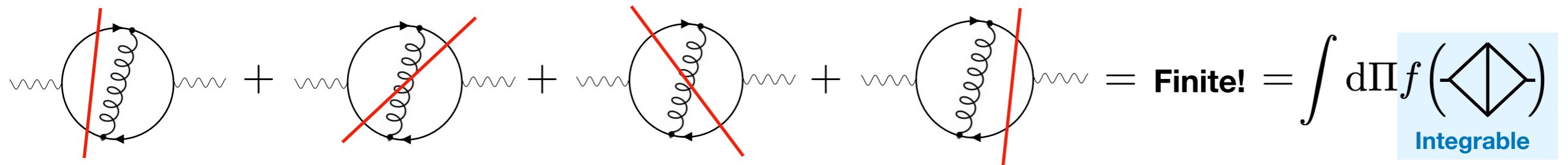


The diagram shows a sequence of four Feynman diagrams representing a sum of terms. Each diagram consists of a circular loop with a wavy line entering from the left and another wavy line exiting to the right. Inside the loop, there is a wavy line with an arrow pointing clockwise. A red diagonal line is drawn across each loop, representing a cut. The red line is vertical in the first and fourth diagrams, and diagonal in the second and third. The diagrams are separated by plus signs. To the right of the fourth diagram is an equals sign, followed by the word "Finite!" in bold, another equals sign, an integral symbol $\int d\Pi$, and a function f applied to a diamond-shaped diagram with a vertical line through its center.

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Diamond Diagram})$$

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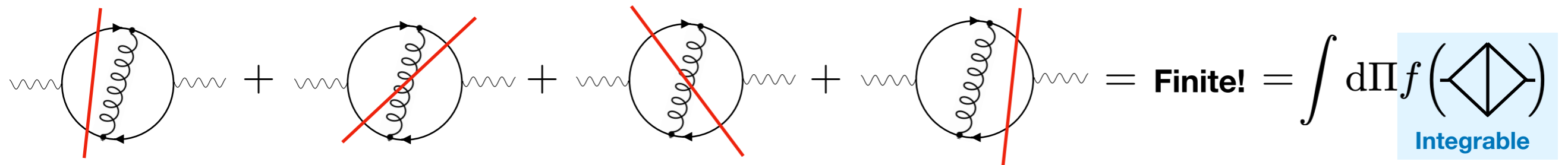


The diagram illustrates the KLN cancellation mechanism. It shows four Feynman diagrams in a row, each representing a different phase of a loop process. Each diagram consists of a circular loop with a wavy line entering from the left and another wavy line exiting to the right. Inside the loop, there is a wavy line with an arrow pointing clockwise. A red diagonal line is drawn across each loop, representing a cut in the propagator. The red line is vertical in the first and fourth diagrams, and diagonal in the second and third. The diagrams are separated by plus signs. To the right of the fourth diagram is an equals sign followed by the word "Finite!". This is followed by another equals sign and an integral over phase space, $\int d\Pi$, multiplied by a function f of a diamond-shaped diagram. The diamond diagram has a vertical line through its center and is enclosed in parentheses. Below the diamond diagram, the word "Integrable" is written in blue text on a light blue background.

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable})$$

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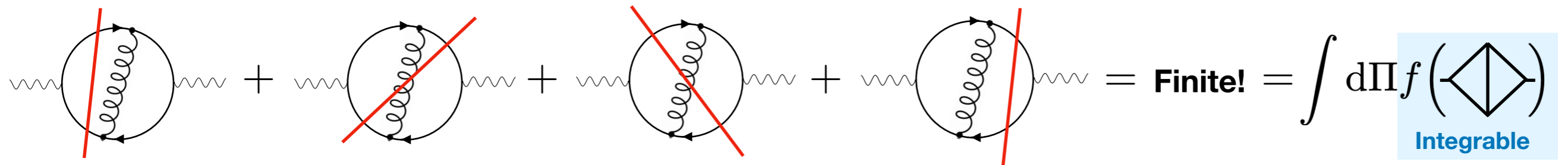
The diagram shows a sequence of four Feynman diagrams representing a loop integral with a wavy external line on the left and a wavy external line on the right. Each diagram is a circle with a wavy line inside. A red diagonal line is drawn across each circle, representing a singularity. The four diagrams are summed together, and the result is labeled "Finite!". This is followed by an equals sign and an integral over phase space, $\int d\Pi$, of a function f applied to a diamond-shaped diagram with a vertical line through it. The word "Integrable" is written in blue below the diamond diagram.

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable})$$

Rough idea:

A local KLN cancellation mechanism

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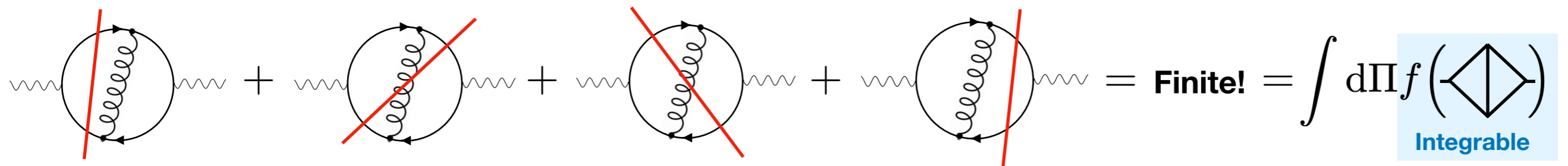

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable})$$

Rough idea:

$$\sum_{i=1}^4 \int d\Pi_i f_i$$

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$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable Diagram})$$

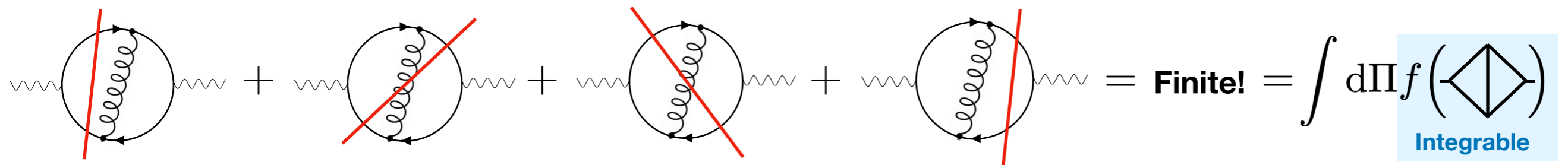
Rough idea:

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Different phase space measure

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The diagram illustrates the KLN cancellation mechanism. It shows four Feynman diagrams, each consisting of a circular loop with a wavy line entering from the left and exiting to the right. A red diagonal line is drawn through the loop in each diagram, representing a singularity. The diagrams are summed together, and the result is shown to be finite and equal to an integral over phase space of a diamond-shaped diagram. The diamond-shaped diagram is labeled as "Integrable".

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable})$$

Rough idea:

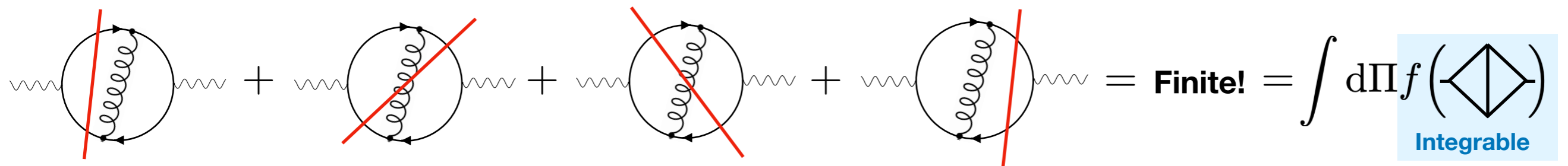
$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}}$$

**Non-integrable
singularities**

Different phase space measure

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The diagram illustrates the KLN cancellation mechanism. It shows four Feynman diagrams, each consisting of a loop with a wavy line and a red diagonal line through it. The diagrams are summed together, and the result is a finite integral over phase space of a diamond-shaped diagram. The diamond-shaped diagram is labeled "Integrable".

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable})$$

Rough idea:

$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}} = \int d\Pi \sum_{i=1}^4 g_i$$

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The diagram illustrates the KLN cancellation mechanism. It shows four Feynman diagrams, each consisting of a loop with a wavy line and a red diagonal line through it. The diagrams are summed together, and the result is labeled "Finite!". This is equated to an integral over phase space $\int d\Pi$ of a function f applied to a diamond-shaped diagram. The diamond diagram is labeled "Integrable".

Rough idea:

$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}} = \underbrace{\int d\Pi \sum_{i=1}^4 g_i}_{\text{No non-integrable singularities}}$$

Different phase space measure

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The diagram illustrates the KLN cancellation mechanism. It shows four Feynman diagrams, each consisting of a loop with a wavy line and a red diagonal line through it. These diagrams are summed together, resulting in a finite result, which is represented as an integral over phase space $\int d\Pi$ of a function f applied to a diamond-shaped diagram. The diamond-shaped diagram is labeled as "Integrable".

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Different phase space measure

Problem: $d\Pi_i$ has to be aligned

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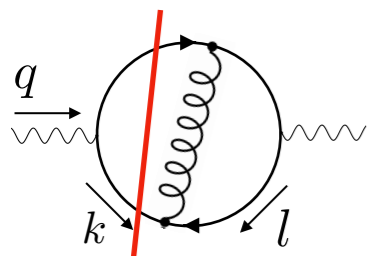
The diagram shows four Feynman diagrams, each consisting of a circle with a wavy line on the left and right, and a vertical red line passing through a loop inside. The diagrams are summed together, followed by an equals sign and the word "Finite!". This is followed by an integral over phase space $\int d\Pi$ of a function f applied to a diamond-shaped diagram. The diamond diagram is highlighted with a blue box and the word "Integrable" below it.

Rough idea:

$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}} = \underbrace{\int d\Pi \sum_{i=1}^4 g_i}_{\text{No non-integrable singularities}}$$

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Rough idea:

$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}} = \underbrace{\int d\Pi \sum_{i=1}^4 g_i}_{\text{No non-integrable singularities}}$$

Different phase space measure

Problem: $d\Pi_i$ has to be aligned

$$\text{Diagram} = \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2}$$

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Our approach instead **combines** singularities of loop and phase-space integrals at the local level through KLN

$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Diamond})$

Integrable

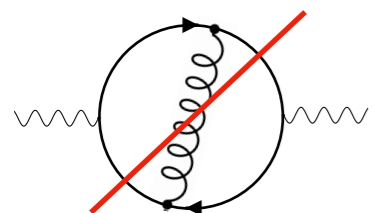
Rough idea:

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Different phase space measure

Problem: $d\Pi_i$ has to be aligned

$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2}$$



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Rough idea:

$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}} = \underbrace{\int d\Pi \sum_{i=1}^4 g_i}_{\text{No non-integrable singularities}} \quad \text{Different phase space measure}$$

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$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k - q)^2) \frac{N}{l^2(l + q)^2(k + l)^2}$$

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Problem 1:
Different number of deltas

$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k+l)^2) \delta^{(+)}((l+q)^2) \frac{N}{l^2(k-q)^2}$$

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Problem 2:
Too few energy variables to solve the deltas

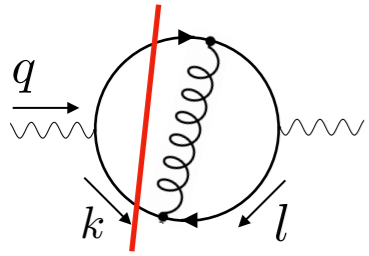
Problem 1: Different number of deltas

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Observation:

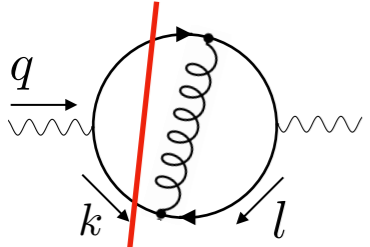
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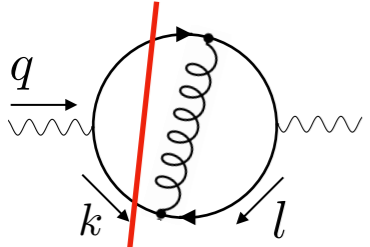


The diagram shows a triangle loop with a wavy line entering from the left with momentum q and another wavy line exiting to the right. The loop consists of two fermion lines (solid lines with arrows) and one boson line (wavy line). The bottom-left vertex has momentum k and the bottom-right vertex has momentum l . A vertical red line is drawn through the loop, representing a cut.

$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

Problem 1: Different number of deltas

Observation:



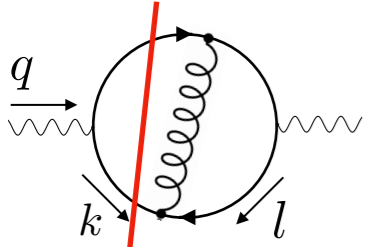
The diagram shows a triangle loop with a vertical red line through it. The left vertex has an incoming wavy line labeled q . The bottom-left vertex has an outgoing wavy line labeled k . The bottom-right vertex has an outgoing wavy line labeled l . The top vertex is connected to the bottom-left vertex by a straight line with an arrow pointing up. The right side of the loop is a wavy line. The bottom side is a straight line with an arrow pointing right.

$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Problem 1: Different number of deltas

Observation:



The diagram shows a triangle loop with a vertical cut. The left vertex has an incoming wavy line with momentum q . The bottom-left vertex has an outgoing wavy line with momentum k . The bottom-right vertex has an outgoing wavy line with momentum l . The top vertex is connected to the bottom-left vertex by a wavy line, and to the bottom-right vertex by a wavy line. A vertical red line is drawn through the loop, representing a cut.

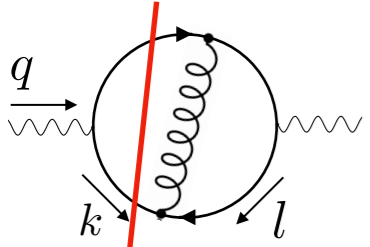
$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Solution 1: Perform integration over l^0

Problem 1: Different number of deltas

Observation:



The diagram shows a triangle loop with a vertical red line through it. The left vertex has an incoming wavy line with momentum q . The bottom-left vertex has an outgoing wavy line with momentum k . The bottom-right vertex has an outgoing wavy line with momentum l . The top vertex is connected to the bottom-left vertex by a straight line with momentum k , to the bottom-right vertex by a straight line with momentum l , and to itself by a wavy line. The red line is vertical and passes through the top vertex and the wavy line.

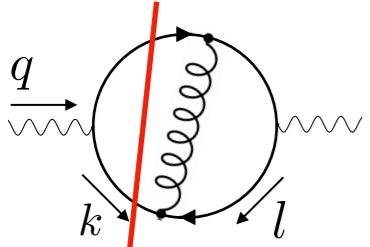
$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT...)

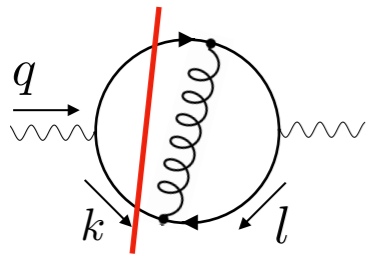
Problem 1: Different number of deltas

Observation:


$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

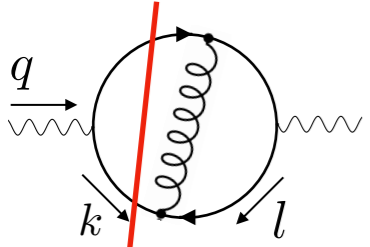
unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT...)


$$=$$

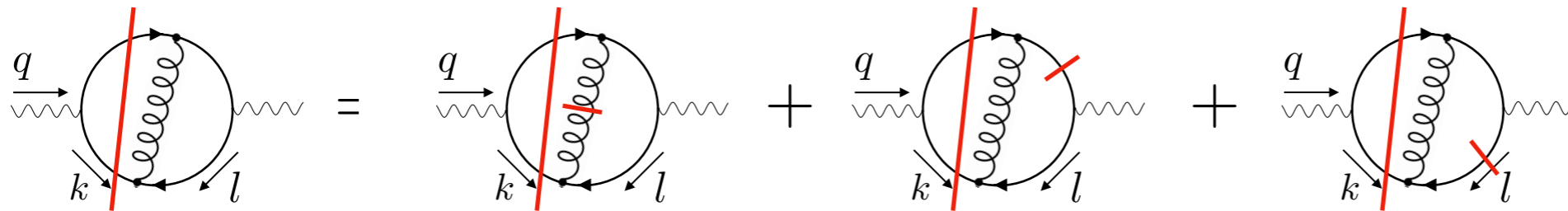
Problem 1: Different number of deltas

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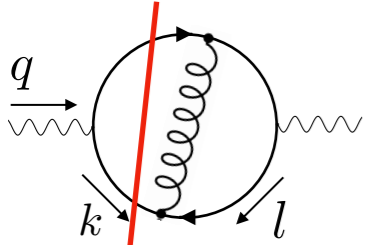
unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT...)


$$= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3}$$

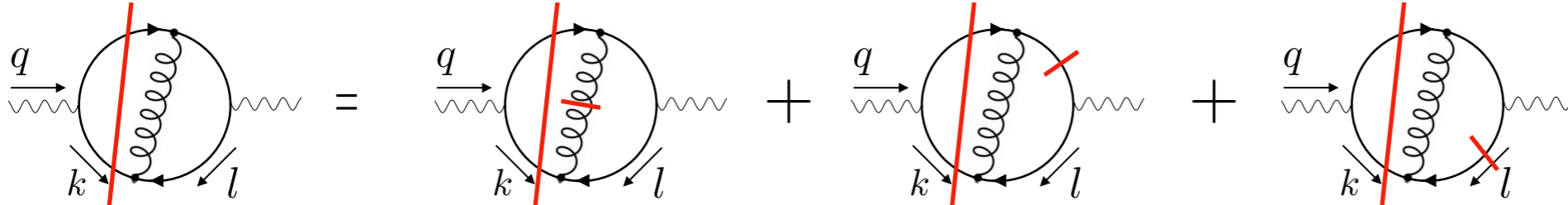
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unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT...)



