Local Unitarity

First applications at NNLO and N3LO

Zeno Capatti

In collaboration with:

Valentin Hirschi
Dario Kermanschah
Andrea Pelloni
Ben Ruijl

A talk for ACAT 2022
Bari, October 25th, 2022

Photo: posing in front of Monte Carlo's casino, 2021

> *"It was at that time that I suggested an obvious name for the statistical method—a suggestion not unrelated to the fact that Stan had an uncle who would borrow money from relatives because he "just had to go to Monte Carlo." The name seems to have endured."*

Nicholas Metropolis, "The beginning of the Monte Carlo method",
Los Alamos Science Special Issue 1987

Local Unitarity: framing the problem

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A cross-section admits a perturbative expansion when $\alpha < 1$

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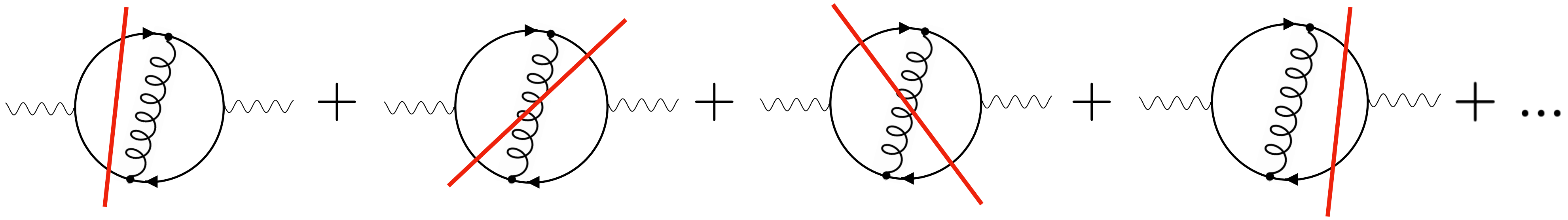

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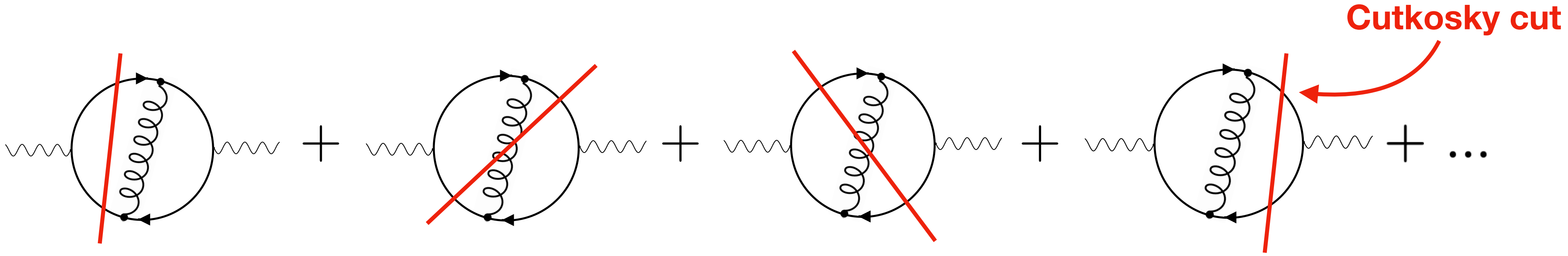

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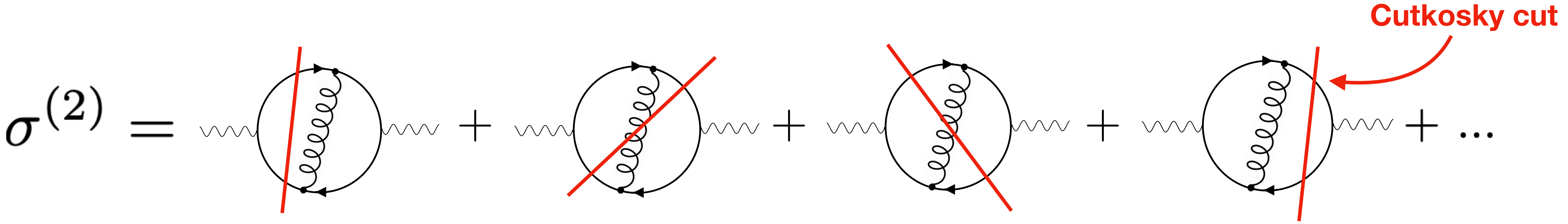

Interference diagrams themselves can be represented as integrals of amplitudes

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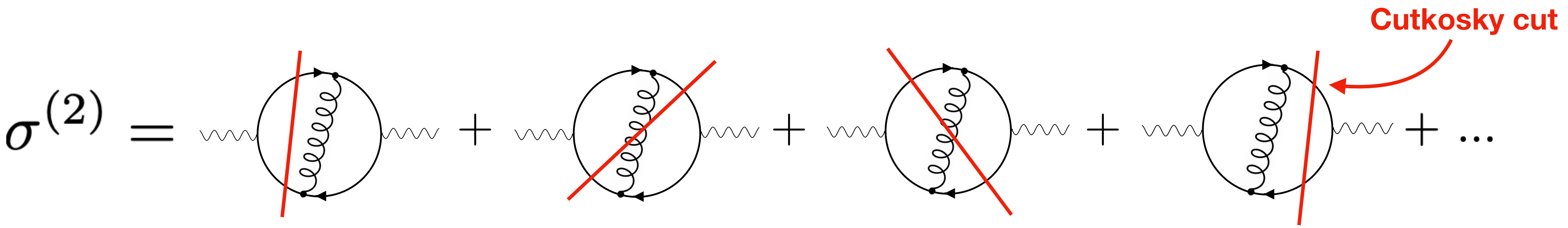

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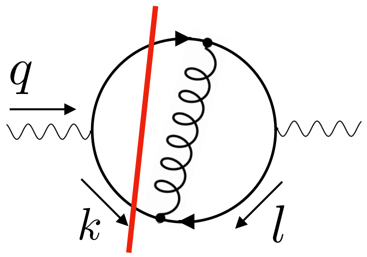

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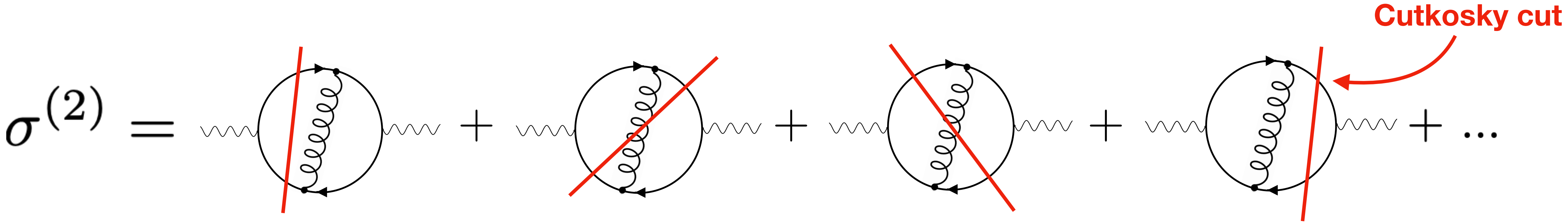

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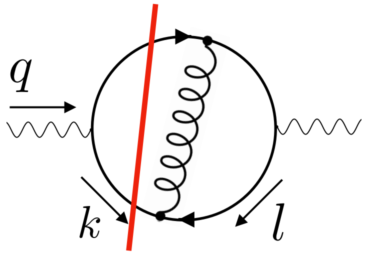

$$= \int \mathrm{d}^4 k \, \delta^+(k^2) \, \delta^+((q-k)^2)$$

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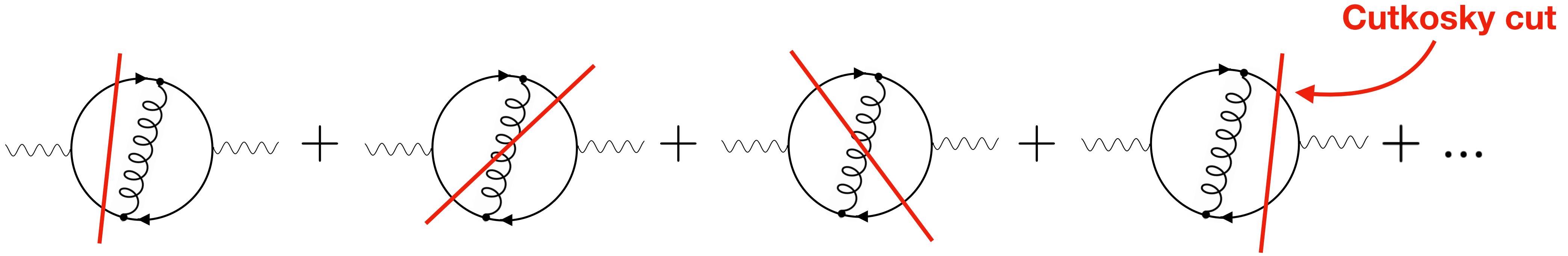

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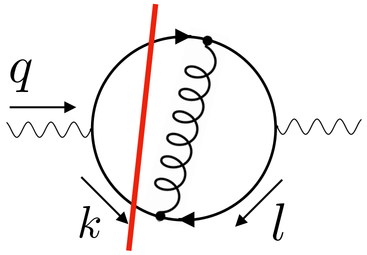

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Phase space integral

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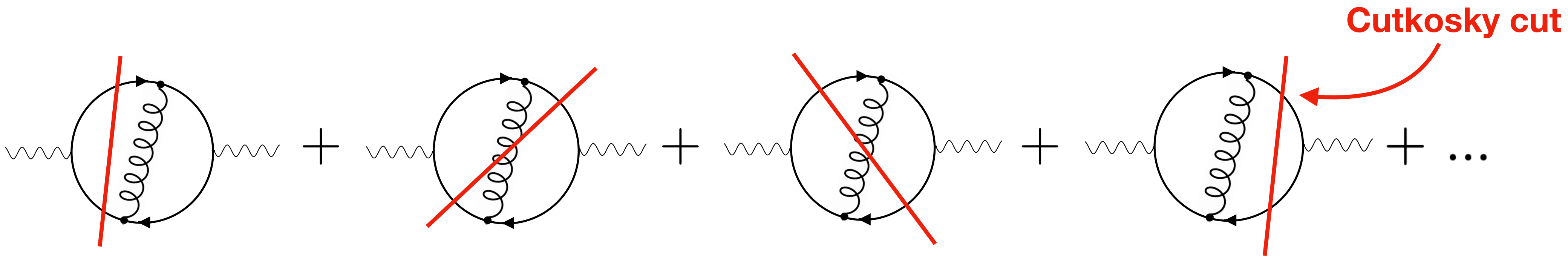

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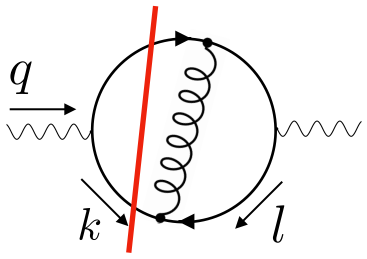

$$= \int \mathrm{d}^4k\, \delta^+(k^2)\, \delta^+((q-k)^2) \int \mathrm{d}^4 l \frac{N(k,l,q)}{l^2(l+q)^2(k+l)^2}$$

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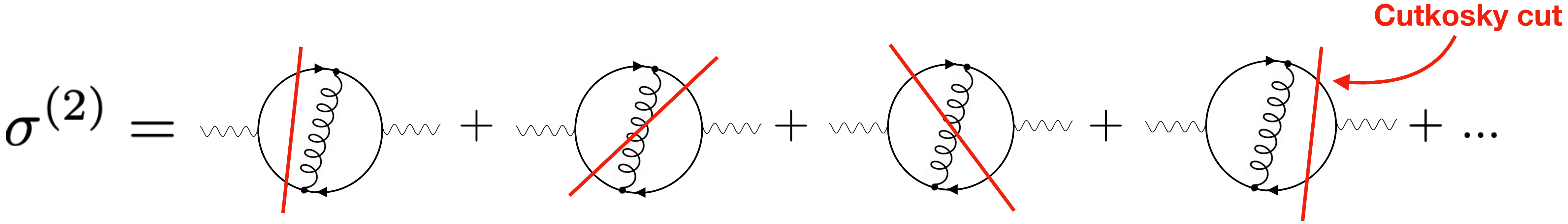

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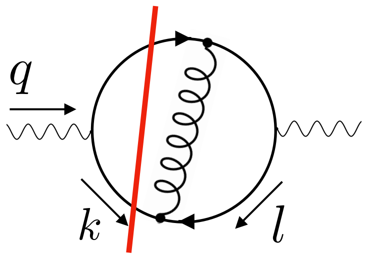

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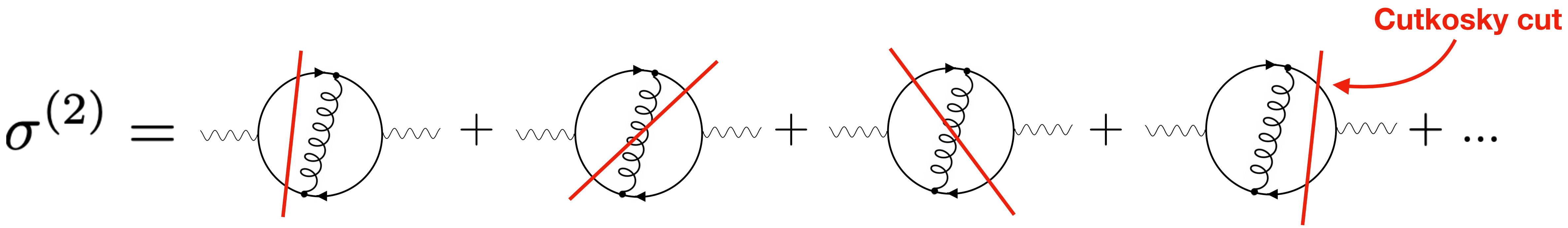

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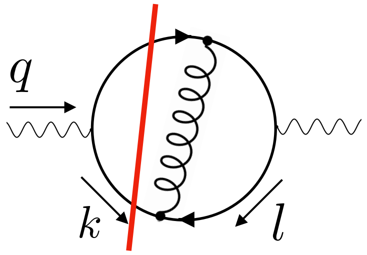

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Problem: both types of integrals have **infrared** (collinear, soft) **divergences** and thresholds

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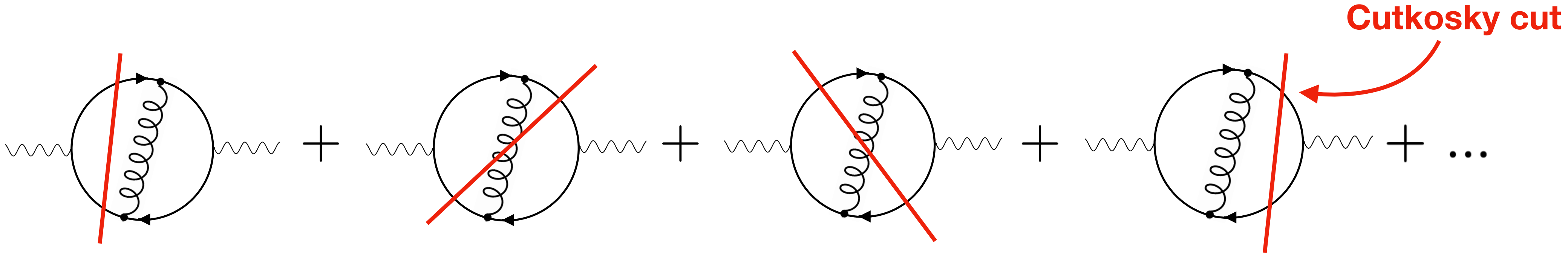

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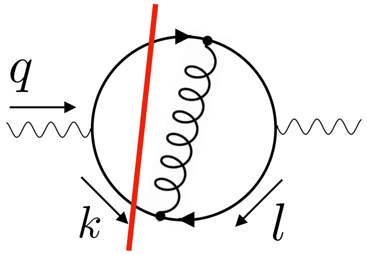

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Many good methods around to deal with this that work **either** for loop or for phase-space integrals

A local KLN cancellation mechanism

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Our approach instead **combines** singularities of loop and phase-space integrals at the local level through KLN

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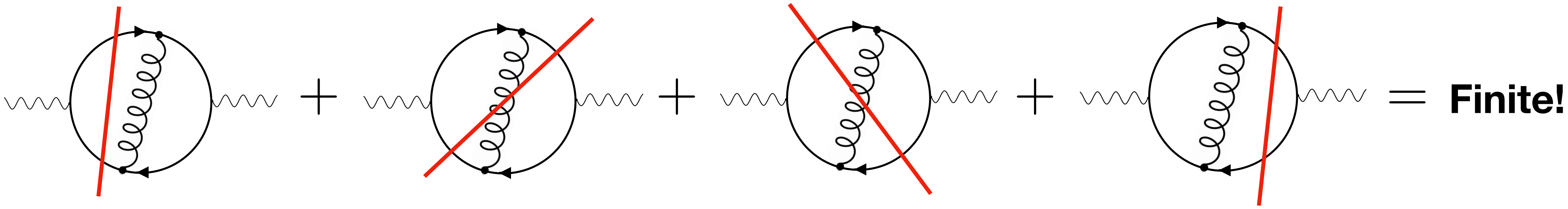

= Finite!

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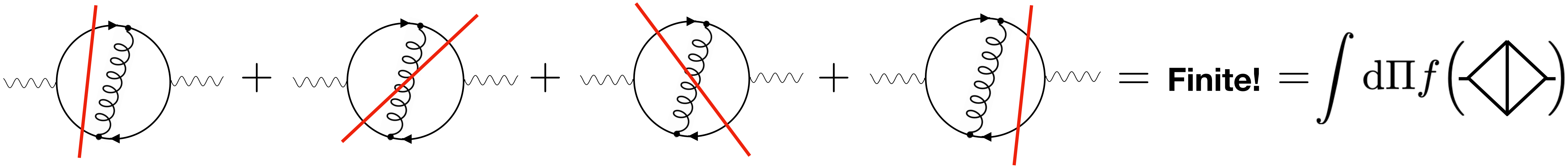

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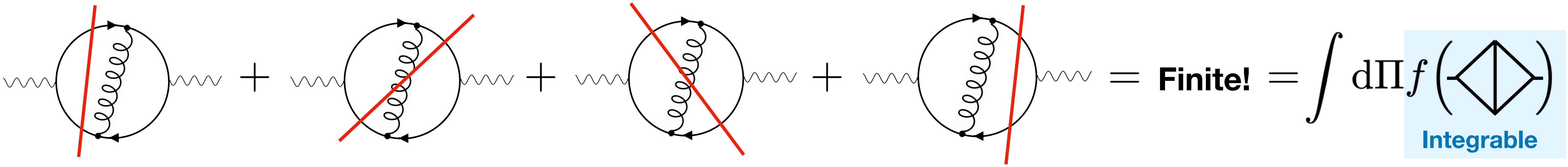

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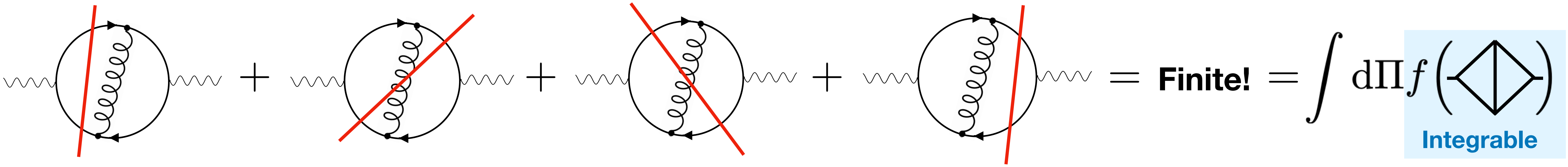

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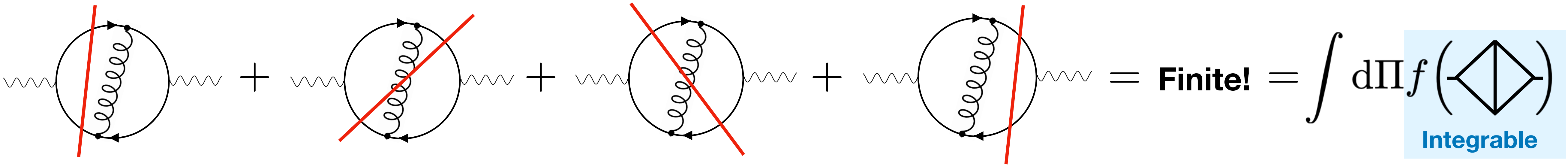

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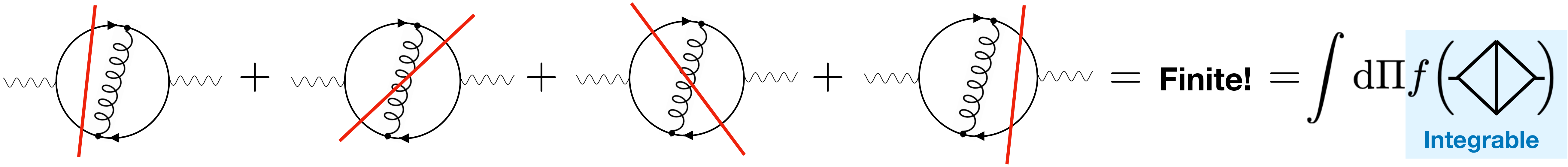

