

Recent Progress in Two-Loop Five-Point Calculations

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based on [[arXiv:2005.04195](#)], [[arXiv:2110.07541](#)]



Motivation

- ▶ **Experimental precision** projected to reach 1% level.

$$\sigma \sim \sigma_{LO} + \alpha_S \delta\sigma_{NLO} + \alpha_S^2 \delta\sigma_{NNLO} + \mathcal{O}(\alpha_S^3).$$

- ▶ Need **two-loop amplitudes** to match with NNLO precision.
- ▶ Must develop techniques for complex **multi-scale amplitudes**.
- ▶ Current theory frontier is **two-loop five-point** scattering.

The Road to Two-Loop Five-Point Master Integrals

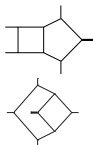
$$\mathcal{I}_j = \frac{N_j(\ell)}{\prod_{k \in \Gamma_j} D_k(\ell)} \xrightarrow[\text{algebraic complexity}]{\text{pure basis}} d\mathcal{I}_j = \epsilon M_{jk} \mathcal{I}_k \xrightarrow[\text{analytic complexity}]{\text{solution method}} \text{numeric values}$$

Solution approaches:

- ▶ (Integrals of) GPLs [Canko, Kardos, Papadopolous, Syrrakos, Wever]
- ▶ Series expansions: [Abreu, Ita, Moriello, BP, Tschernow, Zeng] Alessandro/Tobias
- ▶ **Pentagon functions**: [Chicherin, Henn, lo Presti, Sotnikov, Zoia] Simone.

Available Families:

- ▶ **Massless**. [Abreu, Chicherin, Dixon, Gehrmann, Henn, Herrmann, lo Presti, Mitev, BP, Papadopoulos, Tommasini, Wasser, Wever, Zeng, Zoia].
- ▶ **One-mass planar**. [Abreu, Canko, Chicherin, Ita, Moriello, BP, Papadopoulos, Syrrakos, Sotnikov, Tschernow, Tommasini, Wever, Zeng, Zoia].
- ▶ **One-mass non-planar (subset)**: [Abreu, Ita, Kardos, BP, Papadopoulos, Smirnov, Syrrakos, Tschernow, Wever]



The Road to Two-Loop Five-Point Amplitudes

$$\mathcal{A} = \sum_j \frac{N_j(\ell)}{\prod_{k \in \Gamma_j} D_k(\ell)} \xrightarrow{\text{IBPs}} \sum_j C_j \mathcal{I}_j \xrightarrow{\text{UV/IR Renorm}} \sum_i r_i h_i \xrightarrow{\text{partial fractions}} \text{compact expressions}$$

intermediate complexity
cancellation
simplification

Lessons learned:

- ▶ IBP insights from **algebraic geometry**. [Gluza, Kajda, Kosower '11; Ita '15; Zhang, Larsen '15, Abreu, Febres Cordero, Ita, Jaquier, BP, Zeng '17]
- ▶ **Bypass complexity**: \mathbb{F}_p IBPs + reconstruct remainder. [Schabinger, von Manteuffel '14; Peraro '16]
- ▶ **Multivariate partial** fractions of result for compact expressions. [Abreu, Dormans, Ita, Febres Cordero, BP, Sotnikov '19; von Manteuffel, Heller '21]

Open questions:

- ▶ How to combat large sample growth?: $\sim \mathcal{O}(10^6)$ for $W + jj$
- ▶ How to build **compact Ansätze**?

Status of Two-loop Five-Point Calculations

Many QCD amplitudes **available**:

- ▶ $pp \rightarrow jjj$, $pp \rightarrow \gamma\gamma j$, $pp \rightarrow \gamma\gamma\gamma$: [Abreu, Agarwal, Badger, Bronnum-Hansen, Buccioni, Chawdhry, Chicherin, Czakon, Dormans, Febres Cordero, Gehrmann, Hartano, Henn, Heinrich, Ita, von Manteuffel, Marcoli, Mitov, Moodie, BP, Peraro, Poncelet, Sotnikov, Tancredi, Wasser, Zhang, Zoia]
- ▶ $pp \rightarrow Wb\bar{b}$, $pp \rightarrow Hb\bar{b}$, $pp \rightarrow Wjj$: [Abreu, Badger, Febres Cordero, Hartanto, Ita, Klinkert, Kryz, BP, Sotnikov, Zoia] **This talk**.

