Loop amplitudes at the precision frontier

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based on work with:
Aylett-Bullock, Brønnum-Hansen, Becchetti, Butter, Chaubey, Hartanto, Luchmann, Marcoli, Marzucca, Moodie, Peraro, Pitz, Plehn, Chicherin, Gehrmann, Henn, Zoia

ACAT, Bari
27th Ocotber 2022

European Research Council
Established by the European Commission

motivation:

where is the precision frontier?

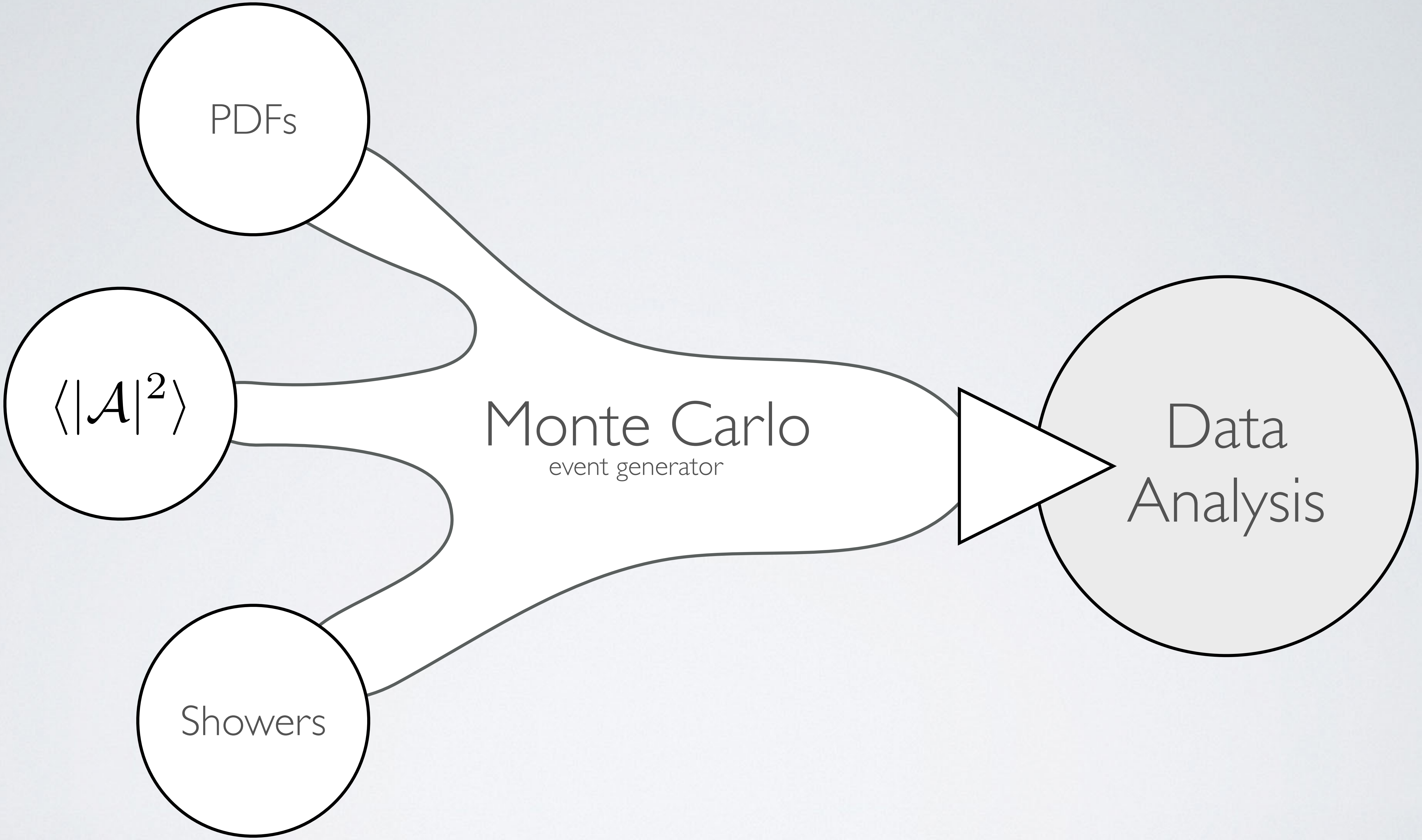
part I:

finite field arithmetic and two-loop amplitudes

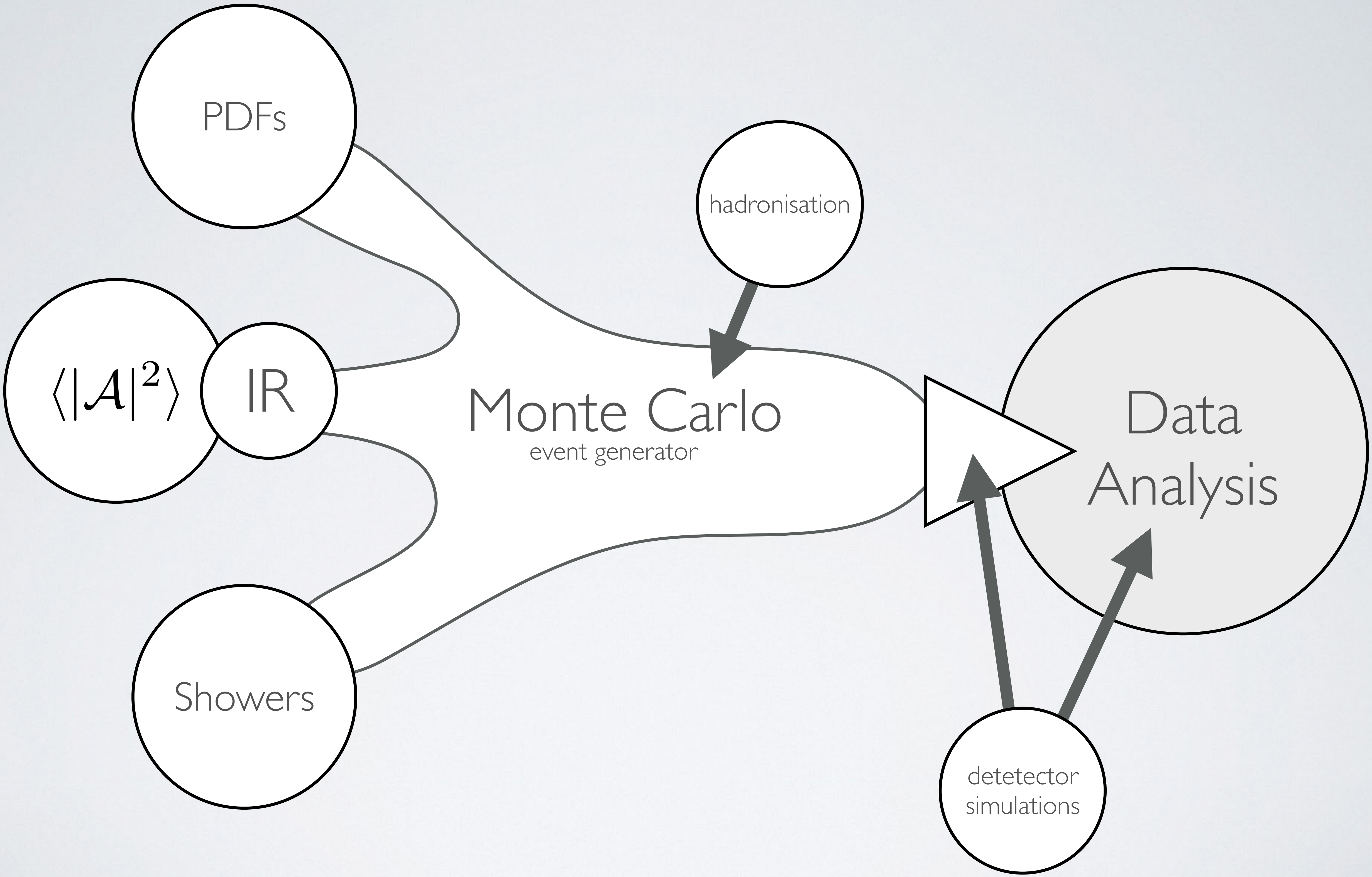
part II:

amplitude neural networks

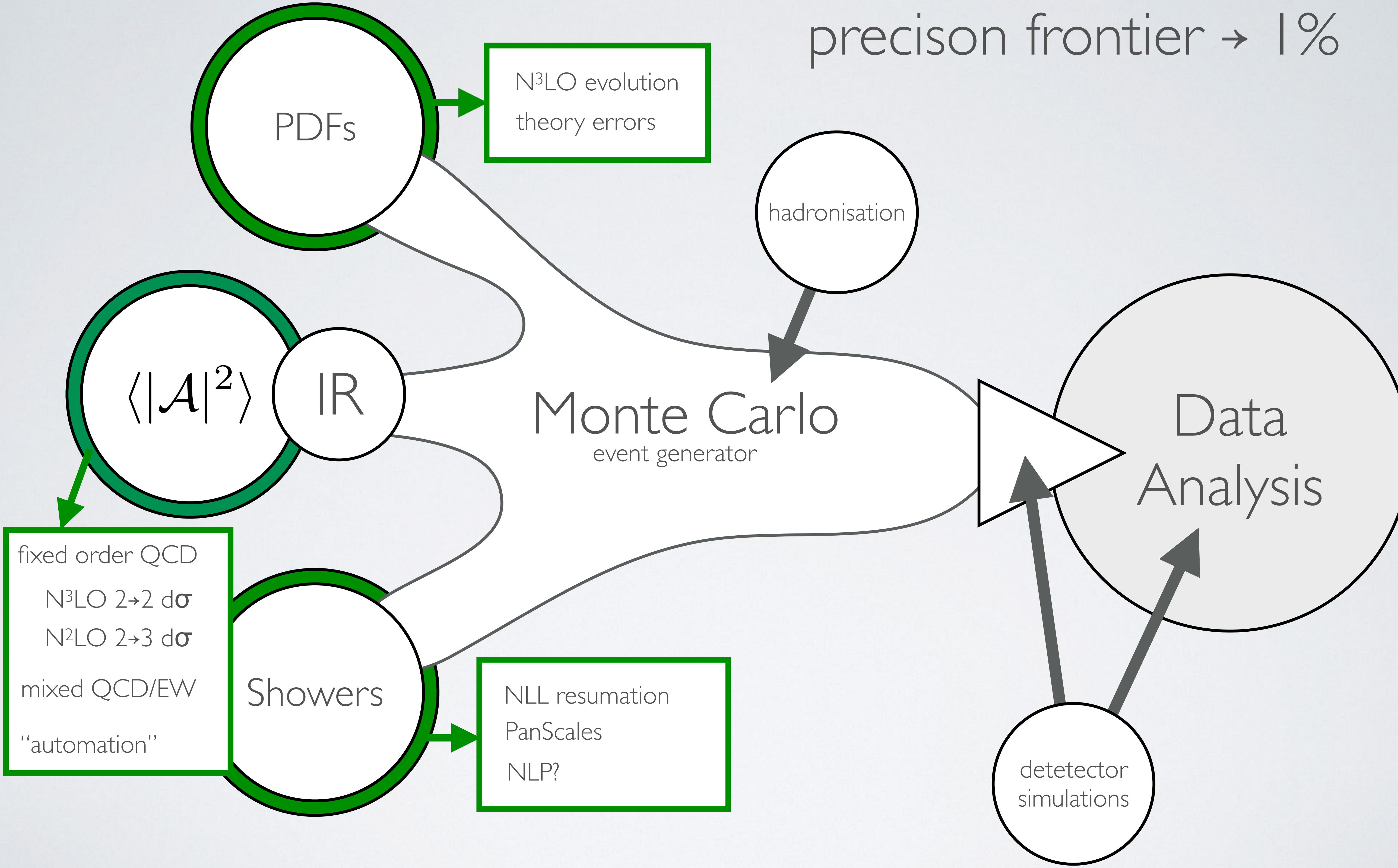
from theory to experiment



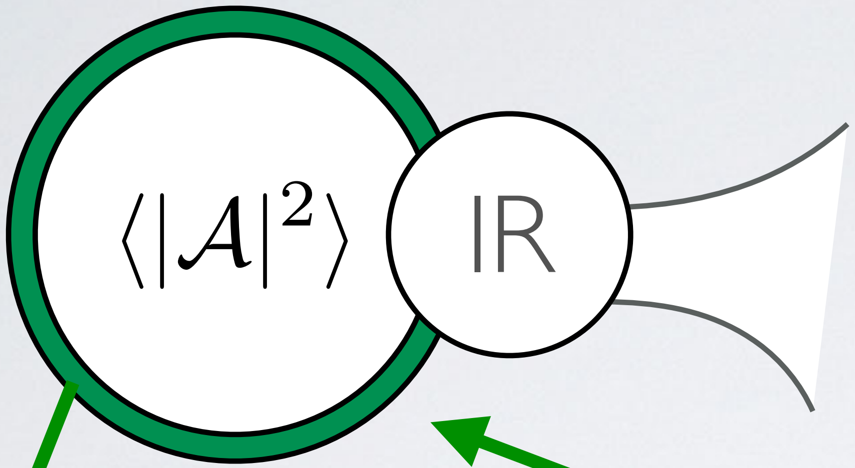
from theory to experiment



from theory to experiment:
precison frontier → 1%



from theory to experiment: precison frontier → 1%



new technology required!

reduction techniques

strenuous computer alegbra

challenging integration

fixed order QCD
N^{3}LO 2→2 dσ
N^{2}LO 2→3 dσ

mixed QCD/EW

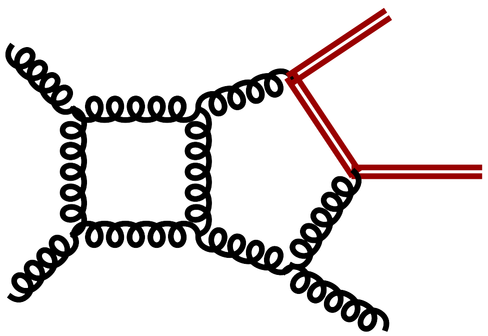
"automation"

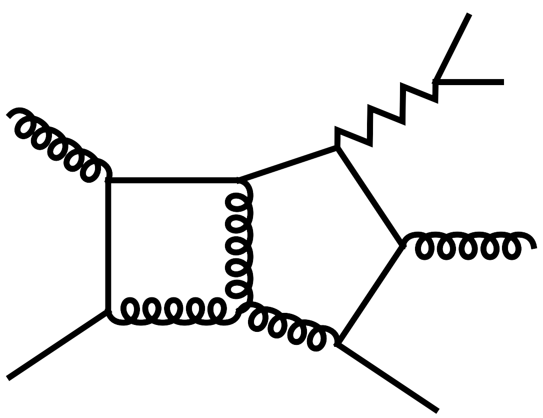
$$d\sigma = d\sigma^{\mathrm{LO}} + \alpha_s d\sigma^{\mathrm{NLO}} + \alpha_s^2 d\sigma^{\mathrm{NNLO}}$$

