

SINGULAR AND p -ADIC PHASE SPACE: A PS GENERATOR FOR THEORY COMPUTATIONS

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Source code at github.com/GDeLaurentis/lips

Interactive notebook at mybinder.org/v2/gh/GDeLaurentis/lips/HEAD

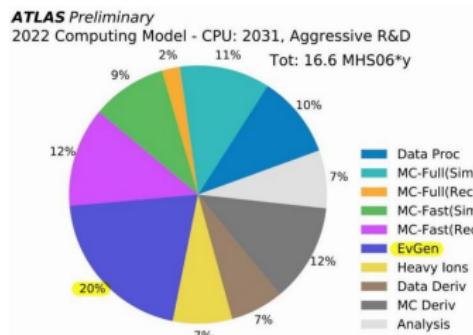
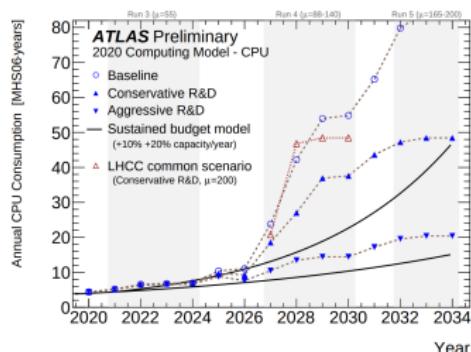
Based on [\[DL, Maître - 2019\]](#) and [\[DL, Page - 2022\]](#)

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COMPUTATIONAL CHALLENGES

- ▶ Projections for computations at the LHC [ATLAS - 2022]



- ▶ Bottlenecks (see e.g. 3j @ NNLO [Czakon, Mitov, Poncelet - 2021]):
 1. Availability of virtual (two-loop) amplitudes;
 2. Efficiency of real(-virtual) radiation.
- ▶ Power of analytics: e.g. 1-loop H+4-parton [Budge, Campbell, DL, Ellis, Seth - 2020] 100× faster than numerical programs [Bonciani et al. - 2022]

HIGH-MULTIPLICITY MULTI-LOOP AMPLITUDES

- ▶ Study analytic properties of scattering amplitudes with $P^\mu \in \mathbb{C}$, e.g. rational vector space of the 2-loop amplitude $\mathcal{A}_{q\bar{q} \rightarrow \gamma\gamma\gamma}^{(2, N_f)}$:

$$\tilde{r}_0^{(2, N_f)} = \left(\frac{8/3 \langle 23 \rangle [23] \langle 24 \rangle [34]}{\langle 15 \rangle \langle 34 \rangle \langle 45 \rangle \langle 4 | 1 + 5 | 4 \rangle} \right)$$

$$+ (45 \rightarrow 54)$$

$$\tilde{r}_1^{(2, N_f)} = \tilde{r}_0^{(2, N_f)}(345 \rightarrow 453)$$

$$\tilde{r}_2^{(2, N_f)} = \tilde{r}_0^{(2, N_f)}(345 \rightarrow 534)$$

$$\tilde{r}_3^{(2, N_f)} = \left(\frac{8/3 \langle 13 \rangle [13] \langle 24 \rangle [45]}{\langle 13 \rangle \langle 34 \rangle \langle 45 \rangle \langle 4 | 1 + 3 | 4 \rangle} \right)$$

+ (45 → 54)

$$+ \left(\frac{-8/3\langle 12\rangle[13]\langle 23\rangle^2}{\langle 13\rangle\langle 24\rangle\langle 25\rangle\langle 34\rangle\langle 35\rangle} \right)$$

$$\tilde{r}_4^{(2, N_f)} = \tilde{r}_3^{(2, N_f)}(345 \rightarrow 453)$$

$$\tilde{r}_5^{(2,N_f)} = \tilde{r}_3^{(2,N_f)}(345 \rightarrow 534)$$

Phenylalanine, tyrosine, and tryptophan are the only amino acids that contain aromatic rings. Tyrosine and phenylalanine are hydroxylated derivatives of the aromatic amino acid, phenylalanine. Tyrosine is hydroxylated at the para position of the ring, while phenylalanine is hydroxylated at the meta position. Tryptophan is a tryptamine derivative, which contains an indole ring system.

the same time, the number of species per genus was higher than in the other groups. The mean number of species per genus was 1.5, which is significantly higher than the mean of 0.8 for the other groups ($t = 2.7$, $p < 0.05$). The mean number of species per genus was also higher than the mean of 1.2 for the *Leptospiraceae* ($t = 1.9$, $p < 0.05$) and the mean of 1.0 for the *Neurospilidae* ($t = 2.0$, $p < 0.05$). The mean number of species per genus was lower than the mean of 2.0 for the *Micrococcaceae* ($t = -1.7$, $p < 0.05$) and the mean of 2.2 for the *Microbacteriaceae* ($t = -1.8$, $p < 0.05$).

[...]