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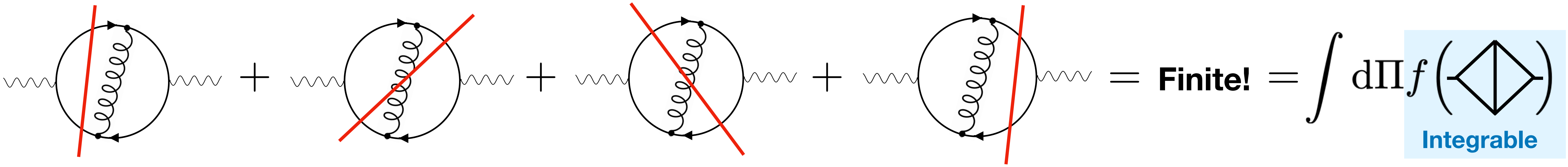

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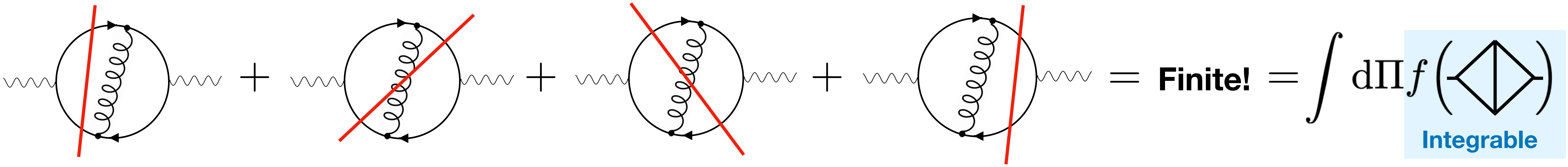

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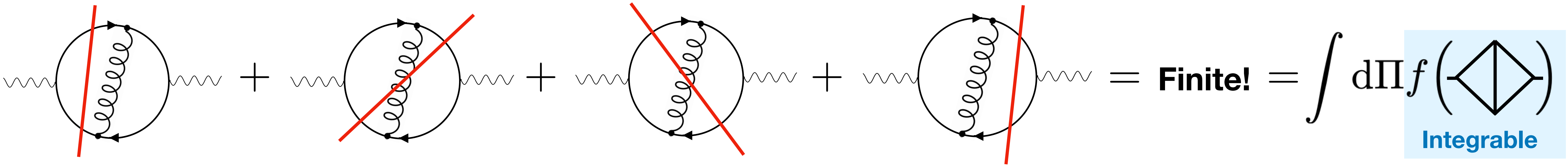

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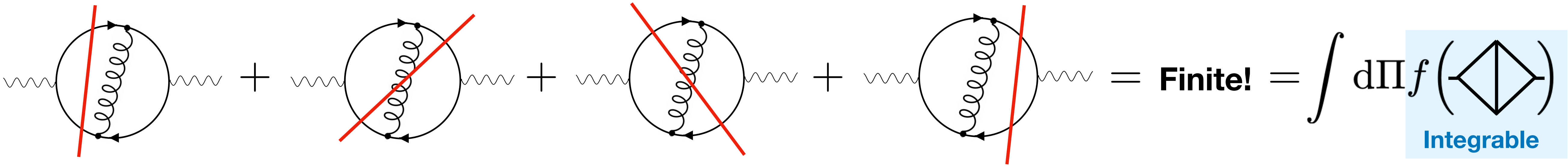

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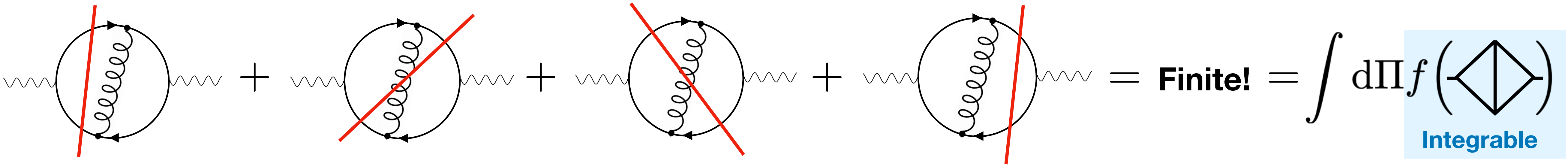

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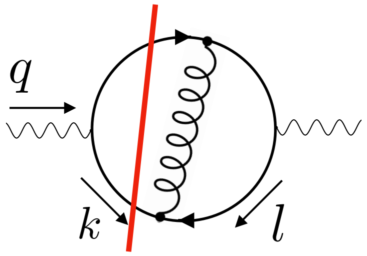

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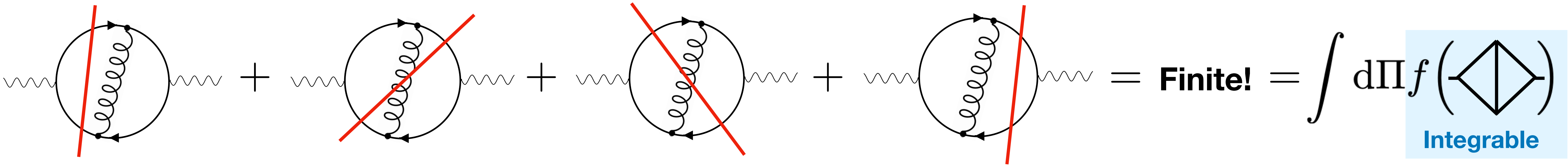

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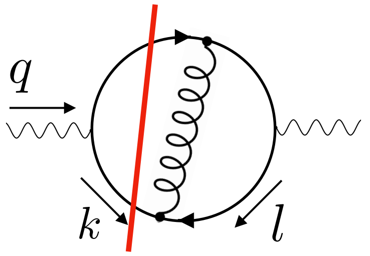

$$= \int \mathrm{d}^4k\mathrm{d}^4l\, \delta^{(+)}(k^2)\, \delta^{(+)}((k-q)^2)\ \frac{N}{l^2(l+q)^2(k+l)^2}$$

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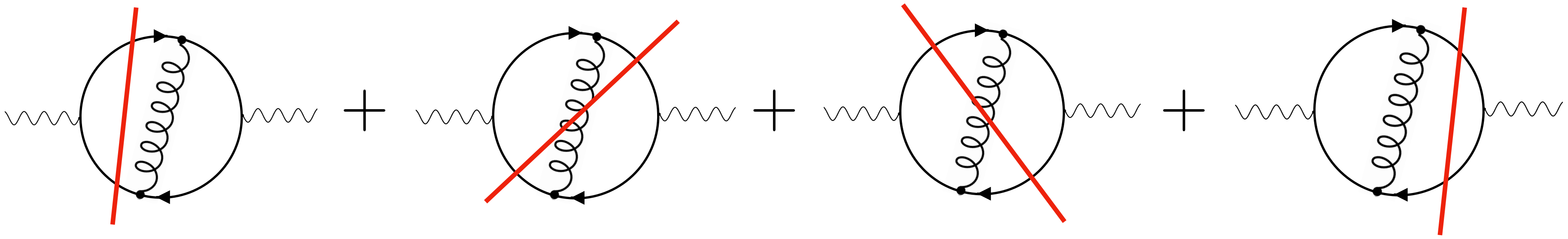

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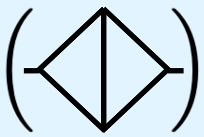

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Integrable

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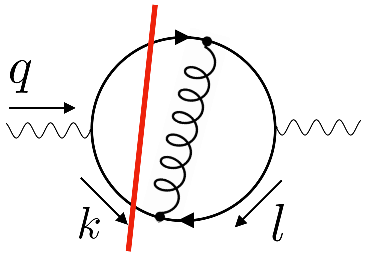

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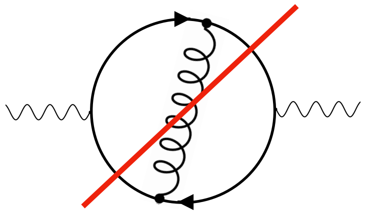

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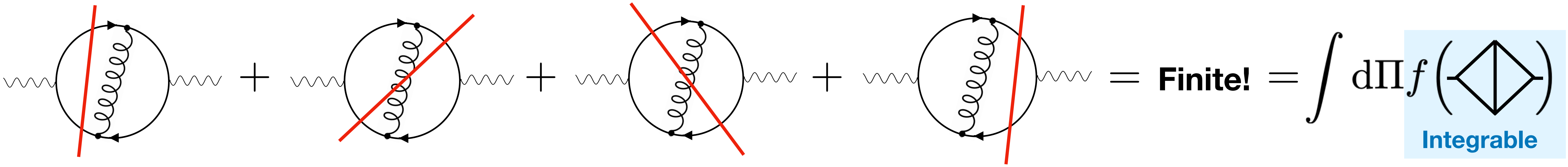

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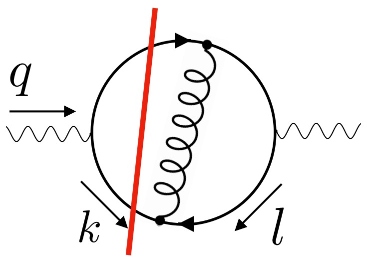

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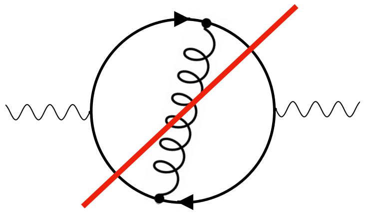

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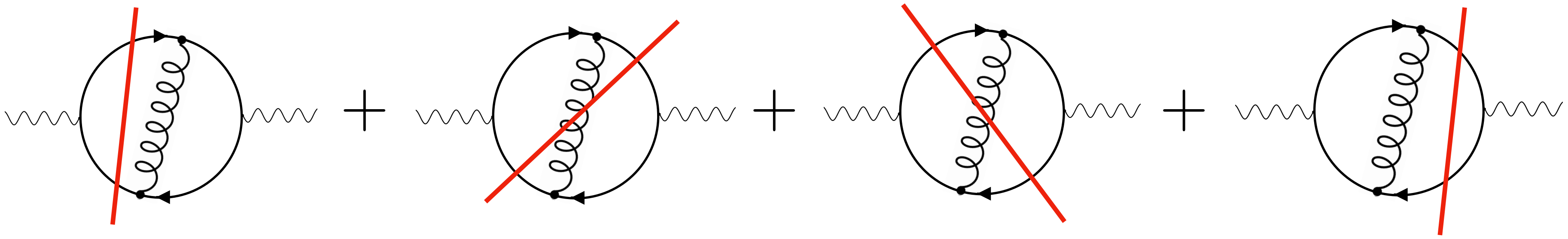

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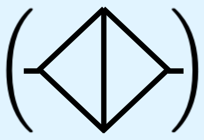

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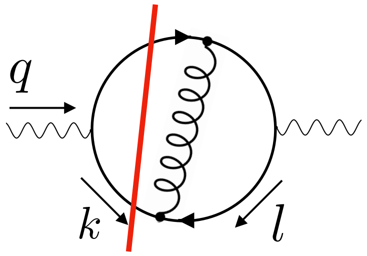

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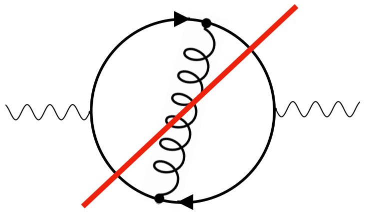
Problem 1:
Different number of deltas



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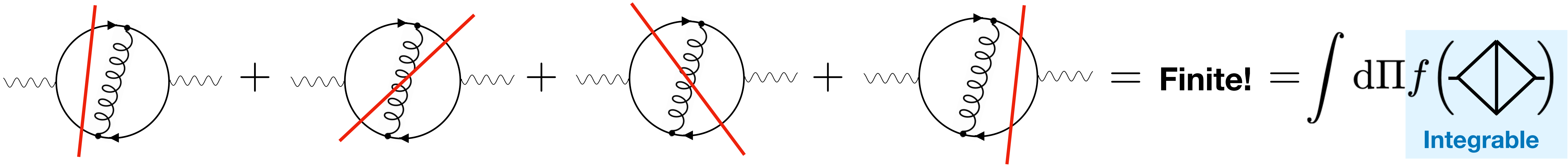

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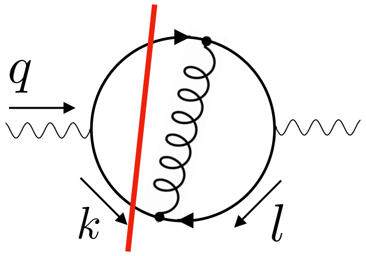
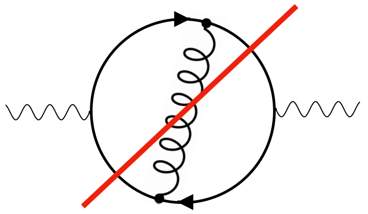


$$= \int \mathrm{d}^4k\mathrm{d}^4l\, \delta^{(+)}(k^2)\, \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2}$$



$$= \int \mathrm{d}^4k\mathrm{d}^4l\, \delta^{(+)}(k^2)\, \delta^{(+)}((k+l)^2)\, \delta^{(+)}((l+q)^2) \frac{N}{l^2(k-q)^2}$$

Problem 1:
Different number of deltas

Problem 2:
Too few energy variables to solve the deltas

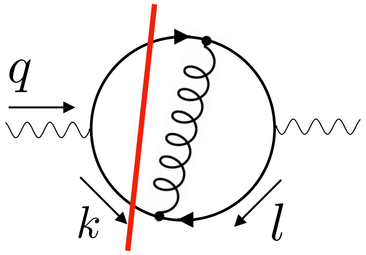
Problem 1: Different number of deltas

Problem 1: Different number of deltas

Observation:

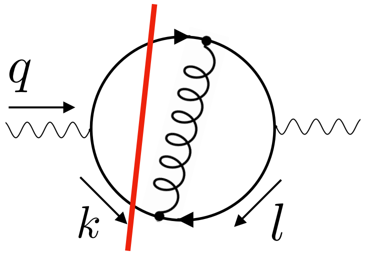
Problem 1: Different number of deltas

Observation:



Problem 1: Different number of deltas

Observation:



$$= \int \mathrm{d}^4k\, \delta^+(k^2)\, \delta^+((q-k)^2) \int \mathrm{d}^4l \frac{N(k,l,q)}{l^2(l+q)^2(k+l)^2}$$

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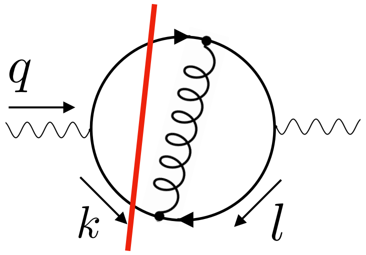

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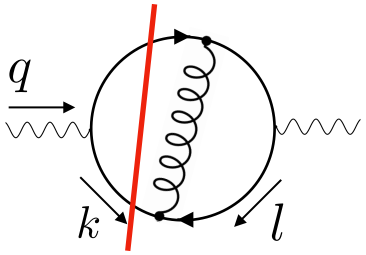
unconstrained integration over l^0

Problem 1: Different number of deltas

Observation:



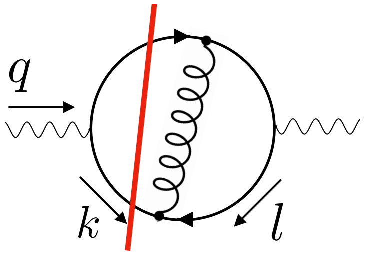
$$= \int \mathrm{d}^4k\, \delta^+(k^2)\, \delta^+((q-k)^2) \int \mathrm{d}^4l \frac{N(k,l,q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Solution 1: Perform integration over l^0

Problem 1: Different number of deltas

Observation:



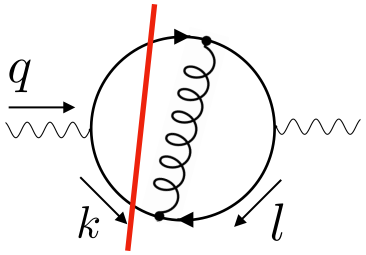
$$= \int \mathrm{d}^4k\, \delta^+(k^2)\, \delta^+((q-k)^2) \int \mathrm{d}^4 l \frac{N(k,l,q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT…)

Problem 1: Different number of deltas

Observation:



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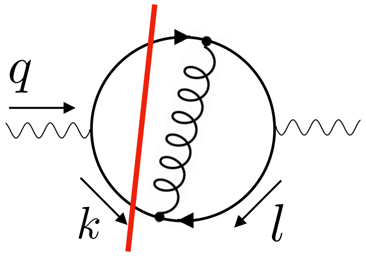
unconstrained integration over l^0

Solution 1: Perform integration over l^0

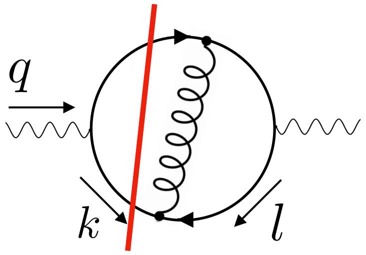
(LTD, cLTD, TOPT…)



$=$

Problem 1: Different number of deltas

Observation:



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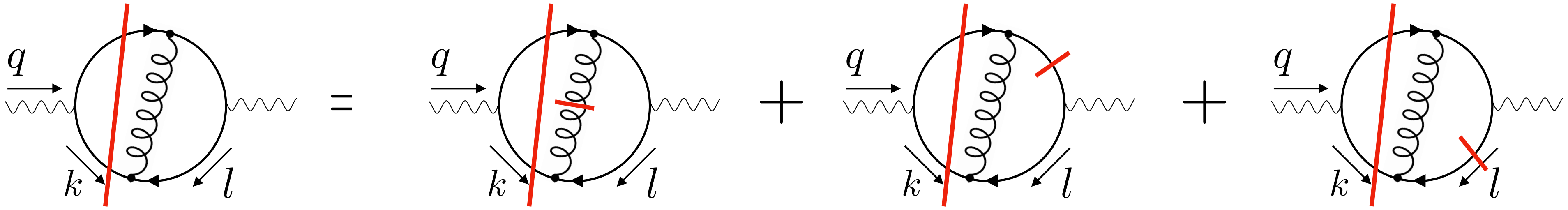
unconstrained integration over l^0

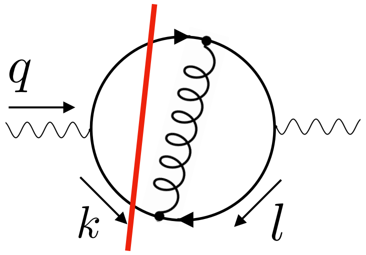
Solution 1: Perform integration over l^0

(LTD, cLTD, TOPT…)



Problem 1: Different number of deltas

Observation:

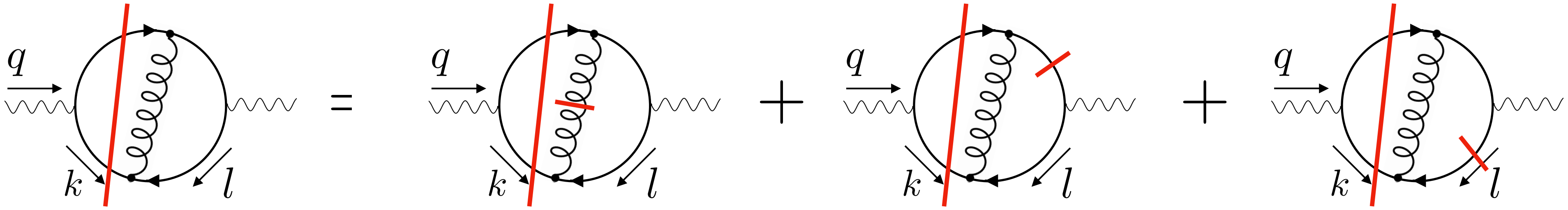


$$= \int \mathrm{d}^4k \, \delta^+(k^2) \, \delta^+((q-k)^2) \int \mathrm{d}^4 l \frac{N(k,l,q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Solution 1: Perform integration over l^0

(LTD, cLTD, TOPT…)



Catani, Gleisberg, Krauss, Rodrigo, Winter
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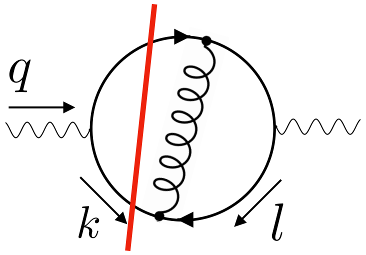
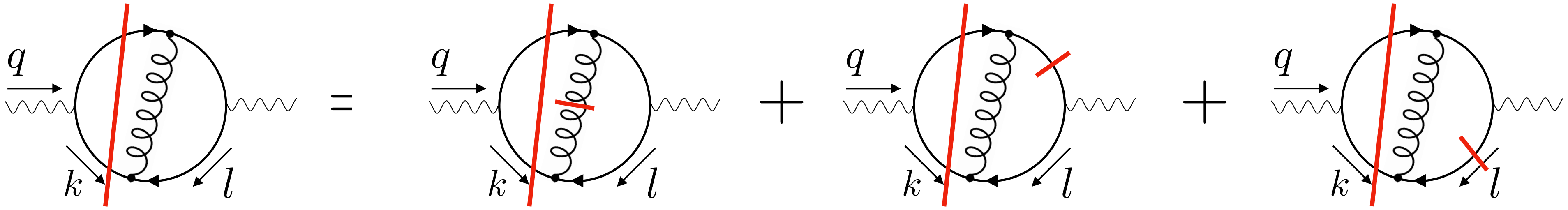


$$= \int \mathrm{d}^4k\, \delta^+(k^2)\, \delta^+((q-k)^2) \int \mathrm{d}^4l \frac{N(k,l,q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT…)