~10-30 % (NLO)

~1-10 % (NNLO)

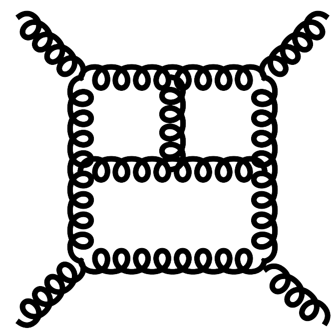
multiplicity frontier

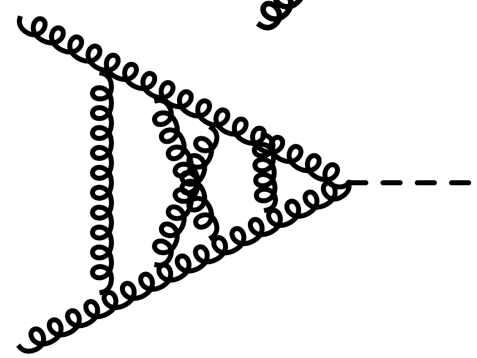




N²LO 2→3 (pp→3j pp→W2j, pp→ttj,…)

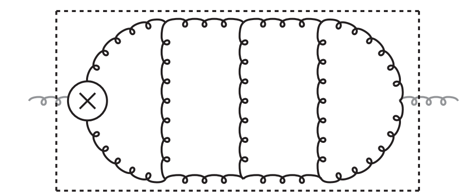
loop frontier

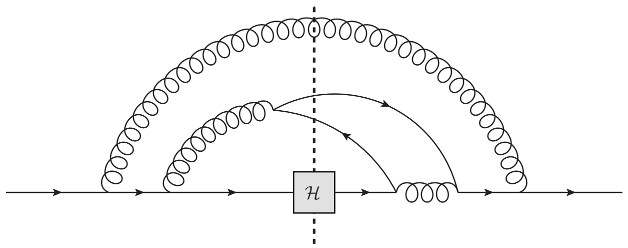




N³LO 2→2, N⁴LO 2→1 (gg→H)

IR frontier





N³LO splitting functions, analytic resummation, SCET, beam functions etc.

the precision wishlist

latest update LH2021 Huss, Huston, Jones, Pellen [2207.02122]

process	known	desired
$pp \to H$	$\mathrm{N^3LO_{HTL}}$ $\mathrm{NNLO^{(t)}_{QCD}}$ $\mathrm{N^{(1,1)}LO^{(HTL)}_{QCD\otimes EW}}$	$\mathrm{N^4LO_{HTL}}$ (incl.) $\mathrm{NNLO^{(b,c)}_{QCD}}$
$pp \to H + j$	$\mathrm{NNLO_{HTL}}$ $\mathrm{NLO_{QCD}}$ $\mathrm{N^{(1,1)}LO_{QCD\otimes EW}}$	$\mathrm{NNLO_{HTL} \otimes NLO_{QCD} + NLO_{EW}}$
$pp \to H + 2j$	$\mathrm{NLO_{HTL} \otimes LO_{QCD}}$ $\mathrm{N^3LO^{(VBF^*)}_{QCD}}$ (incl.) $\mathrm{NNLO^{(VBF^*)}_{QCD}}$ $\mathrm{NLO^{(VBF)}_{EW}}$	$\mathrm{NNLO_{HTL} \otimes NLO_{QCD} + NLO_{EW}}$ $\mathrm{N^3LO^{(VBF^*)}_{QCD}}$ $\mathrm{NNLO^{(VBF)}_{QCD}}$

process	known	desired
$pp \to 2\,\mathrm{jets}$	$\mathrm{NNLO_{QCD}}$ $\mathrm{NLO_{QCD} + NLO_{EW}}$	$\mathrm{N^3LO_{QCD} + NLO_{EW}}$
$pp \to 3\,\mathrm{jets}$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$	

Table 2: Precision wish list: jet final states.

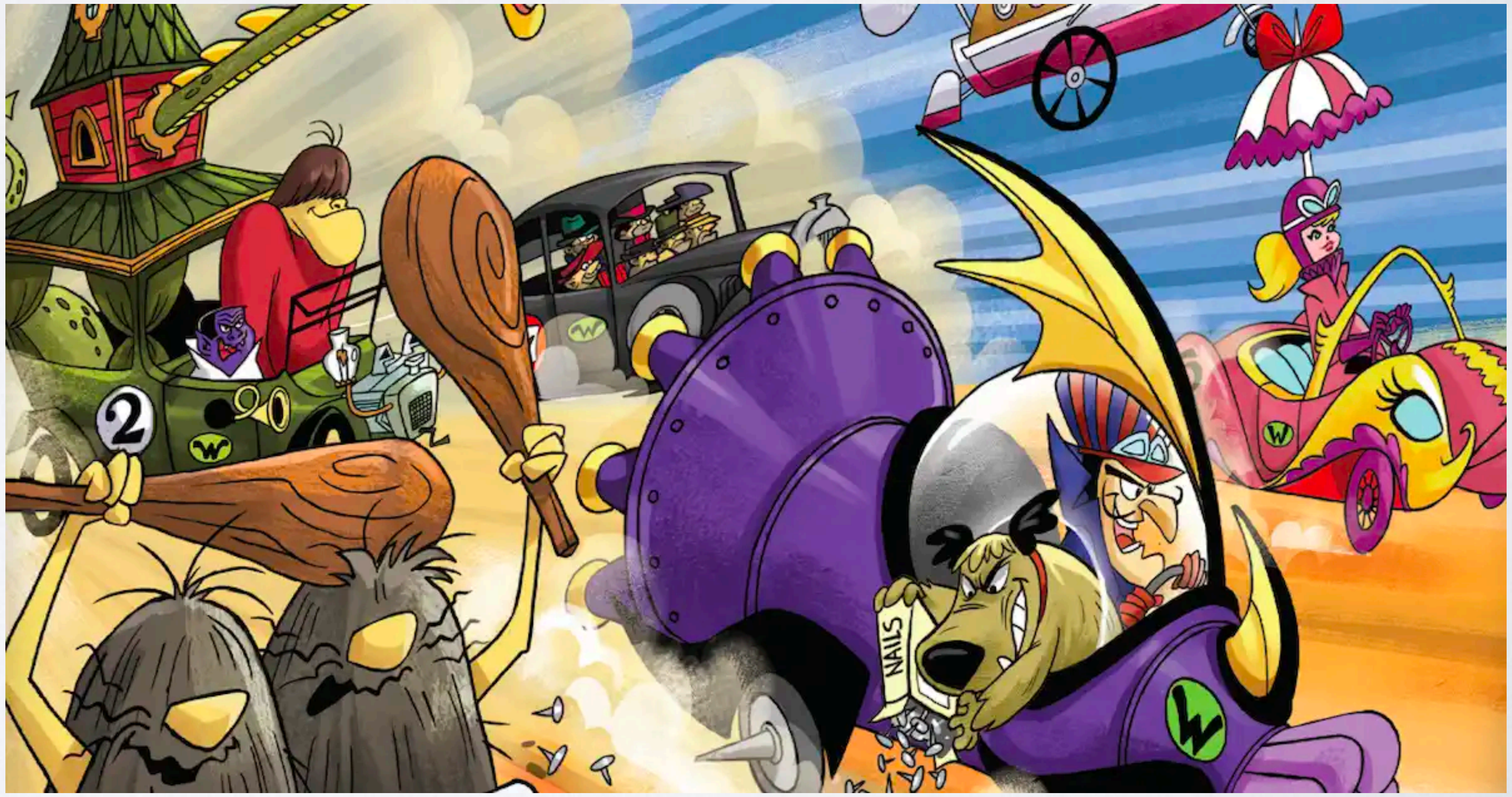
$pp \to V + 2j$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (QCD component) $\mathrm{NLO_{QCD} + NLO_{EW}}$ (EW component)	$\mathrm{NNLO_{QCD}}$
$pp \to V + b\bar{b}$	$\mathrm{NLO_{QCD}}$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$
$pp \to VV' + 1j$	$\mathrm{NLO_{QCD} + NLO_{EW}}$	$\mathrm{NNLO_{QCD}}$