[Abreu, DL, Ita, Klinkert, Page, Sotnikov - to appear]

[Abreu, Page, Pascual, Sotnikov - 2020]

FLASH OVERVIEW OF LIPS

- ▶ n -point phase space over \mathbb{F} , with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{Q}[i], \mathbb{F}_p, \mathbb{Q}_p\}$

```
Particles(multiplicity, field=Field(name, prime, digits))
```

e.g. $\mathbb{C} \rightarrow \text{Field("mpc", 0, 300)}$

$\mathbb{F}_p \rightarrow \text{Field("finite field", 2147483647, 1)}$

[von Manteuffel, Schabinger - 2014], [Peraro - 2016]

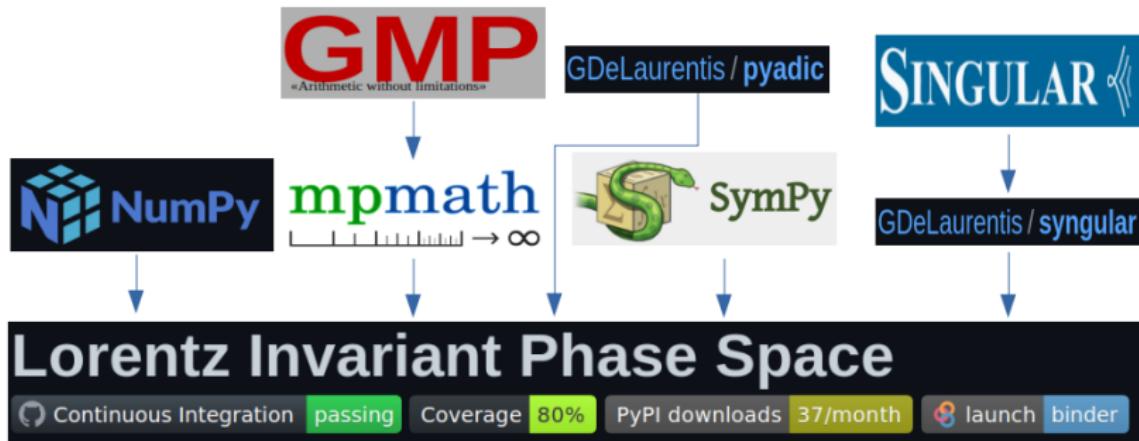
$\mathbb{Q}_p \rightarrow \text{Field("padic", 2147483647, 3)}$

- ▶ On-the-fly evaluation of arbitrary spinor expressions

```
Particles(5, Q13, seed=0)("-8/3s23(24)[34])/((15)<34><45>(4|1+5|4))  
>>> 11 + 12*13 + 2*13^2 + O(13^3)
```

- ▶ Generation of singular phase space configurations

DEPENDENCIES GRAPH¹²³⁴



¹Charles R. Harris et al. "Array programming with NumPy". In: *Nature* 585 (2020), pp. 357–362. doi: [10.1038/s41586-020-2649-2](https://doi.org/10.1038/s41586-020-2649-2).

²Fredrik Johansson et al. *mpmath: a Python library for arbitrary-precision floating-point arithmetic (version 0.18)*. <http://mpmath.org/>. 2013.

³Aaron Meurer et al. "SymPy: symbolic computing in Python". In: *PeerJ Computer Science* 3 (Jan. 2017), e103. ISSN: 2376-5992.

⁴Wolfram Decker et al. *SINGULAR 4-3-0 — A computer algebra system for polynomial computations*. <http://www.singular.uni-kl.de>. 2022.

p -ADIC NUMBERS AND FINITE FIELDS

pyAdic

Continuous Integration passing Coverage 91% PyPI downloads 107/month launch binder

- ▶ Finite field class: `ModP(number, prime)`
- ▶ p -adic number class: `PAdic(number, prime, digits)`
- ▶ \mathcal{O} -term tracking: `pyadic.padic.fixed_relative_precision`
By default this set to `False`. `True` emulates behavior of `float`.
- ▶ Supported operations:
 1. arithmetic operations (`+`, `-`, `*`, `/`),
 2. square root (`finite_field_sqrt`, `padic_sqrt`)
 3. p -adic logarithm (`padic_log`).
- ▶ Field extensions (at present only by a single `sqrt`)

ALGEBRAIC GEOMETRY

An Object-Oriented Python Interface to Singular

 Continuous Integration  passing  Coverage 92%  PyPI downloads 56/month  launch  binder

- ▶ Polynomial ring class: `Ring`
Syntax: `Ring(coeffs, variables, monomial_ordering)`
- ▶ Ideal class: `Ideal`
Syntax: `Ideal(Ring, generators (list of str))`
- ▶ Polynomial quotient ring class: `QuotientRing`
Syntax: `QuotientRing(Ring, Ideal)`
- ▶ Several ideal operations already interfaced, e.g.:
 1. Addition, quotient (`+`, `/`)
 2. Gröbner bases: (`groebner_basis`)
 3. Primary decomposition (`primary_decomposition`)
 4. Etc...