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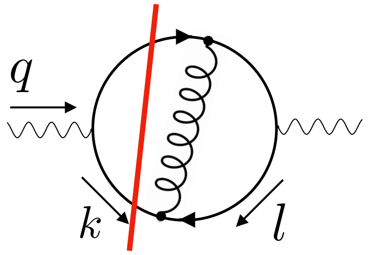
Observation: Now real and virtual contributions have the same amount of deltas

Problem 1: Different number of deltas

Observation:

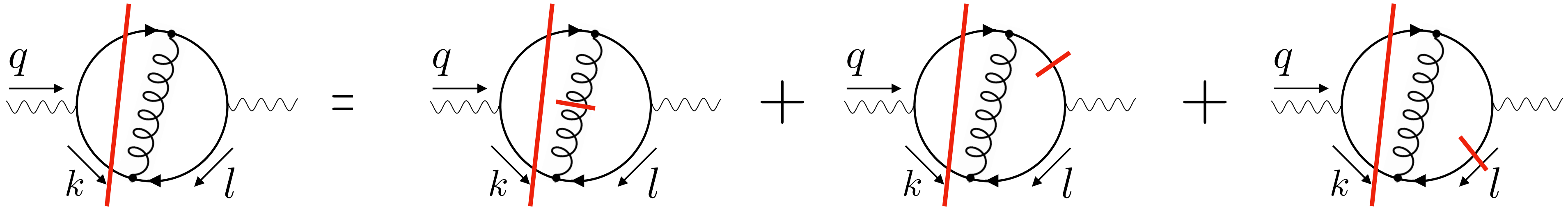


$$= \int \mathrm{d}^4k\, \delta^+(k^2)\, \delta^+((q-k)^2) \int \mathrm{d}^4l \frac{N(k,l,q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Solution 1: Perform integration over l^0

(LTD, cLTD, TOPT…)



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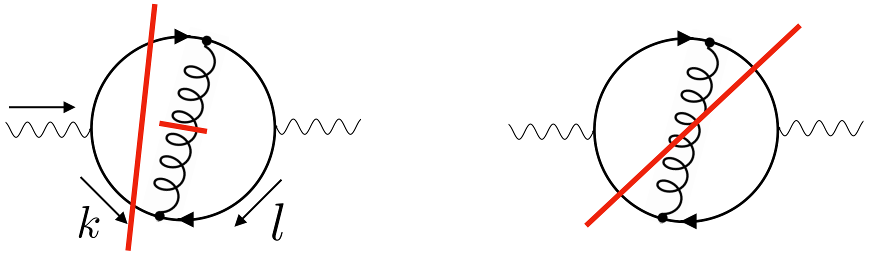
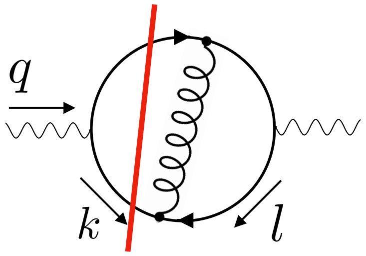
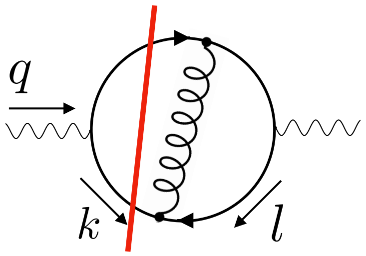


both have three cut lines!



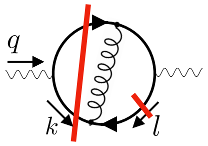
$$= \int \mathrm{d}^4k\mathrm{d}^4l\,\delta^{(+)}(k^2)\,\delta^{(+)}((k-q)^2)\;\frac{N}{l^2(l+q)^2(k+l)^2}\times$$

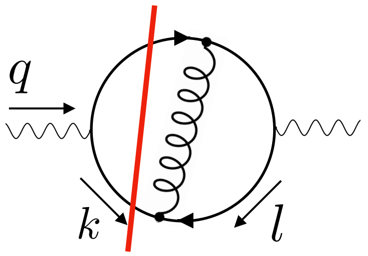
$$\times\Big[l^2\delta^{(+)}(l^2)+(l+q)^2\delta^{(+)}((l+q)^2)+(k+l)^2\delta^{(+)}((k+l)^2)\Big]$$



$$= \int \mathrm{d}^4k\mathrm{d}^4l\, \delta^{(+)}(k^2)\, \delta^{(+)}((k-q)^2)\ \ \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

$$\times \left[l^2\delta^{(+)}(l^2) + (l+q)^2\delta^{(+)}((l+q)^2) + (k+l)^2\delta^{(+)}((k+l)^2)\right]$$





$$= \int \mathrm{d}^4k\,\mathrm{d}^4l\,\delta^{(+)}(k^2)\,\delta^{(+)}((k-q)^2)\ \frac{N}{l^2(l+q)^2(k+l)^2}\times$$

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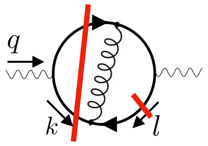

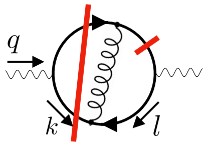

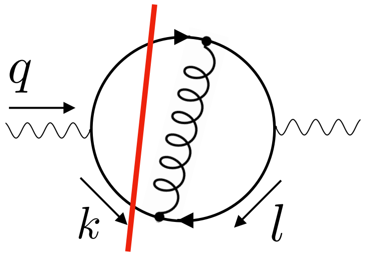

$$= \int \mathrm{d}^4k\,\mathrm{d}^4l\,\delta^{(+)}(k^2)\,\delta^{(+)}((k-q)^2)\ \frac{N}{l^2(l+q)^2(k+l)^2}\times$$

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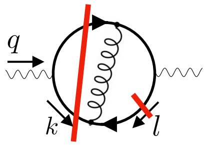
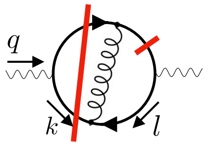
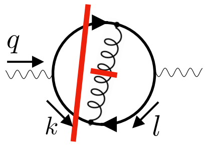
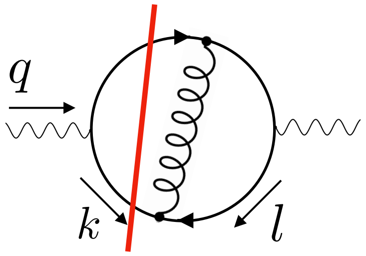




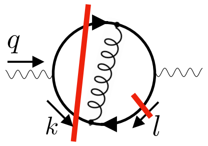


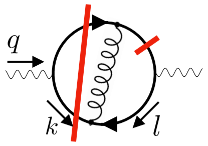
Problem 2: Too few energy variables to solve the deltas

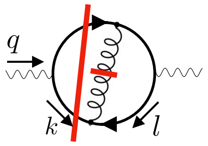


$$= \int \mathrm{d}^4k \mathrm{d}^4l \, \delta^{(+)}(k^2) \, \delta^{(+)}((k-q)^2) \; \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

$$\times \left[l^2 \delta^{(+)}(l^2) + (l+q)^2 \delta^{(+)}((l+q)^2) + (k+l)^2 \delta^{(+)}((k+l)^2) \right]$$







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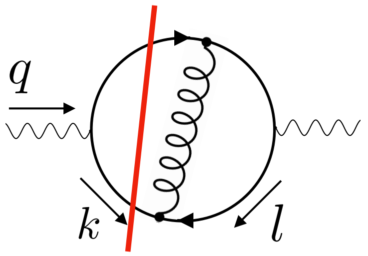
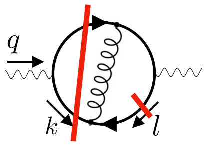
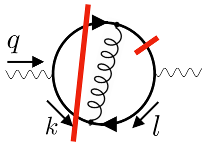
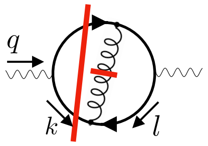


$$= \int \mathrm{d}^4k\,\mathrm{d}^4l\, \delta^{(+)}(k^2)\, \delta^{(+)}((k-q)^2)\, \frac{N}{l^2(l+q)^2(k+l)^2} \times$$

$$\times \left[l^2\delta^{(+)}(l^2) + (l+q)^2\delta^{(+)}((l+q)^2) + (k+l)^2\delta^{(+)}((k+l)^2) \right]$$

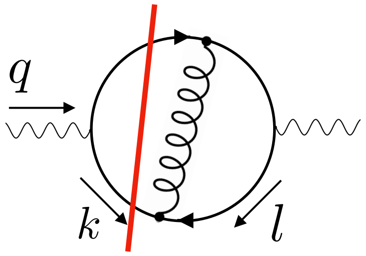






Two deltas, one energy integration

Problem 2: Too few energy variables to solve the deltas



$$= \int \mathrm{d}^4k\,\mathrm{d}^4l\,\delta^{(+)}(k^2)\,\delta^{(+)}((k-q)^2)\,\frac{N}{l^2(l+q)^2(k+l)^2}\times$$

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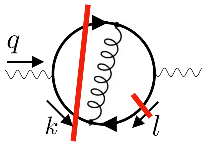
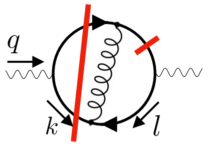
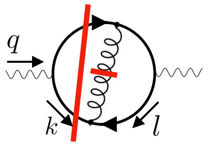
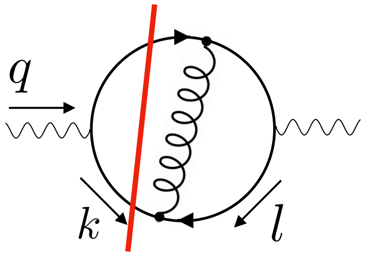






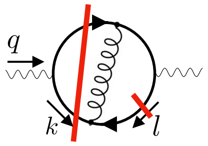
Two deltas, one energy integration

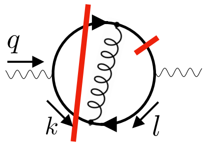
Problem 2: Too few energy variables to solve the deltas

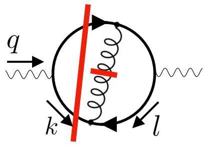


$$= \int \mathrm{d}^4k\,\mathrm{d}^4l\,\delta^{(+)}(k^2)\,\delta^{(+)}((k-q)^2)\,\frac{N}{l^2(l+q)^2(k+l)^2}\times$$

$$\times\Big[l^2\delta^{(+)}(l^2)+(l+q)^2\delta^{(+)}((l+q)^2)+(k+l)^2\delta^{(+)}((k+l)^2)\Big]$$



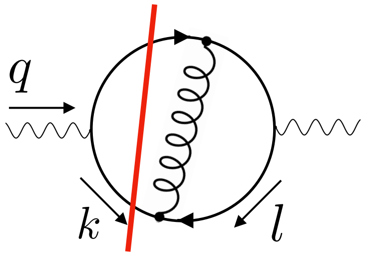




Two deltas, one energy integration

Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one

Problem 2: Too few energy variables to solve the deltas



$$= \int \mathrm{d}^4k\,\mathrm{d}^4l\,\delta^{(+)}(k^2)\,\delta^{(+)}((k-q)^2)\,\frac{N}{l^2(l+q)^2(k+l)^2}\times$$

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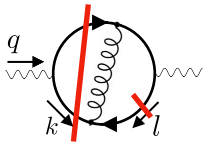
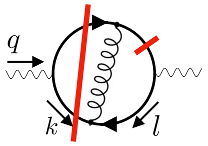
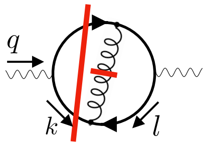
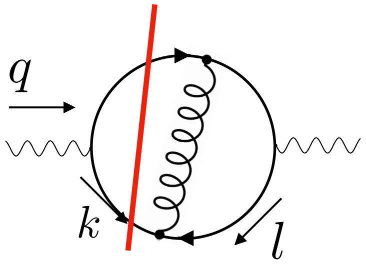






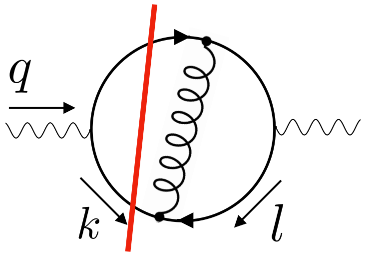
Two deltas, one energy integration

Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one



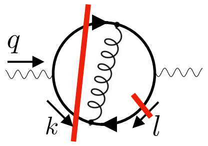
$$\approx \mathrm{d}^2\Omega_k\,\mathrm{d}^3\vec{l}$$

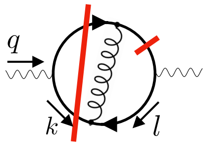
Problem 2: Too few energy variables to solve the deltas

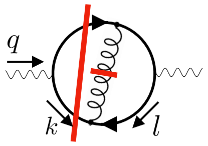


$$= \int \mathrm{d}^4k\,\mathrm{d}^4l\,\delta^{(+)}(k^2)\,\delta^{(+)}((k-q)^2)\,\frac{N}{l^2(l+q)^2(k+l)^2}\times$$

$$\times\left[l^2\delta^{(+)}(l^2)+(l+q)^2\delta^{(+)}((l+q)^2)+(k+l)^2\delta^{(+)}((k+l)^2)\right]$$

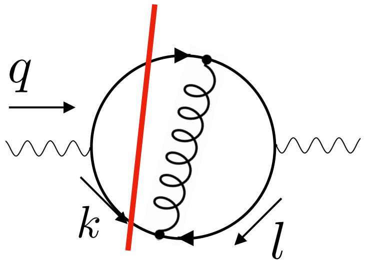






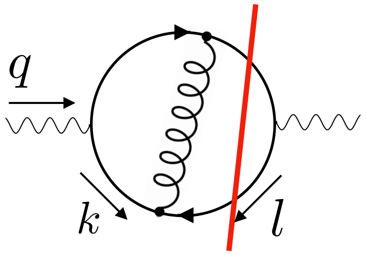
Two deltas, one energy integration

Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one



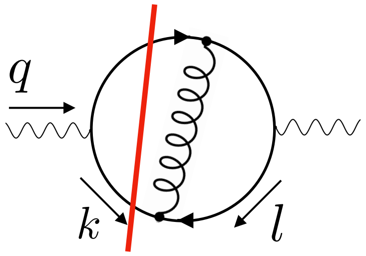
$$\approx \mathrm{d}^2\Omega_k\mathrm{d}^3\vec{l}$$

$$\Rightarrow$$



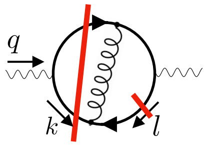
$$\approx \mathrm{d}^3\vec{k}\mathrm{d}^2\Omega_l$$

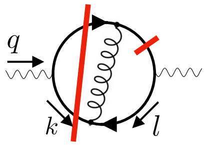
Problem 2: Too few energy variables to solve the deltas

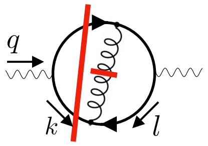


$$= \int \mathrm{d}^4k\,\mathrm{d}^4l\,\delta^{(+)}(k^2)\,\delta^{(+)}((k-q)^2)\;\frac{N}{l^2(l+q)^2(k+l)^2}\times$$

$$\times\left[l^2\delta^{(+)}(l^2)+(l+q)^2\delta^{(+)}((l+q)^2)+(k+l)^2\delta^{(+)}((k+l)^2)\right]$$

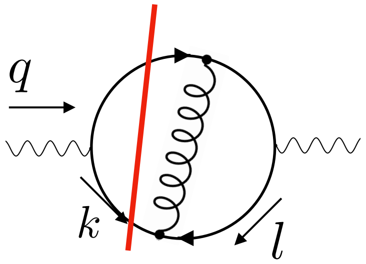




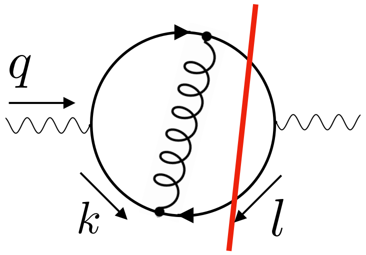


Two deltas, one energy integration

Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one



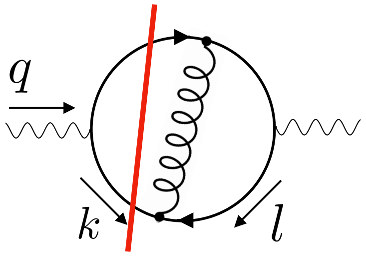
$$\approx \mathrm{d}^2\Omega_k\,\mathrm{d}^3\vec{l} \qquad \Rightarrow$$



$$\approx \mathrm{d}^3\vec{k}\,\mathrm{d}^2\Omega_l$$

Integration measure is mis-aligned!

Problem 2: Too few energy variables to solve the deltas



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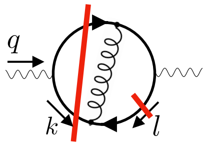
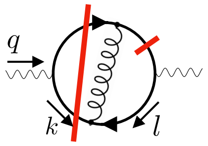
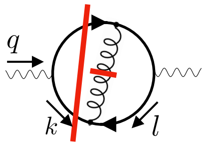
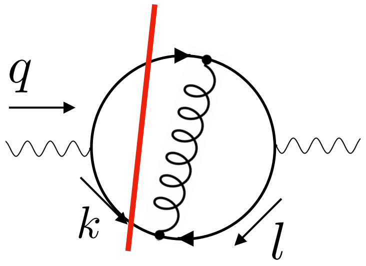




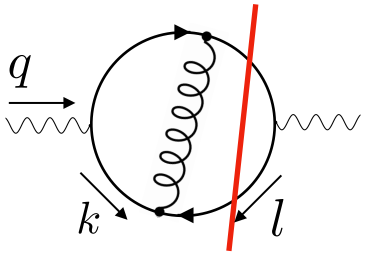


Two deltas, one energy integration

Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one



$$\approx \mathrm{d}^2\Omega_k\mathrm{d}^3\vec{l} \quad \Rightarrow$$

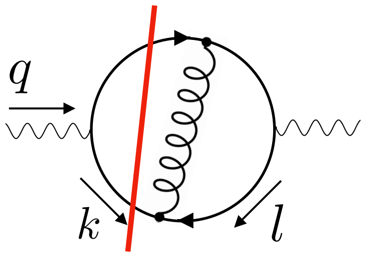


$$\approx \mathrm{d}^3\vec{k}\mathrm{d}^2\Omega_l$$

Integration measure is mis-aligned!

For the real contributions, problem known in phase-space subtraction methods

Problem 2: Too few energy variables to solve the deltas



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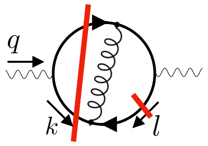

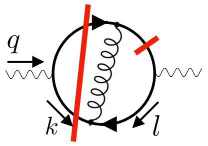

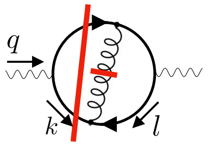

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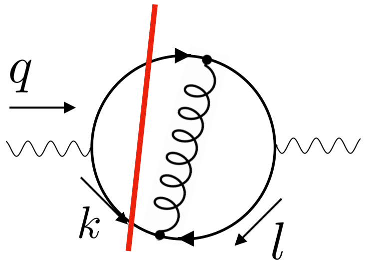
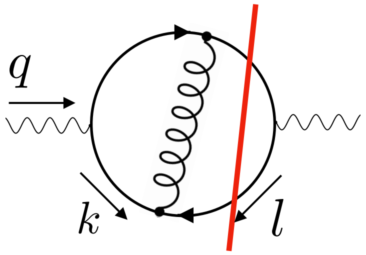


$$\approx \mathrm{d}^2\Omega_k\mathrm{d}^3\vec{l} \quad \Rightarrow$$



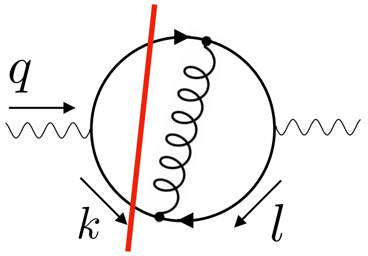
$$\approx \mathrm{d}^3\vec{k}\mathrm{d}^2\Omega_l$$

Integration measure is mis-aligned!