Catani, Gleisberg, Krauss, Rodrigo, Winter
arXiv: [0804.3170](https://arxiv.org/abs/0804.3170) (2008)

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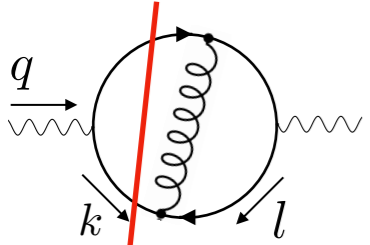
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Problem 1: Different number of deltas

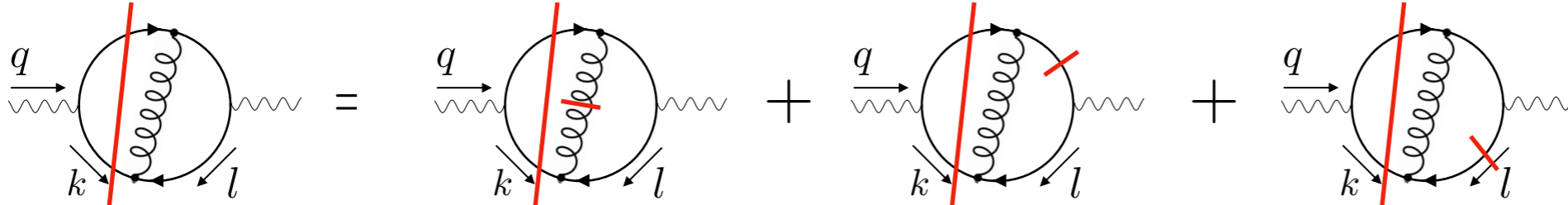
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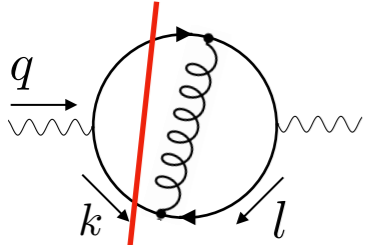
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Observation: Now real and virtual contributions have the same amount of deltas

Problem 1: Different number of deltas

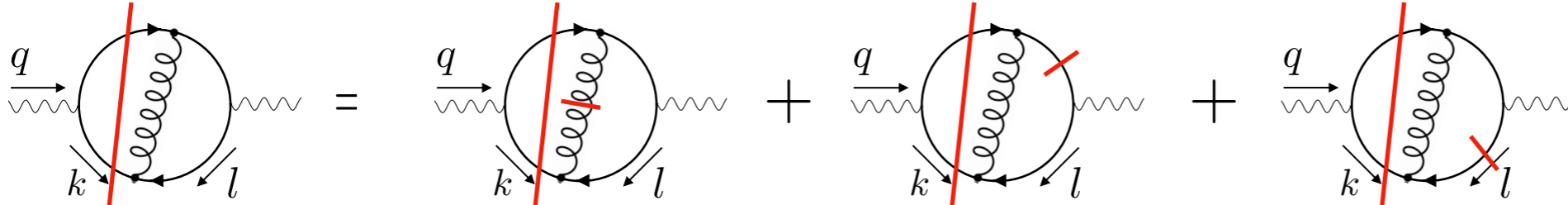
Observation:



$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT...)



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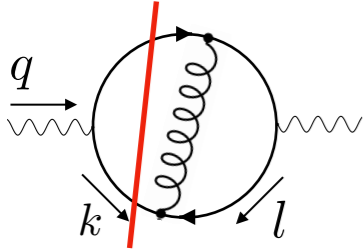
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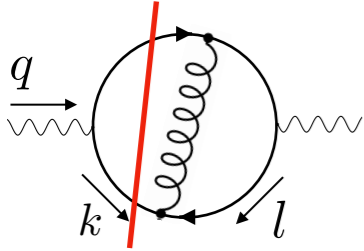


both have three cut lines!



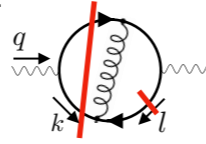
The diagram shows a circular loop with a wavy line entering from the left and another wavy line exiting to the right. The loop is divided by a vertical red line. The top arc of the loop is a fermion line with an arrow pointing clockwise. The bottom arc is a fermion line with an arrow pointing counter-clockwise. The left vertex is labeled with momentum k and the right vertex with momentum l . The external wavy line on the left is labeled with momentum q .

$$\begin{aligned}
 &= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2} \times \\
 &\quad \times \left[l^2 \delta^{(+)}(l^2) + (l+q)^2 \delta^{(+)}((l+q)^2) + (k+l)^2 \delta^{(+)}((k+l)^2) \right]
 \end{aligned}$$

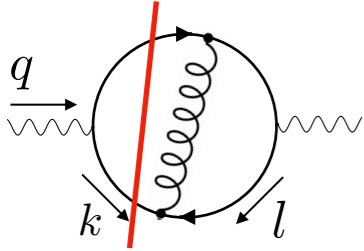


A Feynman diagram showing a triangle loop. An incoming wavy line with momentum q enters from the left. The loop consists of a fermion line (solid line with arrows) and a boson line (wavy line). A vertical red line is drawn through the loop. The momentum of the fermion line is labeled k and the momentum of the boson line is labeled l .

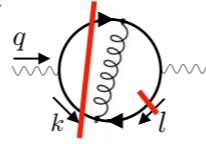
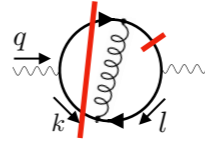
$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k - q)^2) \frac{N}{l^2(l + q)^2(k + l)^2} \times$$

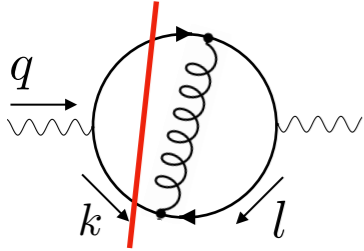
$$\times \left[l^2 \delta^{(+)}(l^2) + (l + q)^2 \delta^{(+)}((l + q)^2) + (k + l)^2 \delta^{(+)}((k + l)^2) \right]$$


A smaller version of the Feynman diagram from above, but with a red 'X' over the wavy boson line, indicating a crossed term.



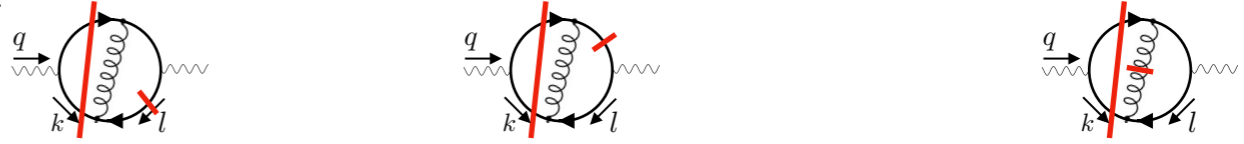
$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

$$\times \left[l^2 \delta^{(+)}(l^2) + (l+q)^2 \delta^{(+)}((l+q)^2) + (k+l)^2 \delta^{(+)}((k+l)^2) \right]$$





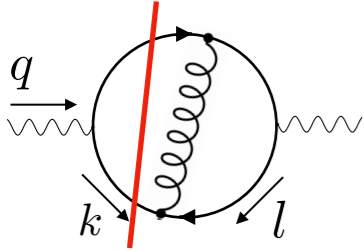
A Feynman diagram showing a triangle loop. The left and right external lines are wavy and labeled with momentum q . The bottom-left and bottom-right internal lines are straight and labeled with momentum k and l respectively. The top internal line is a wavy line. A vertical red line is drawn through the loop, intersecting the bottom-left and top internal lines.

$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

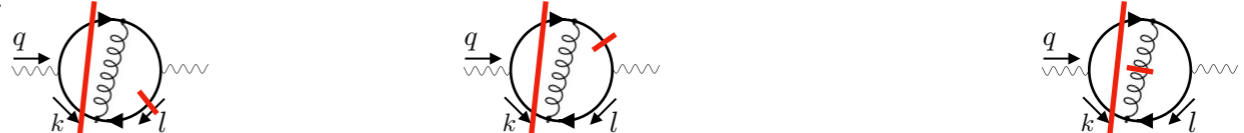
$$\times \left[l^2 \delta^{(+)}(l^2) + (l+q)^2 \delta^{(+)}((l+q)^2) + (k+l)^2 \delta^{(+)}((k+l)^2) \right]$$


Three Feynman diagrams representing the terms in the square brackets. Each diagram is a triangle loop with a vertical red line. The first diagram has a red 'X' on the bottom-right internal line. The second diagram has a red 'X' on the top internal line. The third diagram has a red 'X' on the wavy top internal line.

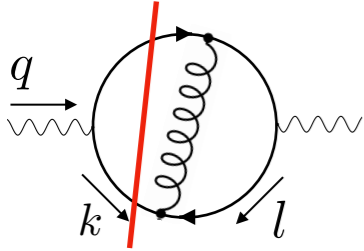
Problem 2: Too few energy variables to solve the deltas



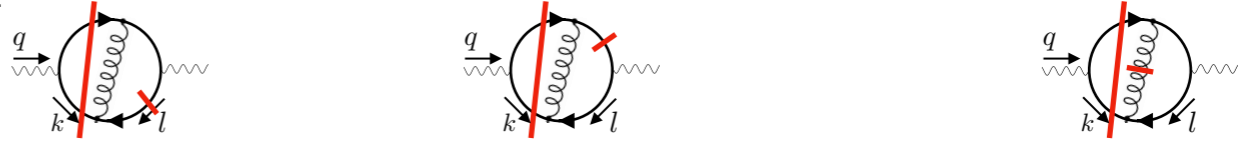
$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

$$\times \left[l^2 \delta^{(+)}(l^2) + (l+q)^2 \delta^{(+)}((l+q)^2) + (k+l)^2 \delta^{(+)}((k+l)^2) \right]$$


Problem 2: Too few energy variables to solve the deltas

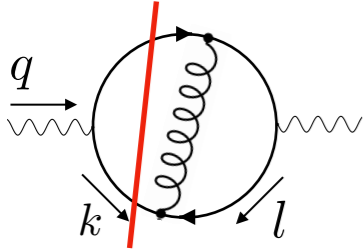


$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k - q)^2) \frac{N}{l^2(l + q)^2(k + l)^2} \times$$

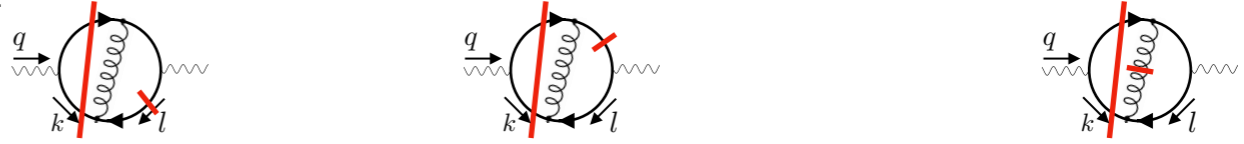
$$\times \left[l^2 \delta^{(+)}(l^2) + (l + q)^2 \delta^{(+)}((l + q)^2) + (k + l)^2 \delta^{(+)}((k + l)^2) \right]$$


Two deltas, one energy integration

Problem 2: Too few energy variables to solve the deltas

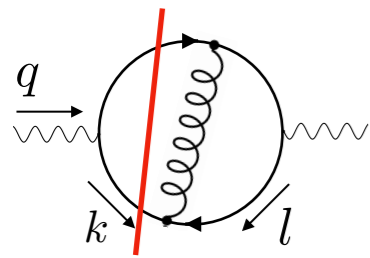


$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k - q)^2) \frac{N}{l^2(l + q)^2(k + l)^2} \times$$

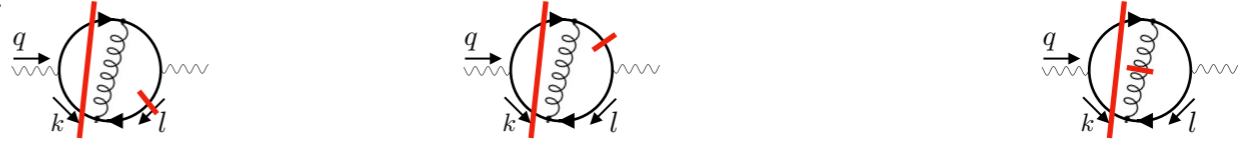
$$\times \left[l^2 \delta^{(+)}(l^2) + (l + q)^2 \delta^{(+)}((l + q)^2) + (k + l)^2 \delta^{(+)}((k + l)^2) \right]$$


Two deltas, one energy integration

Problem 2: Too few energy variables to solve the deltas



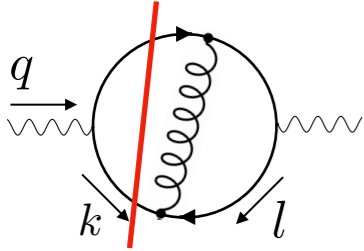
$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k - q)^2) \frac{N}{l^2(l + q)^2(k + l)^2} \times$$

$$\times \left[l^2 \delta^{(+)}(l^2) + (l + q)^2 \delta^{(+)}((l + q)^2) + (k + l)^2 \delta^{(+)}((k + l)^2) \right]$$


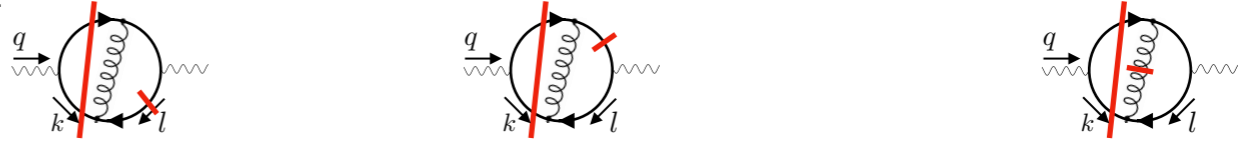
Two deltas, one energy integration

Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one

Problem 2: Too few energy variables to solve the deltas

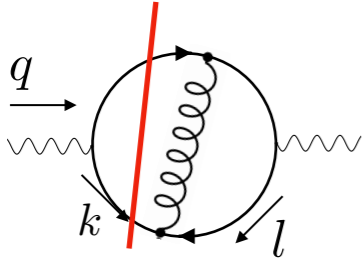


$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

$$\times \left[l^2 \delta^{(+)}(l^2) + (l+q)^2 \delta^{(+)}((l+q)^2) + (k+l)^2 \delta^{(+)}((k+l)^2) \right]$$


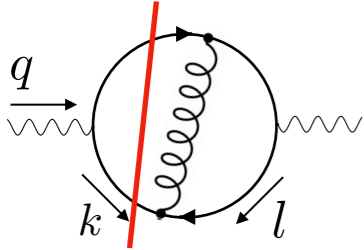
Two deltas, one energy integration

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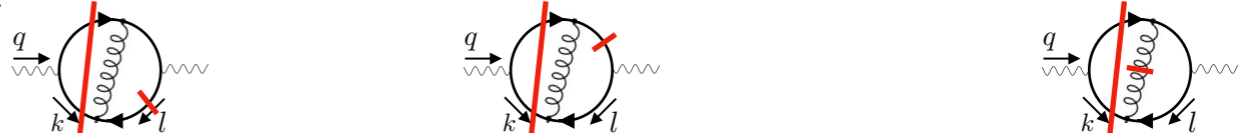


$$\approx d^2\Omega_k d^3\vec{l}$$

Problem 2: Too few energy variables to solve the deltas

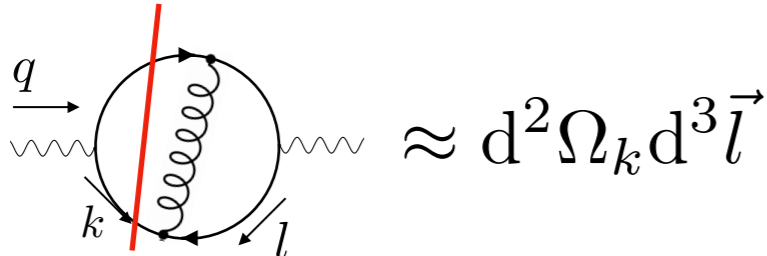


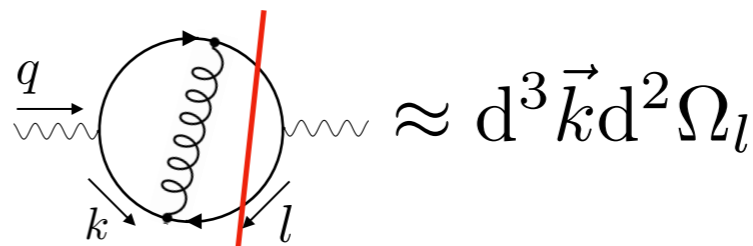
$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

$$\times \left[l^2 \delta^{(+)}(l^2) + (l+q)^2 \delta^{(+)}((l+q)^2) + (k+l)^2 \delta^{(+)}((k+l)^2) \right]$$


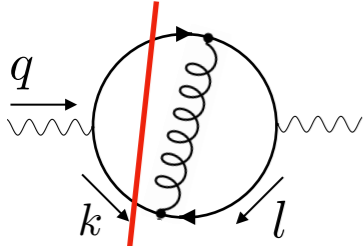
Two deltas, one energy integration

Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one

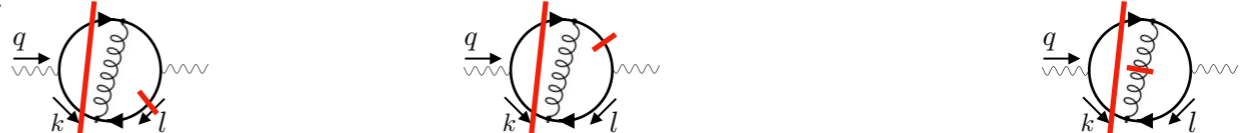


$$\approx d^2 \Omega_k d^3 \vec{l} \quad \Rightarrow \quad \approx d^3 \vec{k} d^2 \Omega_l$$


Problem 2: Too few energy variables to solve the deltas

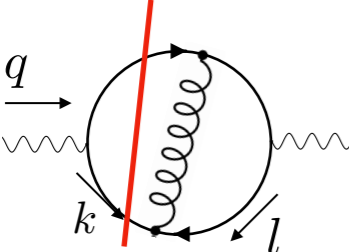


$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

$$\times \left[l^2 \delta^{(+)}(l^2) + (l+q)^2 \delta^{(+)}((l+q)^2) + (k+l)^2 \delta^{(+)}((k+l)^2) \right]$$


Two deltas, one energy integration

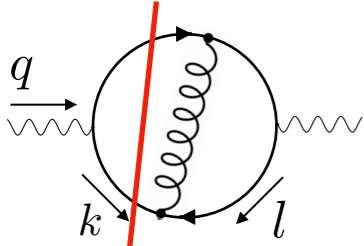
Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one



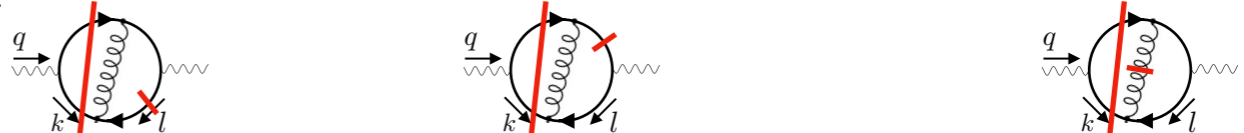
$$\approx d^2\Omega_k d^3\vec{l} \quad \Rightarrow \quad \text{Feynman diagram with delta on the l line} \approx d^3\vec{k} d^2\Omega_l$$

Integration measure is mis-aligned!