C vs Q[i] PHASE SPACE: RATIONAL SPINORS

By default, phase space is massless and satisfies mom. cons.:

- ▶ `Particles` is a **1-indexed** sub-class of `list`;
- ▶ entries are `Particle` objects;
- ▶ `Particles.masses` are zero, `Particles.total_mom` is zero.

Spinors in $\mathbb{F}_p/\mathbb{Q}_p$? But aren't they complex, even with real momenta?

- ▶ The Pauli matrix σ_y is the only one with imaginary entries;
- ▶ Take: $E \in \mathbb{Q}$, $p_x \in \mathbb{Q}$, $p_y \in i\mathbb{Q}$, $p_z \in \mathbb{Q} \implies (P_\mu \sigma^\mu)^{\dot{\alpha}\alpha} \in \mathbb{Q}$
- ▶ In practice, work with rank 1 and/or 2 spinors:

`Particle.r_sp_d`: $\lambda_\alpha \in \mathbb{F}$; `Particle.l_sp_d`: $\tilde{\lambda}_{\dot{\alpha}} \in \mathbb{F}$

`Particle.r2_sp`: $P^{\dot{\alpha}\alpha} \in \mathbb{F}$; `Particle.four_mom`: $P^\mu \in \mathbb{F}$ if $i \in \mathbb{F}$,
e.g. $i \in \mathbb{F}_{2^{31}-19}$, but $i \notin \mathbb{F}_{2^{31}-1}$

PARTIAL FRACTIONS AS IDEAL MEMBERSHIP

- ▶ Imagine we have a numerical program to compute the expression

$$\text{Black Box Function} = \frac{\mathcal{N}}{\mathcal{D}} = \left(\frac{8/3\langle 23\rangle[23]\langle 24\rangle[34]}{\langle 15\rangle\langle 34\rangle\langle 45\rangle\langle 4|1+5|4\rangle} \right) + (45 \rightarrow 54)$$

- ▶ Can we see **numerically** that $\langle 4|1+5|4\rangle$ and $\langle 5|1+4|5\rangle$ can be partial fractioned? Equivalent statements are:

$$\begin{aligned}\mathcal{N} &= a(\lambda, \tilde{\lambda})\langle 4|1+5|4\rangle + b(\lambda, \tilde{\lambda})\langle 5|1+4|5\rangle, \\ \text{or } \mathcal{N} &\in \left\langle \langle 4|1+5|4\rangle, \langle 5|1+4|5\rangle \right\rangle.\end{aligned}$$

- ▶ In the code, `LipsIdeal` subclasses `syngular.Ideal` and uses `Particles.__call__` after `Particles.make_analytical_d` to symbolically compute the expressions.

THE GEOMETRY OF SINGULAR PHASE SPACE

- $\mathcal{N} \in J$ iff \mathcal{N} vanishes on all branches of J ;

```
J = LipsIdeal(5, ("<4|1+5|4]", "<5|1+4|5]"))
```

- There are 5 branches, 3 of which are not related by symmetries:

```
K = LipsIdeal(5, ("<14>", "<15>", "<45>", "[23]"))
```

```
L = LipsIdeal(5, ("<12>", "<13>", "<14>", "<15>",
    "<23>", "<24>", "<25>", "<34>", "<35>", "<45>"))
```

```
M = LipsIdeal(5, ("<4|1+5|4]", "<5|1+4|5]", 
    "|1]<14><15>+|4]<14><45>-|5]<45><15>",
    "|1>[14] [15]+|4>[14] [45]-|5>[45] [15]))
```

- Check it (`&` means ideal intersection \cap , like for `set`):

```
assert J == K & K("12345", True) & L & L("12345", True) & M
("12345", True) means identity permutation, plus swap  $\lambda \leftrightarrow \tilde{\lambda}$ .
```

SINGULAR, p -ADIC PHASE SPACE POINTS

- ▶ Syntax for a phase space point on (or near) a variety

```
Particles._singular_variety(
```

```
    orthogonal directions, valuations, ideal generators)
```

- ▶ Generate a point on each variety:

```
oPs = Particles(5, field=Field("padic", 2**31-1, 3), seed=0)
```

```
oPs._singular_variety(("<4|1+5|4]", "<5|1+4|5]"), (1, 1),
```

```
    generators=M.generators) # and same for K and L
```

- ▶ Use `oPs` to evaluate \mathcal{N} , it's will be proportional to p on all branches.
⇒ partial fraction decomposition is possible.

BEYOND PARTIAL FRACTIONS

- ▶ Even if a partial fraction decomposition is not possible⁵, we can still constrain the numerator if it vanishes somewhere.

⁵without spurious poles

SUMMARY

- ▶ `lips`: a phase space generator for theory computations;
- ▶ `pyadic`: a package for using p -adic numbers;
- ▶ `syngular`: an interface to the algebraic geometry software Singular.

Please feel free to get in touch, open issues, or contribute!

OUTLOOK

- ▶ New two-loop amplitudes relevant for LHC experiments;
- ▶ more concise expressions for high-multiplicity amplitudes.