$pp \to \gamma\gamma + j$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ $+\mathrm{NLO_{QCD}}$ (gg channel)	
$pp \to \gamma\gamma\gamma$	$\mathrm{NNLO_{QCD}}$	$\mathrm{NNLO_{QCD} + NLO_{EW}}$

$pp \to t\bar{t} + j$	$\mathrm{NLO_{QCD}}$ (off-shell effects) $\mathrm{NLO_{EW}}$ (w/o decays)	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ (w/ decays)
$pp \to t\bar{t} + 2j$	$\mathrm{NLO_{QCD}}$ (w/o decays)	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (w/ decays)
$pp \to t\bar{t} + V'$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (w/o decays)	$\mathrm{NNLO_{QCD} + NLO_{EW}}$ (w/ decays)
$pp \to t\bar{t} + \gamma$	$\mathrm{NLO_{QCD}}$ (off-shell effects)	
$pp \to t\bar{t} + Z$	$\mathrm{NLO_{QCD}}$ (off-shell effects)	
$pp \to t\bar{t} + W$	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (off-shell effects)	

$pp \to t\bar{t}t\bar{t}$	Full $\mathrm{NLO_{QCD} + NLO_{EW}}$ (w/o decays)	$\mathrm{NLO_{QCD} + NLO_{EW}}$ (off-shell effects) $\mathrm{NNLO_{QCD}}$

precision measurements



computations at the precision frontier

bare amplitudes

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon, \{p\}) \mathcal{F}_i(\epsilon, \{p\})$$

rational functions

integrals/special functions

finite remainders

$$F^{(L)} = A^{(L),4-2\epsilon} - \sum_{k=1}^{L} I^{(k),4-2\epsilon} A^{(L-k),4-2\epsilon}$$

universal IR/UV poles

[Catani (1998)][Becher, Neubert (2009)]
[Magnea, Gardi (2009)]

computational toolbox

numerical unitarity
all-in-one cuts to master integrals

on-shell methods
hidden simplicity and underlying geometry

momentum twistors
rational kinematics

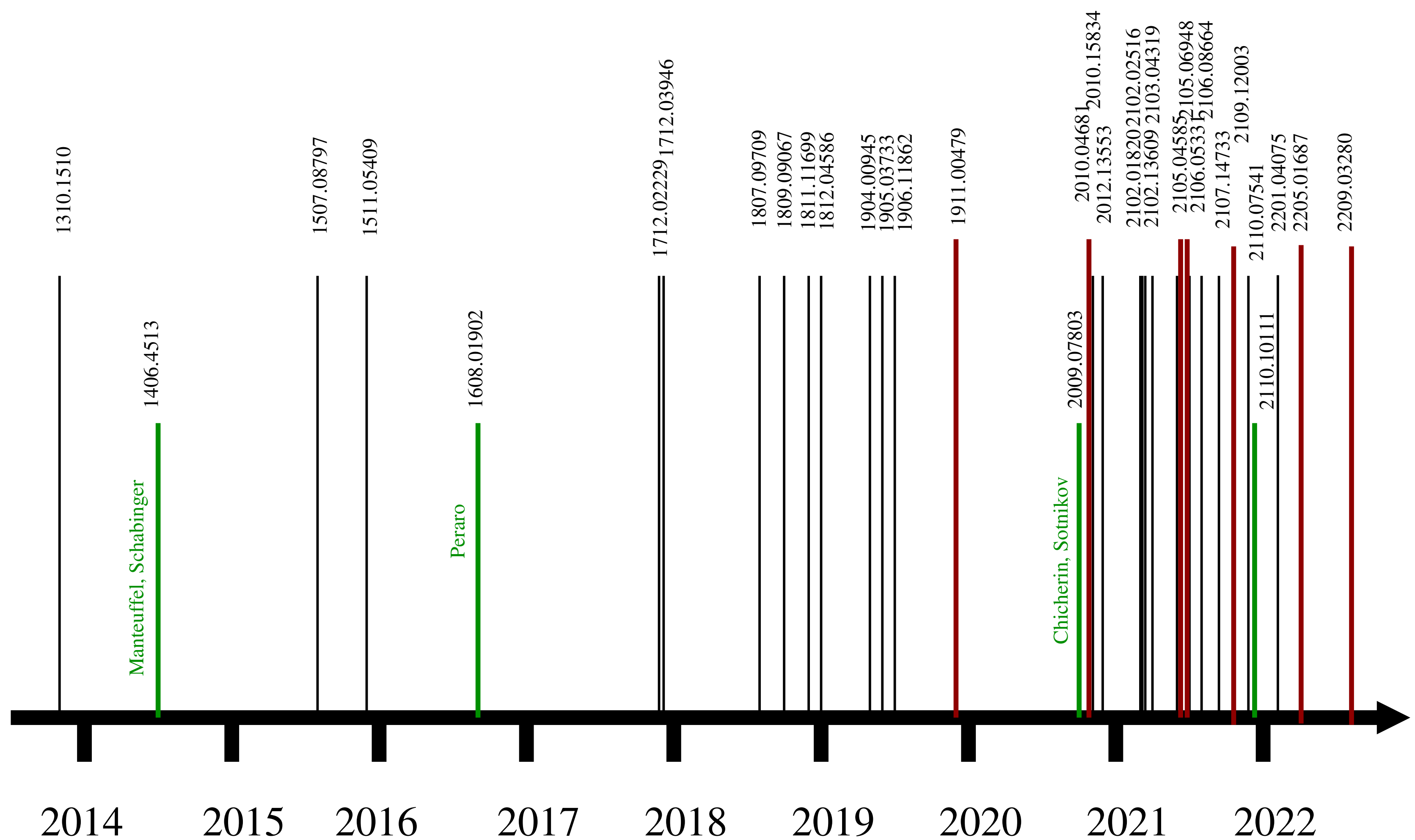
integrand reduction
algebraic reduction

syzygy relations
optimising systems of IBP identities

finite fields
exact numerics - truncated over e.g. prime numbers

recursion relations
reusing common blocks to evaluate diagrams efficiently

computational toolbox



Abreu, Agarwal, SB, Brønnum-Hansen, Buccioni, Chawdhry, Chicherin, Czakon, De Laurentis, Dixon, Dormans, Febres Cordero, Gehrmann, Hartanto, Heinrich, Henn, Herrmann, Ita, Kraus, Kryś, Lo Presti, Mitev, Mitov, Mogull, Ochirov, O'Connell, Page, Papadopoulos, Pascual, Peraro, Poncelet, Ruf, Sotnikov, Tancredi, Tommasini, von Manteuffel, Wasser, Wever, Zeng, Zhang, Zoia, ...