Problem 2: Too few energy variables to solve the deltas

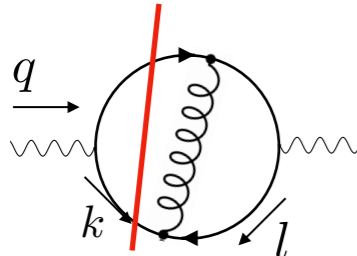


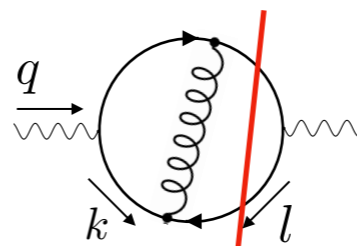
$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

$$\times \left[l^2 \delta^{(+)}(l^2) + (l+q)^2 \delta^{(+)}((l+q)^2) + (k+l)^2 \delta^{(+)}((k+l)^2) \right]$$


Two deltas, one energy integration

Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one

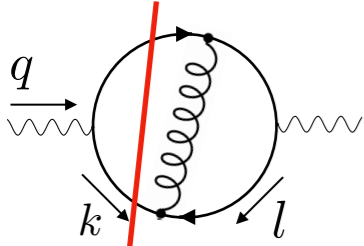


$$\approx d^2 \Omega_k d^3 \vec{l} \quad \Rightarrow \quad \approx d^3 \vec{k} d^2 \Omega_l$$


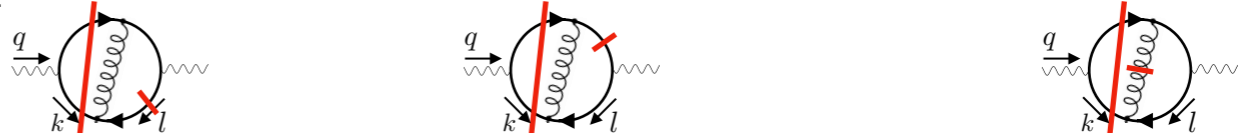
Integration measure is mis-aligned!

For the real contributions, problem known in phase-space subtraction methods

Problem 2: Too few energy variables to solve the deltas

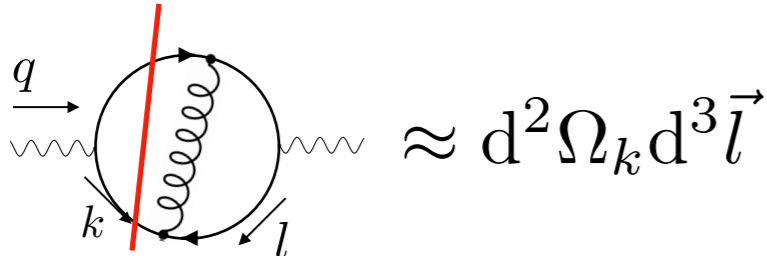


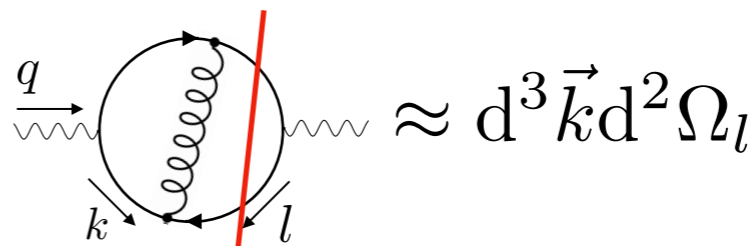
$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

$$\times \left[l^2 \delta^{(+)}(l^2) + (l+q)^2 \delta^{(+)}((l+q)^2) + (k+l)^2 \delta^{(+)}((k+l)^2) \right]$$


Two deltas, one energy integration

Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one

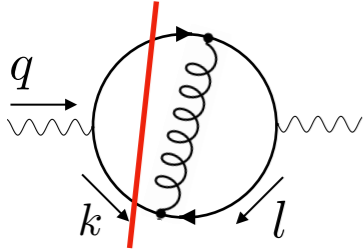


$$\approx d^2 \Omega_k d^3 \vec{l} \quad \Rightarrow \quad \approx d^3 \vec{k} d^2 \Omega_l$$


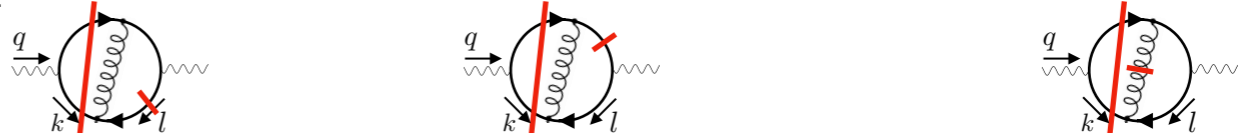
Integration measure is mis-aligned!

For the real contributions, problem known in phase-space subtraction methods (sectoring, mappings...)

Problem 2: Too few energy variables to solve the deltas

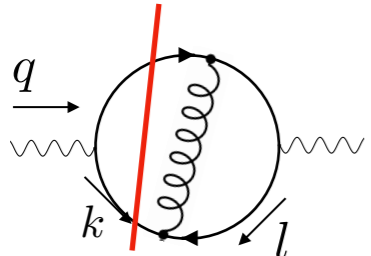


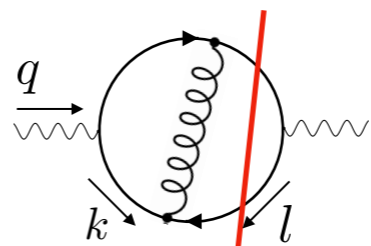
$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

$$\times \left[l^2 \delta^{(+)}(l^2) + (l+q)^2 \delta^{(+)}((l+q)^2) + (k+l)^2 \delta^{(+)}((k+l)^2) \right]$$


Two deltas, one energy integration

Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one



$$\approx d^2 \Omega_k d^3 \vec{l} \quad \Rightarrow \quad \approx d^3 \vec{k} d^2 \Omega_l$$


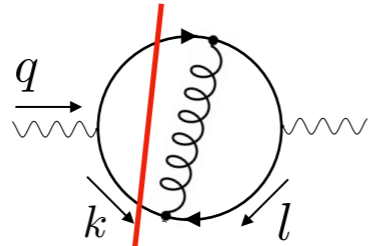
Integration measure is mis-aligned!

For the real contributions, problem known in phase-space subtraction methods

(sectoring, mappings...) **Big obstacle to automation**

After solving all possible deltas using energy integrations

After solving all possible deltas using energy integrations



The diagram shows a circular loop with a vertical red line through it. An incoming wavy line with momentum q enters from the left. An outgoing wavy line exits from the right. Two internal lines, labeled k and l , are shown at the bottom of the loop. A wavy line is also shown inside the loop, representing a self-energy correction.

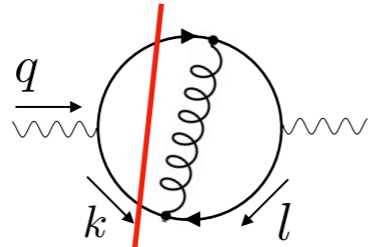
$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

After solving all possible deltas using energy integrations

$$\begin{array}{c} q \\ \text{wavy line} \end{array} \begin{array}{c} \text{circle} \\ \text{with wavy line} \\ \text{and arrows } k, l \\ \text{and a red vertical line} \end{array} = \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

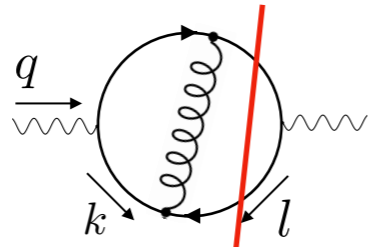
$$\begin{array}{c} q \\ \text{wavy line} \end{array} \begin{array}{c} \text{circle} \\ \text{with wavy line} \\ \text{and arrows } k, l \\ \text{and a red vertical line} \end{array} = \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

After solving all possible deltas using energy integrations



A Feynman diagram showing a circle with a wavy line on the left labeled q and a wavy line on the right. Inside the circle, a vertical red line is drawn. A wavy line labeled k goes from the bottom left to the top of the red line, and another wavy line labeled l goes from the top of the red line to the bottom right. A curly line connects the top and bottom of the red line.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$



A Feynman diagram similar to the first one, but the vertical red line is on the right side. The wavy line labeled k goes from the bottom left to the bottom of the red line, and the wavy line labeled l goes from the top of the red line to the bottom right. The curly line connects the top and bottom of the red line.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

After solving all possible deltas using energy integrations

$$\begin{array}{c}
 \begin{array}{c}
 \text{Diagram 1: A circular loop with a wavy line on the left and a wavy line on the right. A vertical red line is drawn through the loop. The left wavy line is labeled 'q', the bottom-left arrow is labeled 'k', and the bottom-right arrow is labeled 'l'. The loop contains a wavy line and a straight line with an arrow pointing clockwise.} \\
 \end{array}
 \end{array}
 = \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

$$\begin{array}{c}
 \text{Diagram 2: Similar to Diagram 1, but the vertical red line is shifted to the right side of the loop. The left wavy line is labeled 'q', the bottom-left arrow is labeled 'k', and the bottom-right arrow is labeled 'l'. The loop contains a wavy line and a straight line with an arrow pointing clockwise.} \\
 \end{array}
 = \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k})$$

After solving all possible deltas using energy integrations

$$\begin{array}{c}
 \begin{array}{c}
 \text{Diagram 1: A circular loop with a wavy line on the left labeled } q, \text{ and a wavy line on the right. A vertical red line is drawn through the loop. The left side of the loop is labeled } k, \text{ and the right side is labeled } l. \text{ The loop contains a wavy line with an arrow pointing clockwise.} \\
 \text{Diagram 2: A circular loop with a wavy line on the left labeled } q, \text{ and a wavy line on the right. A vertical red line is drawn through the loop. The left side of the loop is labeled } k, \text{ and the right side is labeled } l. \text{ The loop contains a wavy line with an arrow pointing counter-clockwise.}
 \end{array}
 \end{array}
 = \int d^3 \vec{k} d^3 \vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

$$\begin{array}{c}
 \text{Diagram 1} \\
 \text{Diagram 2}
 \end{array}
 = \int d^3 \vec{k} d^3 \vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3 \vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3 \vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k})$$

$$1 = \int dt h(t)$$

After solving all possible deltas using energy integrations

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3\vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k})$$

$$= \int d^3\vec{k} dt t^3 h(t) \delta(t|\vec{k}| - Q_0) f(t\vec{k})$$

$$1 = \int dt h(t)$$

$$\vec{k} \rightarrow t\vec{k}$$

After solving all possible deltas using energy integrations

A Feynman diagram showing a circle with a wavy line labeled q entering from the left and another wavy line exiting to the right. Inside the circle, a vertical red line is drawn. A wavy line labeled k enters from the bottom left, and a wavy line labeled l exits from the bottom right. A vertical wavy line connects the red line to the top of the circle.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

A Feynman diagram similar to the one above, but the vertical wavy line connects the red line to the bottom of the circle.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3\vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k})$$

$$1 = \int dt h(t)$$

$$= \int d^3\vec{k} dt t^3 h(t) \delta(t|\vec{k}| - Q_0) f(t\vec{k}) = \int d^3\vec{k} h\left(\frac{Q_0}{|\vec{k}|}\right) \frac{Q_0^3}{|\vec{k}|^4} f\left(\frac{Q_0}{|\vec{k}|}\vec{k}\right)$$

$$\vec{k} \rightarrow t\vec{k}$$

After solving all possible deltas using energy integrations

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3\vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k})$$

$$1 = \int dt h(t)$$

$$= \int d^3\vec{k} dt t^3 h(t) \delta(t|\vec{k}| - Q_0) f(t\vec{k}) = \int d^3\vec{k} h\left(\frac{Q_0}{|\vec{k}|}\right) \frac{Q_0^3}{|\vec{k}|^4} f\left(\frac{Q_0}{|\vec{k}|}\vec{k}\right)$$

$$\vec{k} \rightarrow t\vec{k}$$

Soper,
arXiv: [9804454](https://arxiv.org/abs/9804454) (1998)

Soper,
arXiv: [0102031](https://arxiv.org/abs/0102031) (2001 @ RADCOR)

ZC, Hirschi, Pelloni, Ruijl
arXiv: [2010.01068](https://arxiv.org/abs/2010.01068) (2020)

After solving all possible deltas using energy integrations

$$\begin{array}{c}
 \text{Diagram 1: A circle with a wavy line labeled } q \text{ on the left and a wavy line on the right. A vertical red line is drawn through the circle. Two arrows labeled } k \text{ and } l \text{ point towards the red line from the left and right respectively. A wavy line is inside the circle, parallel to the red line.} \\
 \int d^3 \vec{k} d^3 \vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}
 \end{array}$$

$$\begin{array}{c}
 \text{Diagram 2: Similar to Diagram 1, but the vertical red line is shifted to the right.} \\
 \int d^3 \vec{k} d^3 \vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}
 \end{array}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3 \vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3 \vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k}) \quad 1 = \int dt h(t)$$

$$= \int d^3 \vec{k} dt t^3 h(t) \delta(t|\vec{k}| - Q_0) f(t\vec{k}) = \int d^3 \vec{k} h\left(\frac{Q_0}{|\vec{k}|}\right) \frac{Q_0^3}{|\vec{k}|^4} f\left(\frac{Q_0}{|\vec{k}|} \vec{k}\right) \quad \vec{k} \rightarrow t\vec{k}$$

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arXiv: [2010.01068](https://arxiv.org/abs/2010.01068) (2020)

In the end:

$$\begin{array}{c}
 \text{Diagram 1} + \text{Diagram 2} = \int d^3 \vec{k} d^3 \vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) \right]
 \end{array}$$

After solving all possible deltas using energy integrations

$$\begin{array}{c}
 \text{Diagram 1: A circular loop with a wavy line labeled } q \text{ entering from the left. Two external wavy lines labeled } k \text{ and } l \text{ exit from the bottom. A vertical red line is drawn through the loop, intersecting the wavy line } q \text{ and the loop boundary.} \\
 \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}
 \end{array}$$

$$\begin{array}{c}
 \text{Diagram 2: Similar to Diagram 1, but the vertical red line is shifted to the right, intersecting the wavy line } q \text{ and the loop boundary at a different position.} \\
 \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}
 \end{array}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3\vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k}) \quad 1 = \int dt h(t)$$

$$= \int d^3\vec{k} dt t^3 h(t) \delta(t|\vec{k}| - Q_0) f(t\vec{k}) = \int d^3\vec{k} h\left(\frac{Q_0}{|\vec{k}|}\right) \frac{Q_0^3}{|\vec{k}|^4} f\left(\frac{Q_0}{|\vec{k}|}\vec{k}\right) \quad \vec{k} \rightarrow t\vec{k}$$

Soper,
arXiv: [9804454](https://arxiv.org/abs/9804454) (1998)

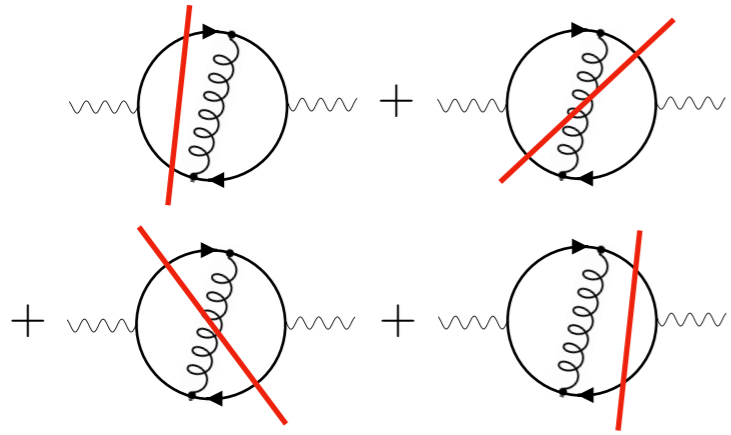
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ZC, Hirschi, Pelloni, Ruijl
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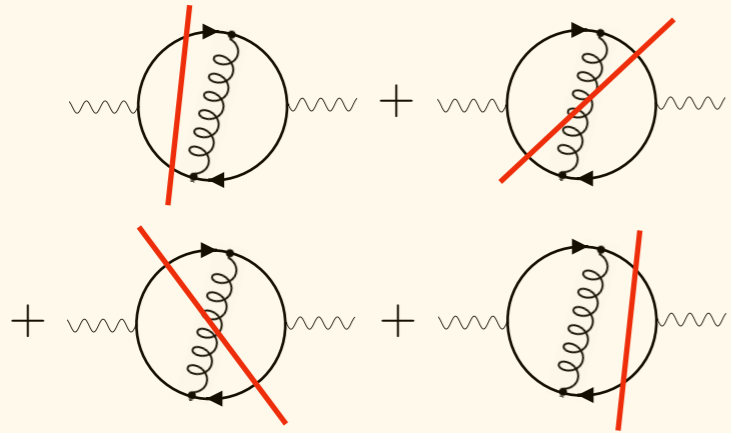
In the end:

$$\begin{array}{c}
 \text{Diagram 1} + \text{Diagram 2} = \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) \right]
 \end{array}$$

Observation: solved deltas, phase-space has same dimensionality (redundant dimension)

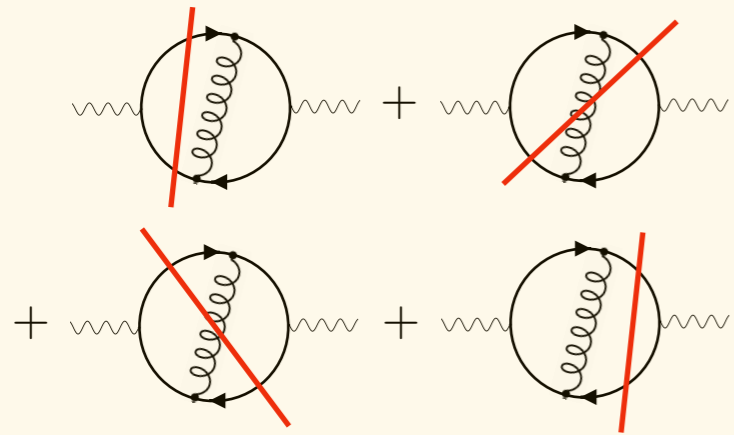


$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) + g_{r,1}(\vec{k}, \vec{l}) + g_{r,2}(\vec{k}, \vec{l}) \right]$$



$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) + g_{r,1}(\vec{k}, \vec{l}) + g_{r,2}(\vec{k}, \vec{l}) \right]$$

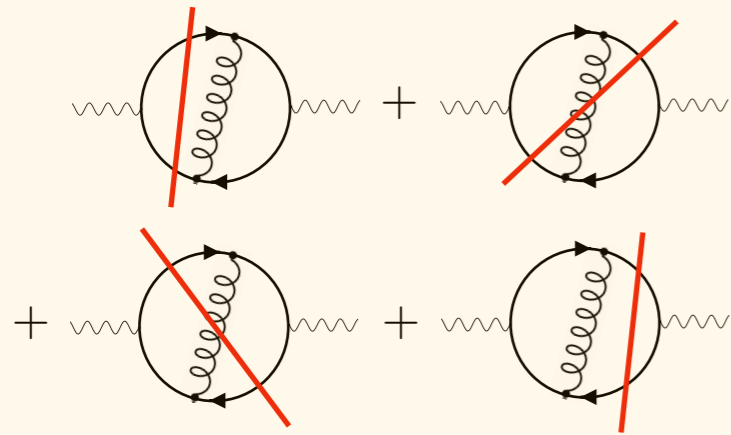
Measure is aligned now!