For the real contributions, problem known in phase-space subtraction methods

(sectoring, mappings…)

Problem 2: Too few energy variables to solve the deltas



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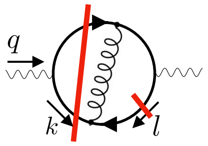
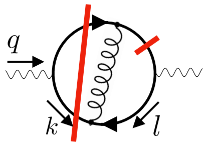
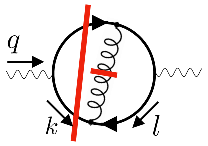
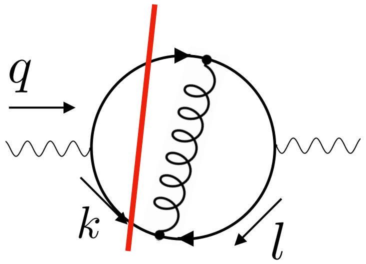




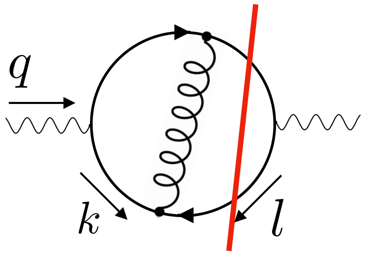


Two deltas, one energy integration

Say we use $|\vec{k}|$ and $|\vec{l}|$ to solve the remaining one



$$\approx \mathrm{d}^2\Omega_k\mathrm{d}^3\vec{l} \qquad \Rightarrow$$



$$\approx \mathrm{d}^3\vec{k}\mathrm{d}^2\Omega_l$$

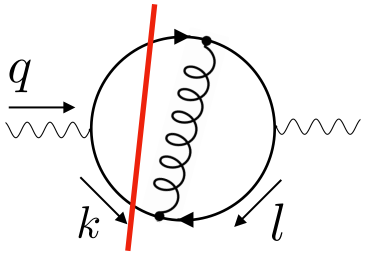
Integration measure is mis-aligned!

For the real contributions, problem known in phase-space subtraction methods

(sectoring, mappings…) **Big obstacle to automation**

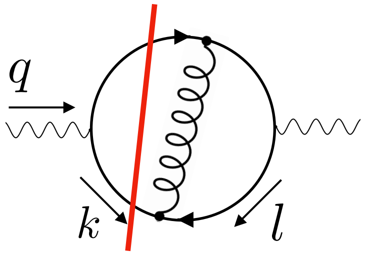
After solving all possible deltas using energy integrations

After solving all possible deltas using energy integrations



$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\,\delta(2|\vec{k}| - Q_0)\,f_{\mathrm{v},1}$$

After solving all possible deltas using energy integrations



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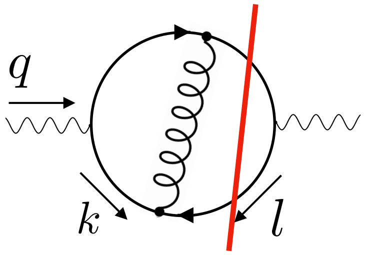
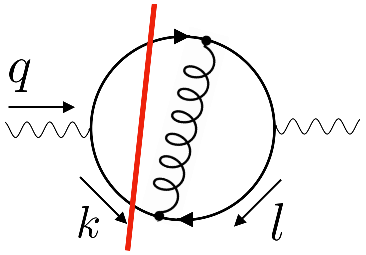
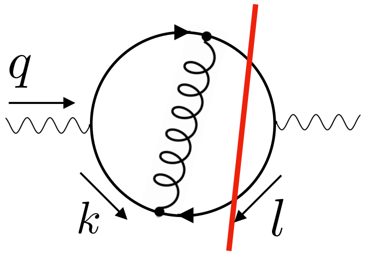


$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\,\delta(2|\vec{l}| - Q_0)\, f_{\mathrm{v},2}$$

After solving all possible deltas using energy integrations



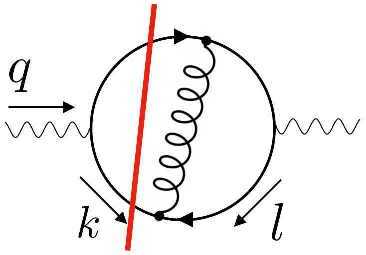
$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\,\delta(2|\vec{k}|-Q_0)\,f_{\mathrm{v},1}$$



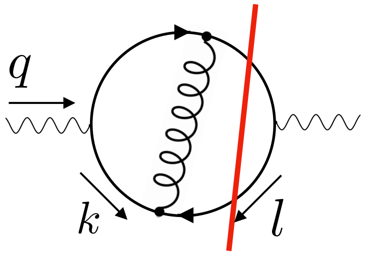
$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\,\delta(2|\vec{l}|-Q_0)\,f_{\mathrm{v},2}$$

Solution 2: Introduce fictitious variable to solve deltas

After solving all possible deltas using energy integrations



$$= \int \mathrm{d}^3\vec{k}\, \mathrm{d}^3\vec{l}\, \delta(2|\vec{k}| - Q_0)\, f_{\mathrm{v},1}$$

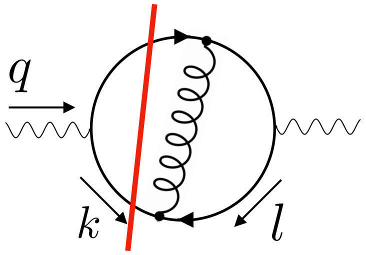


$$= \int \mathrm{d}^3\vec{k}\, \mathrm{d}^3\vec{l}\, \delta(2|\vec{l}| - Q_0)\, f_{\mathrm{v},2}$$

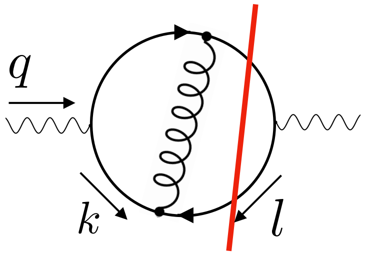
Solution 2: Introduce fictitious variable to solve deltas

$$\int \mathrm{d}^3\vec{k}\, \delta(|\vec{k}| - Q_0) f(\vec{k})$$

After solving all possible deltas using energy integrations



$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\,\delta(2|\vec{k}| - Q_0)\, f_{\mathrm{v},1}$$



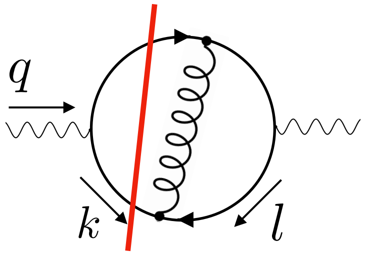
$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\,\delta(2|\vec{l}| - Q_0)\, f_{\mathrm{v},2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int \mathrm{d}^3\vec{k}\,\delta(|\vec{k}| - Q_0) f(\vec{k}) = \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\,\delta(|\vec{k}| - Q_0) f(\vec{k})$$

$$1 = \int \mathrm{d}t\, h(t)$$

After solving all possible deltas using energy integrations



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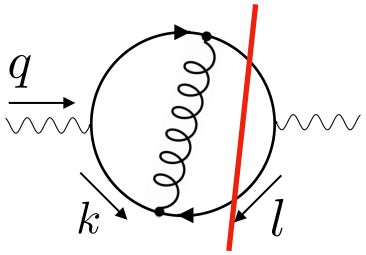

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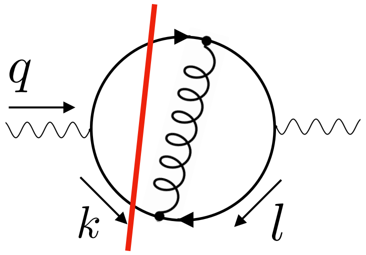
$$1 = \int \mathrm{d}t\, h(t)$$

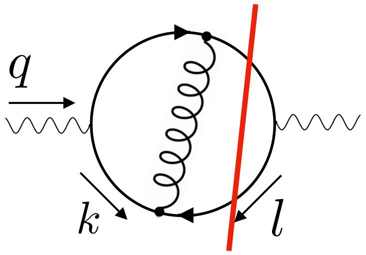
$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}t\, t^3\, h(t)\,\delta(t|\vec{k}| - Q_0) f(t\vec{k})$$

$$\vec{k} \to t\vec{k}$$

After solving all possible deltas using energy integrations



$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\,\delta(2|\vec{k}| - Q_0)\, f_{\mathrm{v},1}$$



$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\,\delta(2|\vec{l}| - Q_0)\, f_{\mathrm{v},2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int \mathrm{d}^3\vec{k}\,\delta(|\vec{k}| - Q_0) f(\vec{k}) = \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0) f(\vec{k})$$

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$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}t\, t^3\, h(t)\, \delta(t|\vec{k}| - Q_0) f(t\vec{k}) = \int \mathrm{d}^3\vec{k}\, h\left(\frac{Q_0}{|\vec{k}|}\right) \frac{Q_0^3}{|\vec{k}|^4} f\left(\frac{Q_0}{|\vec{k}|}\vec{k}\right)$$

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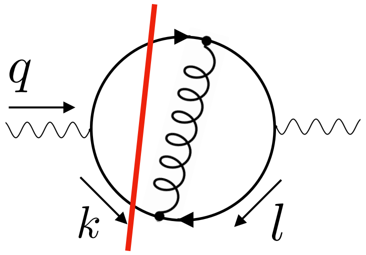
After solving all possible deltas using energy integrations



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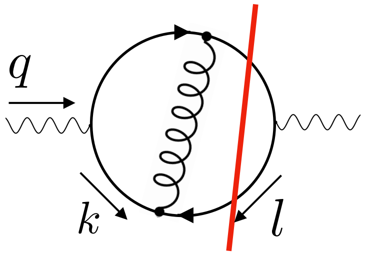

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$$1 = \int \mathrm{d}t\, h(t)$$

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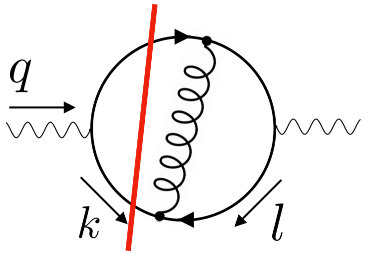
$$\vec{k} \to t\vec{k}$$

Soper,
arXiv: 9804454 (1998)

Soper,
arXiv: 0102031 (2001 @ RADCOR)

ZC, Hirschi, Pelloni, Ruijl
arXiv: 2010.01068 (2020)

After solving all possible deltas using energy integrations



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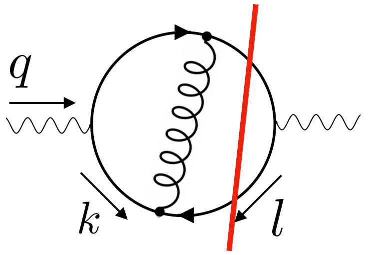


$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\,\delta(2|\vec{l}| - Q_0)\, f_{\mathrm{v},2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int \mathrm{d}^3\vec{k}\,\delta(|\vec{k}| - Q_0) f(\vec{k}) \;= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\,\delta(|\vec{k}| - Q_0) f(\vec{k})$$

$$1 = \int \mathrm{d}t\, h(t)$$

$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}t\, t^3\, h(t)\,\delta(t|\vec{k}| - Q_0) f(t\vec{k}) = \int \mathrm{d}^3\vec{k}\, h\left(\frac{Q_0}{|\vec{k}|}\right)\frac{Q_0^3}{|\vec{k}|^4} f\left(\frac{Q_0}{|\vec{k}|}\vec{k}\right)$$

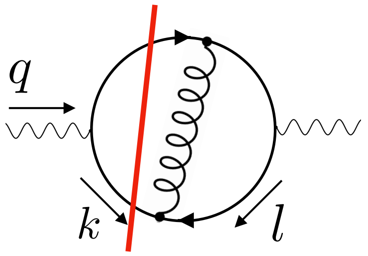
$$\vec{k} \to t\vec{k}$$

Soper,
arXiv: 9804454 (1998)

Soper,
arXiv: 0102031 (2001 @ RADCOR)

ZC, Hirschi, Pelloni, Ruijl
arXiv: 2010.01068 (2020)

In the end:



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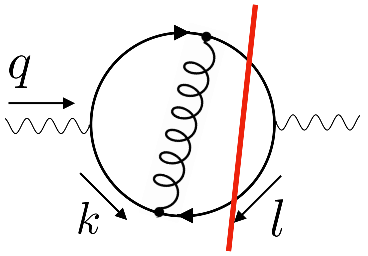
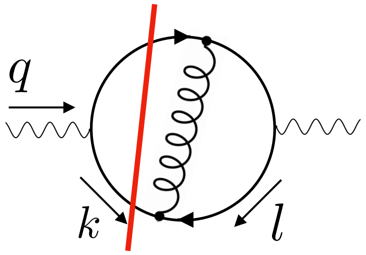
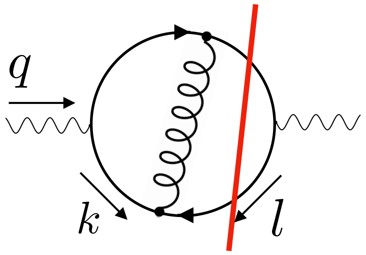


$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\left[g_{\mathrm{v},1}(\vec{k},\vec{l}) + g_{\mathrm{v},2}(\vec{k},\vec{l})\right]$$

After solving all possible deltas using energy integrations



$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\,\delta(2|\vec{k}| - Q_0)\, f_{\mathrm{v},1}$$



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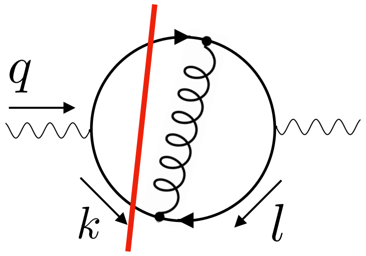

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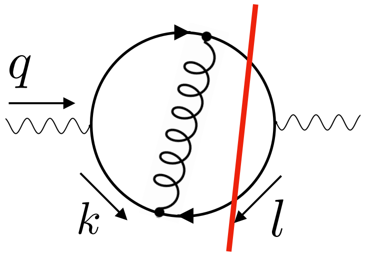
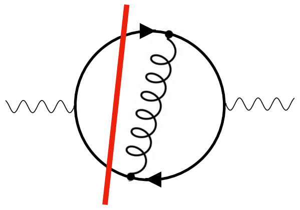


$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\left[g_{\mathrm{v},1}(\vec{k},\vec{l}) + g_{\mathrm{v},2}(\vec{k},\vec{l})\right]$$

Observation: solved deltas, phase-space has same dimensionality (redundant dimension)



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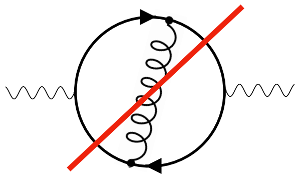

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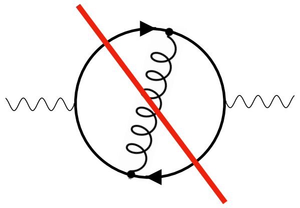

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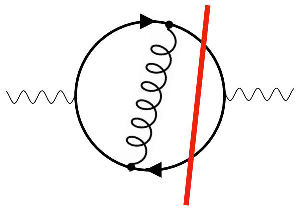
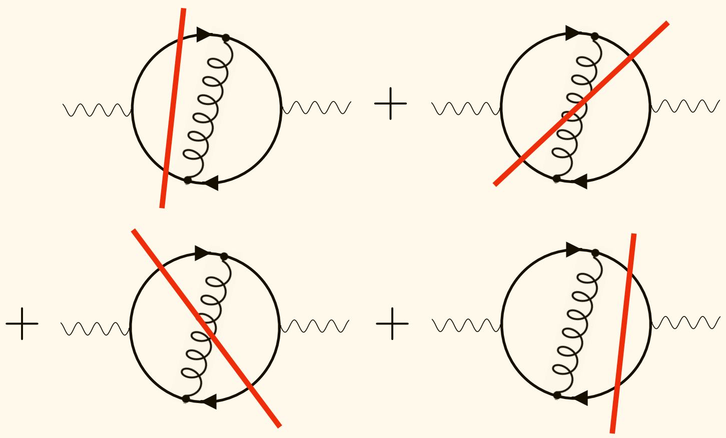


$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{l}\,\left[g_{\mathrm{v},1}(\vec{k},\vec{l}) + g_{\mathrm{v},2}(\vec{k},\vec{l}) + g_{\mathrm{r},1}(\vec{k},\vec{l}) + g_{\mathrm{r},2}(\vec{k},\vec{l})\right]$$



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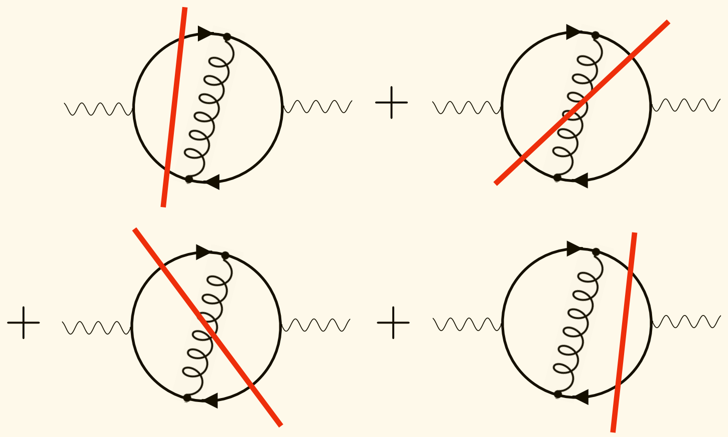
Measure is aligned now!



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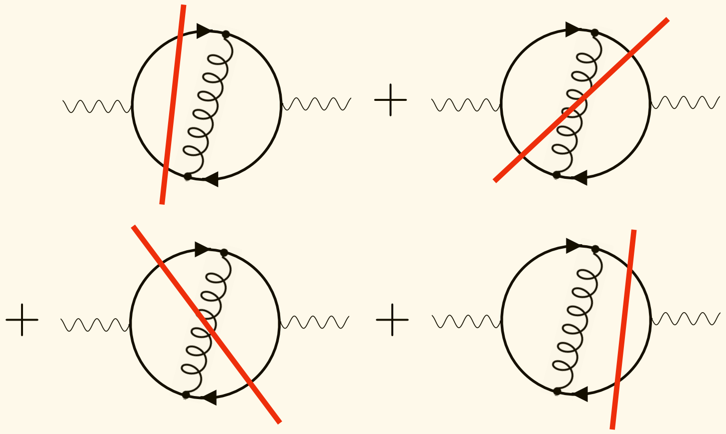

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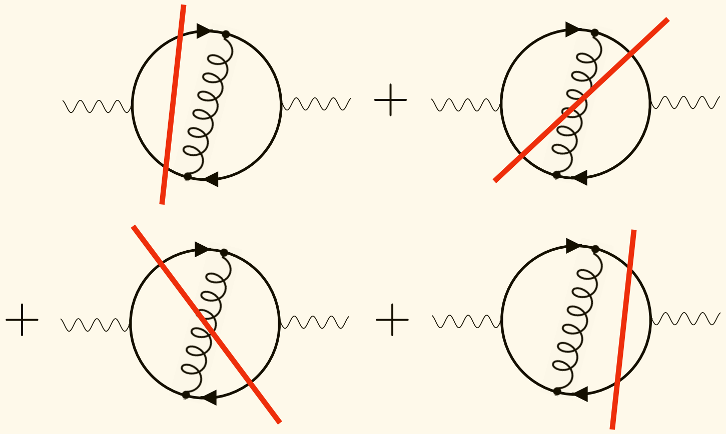
A few comments:

- 3D representations hold at N loops



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- ✦ 3D representations hold at N loops
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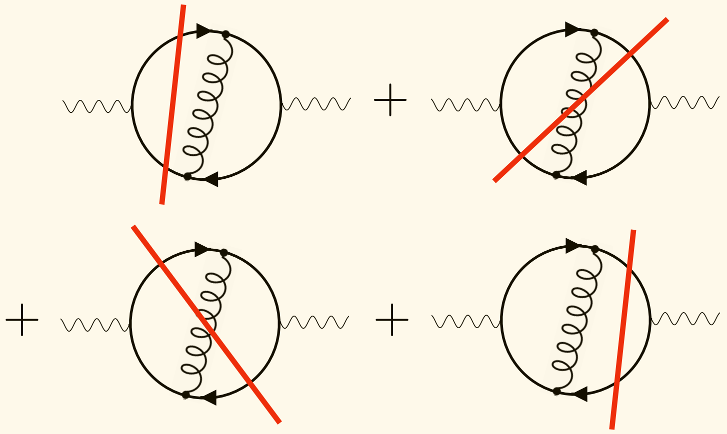

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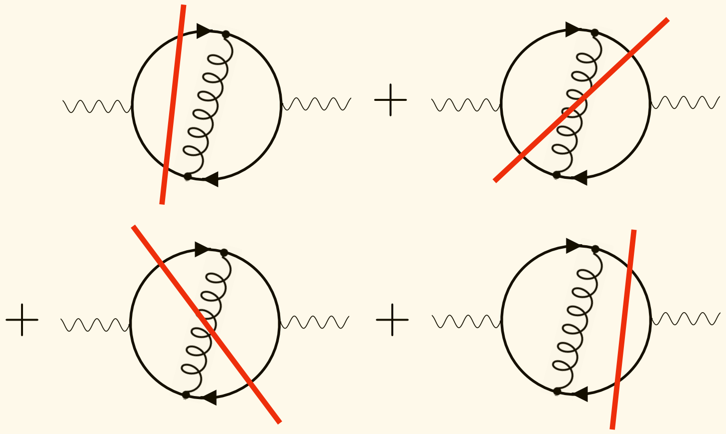

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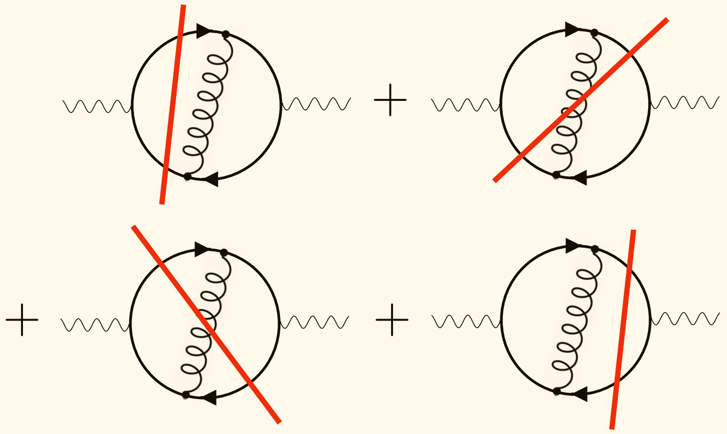

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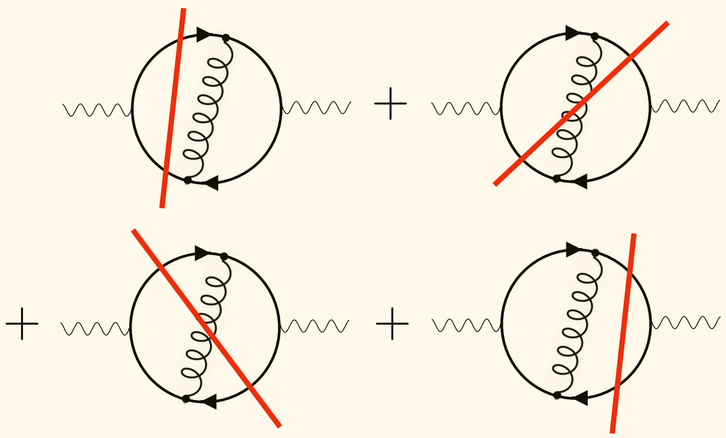
Proof $\Longrightarrow$ ZC, Hirschi, Pelloni, Ruijl
arXiv: 2010.01068 (2020)



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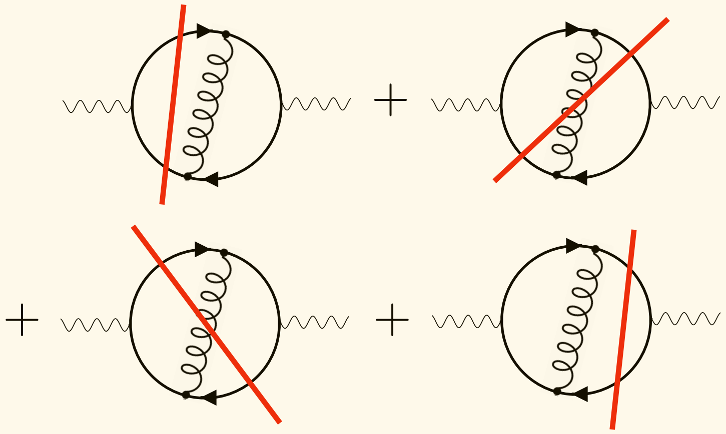
- Deep relationship between interference diagrams and local residues of hyper surfaces



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  Cutkosky result but at the local level

Local IR cancellations, what next?