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part 1:
finite fields arithmetic and two-loop amplitudes

with: Brønnum-Hansen, Becchetti, Chaubey, Hartanto, Marcoli, Marzucca, Moodie, Peraro, Chicherin, Gehrmann, Henn, Zoia

finite field arithmetic

not a new idea - used in many computer algebra systems

solving IBP systems: e.g. FINRED [von Manteuffel], KIRA+FIREFLY [Maierhoefer, Usovitsch, Uwer, Klappert, Lange]

framework for amplitude computations: FINITEFLOW [Peraro (2019)]

```
(* take some, reasonably large, prime number *)
FFPrimeNo[1]
(* all quantities evaluated modulo a prime number *)
Mod[-3,FFPrimeNo[1]]
Mod[87+FFPrimeNo[1],FFPrimeNo[1]]
Solve[b*3==87,Modulus→FFPrimeNo[1]][[1]]
(* already implemented in Mathematica *)
Mod[87/3+FFPrimeNo[1],FFPrimeNo[1]]
```

```
9 223 372 036 854 775 643
9 223 372 036 854 775 640
87
{b → 29}
29
```

NB: multiplicative inverse

extremely efficient solutions to linear algebra systems

other talks at this year's ACAT

De Laurentis

Moodie

Usovitsch

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon,\{p\})\mathcal{F}_i(\epsilon,\{p\})$$

rational functions

multiple numerical (mod prime) evaluations can used to reconstruct complete analytic information

Newton (polynomial) and Thiele (rational) interpolation

```
(* implement the Newton interpolation algorithm *)
NewtonReconstruct[z_, zvalues_List, fvalues_List, primeno_]:=Module[{res,maxdegree,aa,eqs,sol},
maxdegree = Length[zvalues]-1;
res = Sum[aa[r]*Product[(z-zvalues[[i+1]]),{i,0,r-1}],{r,0,maxdegree}];
eqs = Equal@@@Transpose[{res /. ({Rule[z,#]}&/@zvalues),fvalues}];
sol = Solve[eqs,Table[aa[i],{i,0,Length[fvalues]-1}],{Modulus->primeno}];
Return[res/. sol[[1]]];
]
```

```
fff[z_]:=15/2*z+119/6*z^2;
values = {19,44,78};
FFRatMod[fff/@values,FFPrimeNo[0]]
test = NewtonReconstruct[z,values,%,FFPrimeNo[0]]
Collect[%,z,FFRatRec[#,FFPrimeNo[0]]&]
```

{6 148 914 691 236 524 491, 6 148 914 691 236 555 916, 121 251}

6 148 914 691 236 524 491 + 1257 (-19 + z) + 1 537 228 672 809 129 317 (-44 + z) (-19 + z)

$$\frac{15\,z}{2} + \frac{119\,z^2}{6}$$

Rational external kinematics: e.g. Momentum Twistors (Hodges)

Trivial parallelisation of sample points

finite fields for amplitudes

useful features:

- reconstruct exact results using chinese remainder theorem
- extremely efficient solutions to large linear systems
- reconstruct rational functions using Newton/Thiele interpolation
- modular approach in FiniteFlow allows us to link different algorithms and avoid large intermediate steps

QGRAF + FORM/MATHEMATICA + rational phase-space (Momentum Twistors) —pre-processing→ colour ordered helicity amplitudes

$$M^{(2)}(\{p\},\epsilon) = \sum_i c_i(\{p\},\epsilon)\mathcal{F}_i(\{p\},\epsilon)$$

↓ IBPs

$$M^{(2)}(\{p\},\epsilon) = \sum_i d_i(\{p\},\epsilon)\mathrm{MI}_i(\{p\},\epsilon)$$

↓ IR/UV sub + expansion to function basis

$$F^{(2)}(\{p\}) = \sum_i e_i(\{p\})\mathrm{mon}_i(f_j^{(w)})$$

↓ linear relations, univariate apart, polynomial reconstruction

complete reduction setup implemented in FINITEFLOW

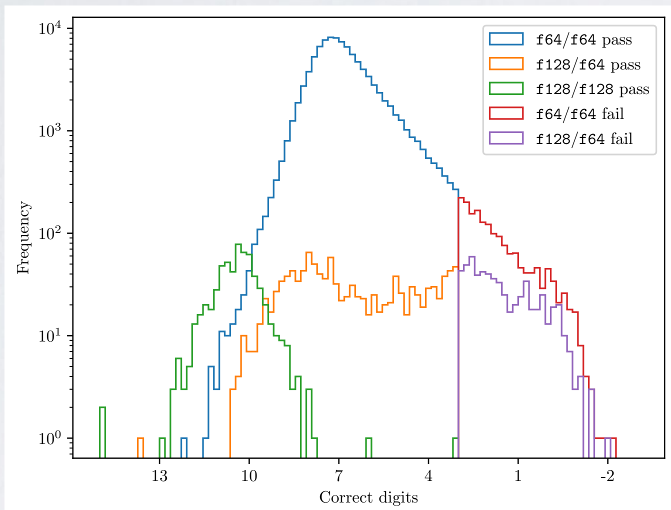
IBPs generated with help from LITERED/ FINITEFLOW

amplitudes → dσ

LCV = Leading Colour Double Virtual

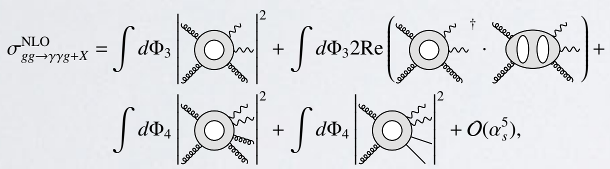
$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon, \{p\}) F_i(\epsilon, \{p\})$$

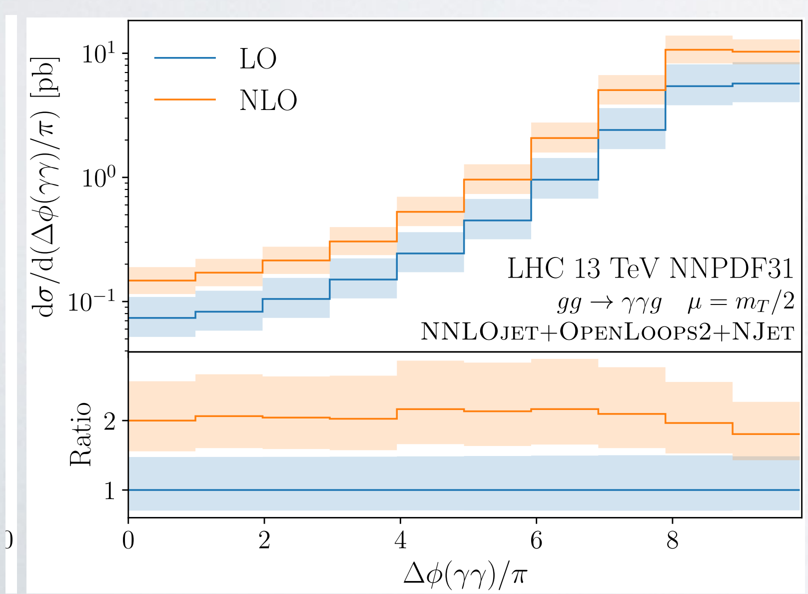
pentagon functions. now with one off-shell leg [Chicherin, Sotnikov, Zoia '21]