$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) + g_{r,1}(\vec{k}, \vec{l}) + g_{r,2}(\vec{k}, \vec{l}) \right]$$

Measure is aligned now!

A few comments:

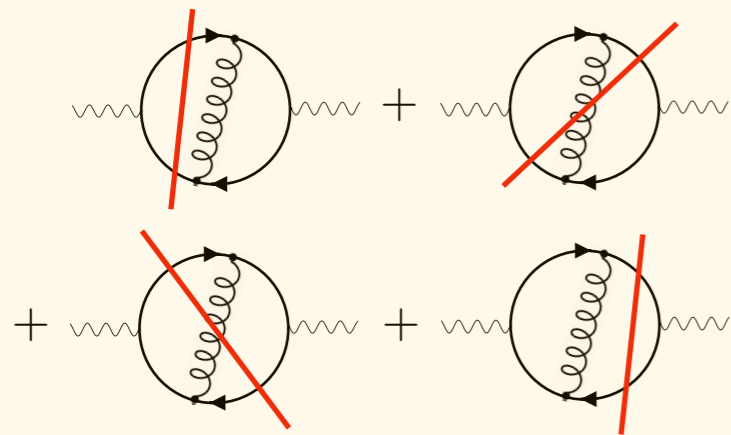


$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) + g_{r,1}(\vec{k}, \vec{l}) + g_{r,2}(\vec{k}, \vec{l}) \right]$$

Measure is aligned now!

A few comments:

- ◆ 3D representations hold at N loops

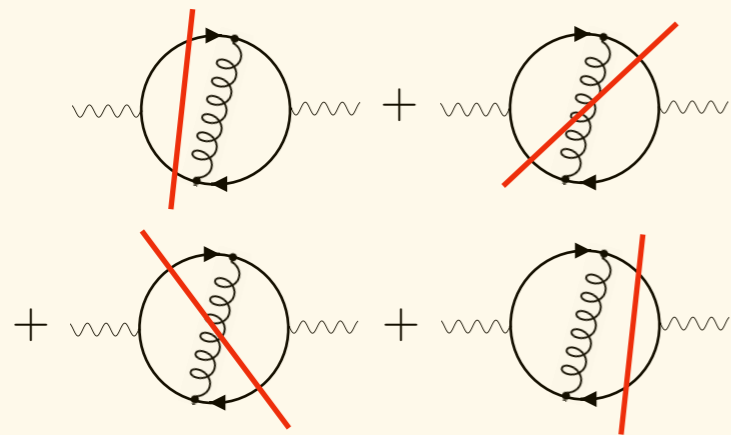


$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) + g_{r,1}(\vec{k}, \vec{l}) + g_{r,2}(\vec{k}, \vec{l}) \right]$$

Measure is aligned now!

A few comments:

- ◆ 3D representations hold at N loops
- ◆ The causal flow is generalised to generic kinematics



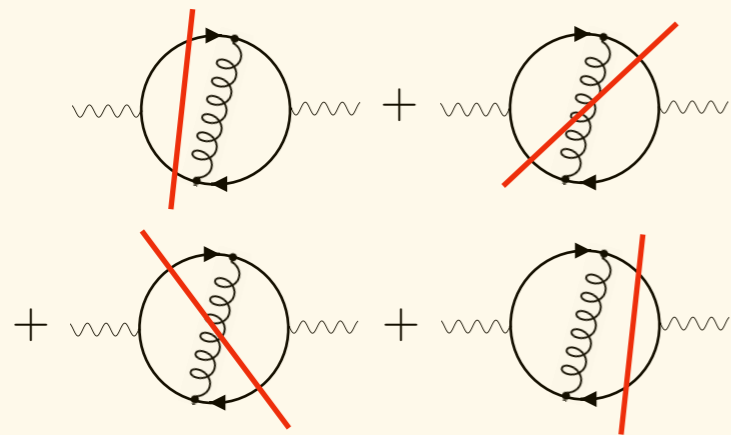
$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) + g_{r,1}(\vec{k}, \vec{l}) + g_{r,2}(\vec{k}, \vec{l}) \right]$$

Measure is aligned now!

A few comments:

- ◆ 3D representations hold at N loops
- ◆ The causal flow is generalised to generic kinematics

$$(\vec{k}, \vec{l}) \rightarrow (t\vec{k}, t\vec{l}) \Rightarrow (\vec{k}, \vec{l}) \rightarrow \vec{\phi}(t, (\vec{k}, \vec{l})) \quad \begin{cases} \partial_t \vec{\phi} = \vec{\kappa} \circ \vec{\phi} \\ \vec{\phi}(0, (\vec{k}, \vec{l})) = (\vec{k}, \vec{l}) \end{cases}$$



$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) + g_{r,1}(\vec{k}, \vec{l}) + g_{r,2}(\vec{k}, \vec{l}) \right]$$

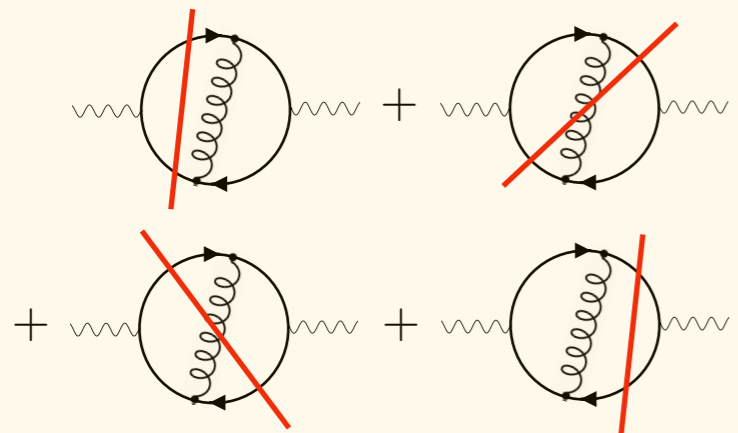
Measure is aligned now!

A few comments:

- ◆ 3D representations hold at N loops
- ◆ The causal flow is generalised to generic kinematics

$$(\vec{k}, \vec{l}) \rightarrow (t\vec{k}, t\vec{l}) \Rightarrow (\vec{k}, \vec{l}) \rightarrow \vec{\phi}(t, (\vec{k}, \vec{l})) \quad \begin{cases} \partial_t \vec{\phi} = \vec{\kappa} \circ \vec{\phi} \\ \vec{\phi}(0, (\vec{k}, \vec{l})) = (\vec{k}, \vec{l}) \end{cases}$$

- ◆ The integrand obtained by aligning the measure this way is Lebesgue-integrable



$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) + g_{r,1}(\vec{k}, \vec{l}) + g_{r,2}(\vec{k}, \vec{l}) \right]$$

Measure is aligned now!

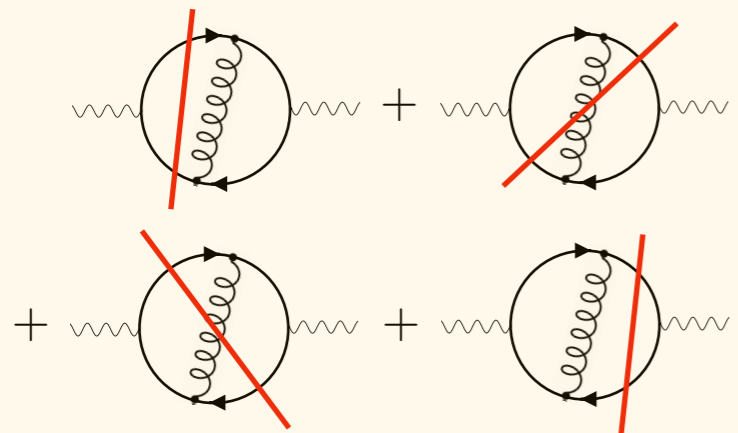
A few comments:

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$$(\vec{k}, \vec{l}) \rightarrow (t\vec{k}, t\vec{l}) \Rightarrow (\vec{k}, \vec{l}) \rightarrow \vec{\phi}(t, (\vec{k}, \vec{l})) \quad \begin{cases} \partial_t \vec{\phi} = \vec{\kappa} \circ \vec{\phi} \\ \vec{\phi}(0, (\vec{k}, \vec{l})) = (\vec{k}, \vec{l}) \end{cases}$$

- ◆ The integrand obtained by aligning the measure this way is Lebesgue-integrable

Proof \Rightarrow [ZC, Hirschi, Pelloni, Ruijl](#)
arXiv: [2010.01068](#) (2020)



$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) + g_{r,1}(\vec{k}, \vec{l}) + g_{r,2}(\vec{k}, \vec{l}) \right]$$

Measure is aligned now!

A few comments:

- ◆ 3D representations hold at N loops
- ◆ The causal flow is generalised to generic kinematics

$$(\vec{k}, \vec{l}) \rightarrow (t\vec{k}, t\vec{l}) \Rightarrow (\vec{k}, \vec{l}) \rightarrow \vec{\phi}(t, (\vec{k}, \vec{l})) \quad \begin{cases} \partial_t \vec{\phi} = \vec{\kappa} \circ \vec{\phi} \\ \vec{\phi}(0, (\vec{k}, \vec{l})) = (\vec{k}, \vec{l}) \end{cases}$$

- ◆ The integrand obtained by aligning the measure this way is Lebesgue-integrable

Proof \Rightarrow [ZC, Hirschi, Pelloni, Ruijl](#)
[arXiv: 2010.01068 \(2020\)](#)

- ◆ Deep relationship between interference diagrams and local residues of hyper surfaces

$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) + g_{r,1}(\vec{k}, \vec{l}) + g_{r,2}(\vec{k}, \vec{l}) \right]$$

Measure is aligned now!

A few comments:

- ◆ 3D representations hold at N loops
- ◆ The causal flow is generalised to generic kinematics

$$(\vec{k}, \vec{l}) \rightarrow (t\vec{k}, t\vec{l}) \Rightarrow (\vec{k}, \vec{l}) \rightarrow \vec{\phi}(t, (\vec{k}, \vec{l})) \quad \begin{cases} \partial_t \vec{\phi} = \vec{\kappa} \circ \vec{\phi} \\ \vec{\phi}(0, (\vec{k}, \vec{l})) = (\vec{k}, \vec{l}) \end{cases}$$

- ◆ The integrand obtained by aligning the measure this way is Lebesgue-integrable

Proof \Rightarrow [ZC, Hirschi, Pelloni, Ruijl](#)
[arXiv: 2010.01068 \(2020\)](#)

- ◆ Deep relationship between interference diagrams and local residues of hyper surfaces
 Cutkosky result but at the local level

Local IR cancellations, what next?



Local IR cancellations, what next?

- **Automated UV renormalisation:**



Local IR cancellations, what next?