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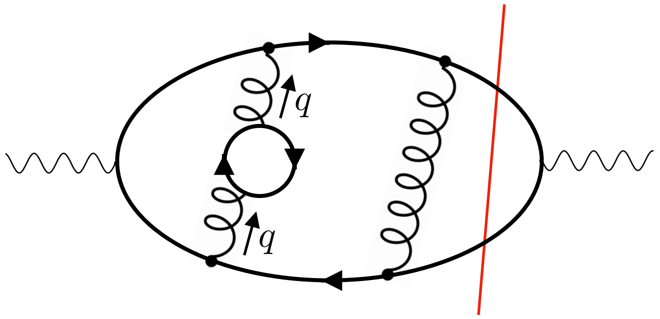

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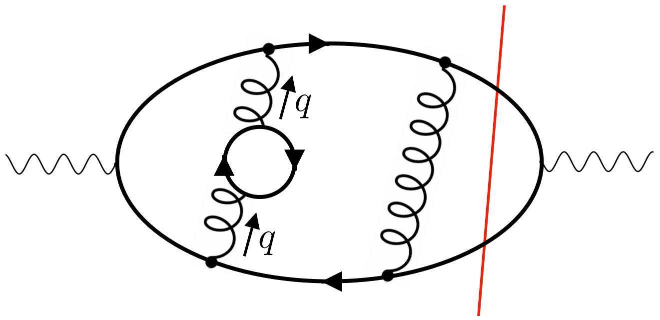

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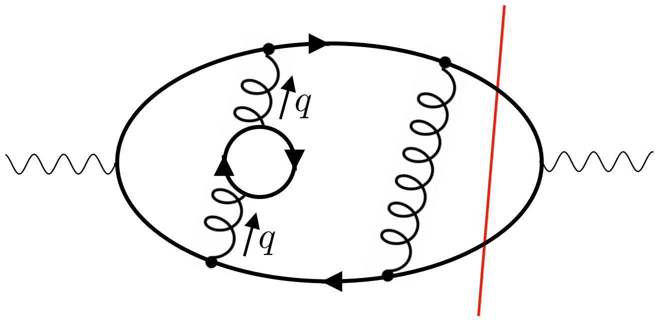

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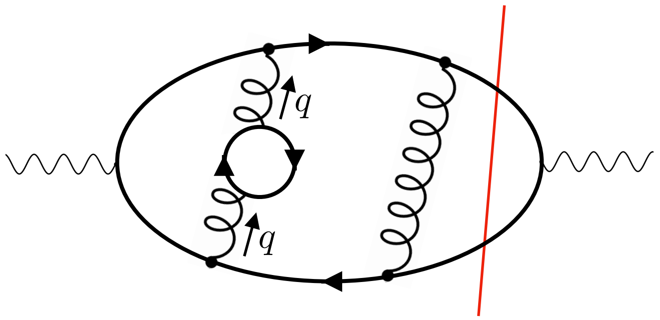

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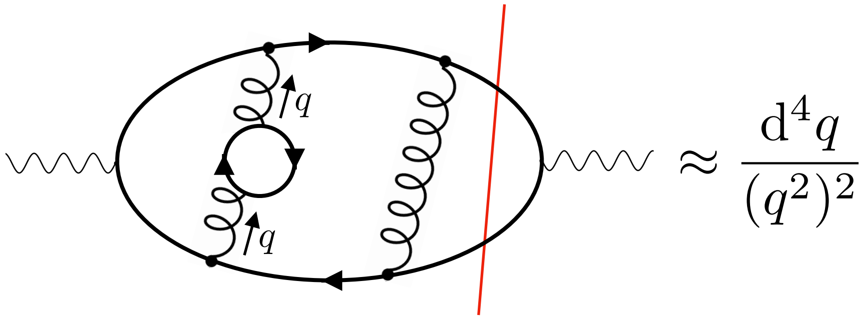

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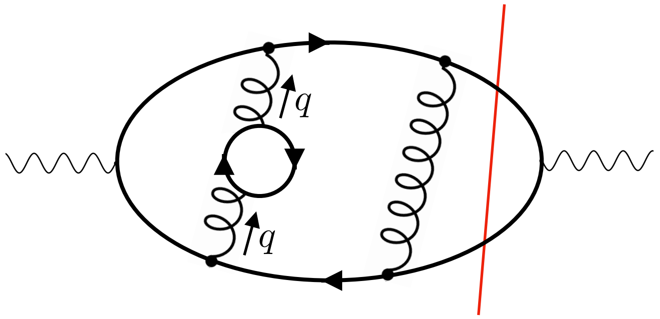

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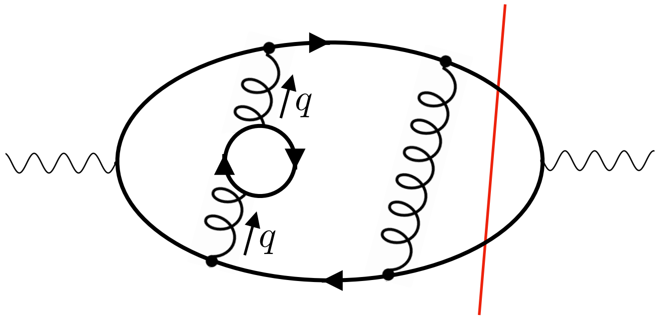

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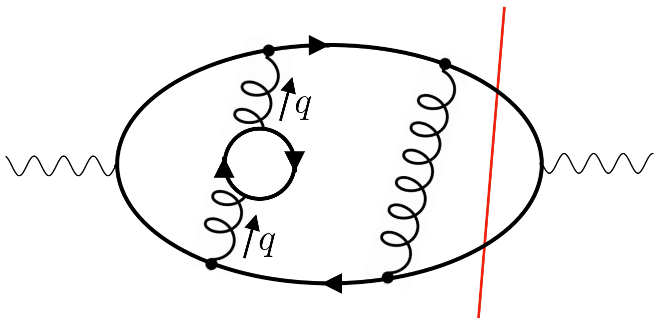

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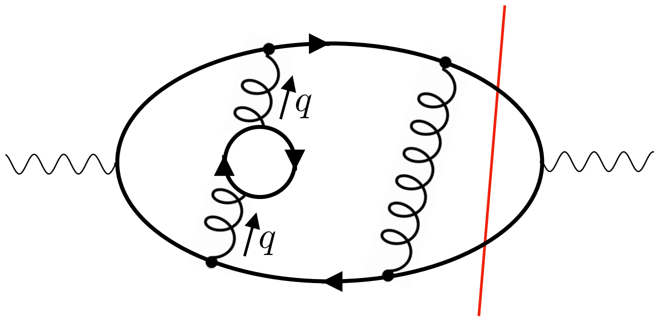

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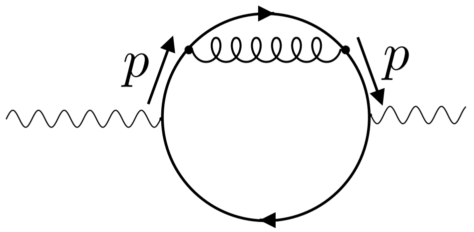

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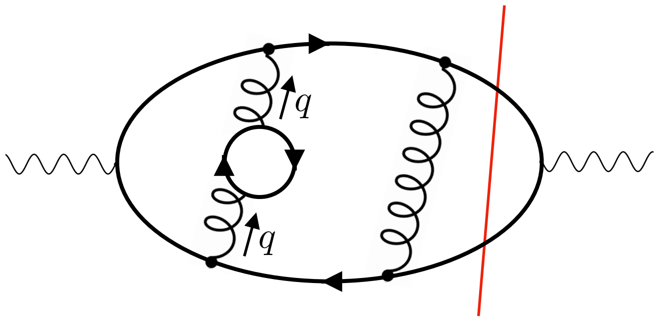

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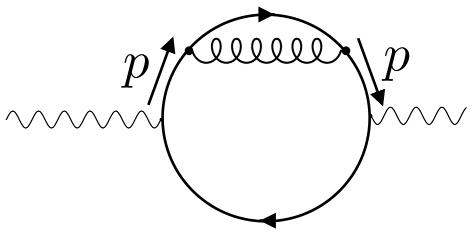

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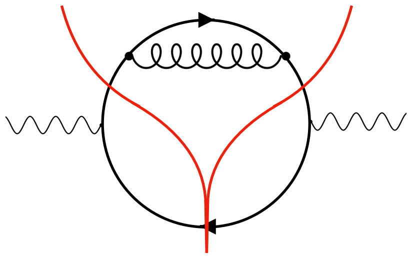

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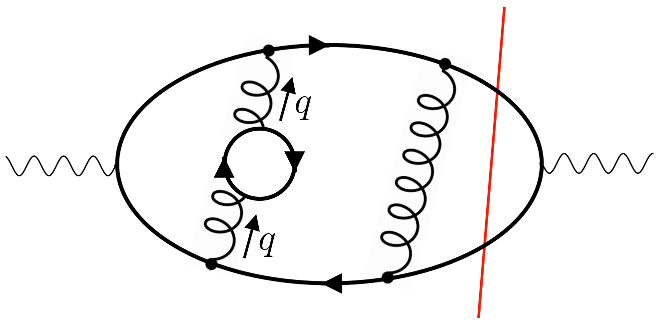

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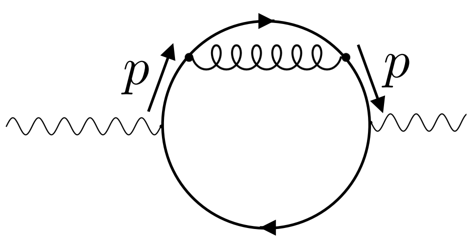

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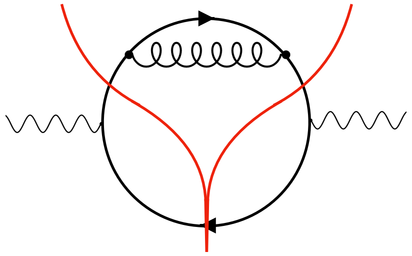

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→ Local subtraction of UV divergences

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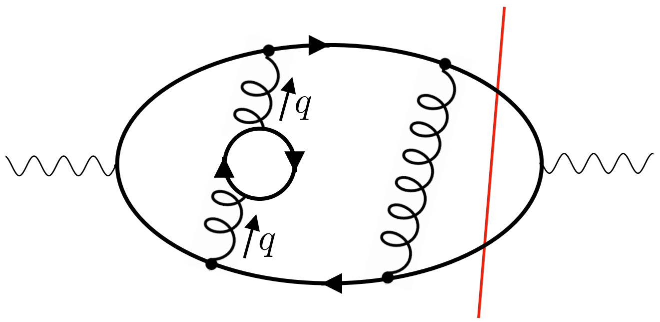

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Localised Renormalisation

→ Minimal analytic complexity: only requires computation of vacuum bubbles

→ Hybrid MSbar+OS schemes

Method is only limited by computation of vacuum diagrams (known up to 4 loops)

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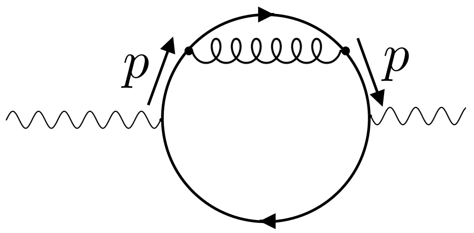

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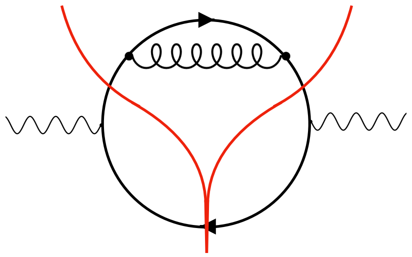

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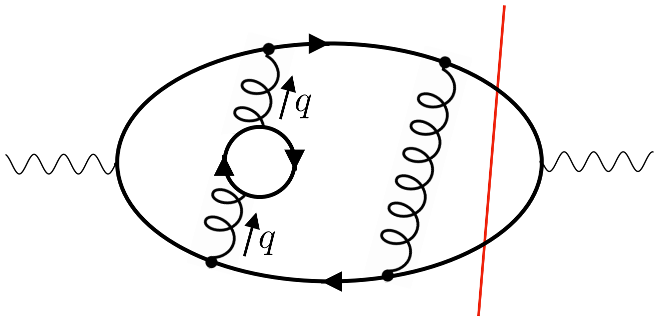

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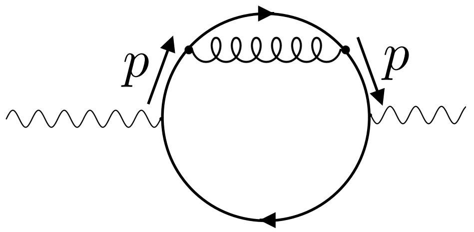

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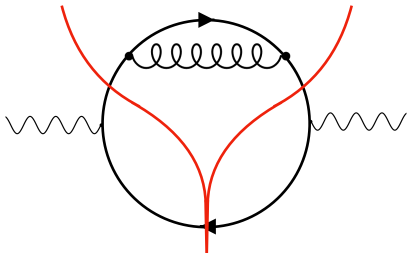

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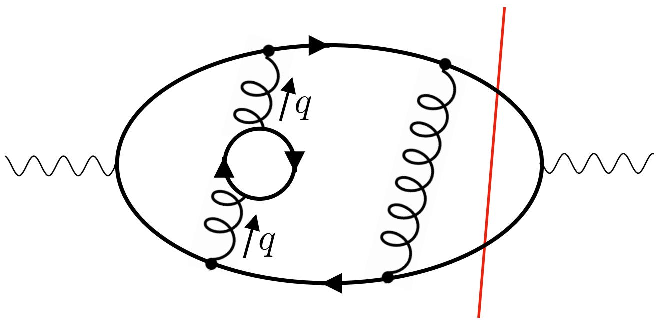

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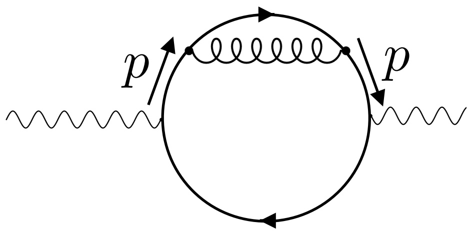

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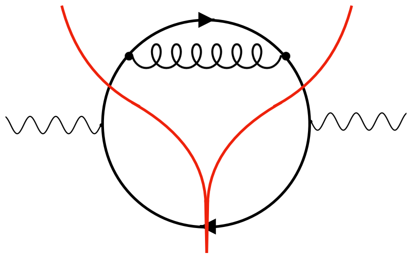

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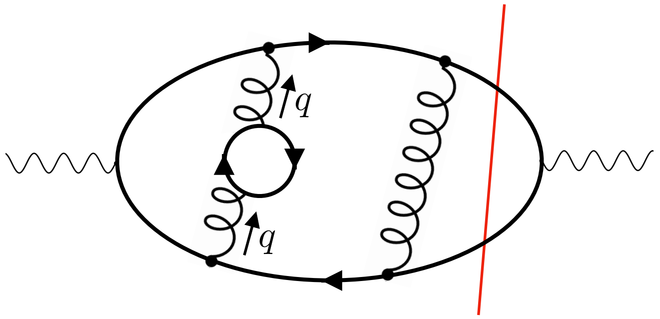

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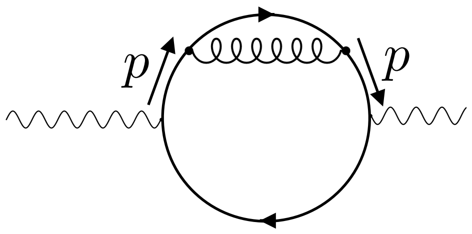

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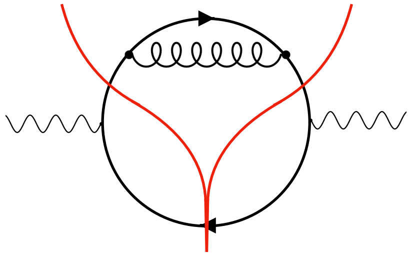

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Generalised cutting rules and LU representation

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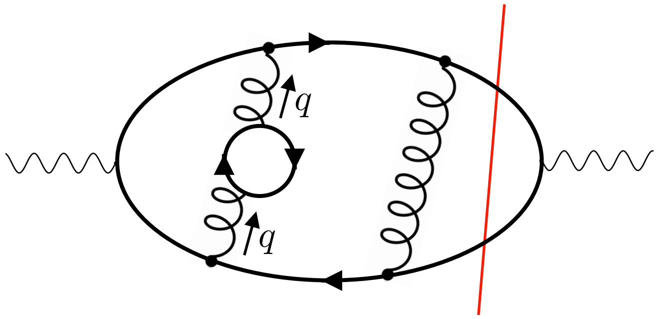

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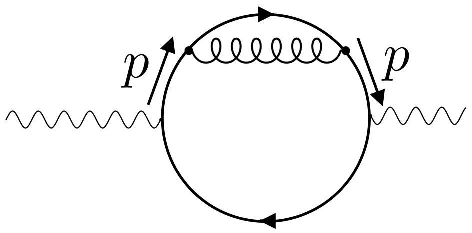

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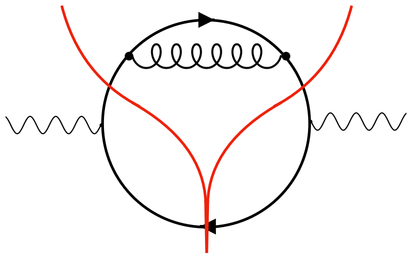

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Everything solved generically in arXiv: 2203.11038