dσ gg→γγg

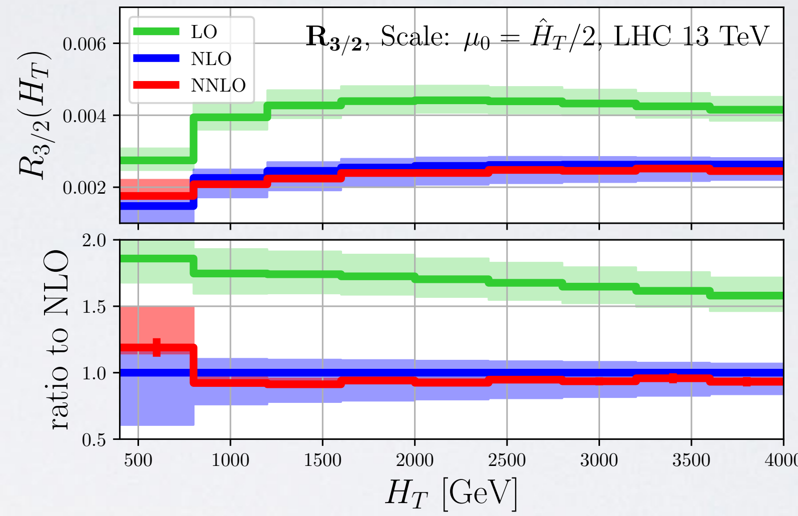
[SB et al. 2109.12003]





dσ pp→3j LCV

[Czakon et al. 2106.05331]

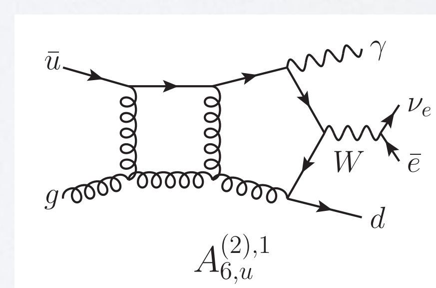


dσ pp→γγj LCV

[Czakon et al. 2105.06940]

A(2) pp→W2j LCV

[Abreu et al. 2110.07541]



A(2) pp→Wγj LCV

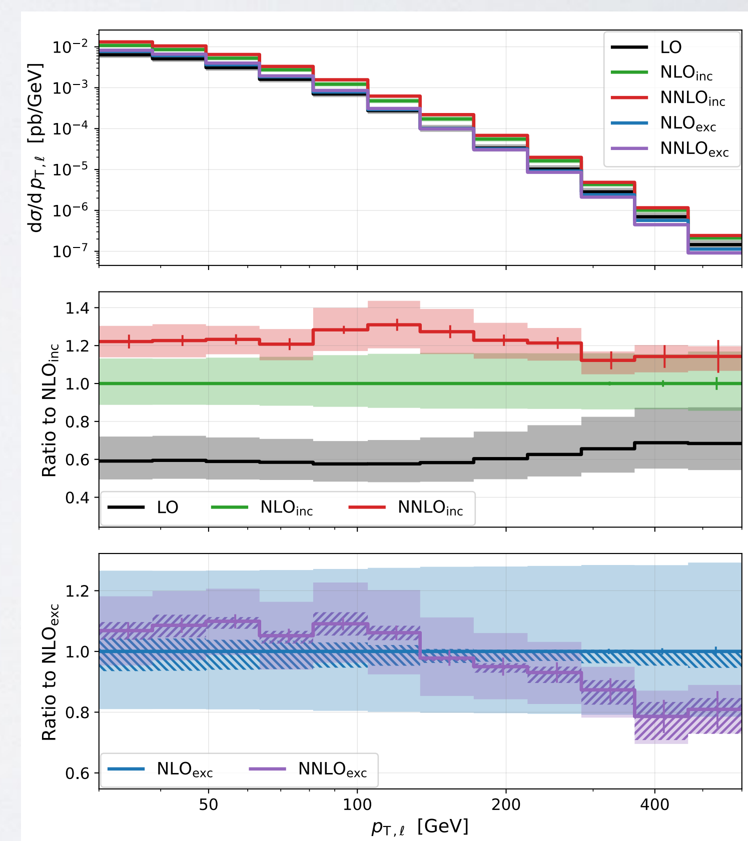
[SB et al. 2201.04075]

dσ gg→3g LCV

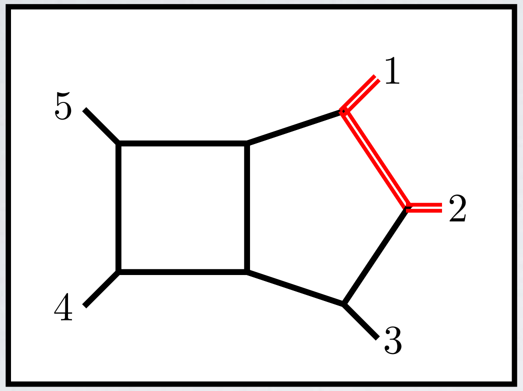
[Chen et al. 2203.13531]

dσ pp→Wbb LCV

[Hartanto et al. 2205.01687]



differential equations for pp→tt+j



[SB, Becchetti, Chaubey, Marzucca (to appear)]

1st steps towards pp→tt+j @ NNLO in QCD

see also 1L pp→tt+j O(**ε**²) [SB, Becchetti, Chaubey, Marzucca]

canonical form [Henn '13] DE of 88 master integrals

$$d\vec{\mathcal{I}}(\vec{x},\varepsilon) = \varepsilon\, dA(\vec{x})\,\vec{\mathcal{I}}(\vec{x},\varepsilon),$$
$$A(\vec{x}) = \sum c_i \log(w_i(\vec{x})).$$

IBP reduction and reconstruction over finite fields: good basis leads to simple reconstruction

$$\mathcal{I}_1 = \epsilon^4\, 8\, d_{23}\, d_{45}\, (d_{12} + m_t^2)\, I_{1,1,1,1,1,1,1,1,-1,0,0},$$
$$\mathcal{I}_2 = \epsilon^4\, \frac{d_{45}}{2\,\mathrm{tr}_5} I^{[11]}_{1,1,1,1,1,1,1,1,0,0,0},$$
$$\mathcal{I}_3 = \epsilon^4\, \frac{d_{45}}{2\,\mathrm{tr}_5} I^{[12]}_{1,1,1,1,1,1,1,1,0,0,0}\,.$$