- **Automated UV renormalisation:**
 - Local subtraction of UV divergences

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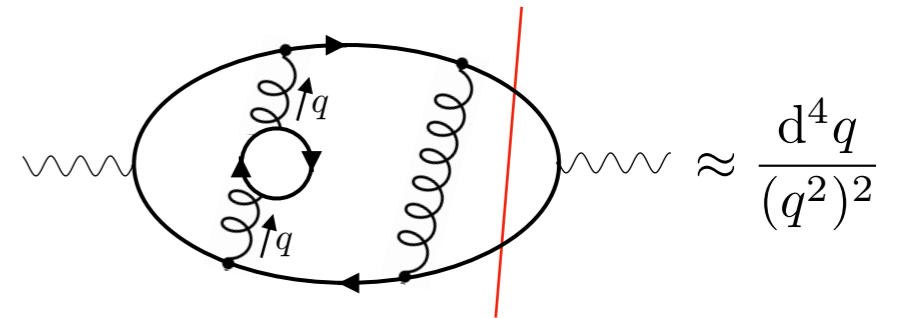
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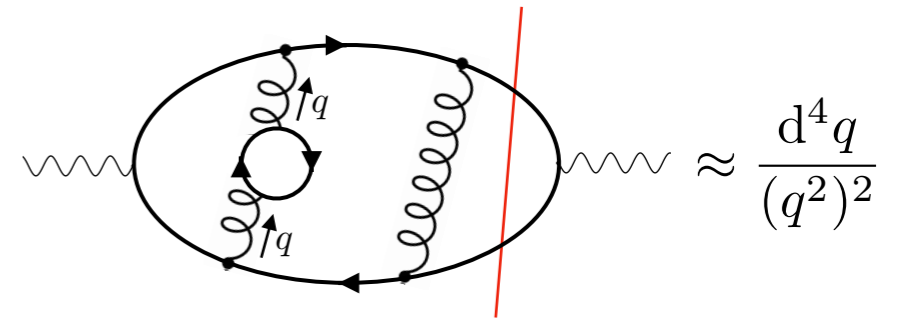
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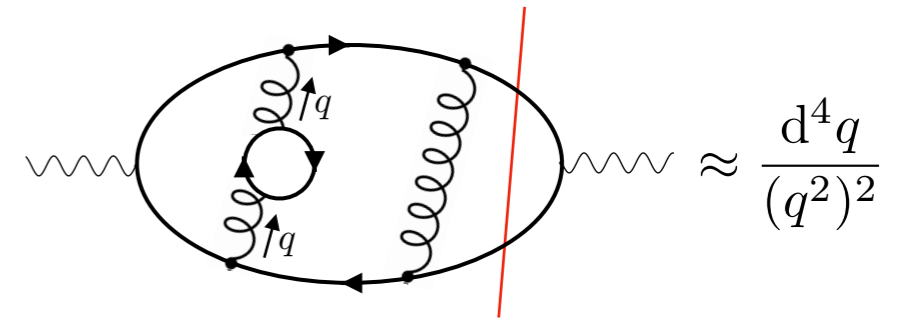
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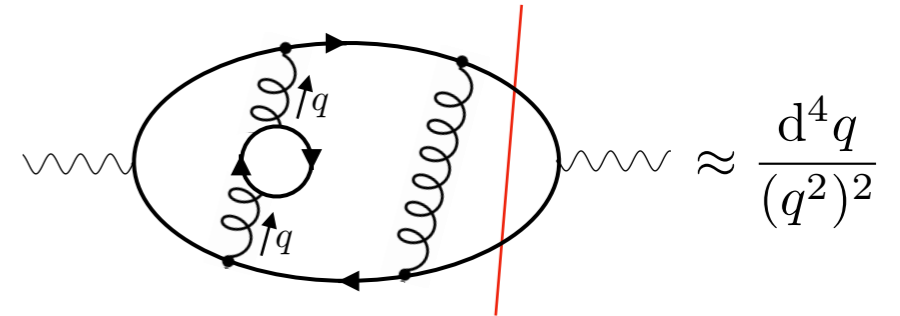
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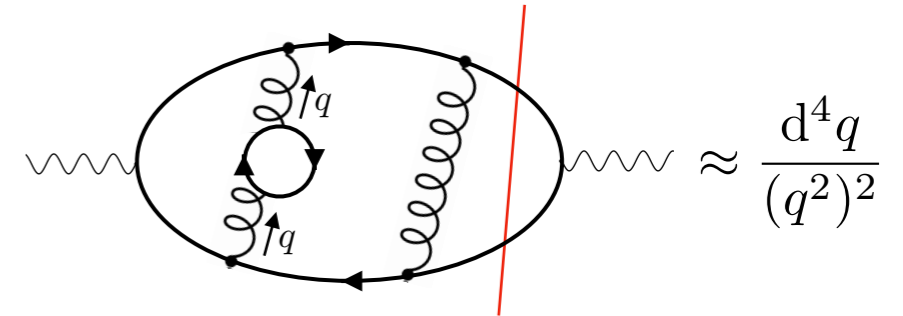
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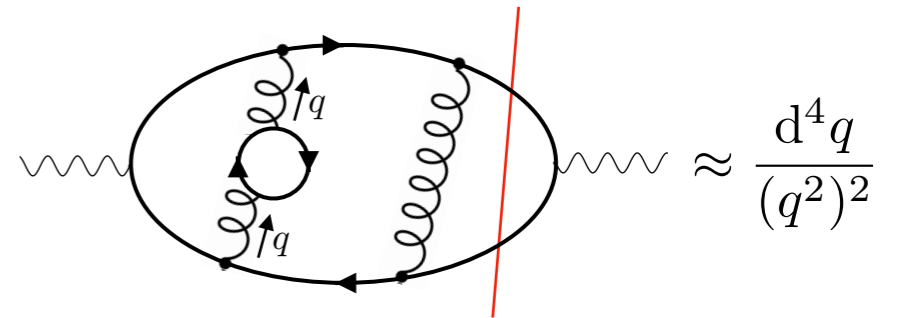
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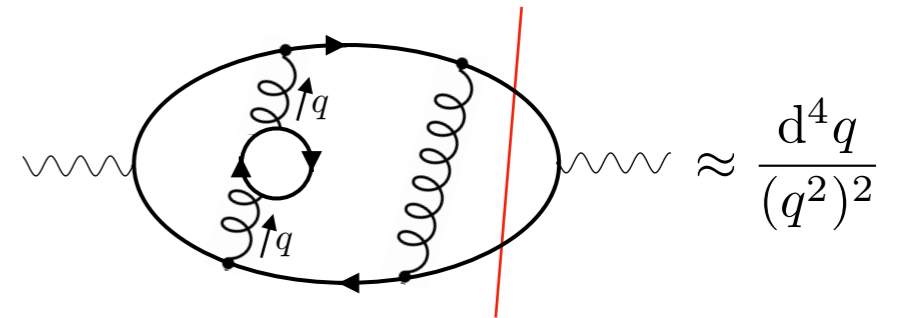
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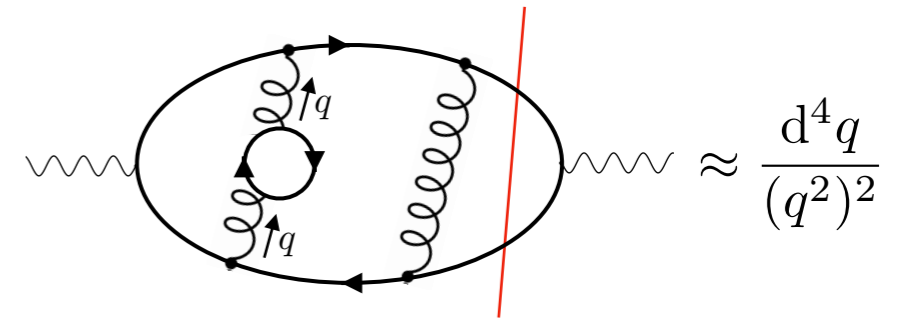
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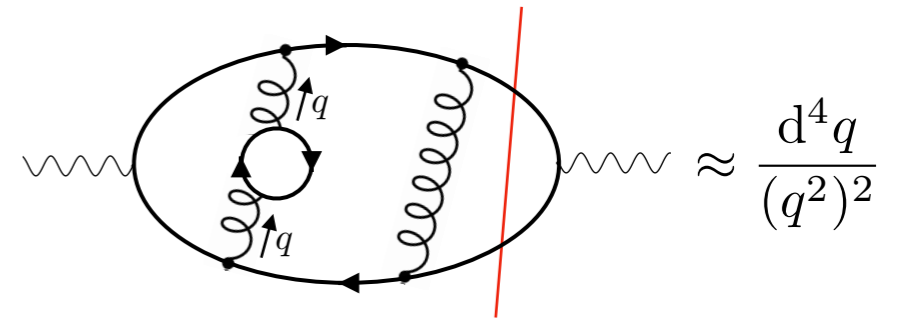
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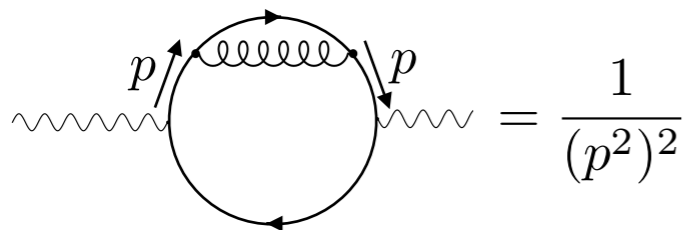
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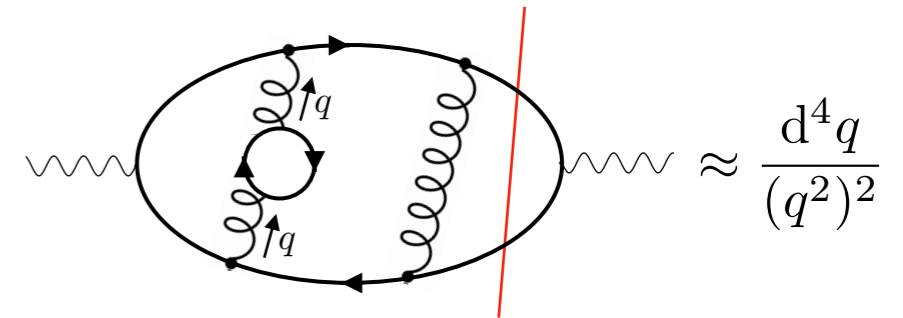
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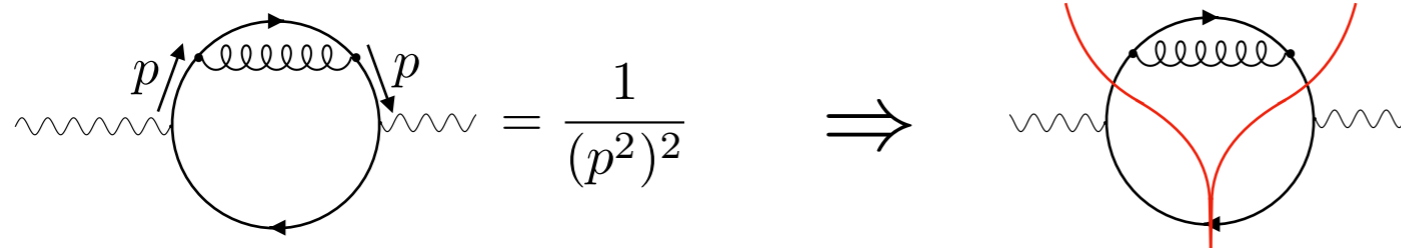
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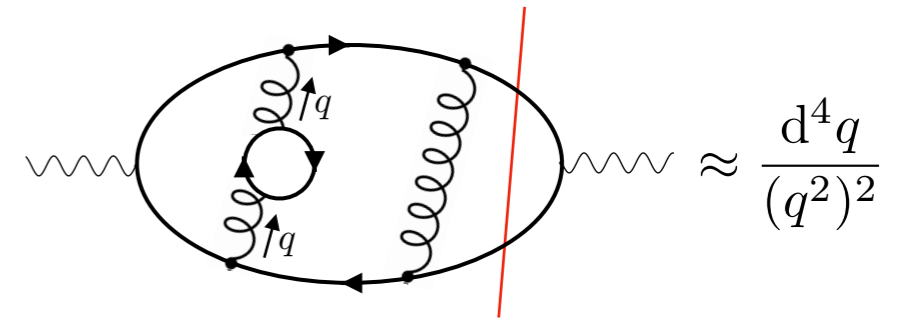
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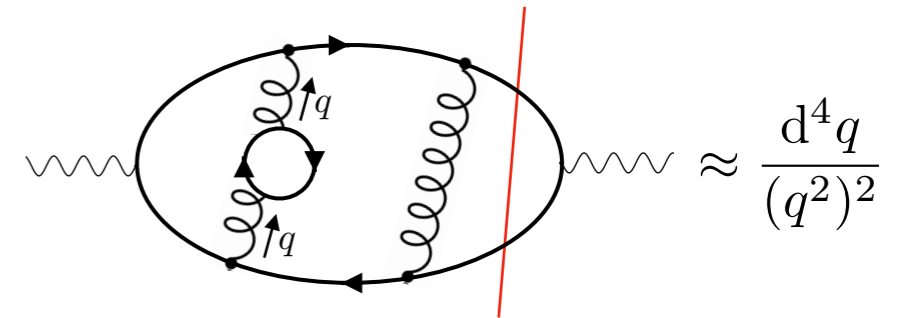
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$$\begin{aligned}
 & \text{Diagram} = \frac{1}{(p^2)^2} \quad \Rightarrow \quad \text{Diagram} = \frac{-2\pi i}{(2-1)!} \frac{1}{(2E_{\vec{p}})^2} \frac{d}{dp^0} \delta(p^0 - E_{\vec{p}})
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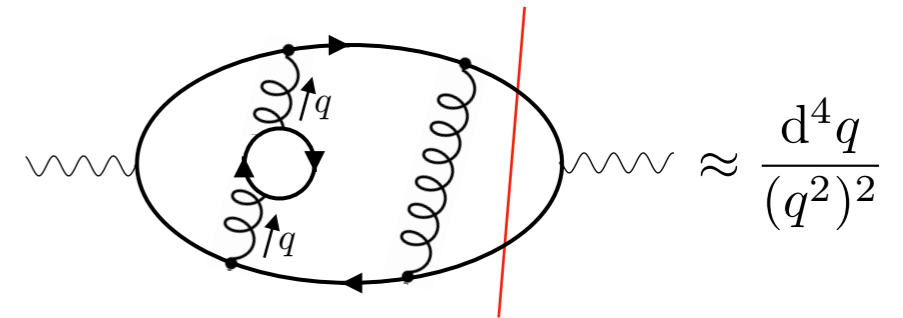
$$= \frac{1}{(p^2)^2} \quad \Rightarrow \quad = \frac{-2\pi i}{(2-1)!} \frac{1}{(2E_{\vec{p}})^2} \frac{d}{dp^0} \delta(p^0 - E_{\vec{p}})$$

→ Retain local IR cancellations: more complicated cancellation patterns

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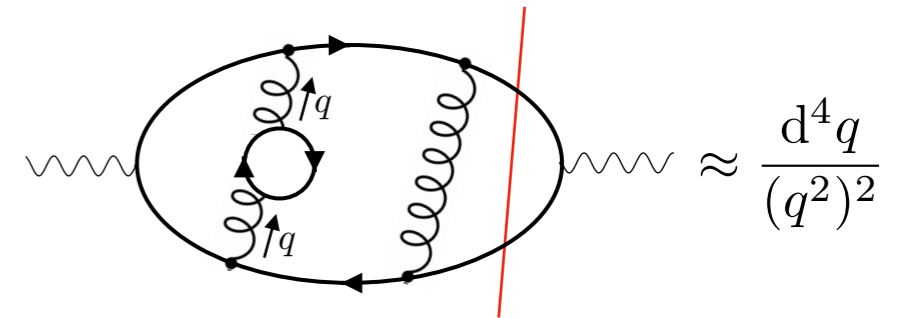
$$\begin{aligned}
 & \text{Bubble}(p) = \frac{1}{(p^2)^2} \quad \Rightarrow \quad \text{Bubble}(p) \text{ with raised propagator} = \frac{-2\pi i}{(2-1)!} \frac{1}{(2E_{\vec{p}})^2} \frac{d}{dp^0} \delta(p^0 - E_{\vec{p}})
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- Untangle mangling of scheme choice (OS vs MSbar) and IR structure

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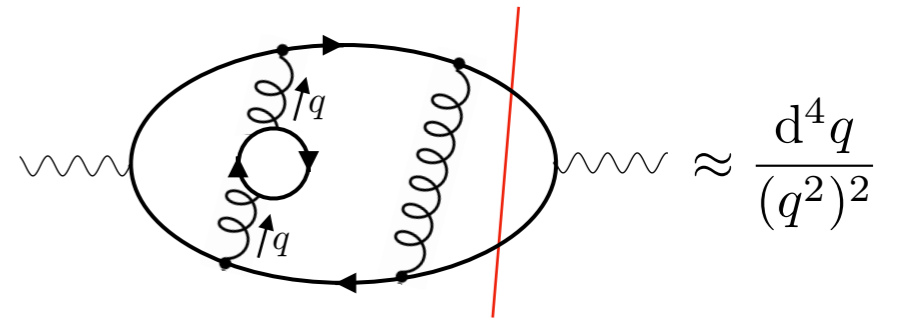
$$\frac{1}{(p^2)^2} \Rightarrow \frac{-2\pi i}{(2-1)!} \frac{1}{(2E_{\vec{p}})^2} \frac{d}{dp^0} \delta(p^0 - E_{\vec{p}})$$

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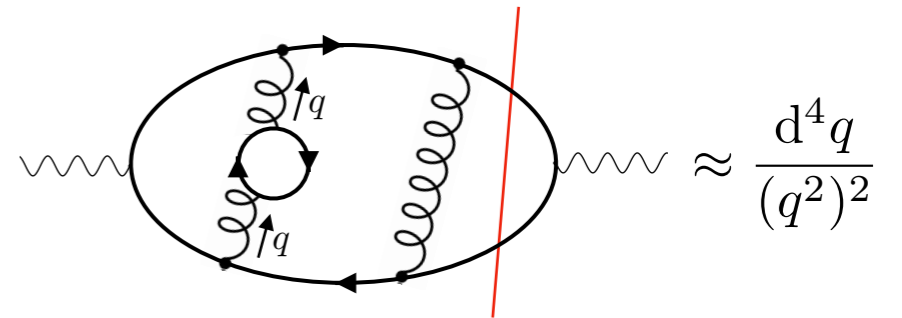
$$\begin{aligned}
 & \text{Diagram: a circle with a wavy line on the left and right, momentum p entering and leaving. The top part of the circle is a wavy line.} = \frac{1}{(p^2)^2} \quad \Rightarrow \quad \text{Diagram: a circle with a wavy line on the left and right, momentum p entering and leaving. The top part of the circle is a wavy line. Two red lines cut through the circle from the top and bottom.} = \frac{-2\pi i}{(2-1)!} \frac{1}{(2E_{\vec{p}})^2} \frac{d}{dp^0} \delta(p^0 - E_{\vec{p}})
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- Generalised cutting rules and LU representation
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Everything solved generically in [arXiv: 2203.11038](https://arxiv.org/abs/2203.11038)

Tests and results

Tests and results

NNLO $e^+e^- \rightarrow jj$

and $e^+e^- \rightarrow t\bar{t}$

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NNLO $e^+e^- \rightarrow jj$ Herzog, Ruijl, Ueda, Vermaseren, Vogt
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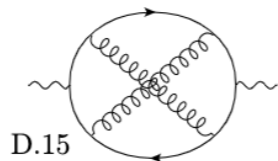
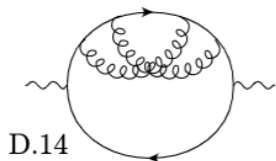
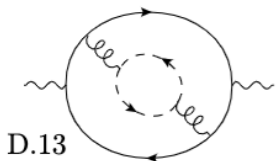
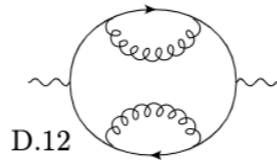
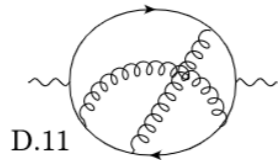
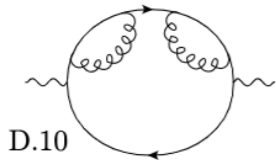
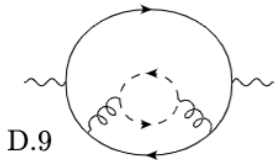
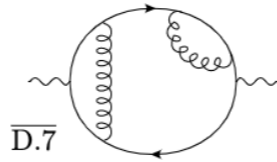
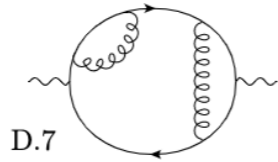
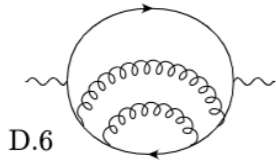
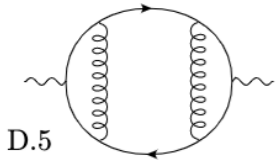
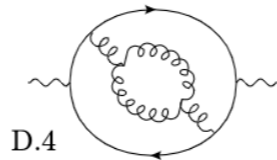
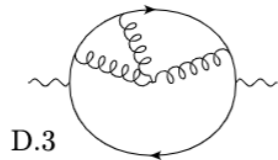
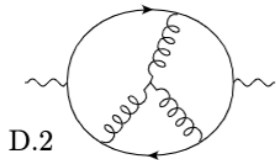
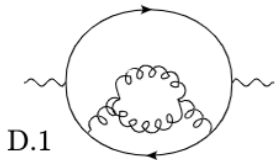
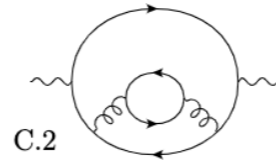
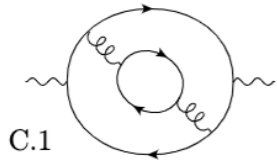
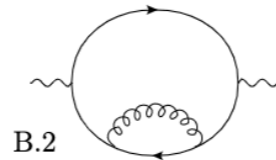
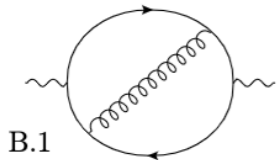
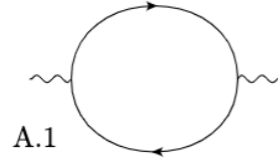
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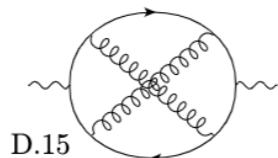
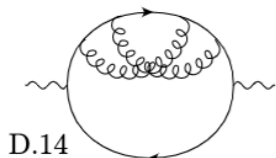
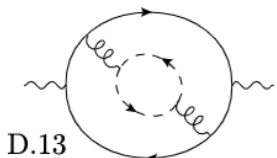
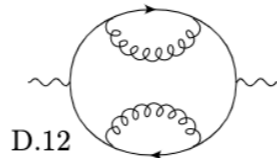
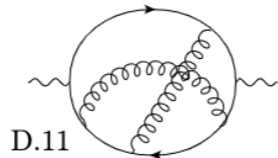
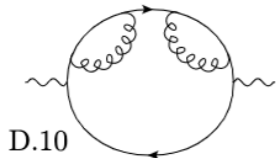
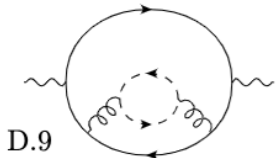
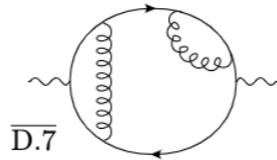
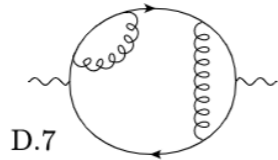
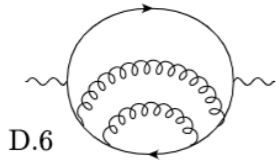
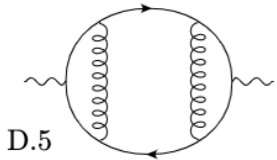
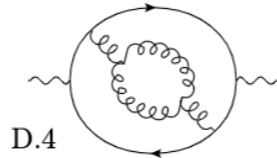
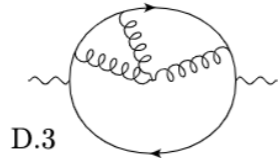
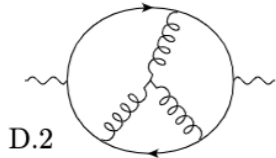
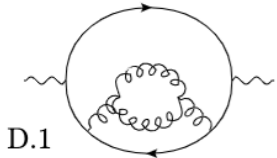
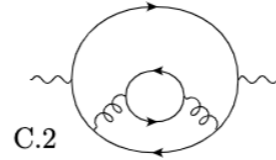
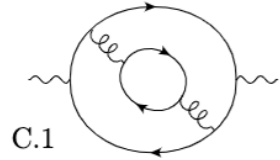
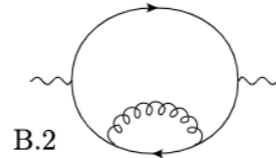
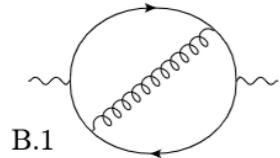
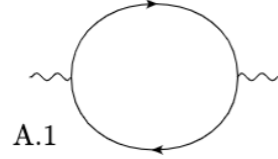
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SG id	Ξ	$\sigma_{\gamma^* \rightarrow t\bar{t}}^{[\alpha_s^{(\overline{\text{MS}})}, m_t^{(\text{OS})}]} [\text{GeV}^{-2}]$ $\mu_r^2 = m_t^2, p_{\gamma^*}^2 = (400 \text{ GeV})^2$	Δ [%]	$\sigma_{\gamma^* \rightarrow t\bar{t}}^{[\alpha_s^{(\overline{\text{MS}})}, m_t^{(\text{OS})}]} [\text{GeV}^{-2}]$ $\mu_r^2 = m_t^2, p_{\gamma^*}^2 = (3000 \text{ GeV})^2$	Δ [%]
LO $\mathcal{O}(\alpha_s^0)$					
A.1	1	$1.387586 \cdot 10^{+00}$	0.0011	$1.509262 \cdot 10^{+01}$	0.000064
Total		$1.387586 \cdot 10^{+00}$	0.0011	$1.509262 \cdot 10^{+01}$	0.000064
NLO $\mathcal{O}(\alpha_s)$					
B.1	1	$2.52705 \cdot 10^{-01}$	0.034	$-6.3725 \cdot 10^{-01}$	0.071
B.2	2	$1.80050 \cdot 10^{-01}$	0.049	$1.22702 \cdot 10^{+00}$	0.039
Total		$4.3276 \cdot 10^{-01}$	0.028	$5.8977 \cdot 10^{-01}$	0.11
Benchmark		$4.32831 \cdot 10^{-01}$	-0.018	$5.9047 \cdot 10^{-01}$	-0.12
NNLO $\mathcal{O}(\alpha_s^2)$ ($n_f = 1$ contribution)					
C.1	1	$-1.0022 \cdot 10^{-03}$	0.17	$2.6658 \cdot 10^{-02}$	0.059
C.2	2	$-4.6982 \cdot 10^{-03}$	0.081	$-8.388 \cdot 10^{-03}$	0.30
Total		$-5.7004 \cdot 10^{-03}$	0.073	$1.8270 \cdot 10^{-02}$	0.16
Benchmark		$-5.6982 \cdot 10^{-03}$	0.038	$1.8296 \cdot 10^{-02}$	-0.15
NNLO $\mathcal{O}(\alpha_s^2)$ (all other contributions)					
D.1	2	$3.8886 \cdot 10^{-02}$	0.031	$6.3163 \cdot 10^{-02}$	0.11
D.2	2	$5.6351 \cdot 10^{-03}$	0.14	$-3.52337 \cdot 10^{-01}$	0.027
D.3	2	$1.76075 \cdot 10^{-02}$	0.055	$5.6646 \cdot 10^{-02}$	0.14
D.4	1	$8.8163 \cdot 10^{-03}$	0.078	$-1.83770 \cdot 10^{-01}$	0.023
D.5	1	$9.200 \cdot 10^{-04}$	0.79	$-7.9531 \cdot 10^{-02}$	0.054
D.6	2	$5.1058 \cdot 10^{-03}$	0.15	$1.1244 \cdot 10^{-02}$	0.51
D.7	2	$6.7284 \cdot 10^{-03}$	0.10	$5.2105 \cdot 10^{-02}$	0.094
D.7	2	$6.7300 \cdot 10^{-03}$	0.10	$5.2171 \cdot 10^{-02}$	0.094
D.9	2	$2.3361 \cdot 10^{-03}$	0.12	$2.520 \cdot 10^{-03}$	0.73
D.10	2	$3.7418 \cdot 10^{-03}$	0.14	$3.4996 \cdot 10^{-02}$	0.11
D.11	2	$2.0845 \cdot 10^{-03}$	0.083	$2.5486 \cdot 10^{-02}$	0.060
D.12	1	$3.5114 \cdot 10^{-03}$	0.12	$2.8263 \cdot 10^{-02}$	0.10
D.13	1	$8.222 \cdot 10^{-04}$	0.19	$-7.994 \cdot 10^{-03}$	0.13
D.14	2	$1.76075 \cdot 10^{-02}$	0.055	$9.106 \cdot 10^{-03}$	0.19
D.15	1	$-7.242 \cdot 10^{-04}$	0.14	$-1.96633 \cdot 10^{-02}$	0.044
Total		$1.04214 \cdot 10^{-01}$	0.024	$-3.0760 \cdot 10^{-01}$	0.061
Benchmark		$1.0386 \cdot 10^{-01}$	0.34	$-3.0818 \cdot 10^{-01}$	-0.19