Tests and results

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and $e^+e^- \to t\bar{t}$

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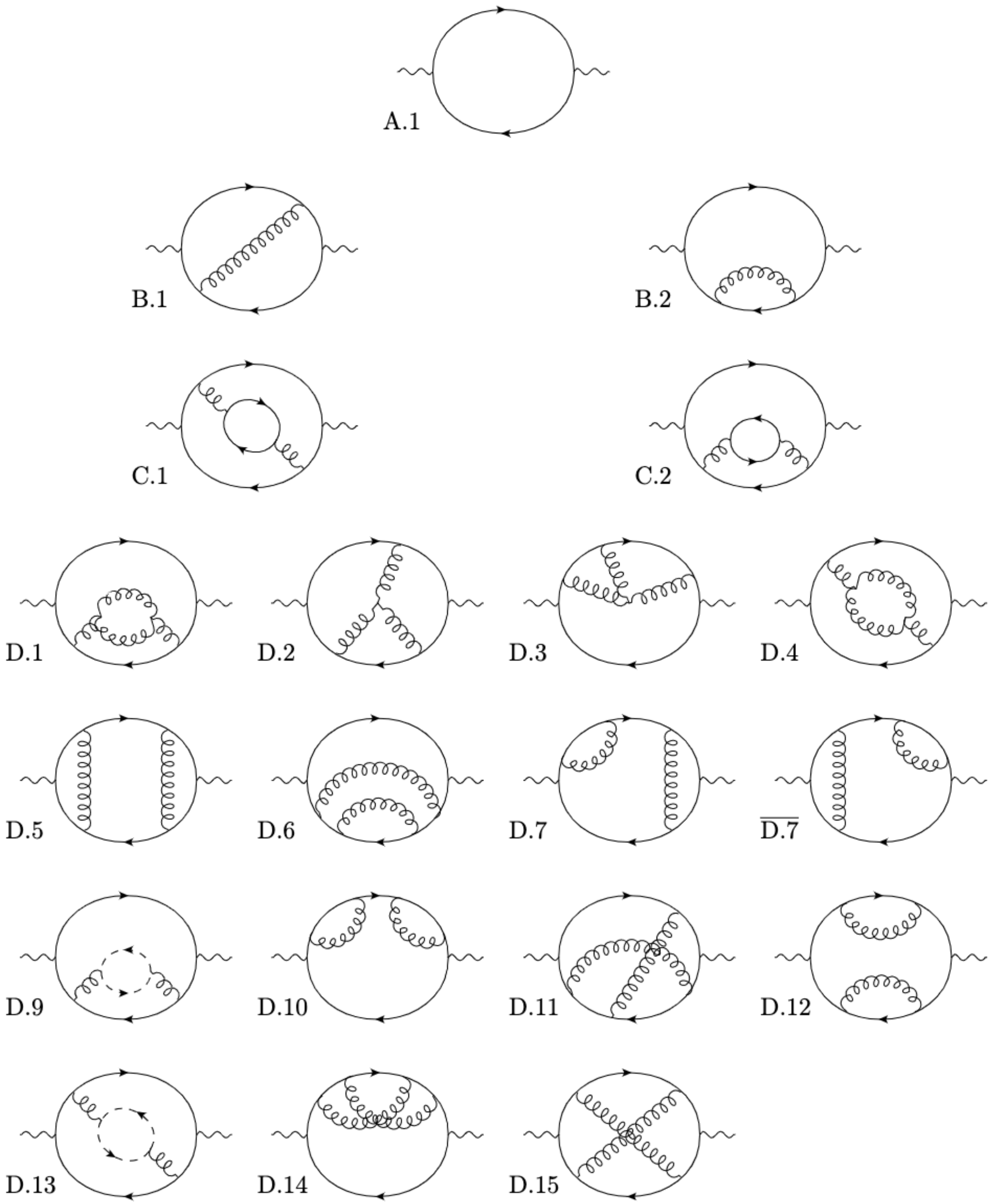

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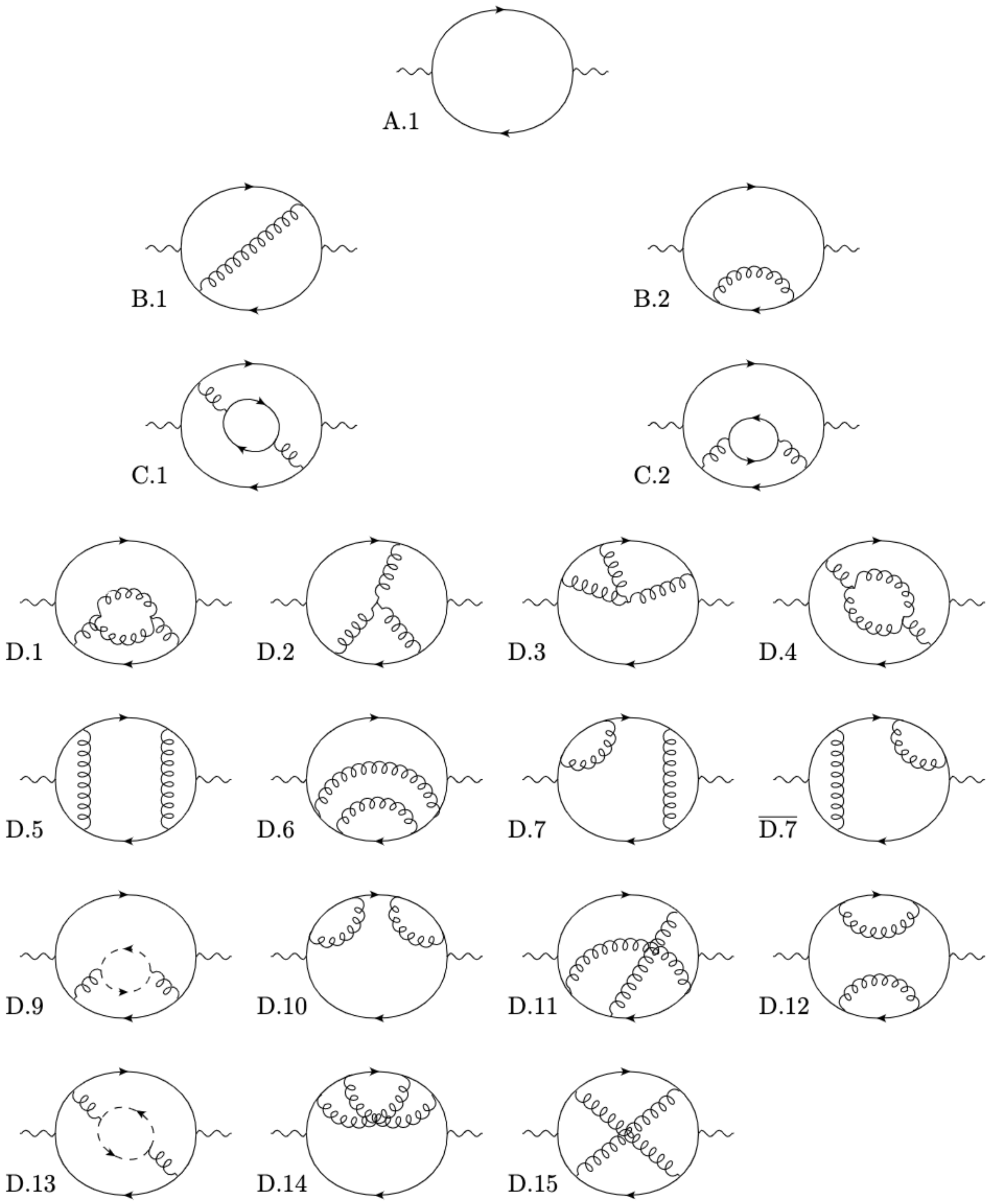

SG id	Ξ	$\sigma_{\gamma^\star\to t\bar{t}}^{[\alpha_s^{(\overline{\mathrm{MS}})},m_t^{(\mathrm{OS})}]}$ [GeV^{-2}] $\mu_r^2=m_t^2,\ p_{\gamma^\star}^2=(400\text{ GeV})^2$	Δ [%]	$\sigma_{\gamma^\star\to t\bar{t}}^{[\alpha_s^{(\overline{\mathrm{MS}})},m_t^{(\mathrm{OS})}]}$ [GeV^{-2}] $\mu_r^2=m_t^2,\ p_{\gamma^\star}^2=(3000\text{ GeV})^2$	Δ [%]
LO		$\mathcal{O}(\alpha_s^0)$			
A.1	1	1.387586 · 10^{+00}	0.0011	1.509262 · 10^{+01}	0.000064
Total		1.387586 · 10^{+00}	0.0011	1.509262 · 10^{+01}	0.000064
NLO		$\mathcal{O}(\alpha_s)$			
B.1	1	2.52705 · 10^{-01}	0.034	-6.3725 · 10^{-01}	0.071
B.2	2	1.80050 · 10^{-01}	0.049	1.22702 · 10^{+00}	0.039
Total		4.3276 · 10^{-01}	0.028	5.8977 · 10^{-01}	0.11
Benchmark		4.32831 · 10^{-01}	-0.018	5.9047 · 10^{-01}	-0.12
NNLO		$\mathcal{O}(\alpha_s^2)$ ($n_f=1$ contribution)			
C.1	1	-1.0022 · 10^{-03}	0.17	2.6658 · 10^{-02}	0.059
C.2	2	-4.6982 · 10^{-03}	0.081	-8.388 · 10^{-03}	0.30
Total		-5.7004 · 10^{-03}	0.073	1.8270 · 10^{-02}	0.16
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D.1	2	3.8886 · 10^{-02}	0.031	6.3163 · 10^{-02}	0.11
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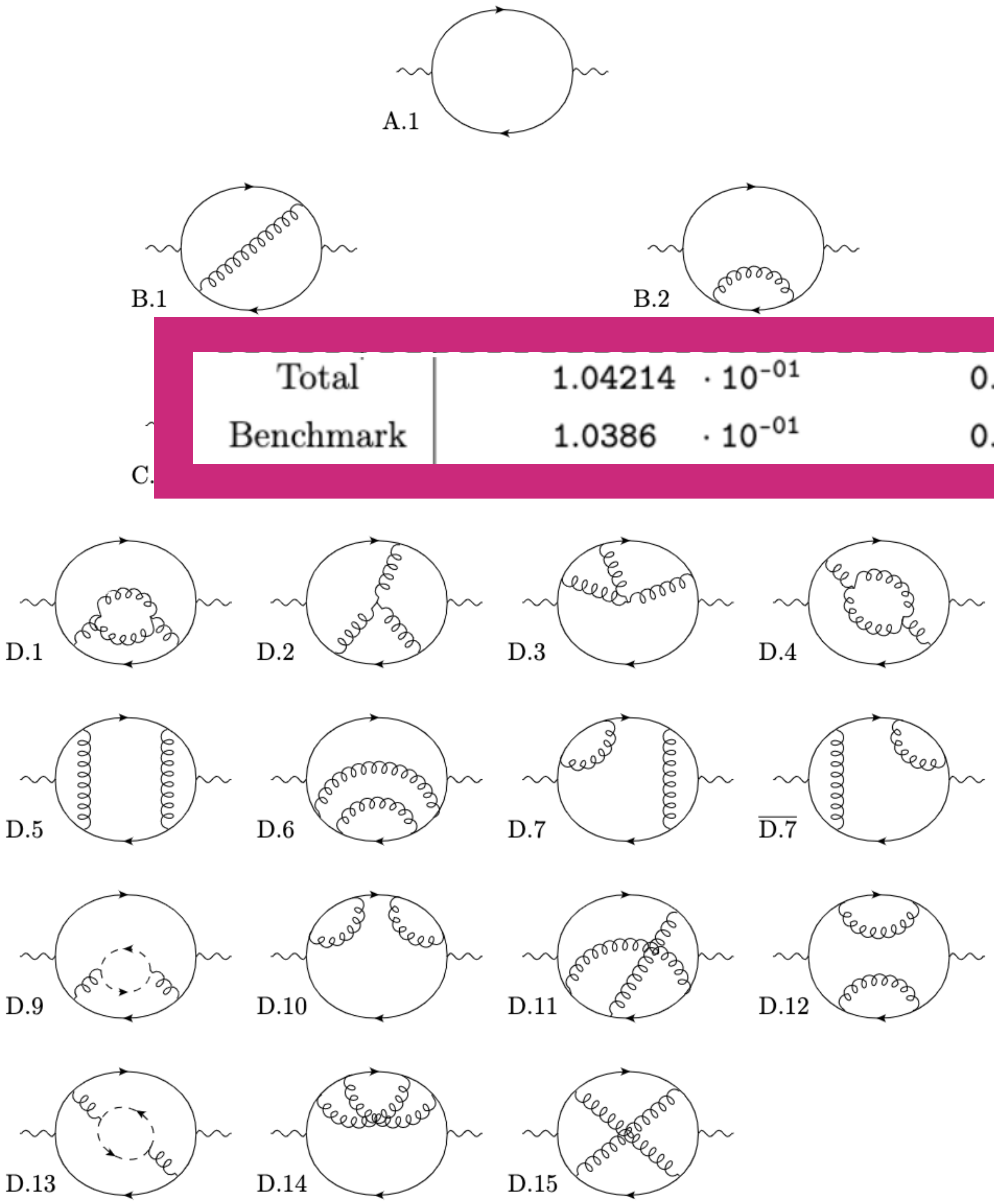

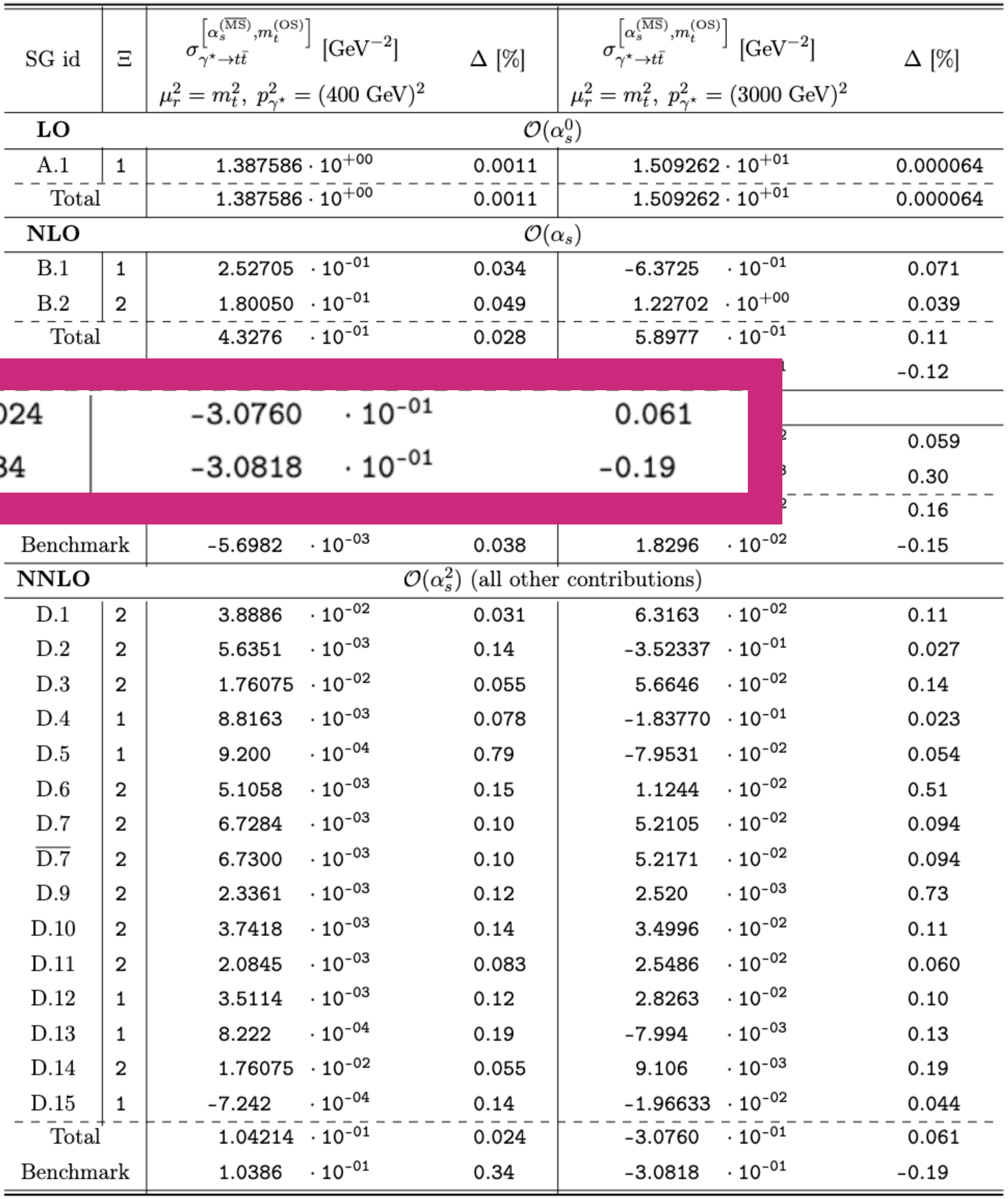

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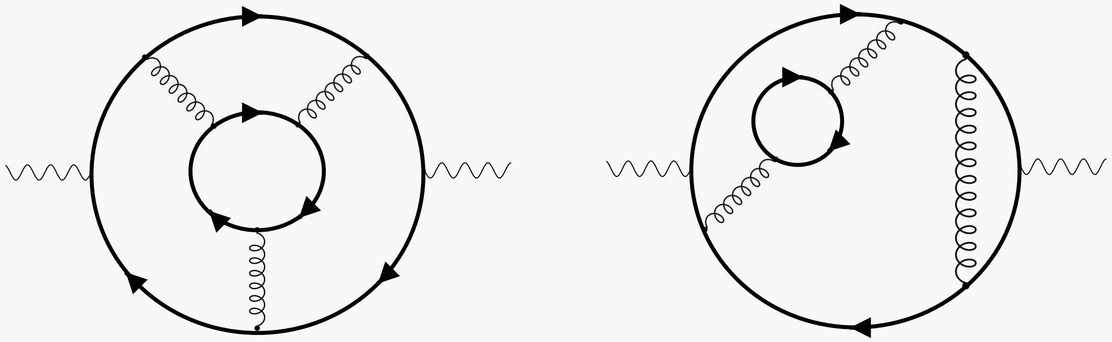

(+ many more)

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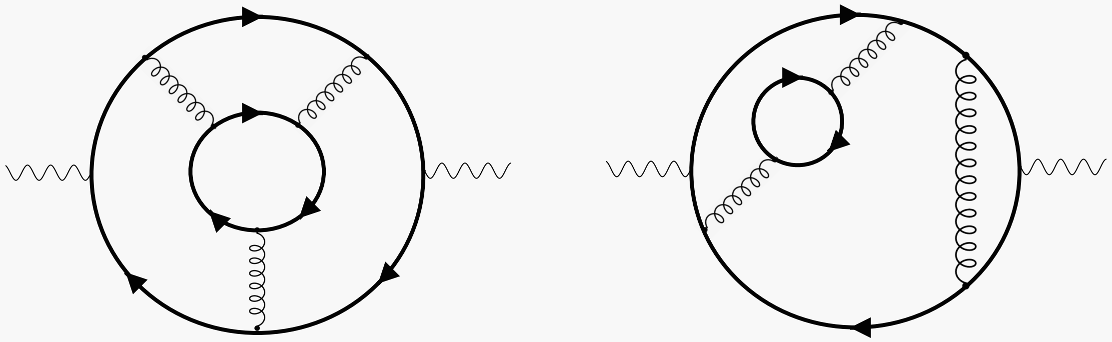

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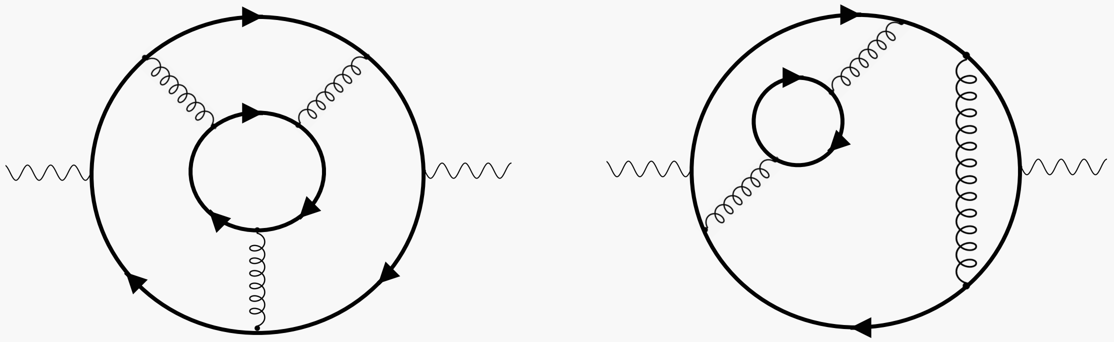

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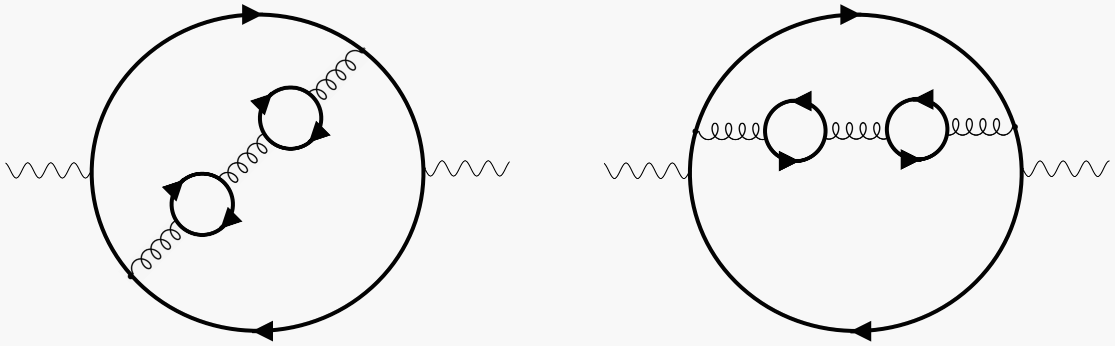