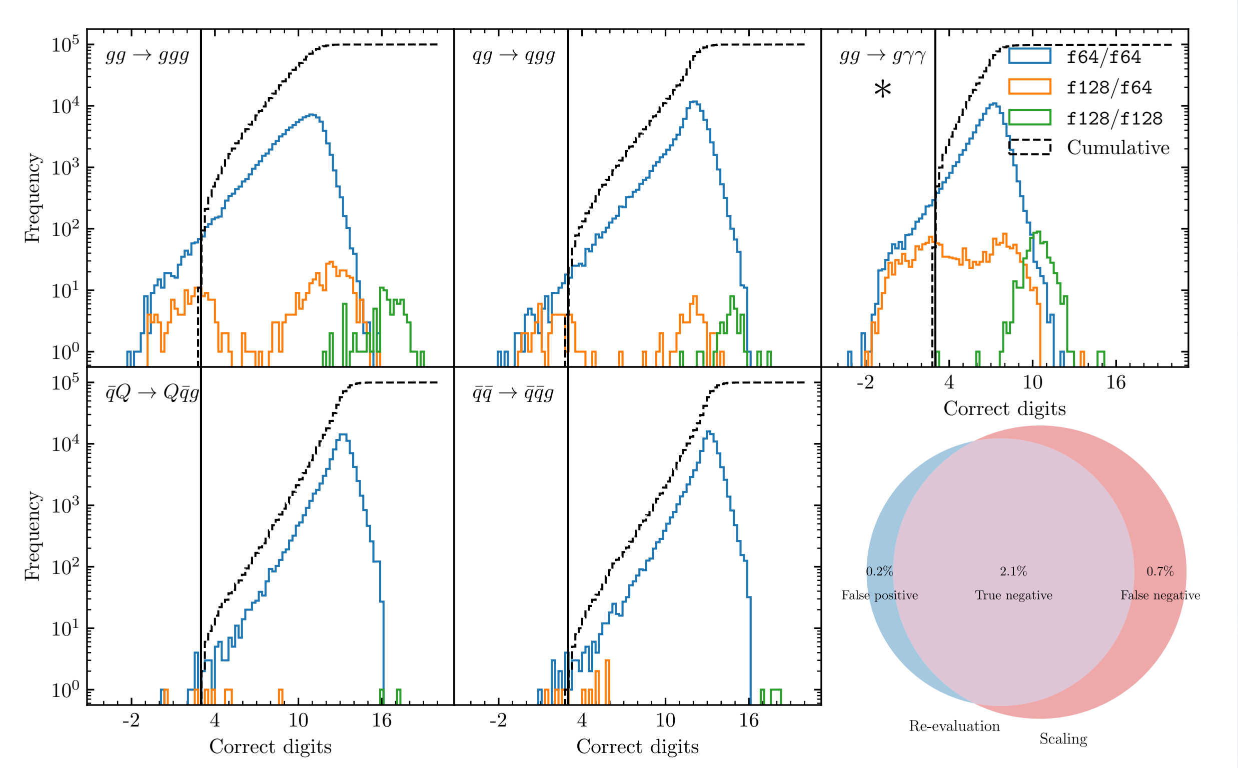
dLog alphabet of 71 letters

numerical evaluation with generalised series expansions (DiffExp)

high precision boundary values (AMFlow)

two-loop five-point processes in NJET

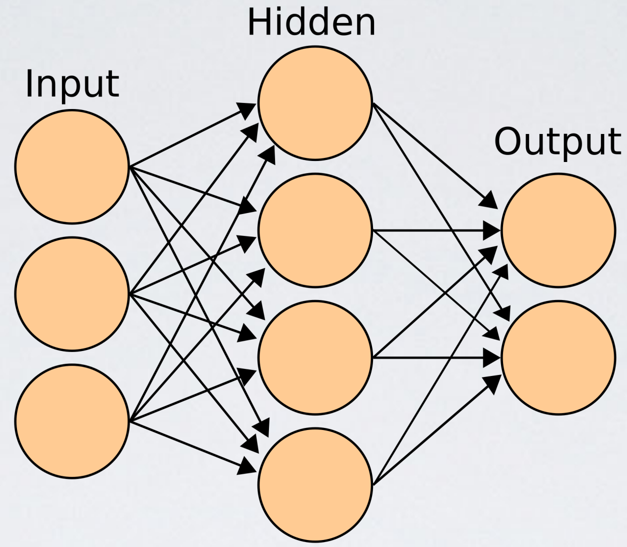
Channel	f64/f64		Evaluation strategy	
	Time (s)	f (%)	Time (s)	f (%)
$gg \to ggg$	1.39	69	1.89	77
$gg \to \bar{q}qg$	1.35	91	1.37	91
$qg \to qgg$	1.34	92	1.57	93
$q\bar{q} \to ggg$	1.34	93	1.38	93
$\bar{q}Q \to Q\bar{q}g$	1.14	99	1.16	99
$\bar{q}\bar{Q} \to \bar{q}\bar{Q}g$	1.36	99	1.39	99
$\bar{q}g \to \bar{q}Q\bar{Q}$	1.36	99	1.39	99
$\bar{q}q \to Q\bar{Q}g$	1.14	99	1.14	99
$\bar{q}g \to \bar{q}q\bar{q}$	1.84	99	1.90	99
$\bar{q}\bar{q} \to \bar{q}\bar{q}g$	1.82	99	1.94	99
$\bar{q}q \to q\bar{q}g$	1.71	99	1.77	99
$gg \to \gamma\gamma g$ *	9	99	26	99

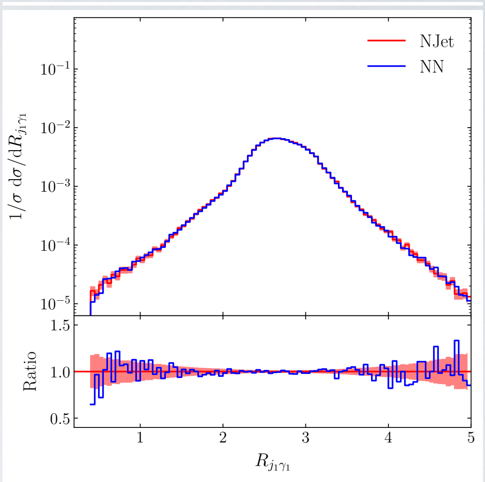


https://bitbucket.org/njet/njet

from Ryan Moodie's slides

$$\langle |\mathcal{A}|^2 \rangle$$





part II: amplitude neural networks

with: Aylett-Bullock, Butter, Luchmann, Moodie, Pitz, Plehn

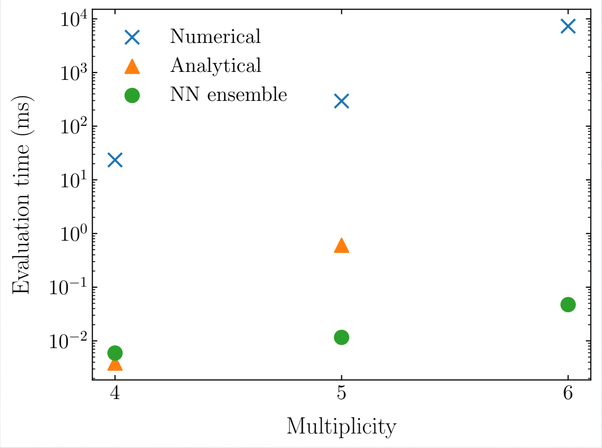
optimising simulations

how can we speed up simulations
with expensive amplitude calls?

other talks and new results!
Bothmann, Butter, Truong, Janssen

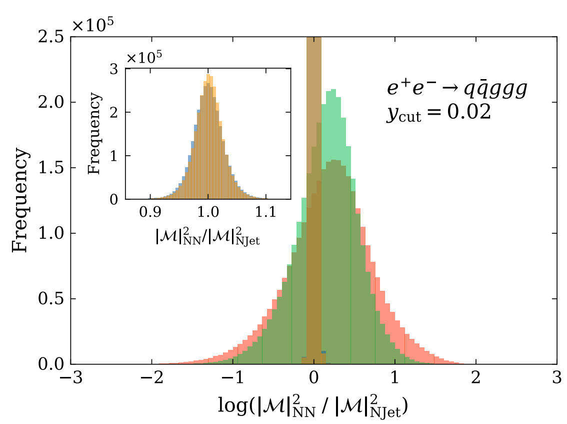
SB, (Aylett-)Bullock [2002.07516]
Aylett-Bullock, SB, Moodie [2106.09474]

huge potential!