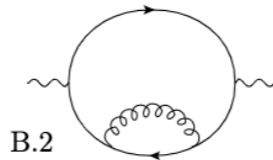
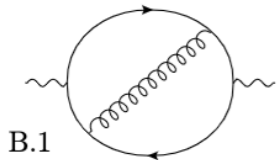
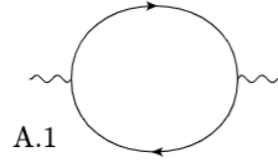
Tests and results

NNLO $e^+e^- \rightarrow jj$
and $e^+e^- \rightarrow t\bar{t}$

Herzog, Ruijl, Ueda, Vermaseren, Vogt
arXiv:1707.01044

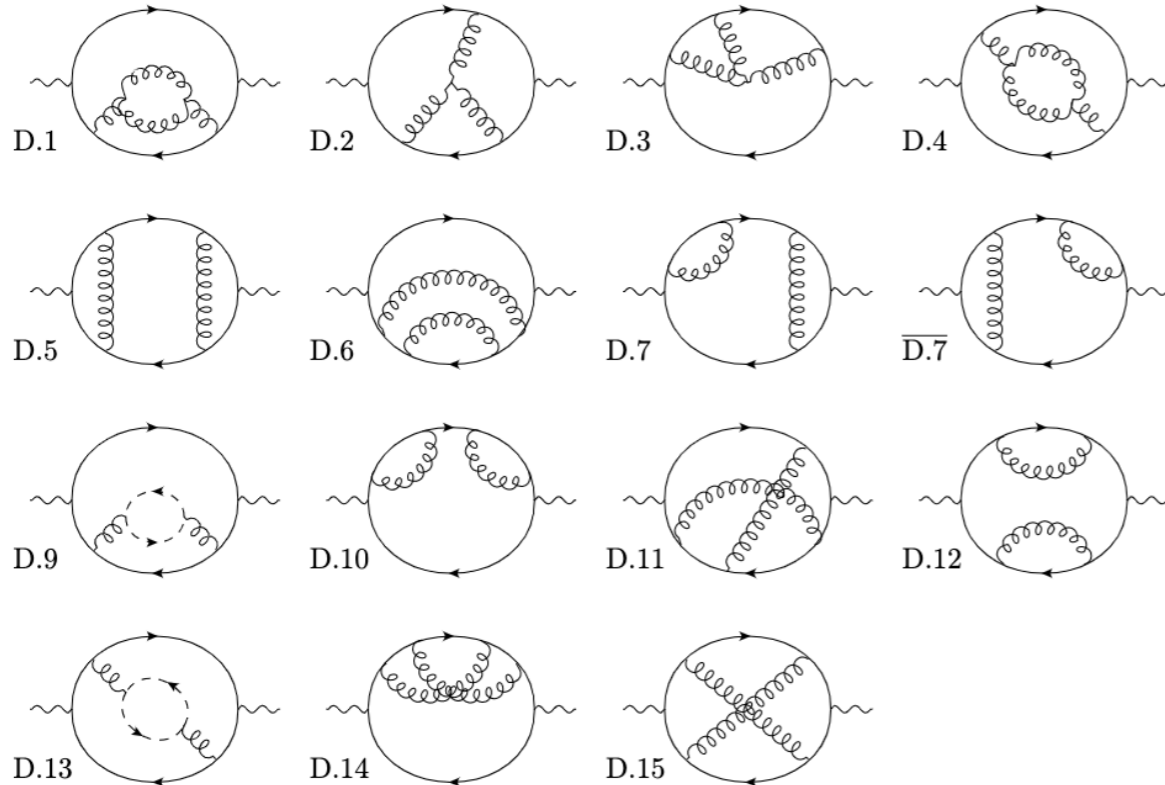
Chetyrkin, Kuehn, Steinhauser,
arXiv:9606230

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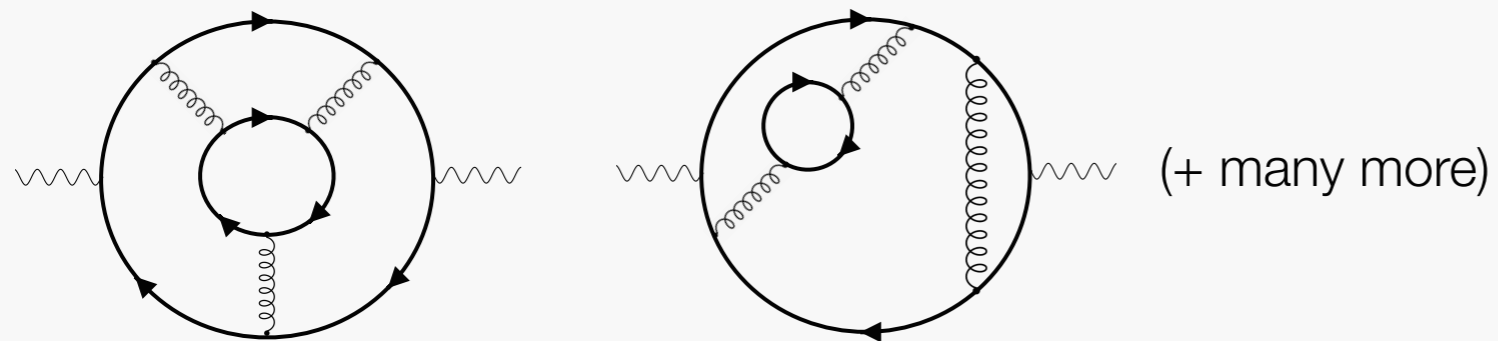
C.



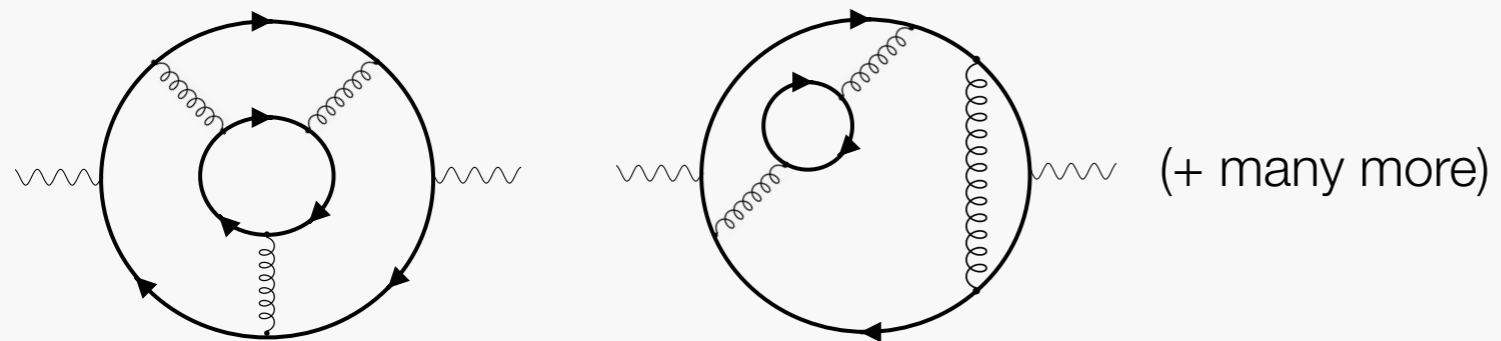
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N_f part @N³LO $e^+e^- \rightarrow jj$

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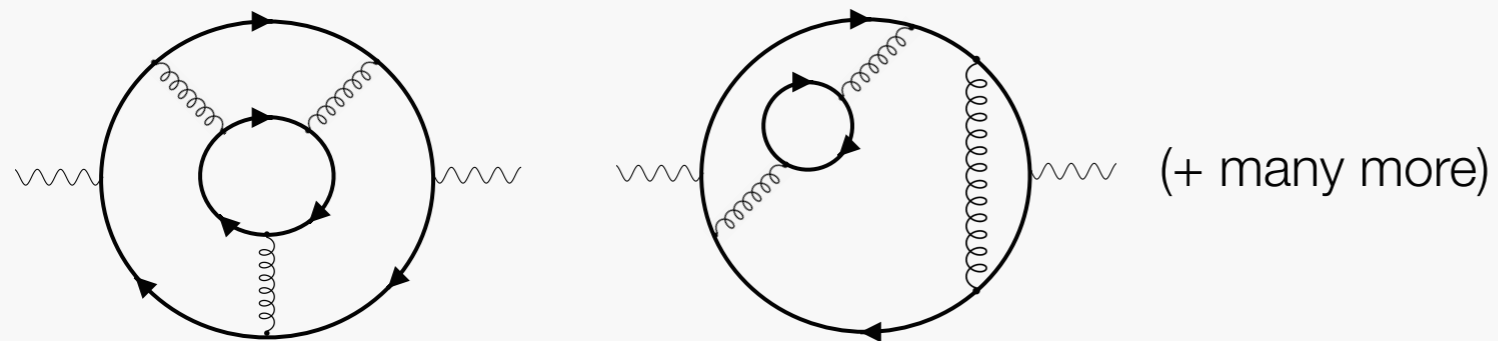


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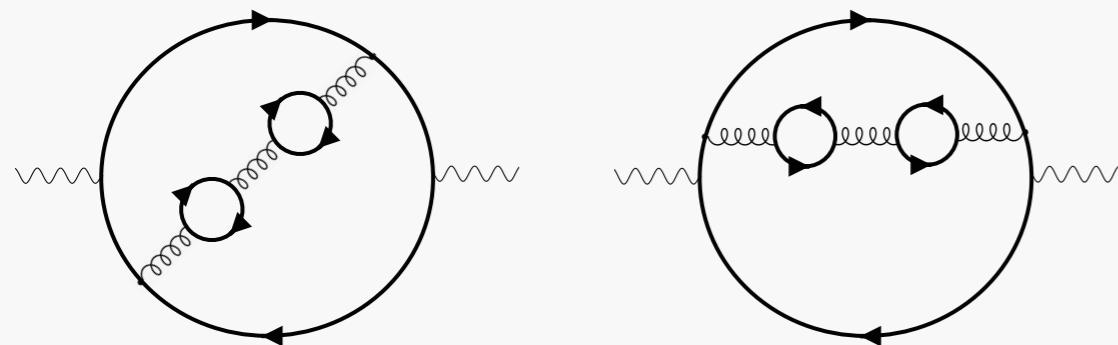


N_f^2 part @N³LO $e^+e^- \rightarrow jj$

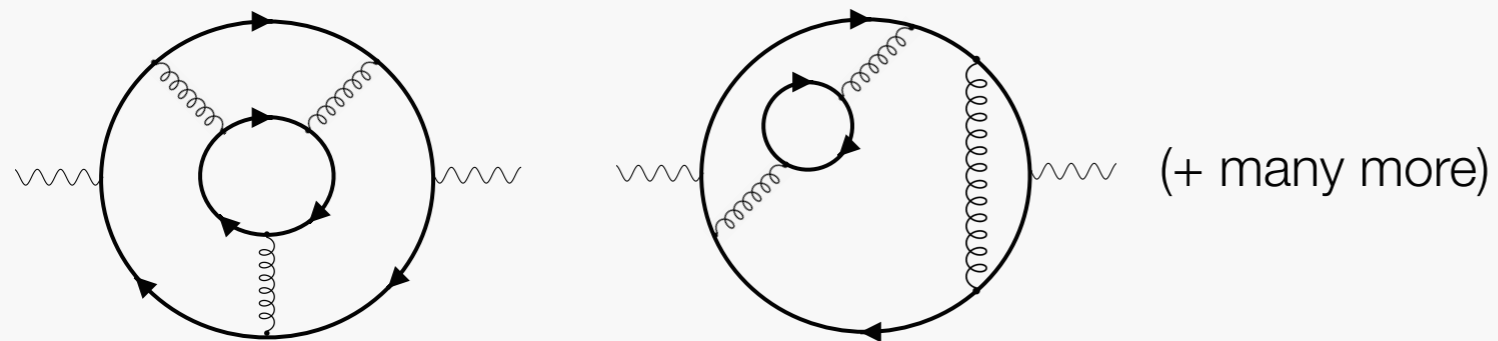
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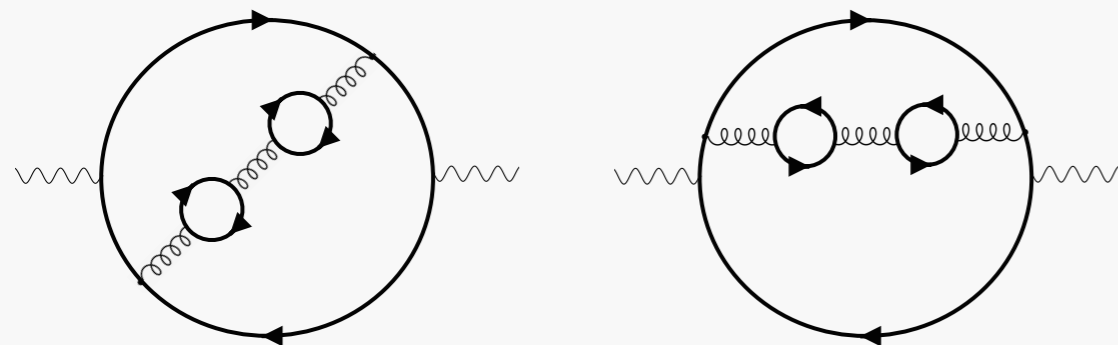
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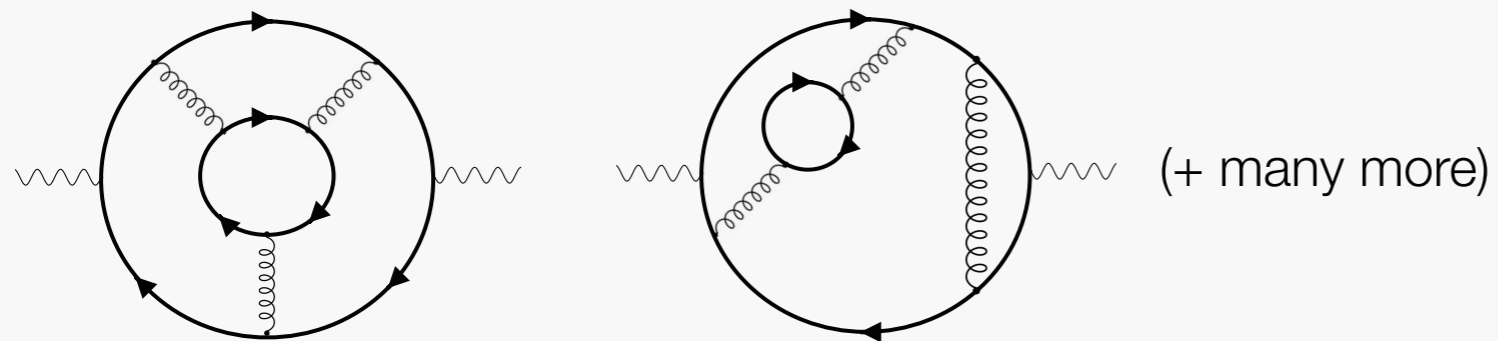