Growing collection of **NNLO predictions**

- ▶ $pp \rightarrow jjj$, $pp \rightarrow \gamma\gamma j$, $pp \rightarrow \gamma\gamma\gamma$ [Badger, Chawdhry, Chen, Czakon, Gehrmann, Glover, Huss, Kallweit, Moodie, Marcoli, Mitov, Poncelet, Sotnikov, Wiesemann] **Matteo's talk**
- ▶ $pp \rightarrow Wb\bar{b}$ [Hartanto, Poncelet, Zoia] **Simone's talk**

Singularities and Ansätze

$$A(p_1, \dots, p_n) = \sum_j \underbrace{C_j(\epsilon, p_1, \dots, p_n)}_{\text{coefficients}} \underbrace{\mathcal{I}_j(p_1, \dots, p_n)}_{\text{master integrals}}$$

- ▶ Both coefficients and integrals are governed by **singularities**.

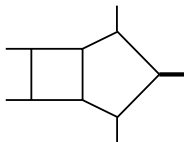
$$d\mathcal{I}_j = \epsilon \sum_{\alpha} M_{jk}^{\alpha} d \log(W_{\alpha}) \mathcal{I}_k, \quad \leftrightarrow \quad C_j = \frac{\mathcal{N}_j(\epsilon, p_1, \dots, p_n)}{\prod_{\alpha} W_{\alpha}^{q_{j\alpha}}}$$

- ▶ Our approach: Construct/fit **Ansätze**, from knowledge of W_{α} .

Focus of this talk: leading color 2-loop **W-plus-4-partons**.

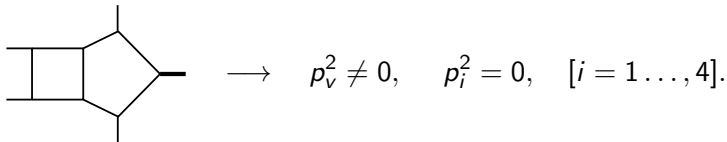
$$\mathcal{A}_g = \text{[Diagram: 2-loop gluon diagram with 4 external quarks]} + \dots, \quad \mathcal{A}_Q = \text{[Diagram: 2-loop quark diagram with 4 external quarks]} + \dots$$

Singularities, Integrals and All That



Scattering Kinematics and Notation

- ▶ Five-point **one-mass** kinematics.



- ▶ Integrals depend on **six independent** Mandelstam variables.

$$\vec{s} = \{p_v^2, s_{v1}, s_{12}, s_{23}, s_{34}, s_{4v}\}.$$

- ▶ Also depend on **square roots** of pentagon and triangle grams.

$$\text{tr}_5 = 4i\epsilon_{\alpha,\beta,\gamma,\delta} p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta.$$

$$\sqrt{\Delta_3} = \sqrt{(p_v^2 - s_{12} - s_{34})^2 - 4s_{12}s_{34}},$$

$$\sqrt{\Delta_3^{\text{nc}}} = \sqrt{(p_v^2 - s_{14} - s_{23})^2 - 4s_{14}s_{23}}.$$

Constructing Differential Equations from their Singularities

- ▶ Pure polylogarithmic integrals \Rightarrow simple differential equation.

$$d\mathcal{I}_j = \epsilon \sum_{\alpha} \underbrace{M_{jk}^{\alpha}}_{\text{rational numbers}} d \log \left(\underbrace{W_{\alpha}}_{\text{algebraic functions}} \right) \mathcal{I}_k.$$

[Henn '13]

- ▶ Letters W_{α} are algebraic kinematic functions via square roots.

$$W_{\alpha} \in \mathbb{Q} \left(\vec{s}, \text{tr}_5, \sqrt{\Delta_3}, \sqrt{\Delta_3^{\text{nc}}} \right).$$

- ▶ Determine M_{jk}^{α} using letters as Ansatz and numerical IBPs.

[Abreu, BP, Zeng '18]

- ▶ Set of W_{α} are computed from cut differential equation.

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

Alphabet Structures

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '21]

- ▶ Alphabet has 58 W_α , **only 49** appear in integrals up to $\mathcal{O}(\epsilon)$.

$$\underbrace{\{W_1, \dots, W_{49}\}}_{\text{relevant}} \cap \underbrace{\{W_{50}, \dots, W_{58}\}}_{\text{irrelevant}}.$$

- ▶ Transforms non-trivially under **Galois group** = $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

$$\text{tr}_5 \rightarrow -\text{tr}_5, \quad \sqrt{\Delta_3} \rightarrow -\sqrt{\Delta_3}, \quad \sqrt{\Delta_3^{\text{nc}}} \rightarrow -\sqrt{\Delta_3^{\text{nc}}}.$$

- ▶ Many W_α (odd and even) are **compactly expressed** with tr_+ .

$$\text{tr}_+(i_1 \cdots i_n) = \text{tr} \left(\left[\frac{1+\gamma_5}{2} \right] \not{p}_{i_1} \cdots \not{p}_{i_n} \right).$$