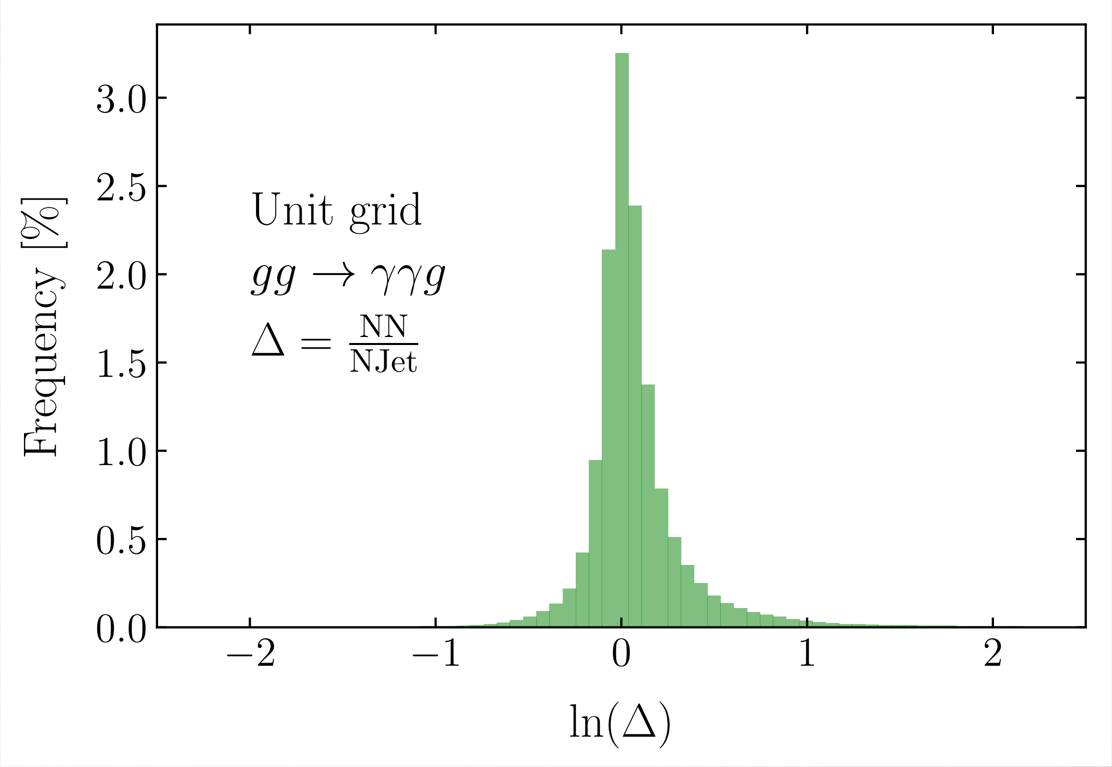


- Single NN does badly
- Understanding IR sectors via FKS improves reliablity (ensemble of networks)
- Error estimates by varying model initialisation
- Various tests suggest single run speed improvements at least x10

factorisation aware approach looks to be working nicely!



Maitre, Truong [2107.06625]



Bayesian Networks

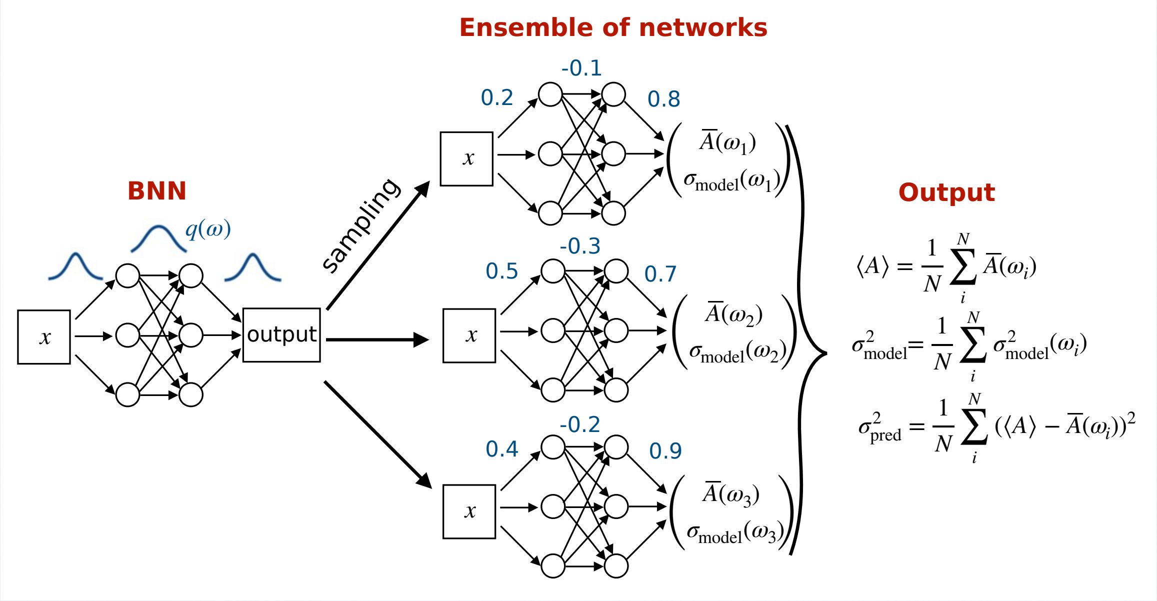
SB, Butter, Luchmann, Pitz, Plehn [2206.14831]

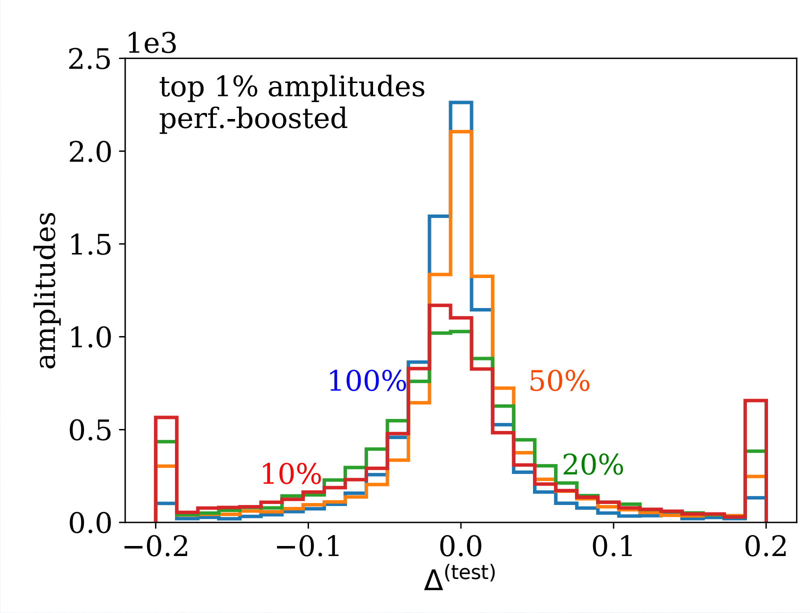
[See talk by Butter]

another experiment with loop amplitudes

gg→γγg, gg→γγgg @(1L)^2

- better defined error estimates
- improved training via loss and performance boosting





outlook

- new theory techniques are essential to meet the precision requirements at the LHC
- finite field arithmetic is making a dent in the 2L 2→3 wishlist
- some progress for amplitudes with internal masses (ttj)
- amplitudes neural networks look to be a promising way to significantly optimise MC simulations