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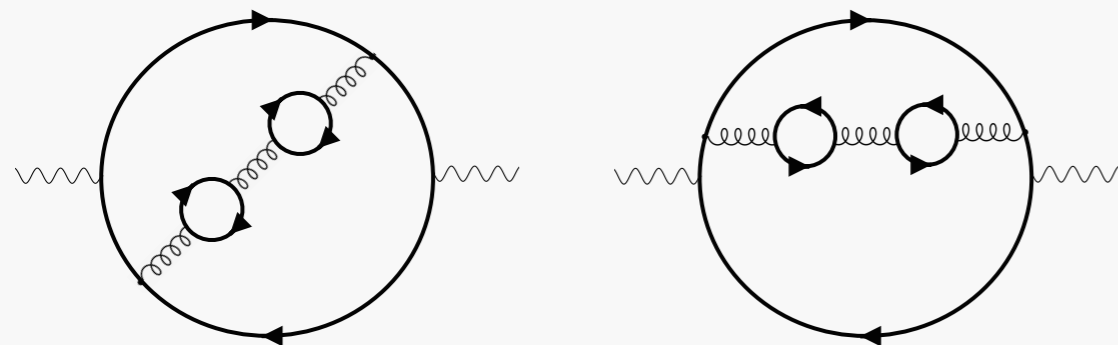


singlet part @N³LO $e^+e^- \rightarrow jj$

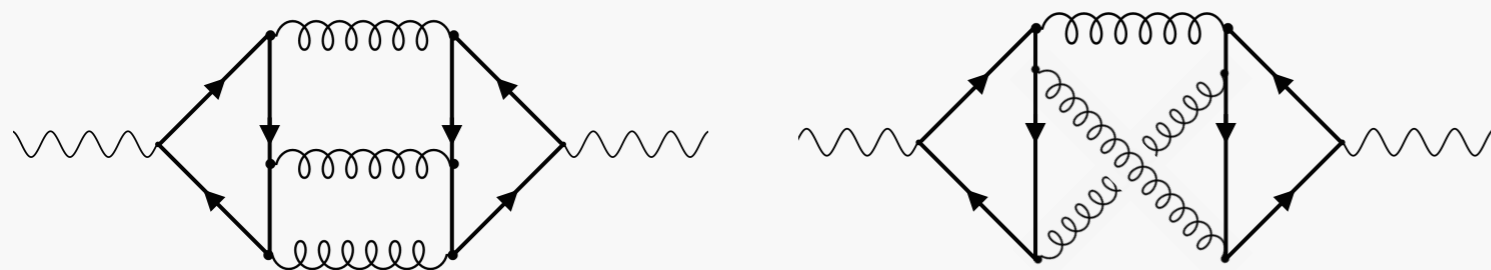
N_f part @N³LO $e^+e^- \rightarrow jj$



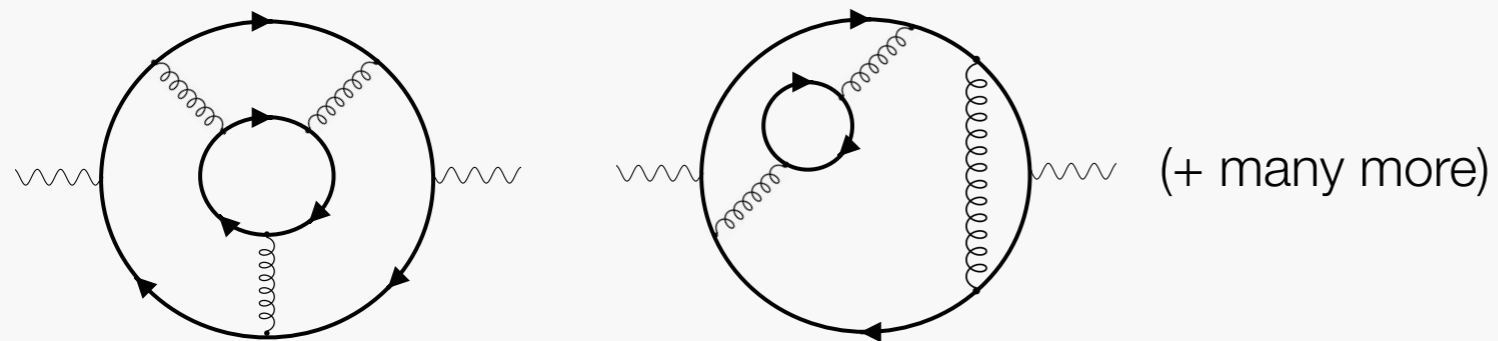
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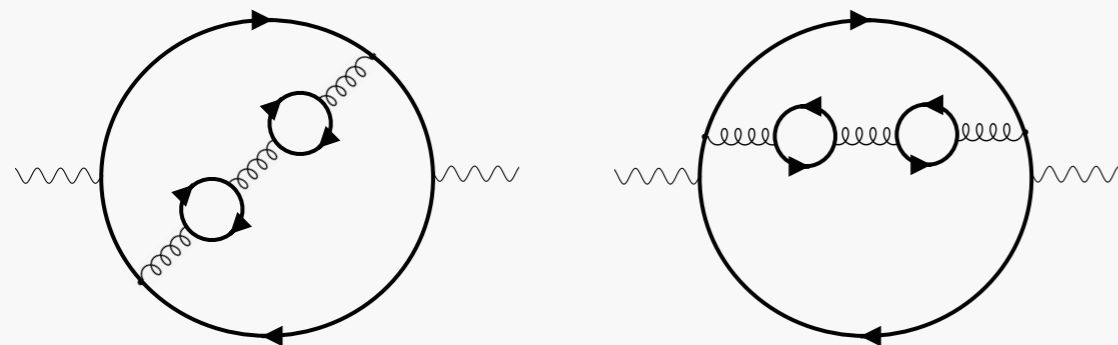


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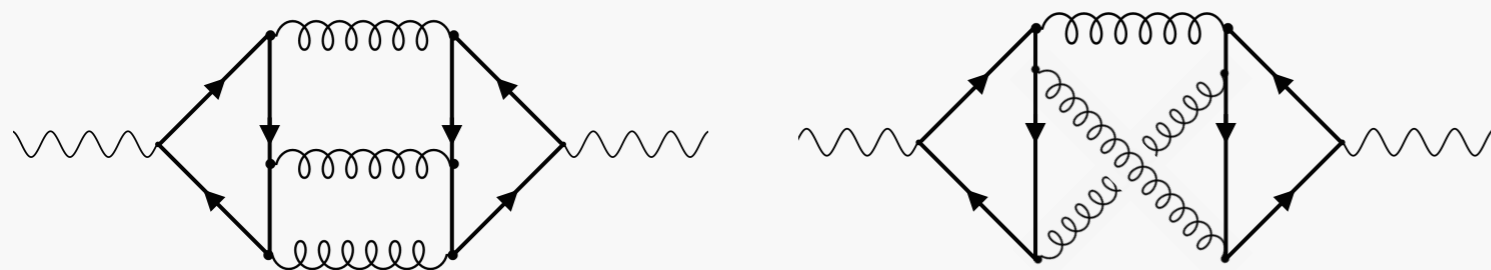
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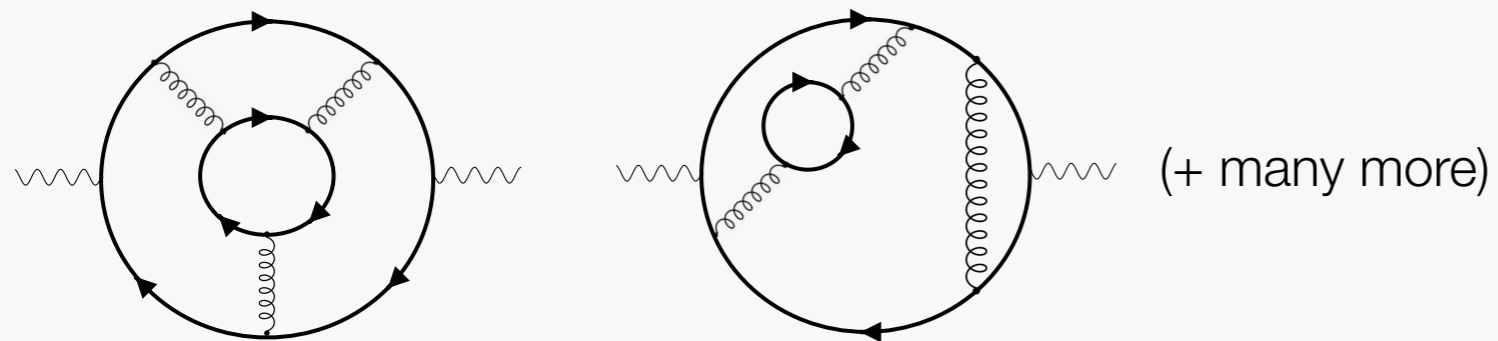
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$$K_{\alpha_s^3}^{\text{singlet}, \text{LU}} = -25.6(1.5)$$

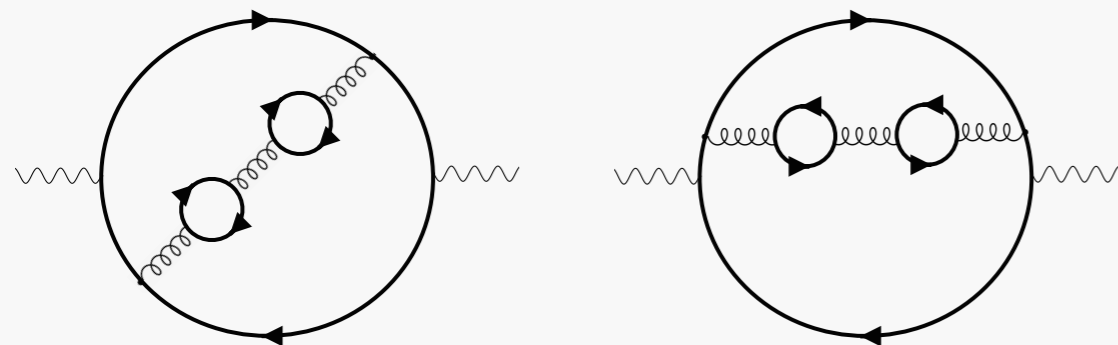
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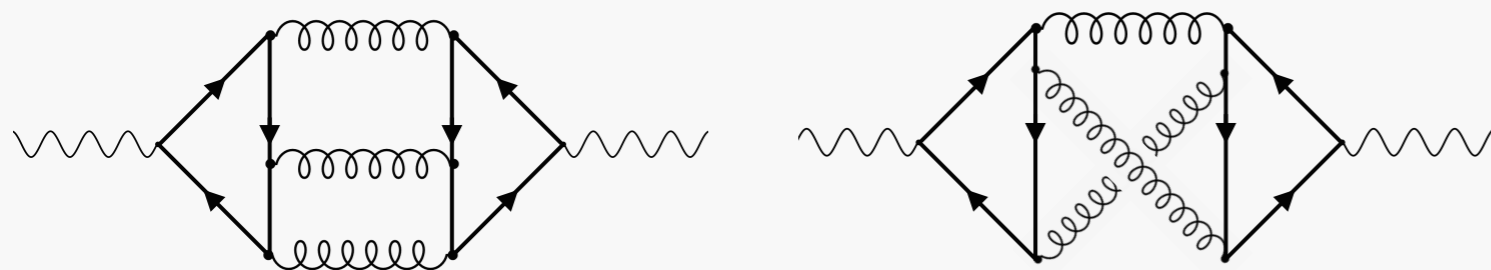
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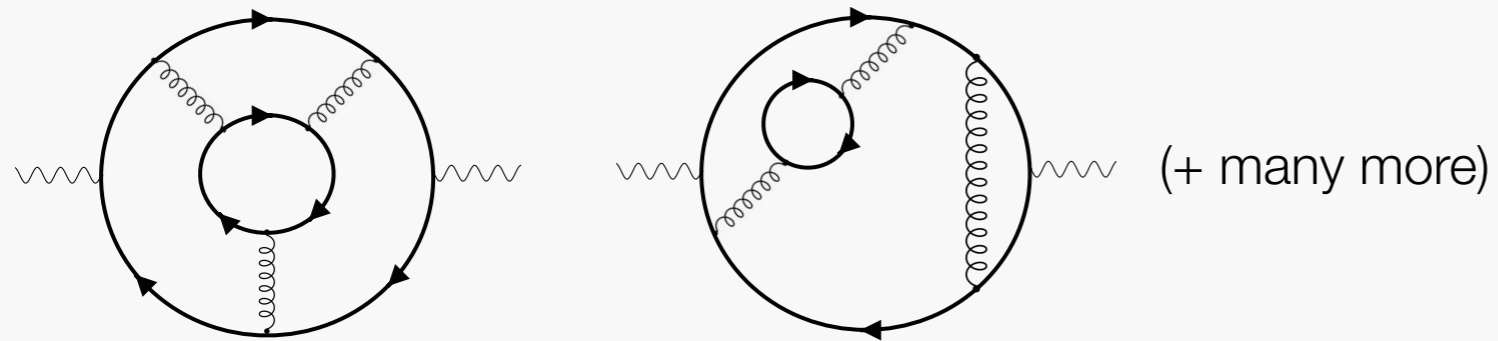
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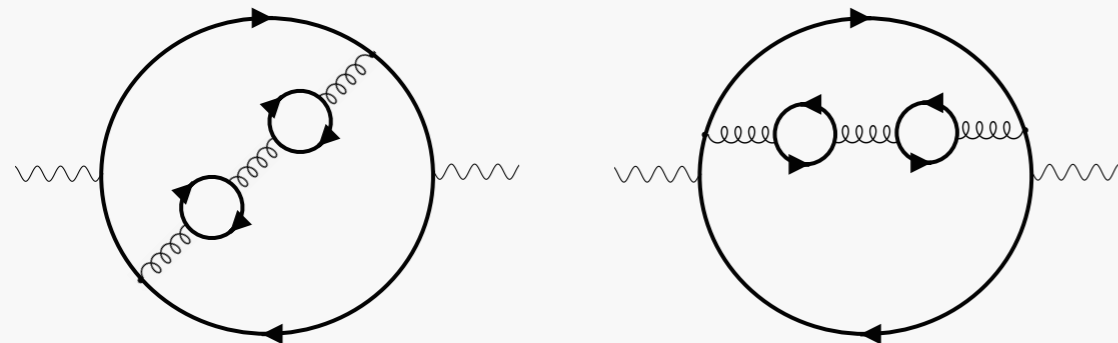
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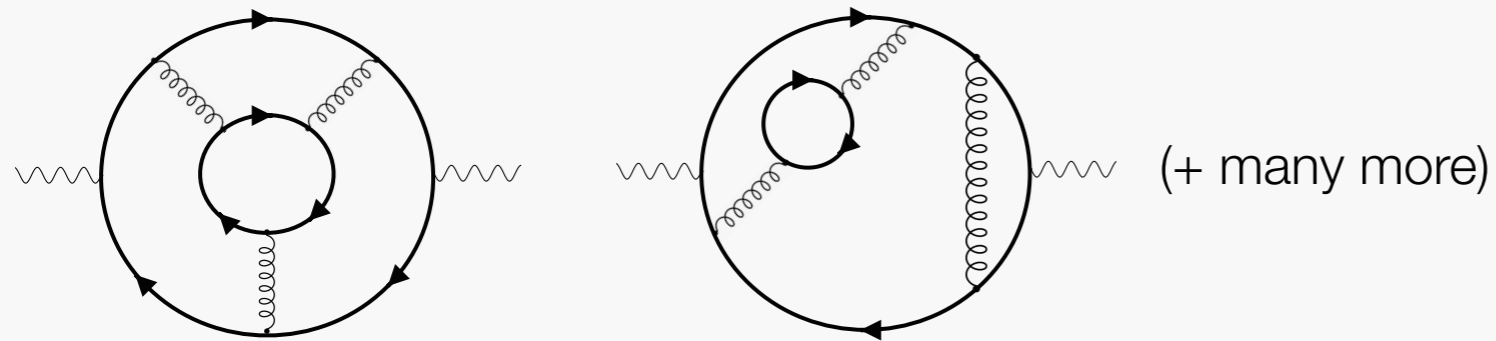
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Benchmarks:

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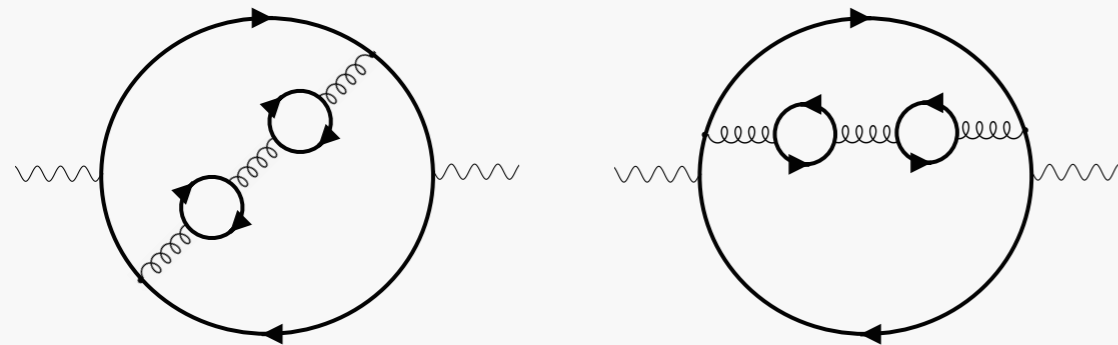
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Testing {

- ✓ N3LO IR cancellations
- ✓ 3-loop UV renormalisation
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Outlook

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Local Unitarity soon ready for automation of processes without initial-state singularities