Galois Invariant Relevant Letters

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '21]

$$\{W_1, \dots, W_6\} \sim \{p_v^2, s_{23}, s_{v1}, s_{4v}, s_{12}, s_{34}\}.$$

$$\{W_7, \dots, W_{13}\} \sim 2 \{p_1 \cdot p_4, p_v \cdot p_1, p_v \cdot p_2, p_1 \cdot p_3\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{14}, \dots, W_{21}\} \sim 2 \{p_1 \cdot p_{23}, p_1 \cdot p_{34}, p_2 \cdot p_{v1}, p_2 \cdot p_{4v}\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{22}, \dots, W_{30}\} \sim \{\text{tr}_+(V1V4), \text{tr}_+(V1V2), \text{tr}_+(V1V3), \text{tr}_+(V1V[3+4]), \\ \text{tr}_+([1+2]3[1+2]V)\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{31}, W_{32}\} \sim \{\text{tr}_+(V123) - \text{tr}_+(V134), \text{tr}_+(V432) - \text{tr}_+(V421)\}.$$

$$\{W_{48}, W_{49}\} = \{\sqrt{\Delta_3}, \text{tr}_5\}.$$

Describes (little-group invariant) **integral/amplitude singularities**.

Galois Non-Trivial Relevant Letters

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '21]

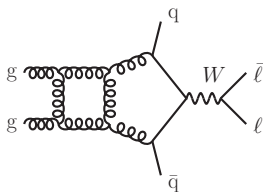
$$\{W_{33}, \dots, W_{36}\} = \left\{ \frac{s_{v1} + s_{v2} + \sqrt{\Delta_3}}{s_{v1} + s_{v2} - \sqrt{\Delta_3}}, \frac{s_{v1} + s_{4v} + \sqrt{\Delta_3^{\text{nc}}}}{s_{v1} + s_{4v} - \sqrt{\Delta_3^{\text{nc}}}}, \frac{s_{v3} + s_{v2} + \sqrt{\Delta_3^{\text{nc}}}}{s_{v3} + s_{v2} - \sqrt{\Delta_3^{\text{nc}}}} \right\} \\ + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{37}, W_{38}, W_{39}\} = \left\{ \frac{s_{v1} - s_{v2} + \sqrt{\Delta_3}}{s_{v1} - s_{v2} - \sqrt{\Delta_3}}, \frac{s_{v1} - s_{4v} + \sqrt{\Delta_3^{\text{nc}}}}{s_{v1} - s_{4v} - \sqrt{\Delta_3^{\text{nc}}}} \right\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{40}, \dots, W_{46}\} = \left\{ \frac{\text{tr}_+(1234)}{\text{tr}_-(1234)}, \frac{\text{tr}_+(V123)}{\text{tr}_-(V123)}, \frac{\text{tr}_+(34V1)}{\text{tr}_-(34V1)}, \frac{\text{tr}_+(V132)}{\text{tr}_-(V132)} \right\} \\ + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$W_{47} = \frac{\Omega^{--} \Omega^{++}}{\Omega^{+-} \Omega^{-+}}, \quad \text{where} \quad \Omega^{\pm\pm} = s_{v1}s_{4v} - s_{v1}s_{12} - s_{4v}s_{34} \pm s_{23}\sqrt{\Delta_3} \pm \text{tr}_5.$$

Provide **no new** singular surfaces.

Ansätze and $W + 4$ -Parton Amplitudes

Singularities and Ansätze for Coefficients

- ▶ Apply analytic reconstruction techniques to **finite remainder**.

$$\mathcal{R}_\kappa = \sum_{i \in B} r_i h_i^*, \quad \text{where } r_i = r_i^+ + \text{tr}_5 r_i^-. \\ *h_i \text{ from [Chicherin, Sotnikov, Zoia '21].}$$



- ▶ Fit Ansatz with \mathbb{F}_p **evaluations** of r_i^\pm from Caravel.

[Abreu, Febres Cordero, Ita, BP, Pascual, Ruf, Sotnikov '20]

- ▶ Singularities known, can consider **common denominator form**.

$$r_i^\pm(\vec{s}) = \frac{n_i^\pm(\vec{s})}{\prod_\alpha W_\alpha^{q_{i\alpha}^\pm}}$$

- ▶ Determine $q_{i\alpha}^\pm$ from **univariate slice**.

[Abreu, Dormans, Febres Cordero, Ita, BP '18] see also [Heller, von Manteuffel '21]

Given known W_α , $q_{i\alpha}^\pm$, can we build Ansatz with **fewer terms**?

Univariate Partial Fractions Ansatz

- ▶ Perform **partial fraction** during reconstruction.