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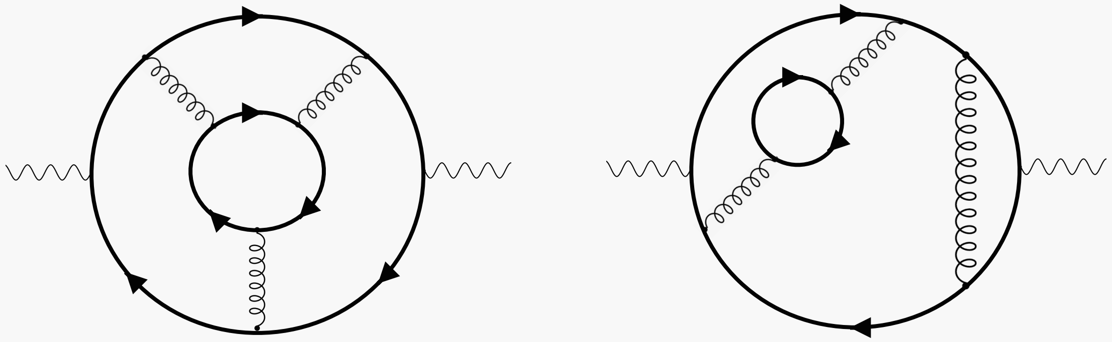

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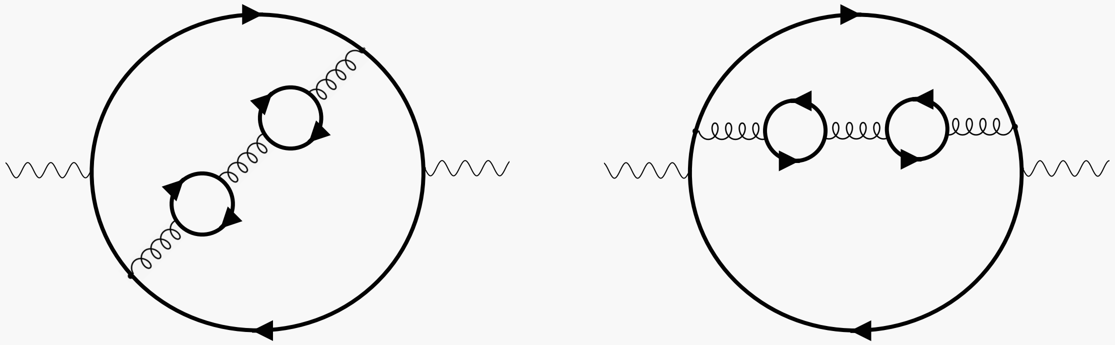

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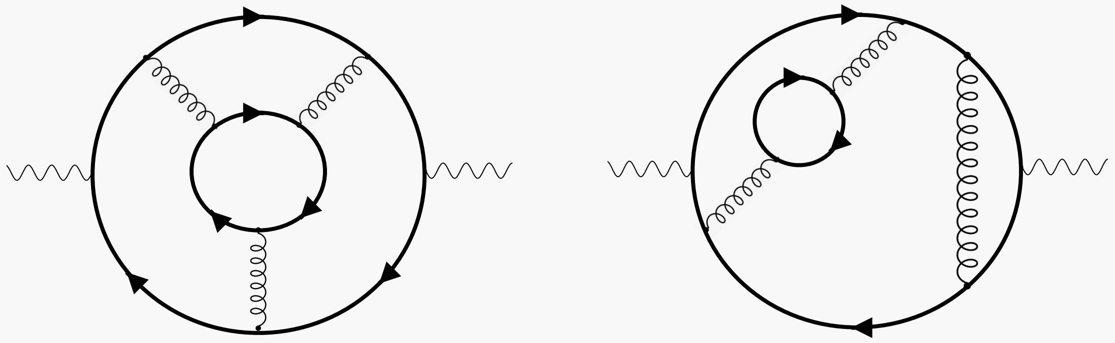

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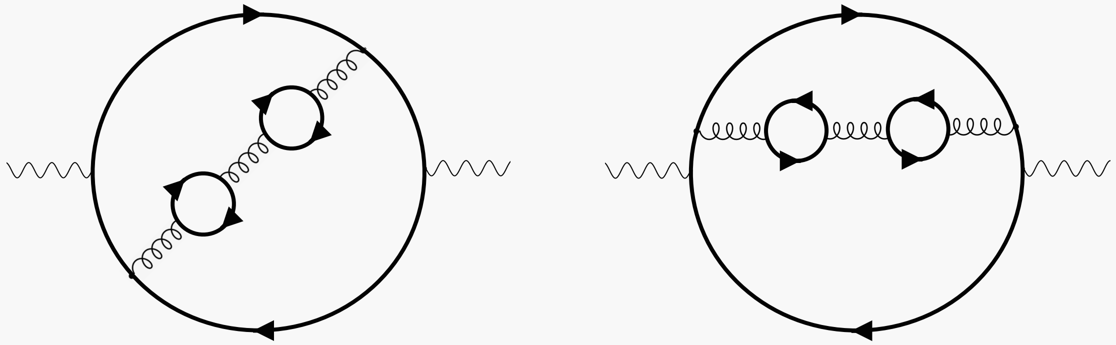

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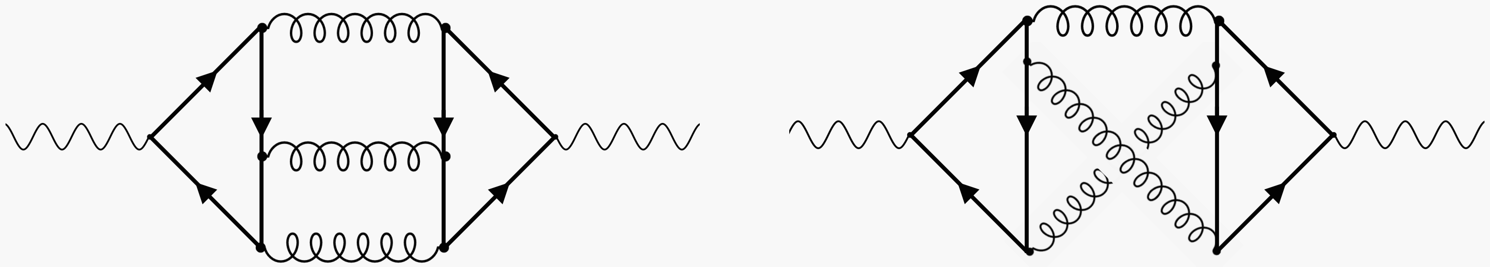

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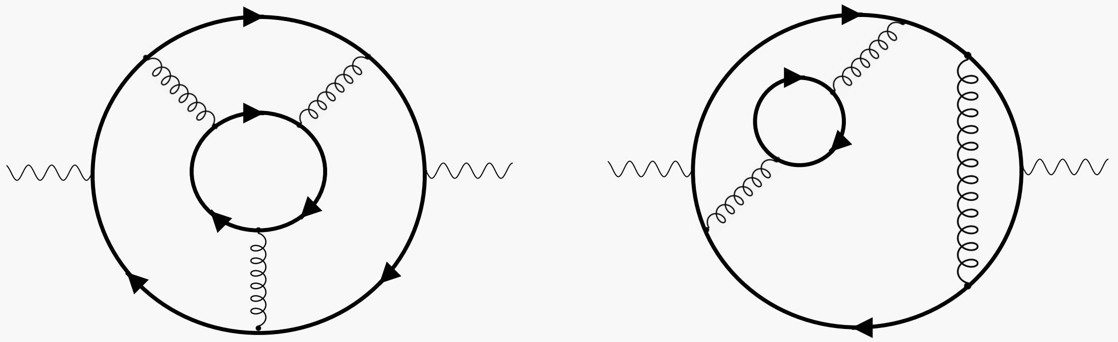
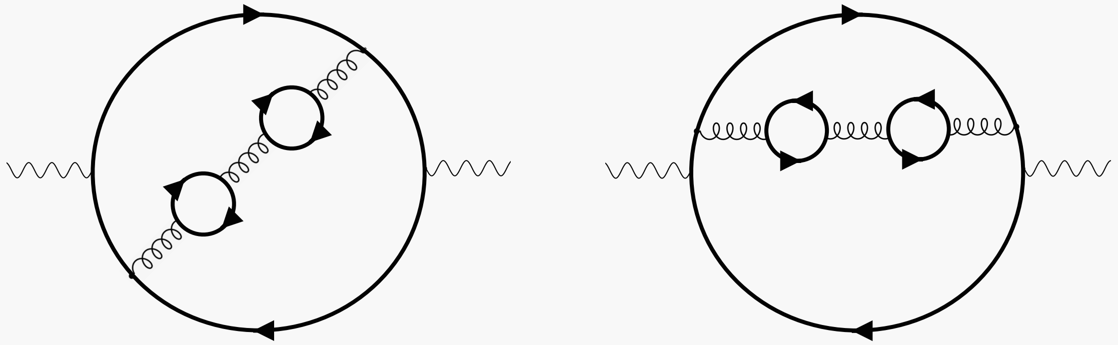


(+ many more)

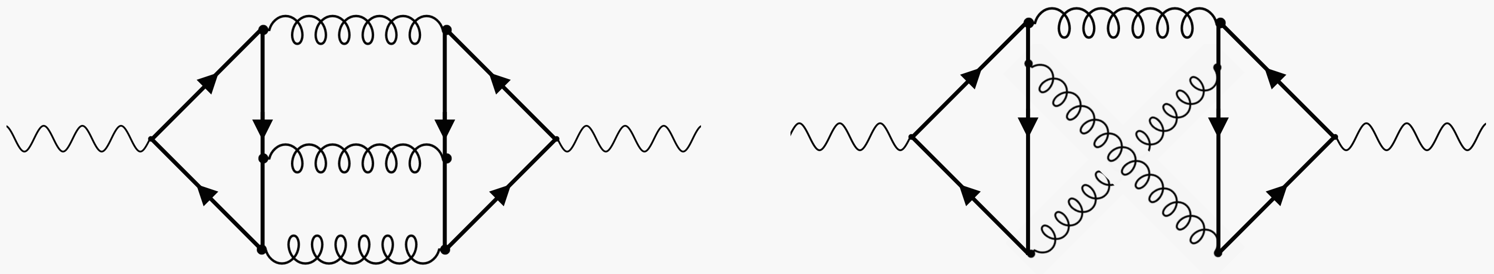
$$K_{\alpha_s^3}^{N_f,\mathrm{LU}} = -77.1(1.7)$$

N_f^2 part @N^{3}LO $e^+e^- \to jj$



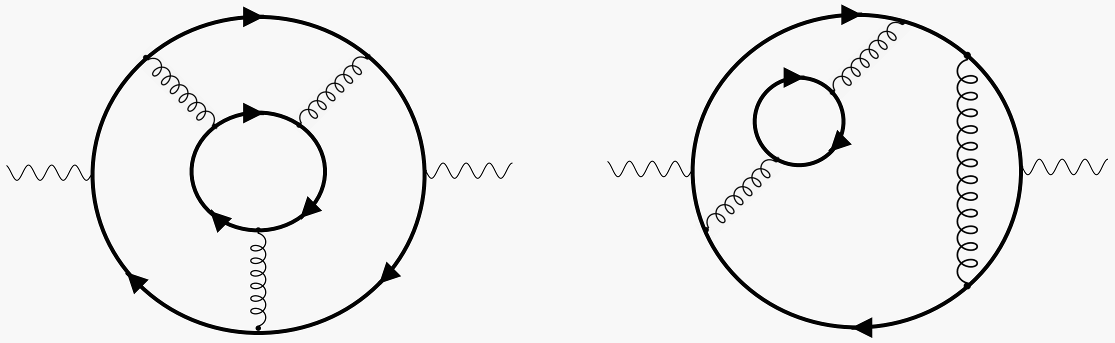
$$K_{\alpha_s^3}^{N_f^2,\mathrm{LU}} = -0.35(24)$$

singlet part @N^{3}LO $e^+e^- \to jj$



$$K_{\alpha_s^3}^{\mathrm{singlet},\mathrm{LU}} = -25.6(1.5)$$

N_f part @N^{3}LO $e^+e^- \to jj$

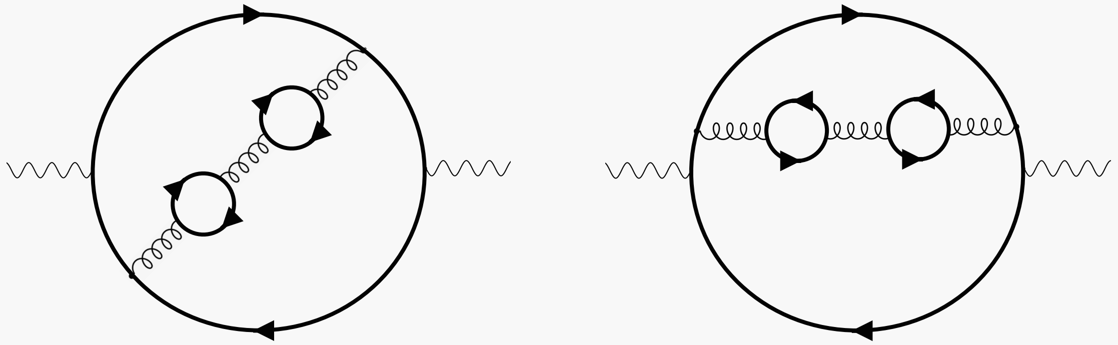


(+ many more)

$$K_{\alpha_s^3}^{N_f,\mathrm{LU}} = -77.1(1.7)$$

$$K_{\alpha_s^3}^{N_f,\mathrm{bm}} = -76.8086$$

N_f^2 part @N^{3}LO $e^+e^- \to jj$



$$K_{\alpha_s^3}^{N_f^2,\mathrm{LU}} = -0.35(24)$$

$$K_{\alpha_s^3}^{N_f^2,\mathrm{bm}} = -0.331415$$

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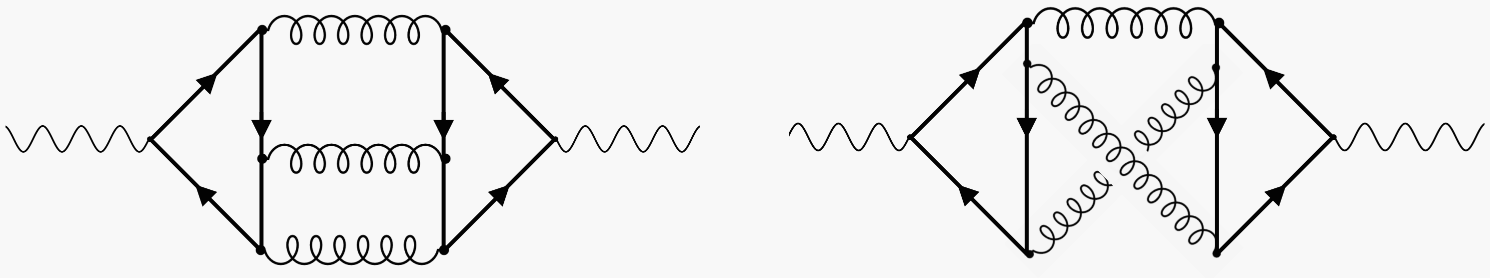
singlet part @N^{3}LO $e^+e^- \to jj$



$$K_{\alpha_s^3}^{\mathrm{singlet},\mathrm{LU}} = -25.6(1.5)$$

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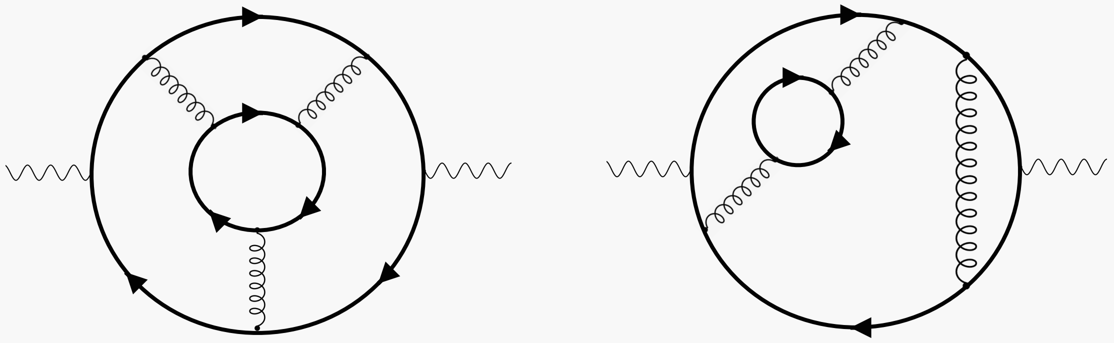

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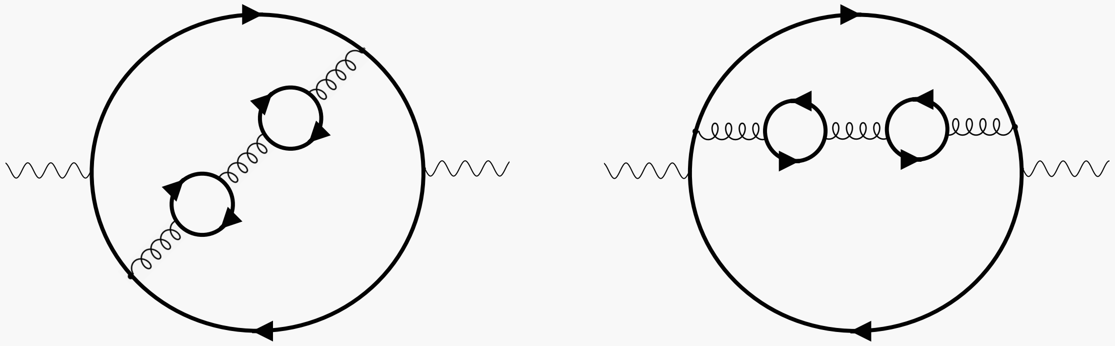

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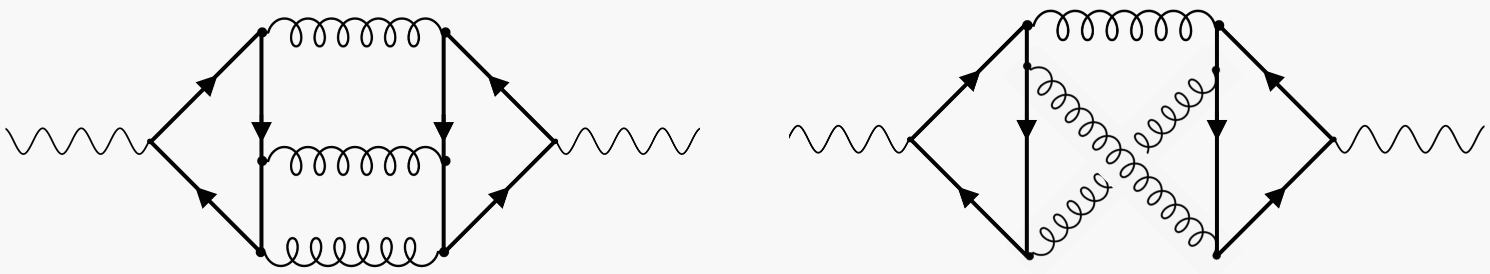

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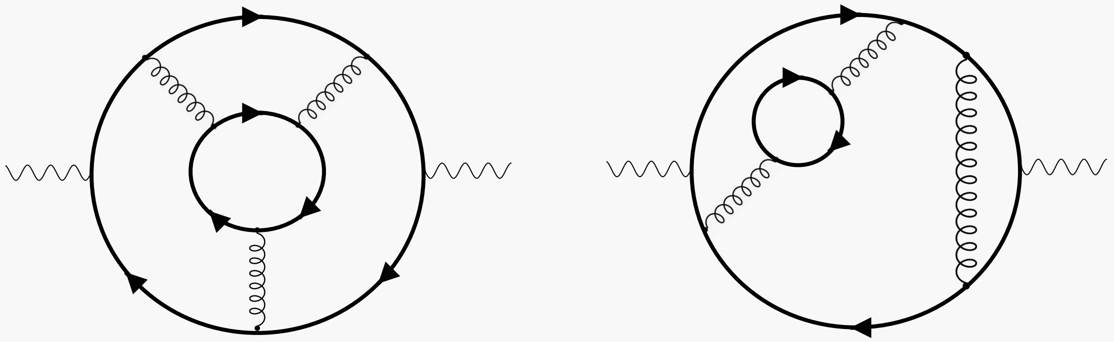
Benchmarks:

Herzog, Ruijl, Ueda, Vermaseren, Vogt :
arXiv:1707.01044

N_f part @N^{3}LO $e^+e^- \to jj$



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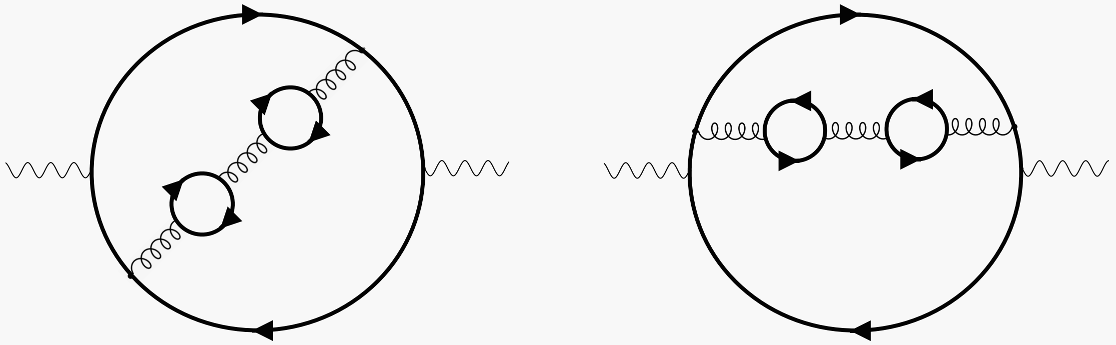