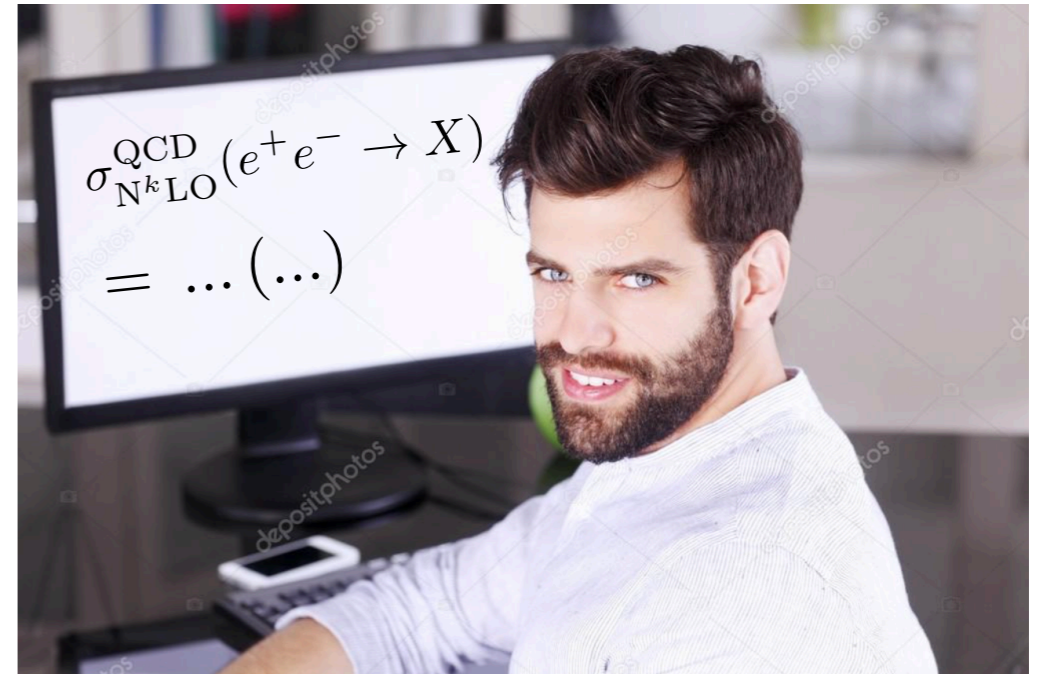
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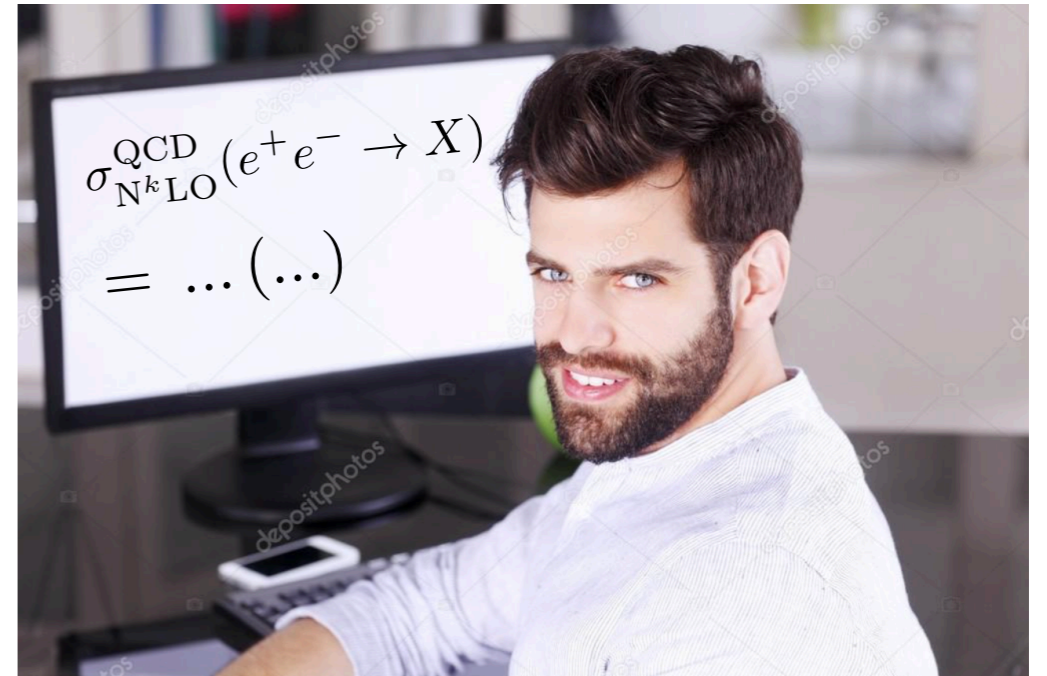
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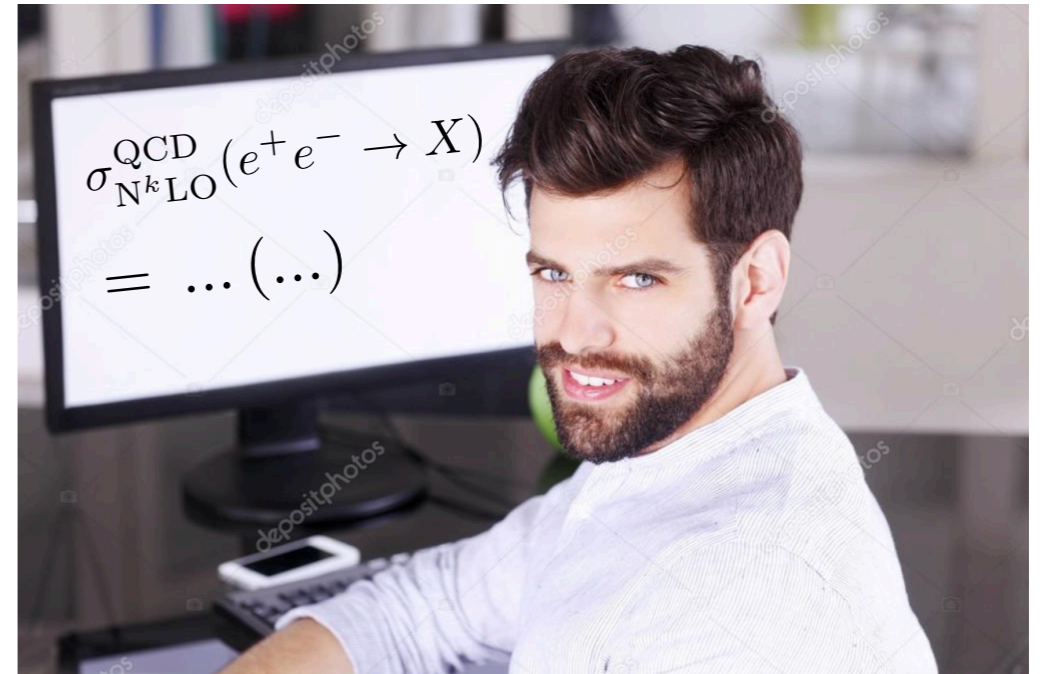
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Future theory work:

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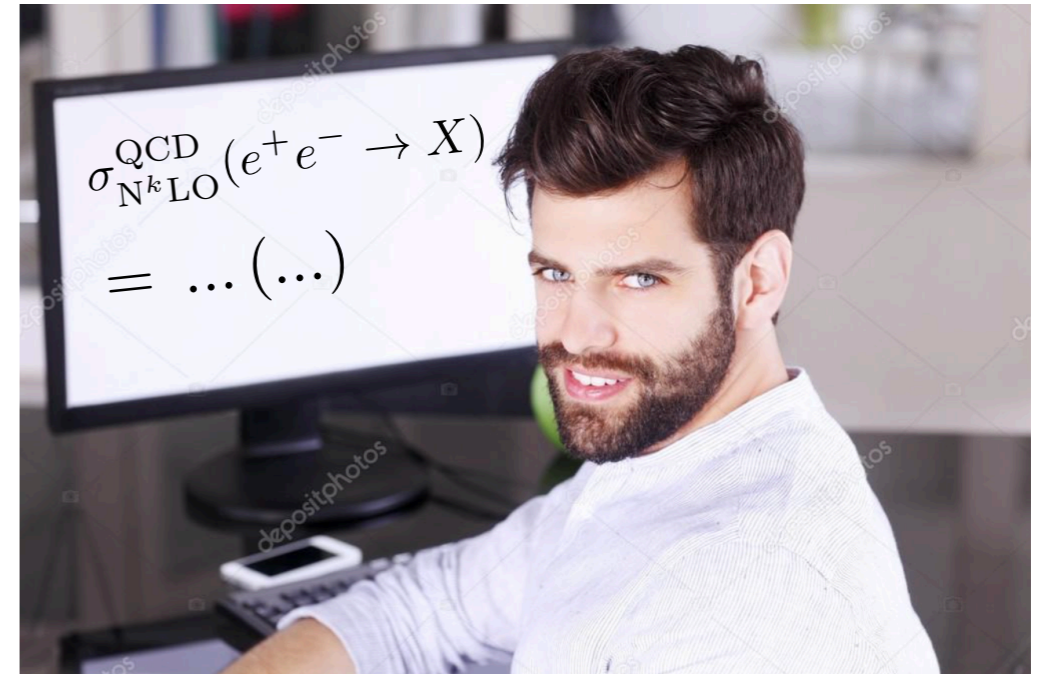


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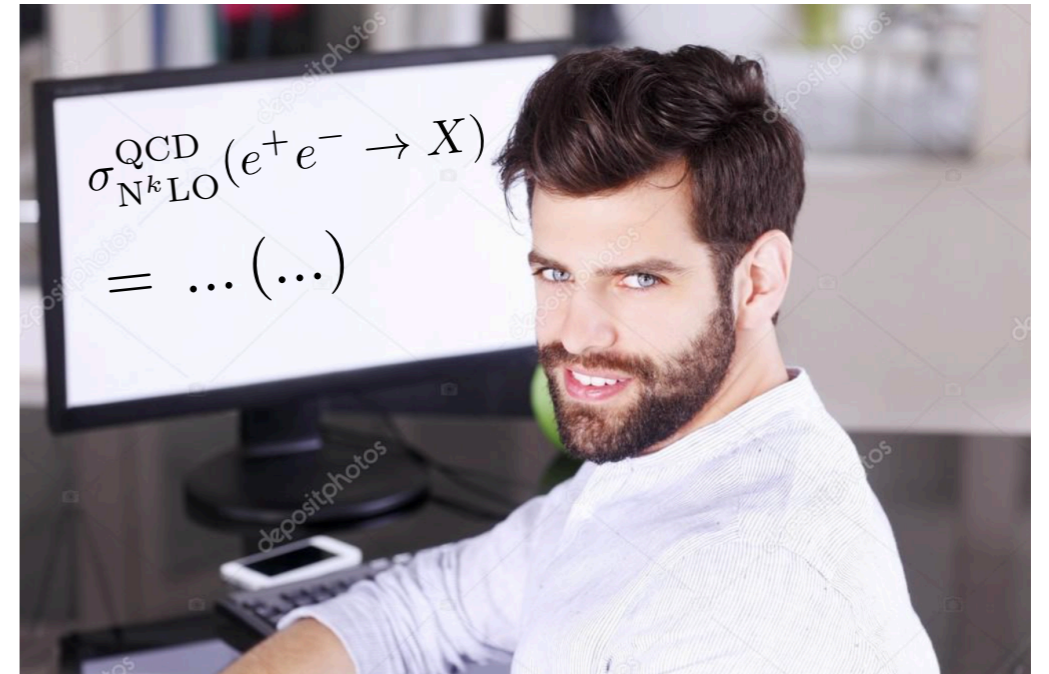
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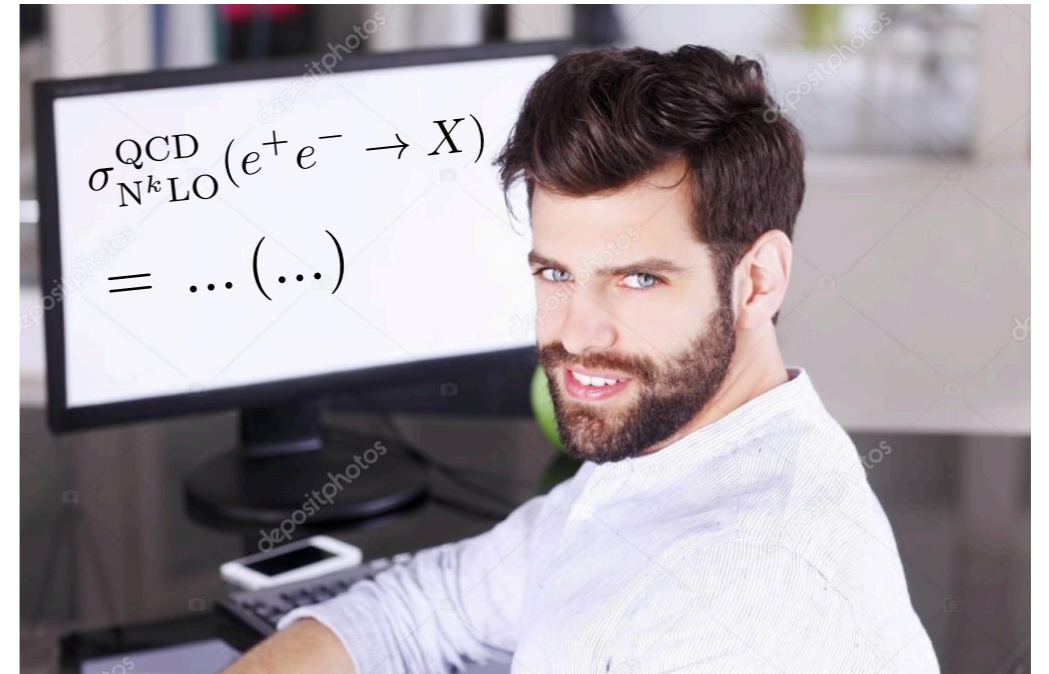
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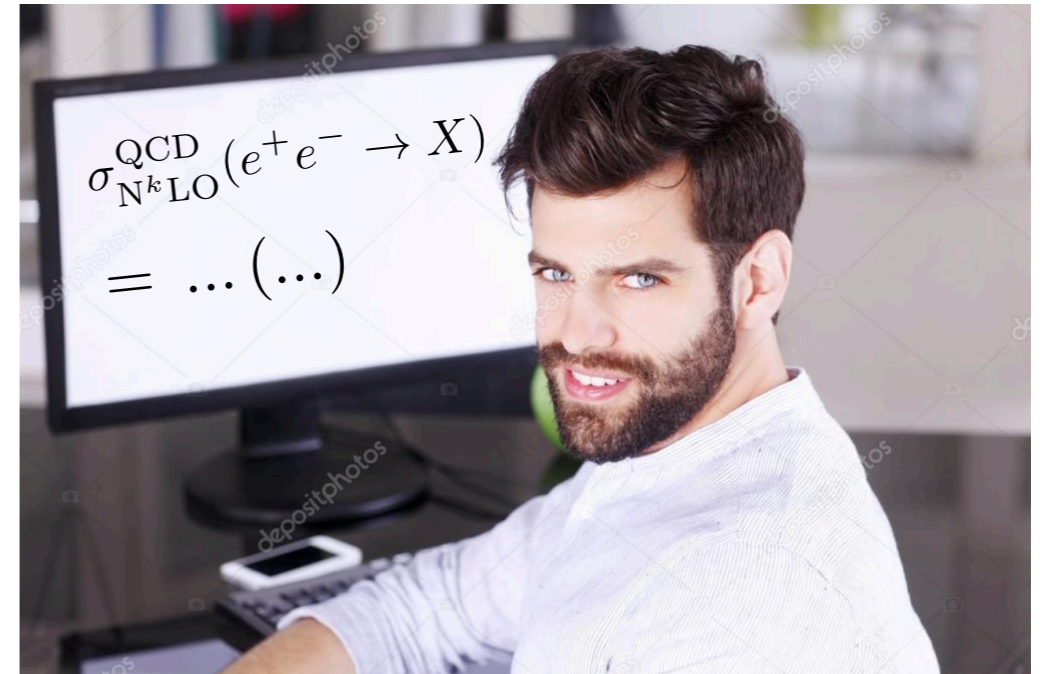
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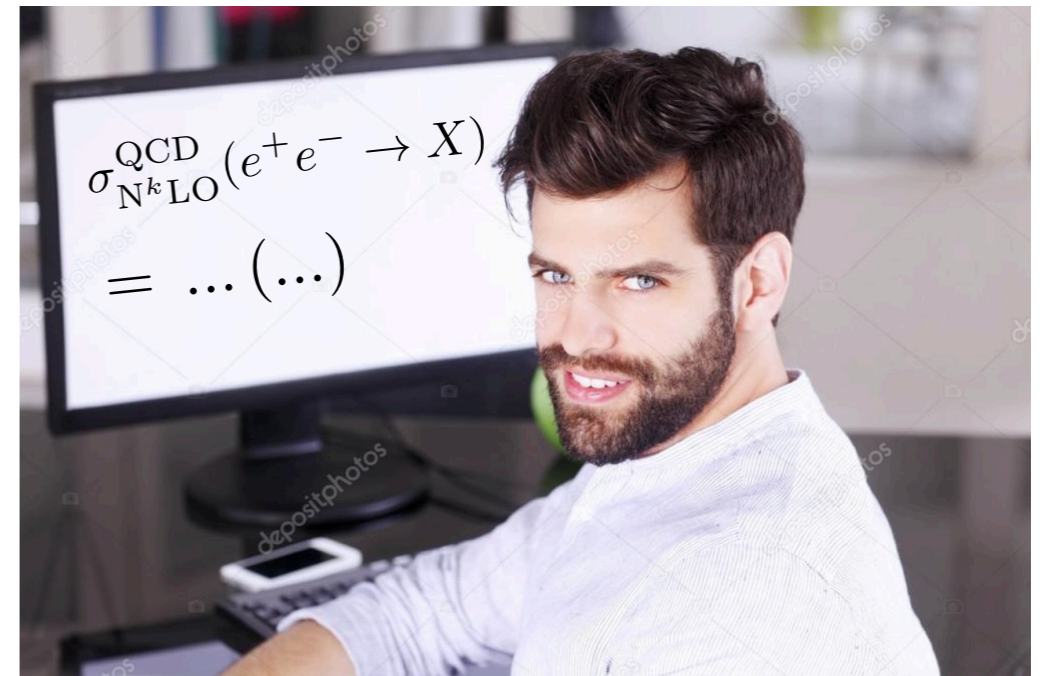
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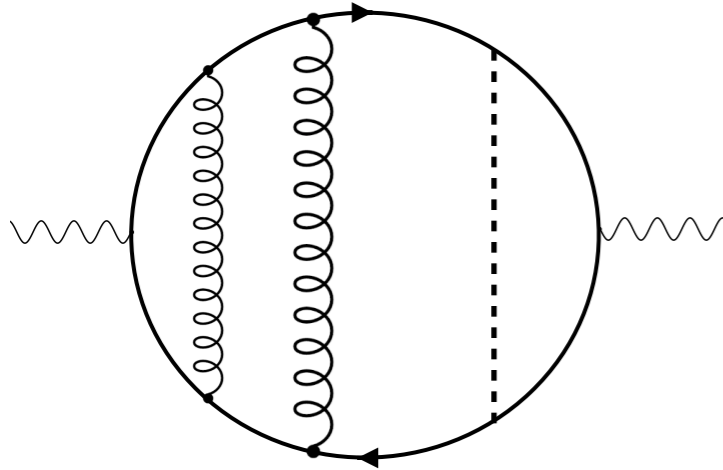
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- **Local PDF counterterms:** develop a competitive subtraction method or integrate existing ones

Thank you!

$$N^2LO \quad \gamma^* \rightarrow t\bar{t}H$$

Top energy distribution
for the supergraph



$$\sqrt{s} = 1\text{TeV}$$