[Badger, Hartanto, Zoia '21] (see also [De Laurentis, BP '22])

- ▶ Partial fraction **w.r.t** s_{34} : $\vec{s} = \underbrace{\{s_{v1}, s_{12}, s_{23}, s_{4v}, p_v^2\}}_{\vec{s}_{\text{rem}}} \cap \{s_{34}\}$.
- ▶ Rational function depends on **simpler polynomials** P_{ij}^\pm as

$$r_i^\pm(\vec{s}) = \sum_j \frac{P_{ij}^\pm(\vec{s}_{\text{rem}})}{\underbrace{W_{k_{ij}}(s_{34}, \vec{s}_{\text{rem}})}_{\text{poles in } s_{34}} \prod_l \underbrace{\overline{W}_l(\vec{s}_{\text{rem}})^{\gamma_{ij}^\pm}}_{\text{spurious \& physical}}}.$$

- ▶ Choose $\vec{s}_{\text{rem}}^{(k)}$ so that P_{ij}^\pm Ansatz matrix in **Vandermonde form**.
[Klappert, Klein, Lange '20; Abreu, Febres Cordero, Ita, Klinkert, BP, Sotnikov '21]
- ▶ Can be solved in $\mathcal{O}(N^2)$ **time**, can adapt to zero coefficients.

Summary of Results [Abreu, Febres Cordero, Ita, Klinkert, BP, Sotnikov '21]

$\mathcal{R}_Q^{N_f^0}$	$p_5 \parallel p_i$	# Terms	n_i^\pm	\sum_j # Terms	P_{ij}^\pm	# Result Terms
+−	2	1700 k		210 k		56 k
−+	3	5500 k		220 k		48 k

$\mathcal{R}_g^{N_f^0}$	$p_5 \parallel p_i$	# Terms	n_i^\pm	\sum_j # Terms	P_{ij}^\pm	# Result Terms
++	2	1100 k		82 k		14 k
+−	2	7000 k		480 k		110 k
−+	3	24 000 k		430 k		99 k

- ▶ Greatly **reduced term/sample count** in partial fraction Ansatz.
- ▶ Reconstructed expression is **yet more sparse**.
- ▶ Analytic results are **publicly available** in ancillary files.

Analytic Structure of The Remainder

Some **Galois invariant** letters **cancel** in amplitude/remainder:

(Also observed in amplitudes of [Badger, Hartanto, Kryz, Zoia])

$$\{W_1, \dots, W_6\} \sim \{p_v^2, s_{23}, s_{v1}, s_{4v}, s_{12}, s_{34}\}.$$

$$\{W_7, \dots, W_{13}\} \sim 2 \{p_1 \cdot p_4, p_v \cdot p_1, p_v \cdot p_2, p_1 \cdot p_3\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{14}, \dots, W_{21}\} \sim 2 \{p_1 \cdot p_{23}, \cancel{p_1 \cdot p_{34}}, p_2 \cdot p_{v1}, p_2 \cdot p_{4v}\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{22}, \dots, W_{30}\} \sim \{\text{tr}_+(V1V4), \text{tr}_+(V1V2), \text{tr}_+(V1V3), \cancel{\text{tr}_+(V1V[3+4])}, \\ \cancel{\text{tr}_+([1+2]3[1+2]V)}\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{31}, W_{32}\} \sim \{\text{tr}_+(V123) - \text{tr}_+(V134), \text{tr}_+(V432) - \text{tr}_+(V421)\}.$$

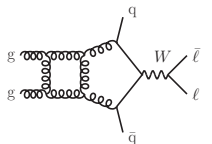
$$\{W_{48}, W_{49}\} = \{\sqrt{\Delta_3}, \cancel{\text{tr}_5}\}.$$

Summary and Conclusions

- ▶ **Large progress** has led to many five-point amplitudes.
- ▶ **Understanding of singularities** helping to construct amplitudes.
- ▶ Leading-colour **W-plus-4-parton** amplitudes are now known.
- ▶ Unexpected analytic cancellations \Rightarrow **Better ways** to compute?

Back to Five-Point Kinematics [Abreu, Febres Cordero, Ita, Klinkert, BP, Sotnikov '21]

- Amplitudes depend on **six points**, but decay is simple.



$$+ \dots = A^\mu J_\mu, \quad \text{where} \quad J_\mu = \bar{u}(p_6) \gamma_\mu v(p_5).$$

- Consider configuration where p_5 is **collinear to p_i** , i.e.

$$p_5^{(i)} = \frac{p_V^2}{2p_i \cdot p_V} p_i, \quad p_6^{(i)} = p_V - \frac{p_V^2}{2p_i \cdot p_V} p_i.$$

- Expand A^μ in terms of collinear **“form factor”** amplitudes:

$$A^\mu = \sum_{i=1}^3 \mathcal{A}^{\{i\}} N_i^\mu, \quad \text{where} \quad \mathcal{A}^{\{i\}} = \mathcal{A}(p_5^{(i)}, p_6^{(i)}).$$

- We reconstruct its remainder $\mathcal{R}^{\{i\}}(s_{V1}, s_{12}, s_{23}, s_{34}, s_{4V}, p_V^2)$.