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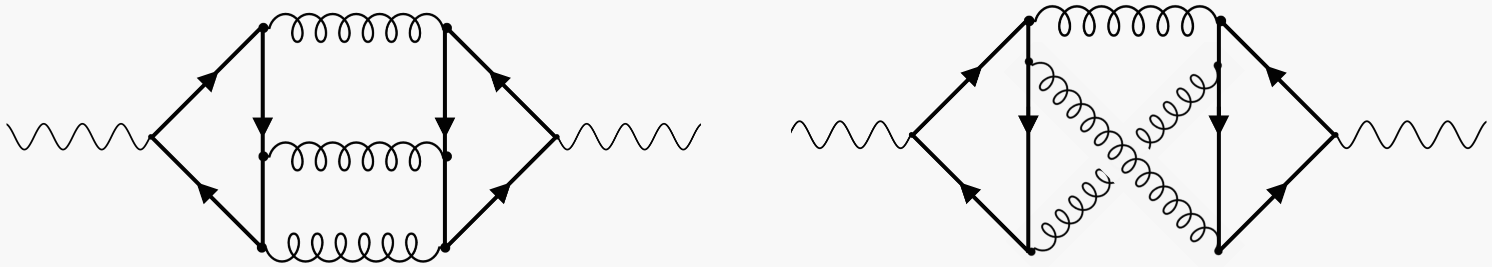

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Testing
- ✓ N3LO IR cancellations
- ✓ 3-loop UV renormalisation
- ✓ 1,2,3-loop self-energies

Outlook

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Local Unitarity soon ready for automation of processes without initial-state singularities

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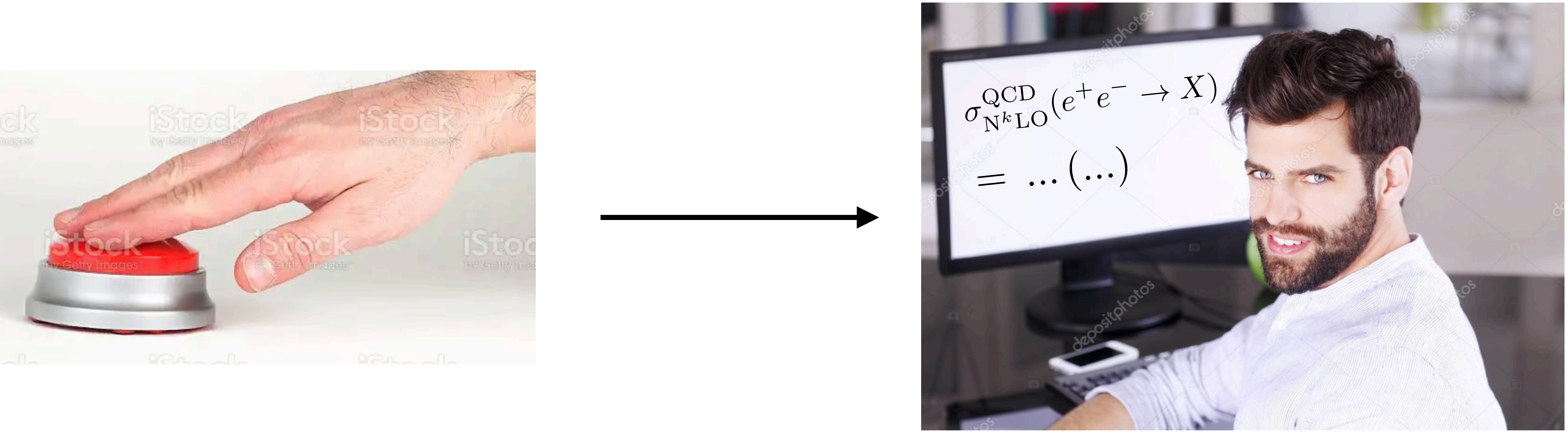

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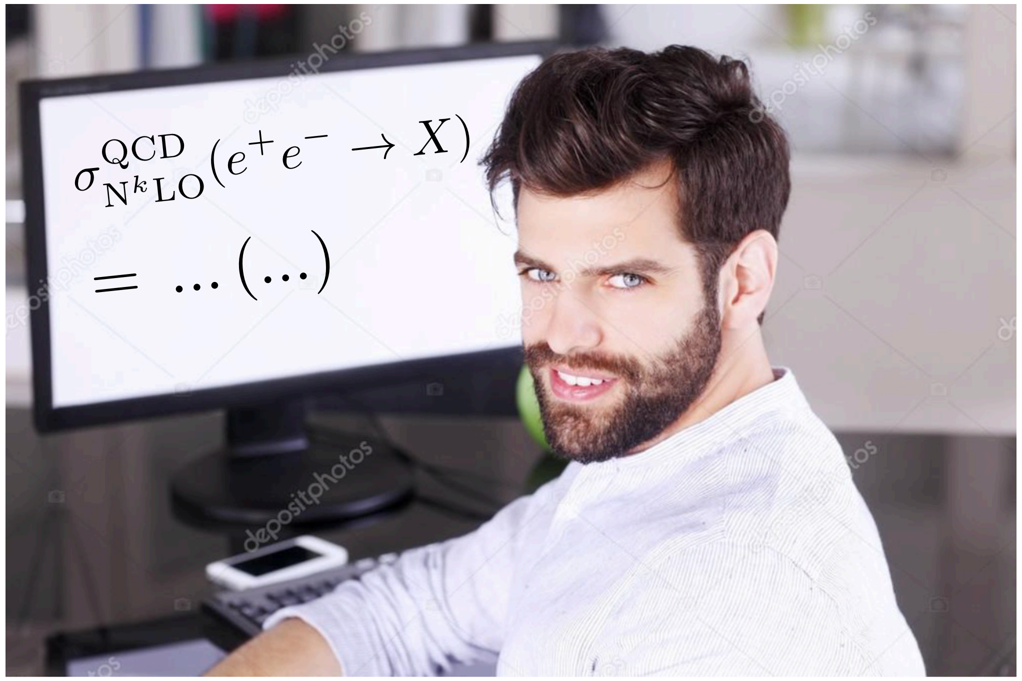

Future theory work:

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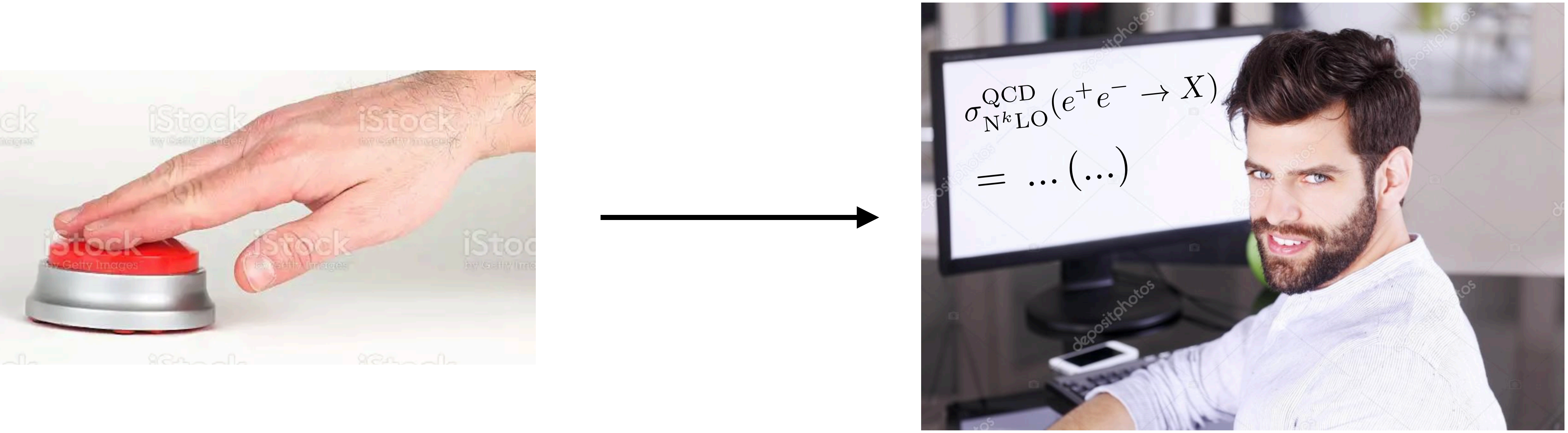

Future theory work:

Resummation: we would like an automatable solution, that only relies on the general factorisation properties of amplitudes

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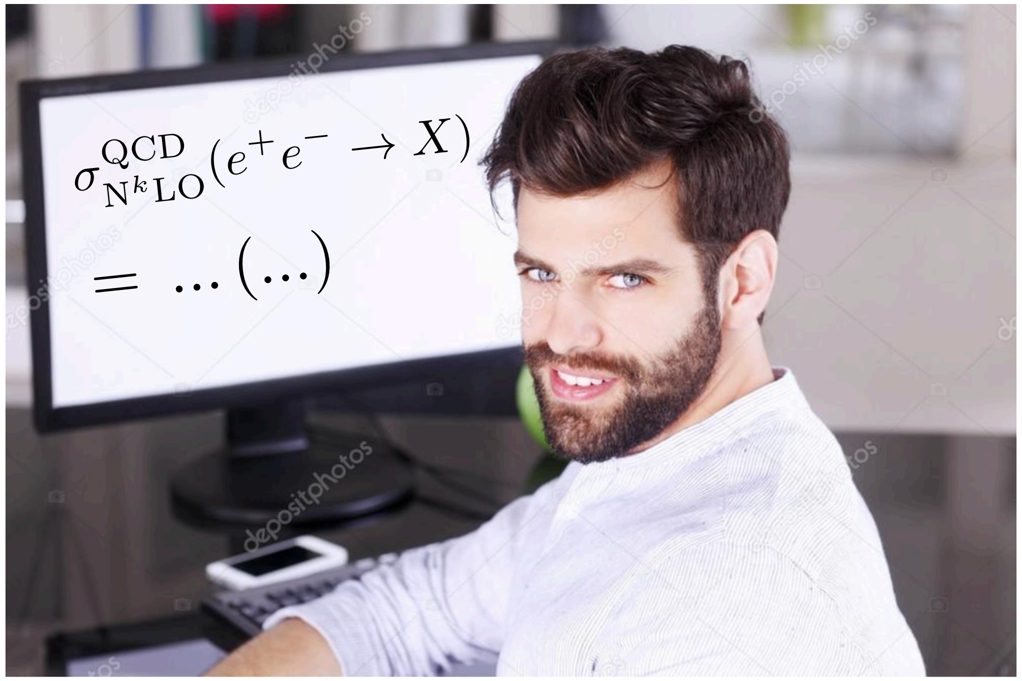

Future theory work:

Resummation: we would like an automatable solution, that only relies on the general factorisation properties of amplitudes Banfi, Salam, Zanderighi, arXiv:0304148

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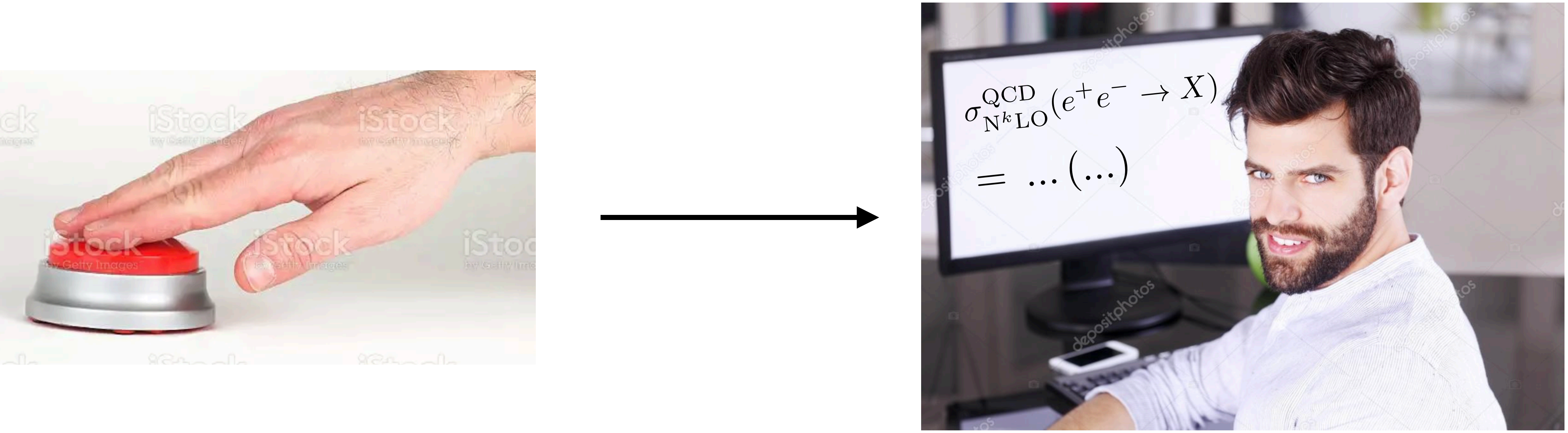

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Initial-state singularities: develop a competitive method for the treatment of initial-state singularities that is compatible with LU

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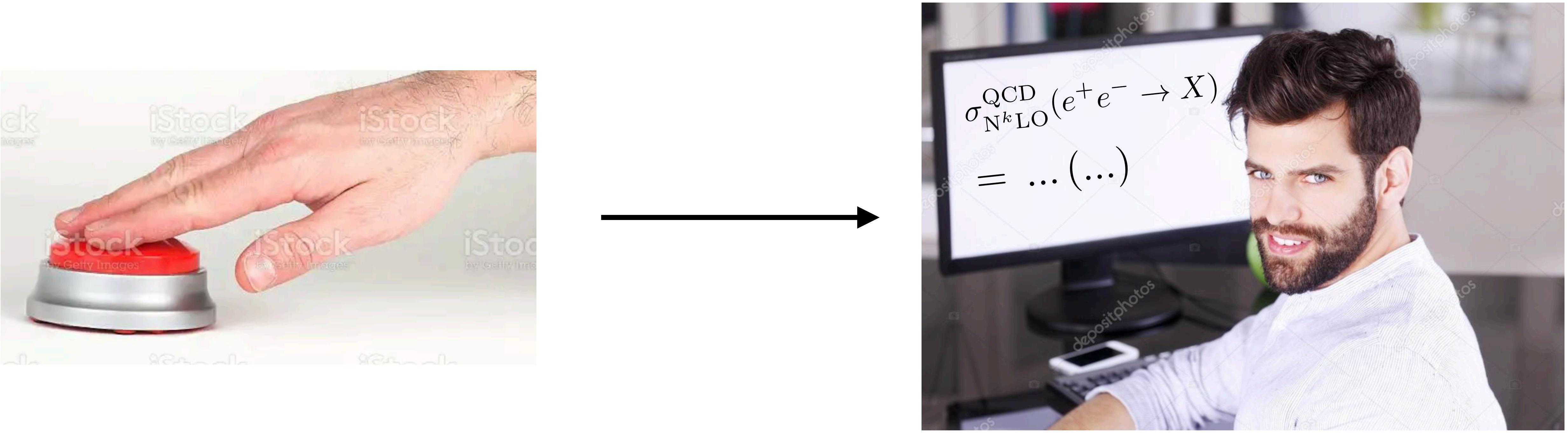

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Initial-state singularities: develop a competitive method for the treatment of initial-state singularities that is compatible with LU

- **KLN for initial states:** the KLN cancellation mechanism can be extended to include initial state singularities, but requires a significant departure from the Parton model

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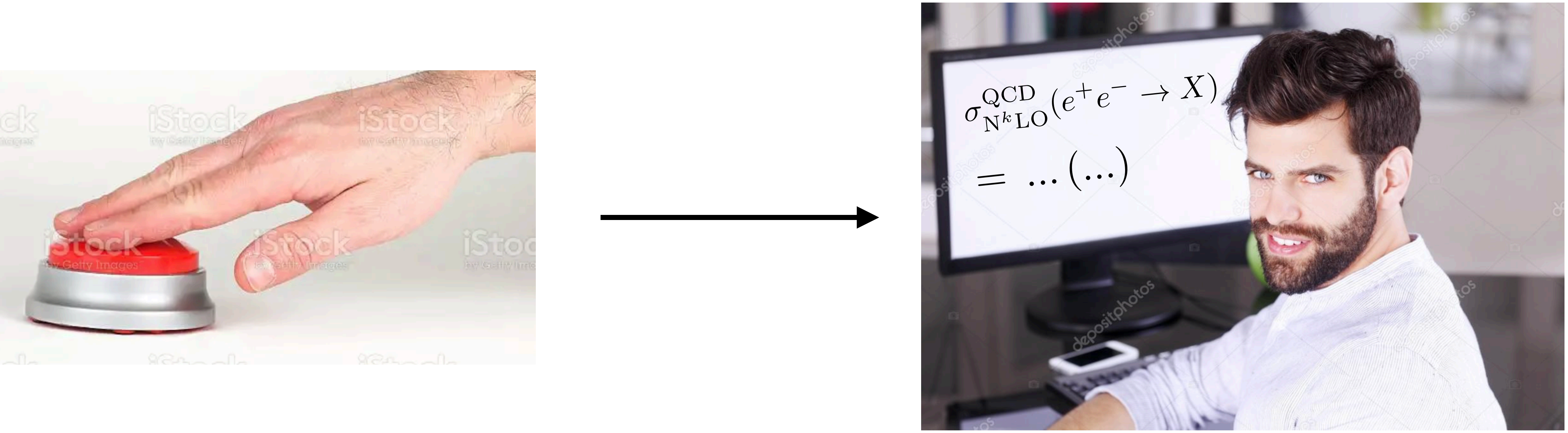

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a lot of physics and cool mathematics!

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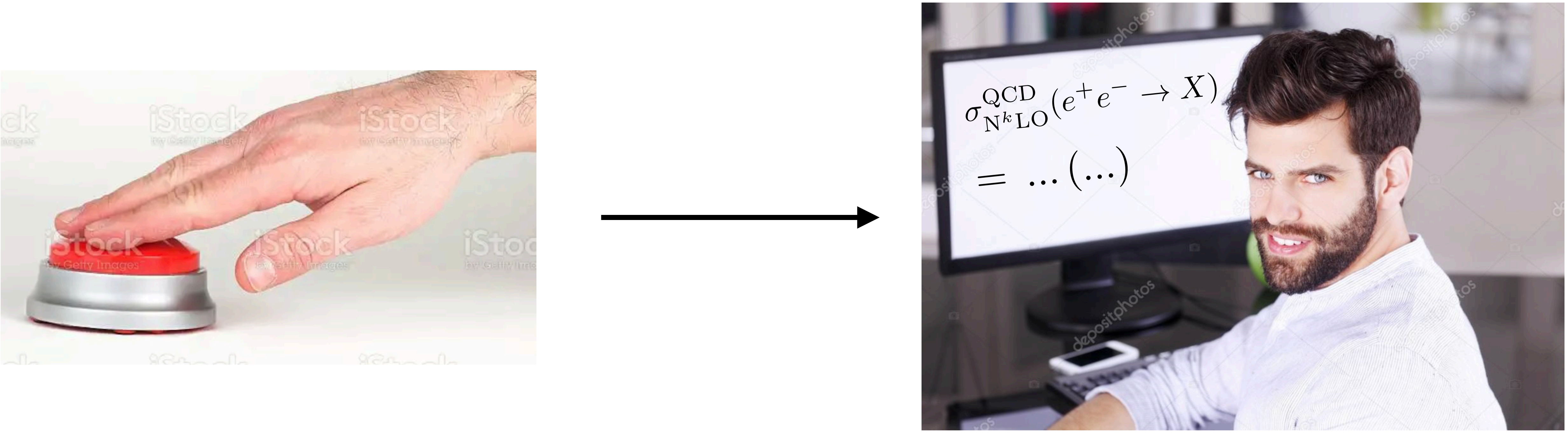

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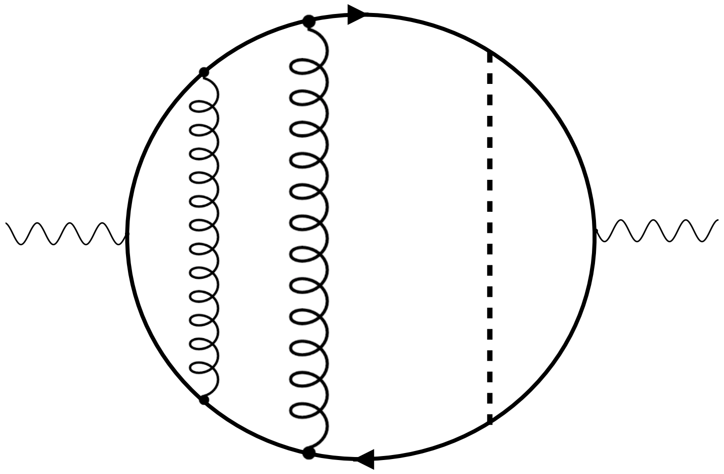
a lot of physics and cool mathematics!

- **Local PDF counterterms:** develop a competitive subtraction method or integrate existing ones

Thank you!

$$\mathrm{N^2LO}\quad \gamma^\star \to t\bar{t}H$$

Top energy distribution for the supergraph



$$\sqrt{s} = 1\mathrm{TeV}$$