Wilson loops with Lagrangian insertion and all-plus amplitudes

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Based on 2202.05596, 2204.00329 with D. Chicherin,
and on previous work 2122.06956 with N. Arkani-Hamed and J. Trnka

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Why are Wilson loops with Lagrangian insertions (in planar maximally supersymmetric Yang-Mills) interesting?

They are well-defined, finite quantities in an interacting QFT, whose integrand is completely known (in principle). Can we understand the integrations, and derive all-order results?

Their similarity to (finite parts of QCD) scattering amplitudes allows us to shed light on the relevant function space, and on novel ways of dealing with infrared divergences.

They display many surprising properties, such as hidden symmetries, positivity properties, and a duality to all-plus amplitudes in pure Yang-Mills.

Outline

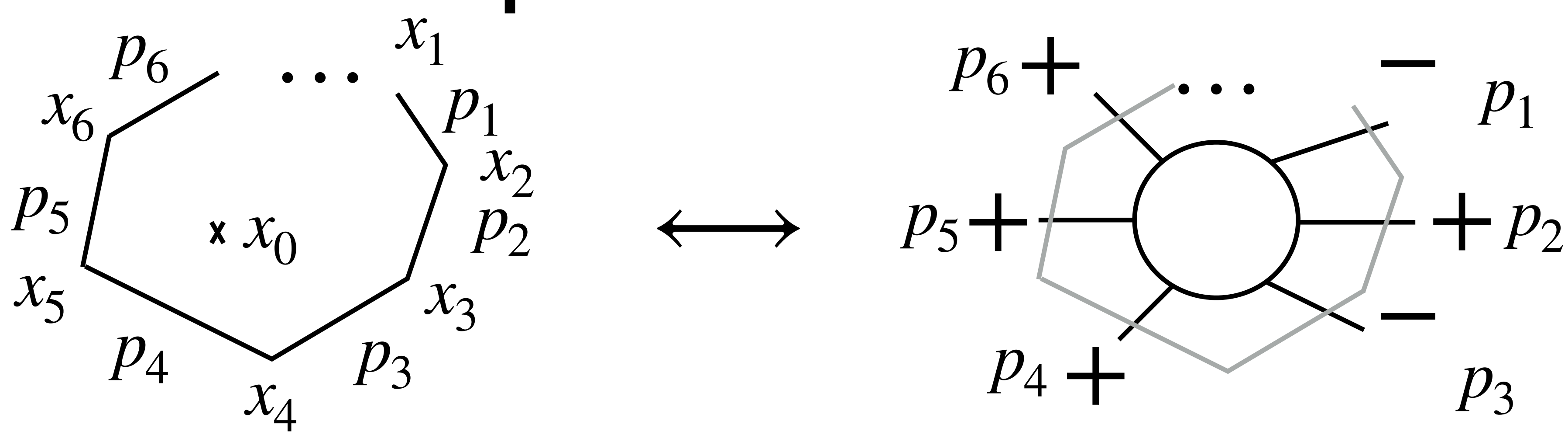
1. Integrand-level duality between Wilson loops and scattering amplitudes

2. Finite Wilson loop with Lagrangian insertion: definition; properties; overview results.

3. Surprising new features

Part 1 : Integrand-level duality between Wilson loops and scattering amplitudes

Wilson loop/MHV scattering amplitude duality in planar N=4 sYM



Null Wilson loop MHV scattering amplitude

Dual variables: $x_{i+1} - x_i = p_i$

(Dual) conformal symmetry in x-space

Conformal symmetry in p-space

$$W_n \sim A_n / A_n^{(0)}$$

Wilson loop has ultraviolet divergences, amplitude has infrared divergences. Better formulate duality at level of integrands!

Integrand for scattering amplitudes

$$M_n = A_n / A_n^{(0)} \qquad M_n = 1 + g^2 M_n^{(1)} + g^4 M_n^{(2)} + \ldots$$

$$M_n^{(L)}(x_1, \ldots, x_n) \sim \int d^D y_1 \ldots d^D y_L \mathcal{I}_n^{(L)}(x_1, \ldots x_n; y_1, \ldots y_L)$$

Integrand is a rational function, defined in D=4. dimensions.

Various methods to obtain integrands:

- Generalized unitarity [Bern et al., 1994]
- Loop-level on-shell recursion; on-shell graphs [Arkani-Hamed et al., 2010]
- Canonical form on Amplituhedron geometry [Arkani-Hamed, Trnka, 2013]
- Bootstrap from symmetry and analyticity [Bourjaily at al.; Eden et al., 2011]

Four-point integrand explicitly known to 10 loops!

Wilson loop integrand via Lagrangian insertion

$$\langle W_n \rangle = 1 + g^2\, W_n^{(1)} + g^4\, W_n^{(2)} + \ldots\,, \qquad g^2 \equiv \frac{N_c g_{\rm YM}^2}{16\pi^2}$$

Lagrangian insertion formula

$$g^2 \partial_{g^2} \langle W_n \rangle \sim \int d^D y \langle W_n \mathscr{L}(y) \rangle$$

One-loop integrand:

$$W_n^{(1)} \sim \int d^D y \underbrace{\langle W_n \mathscr{L}(y) \rangle_{\rm Born}}_{1\text{–loop integrand}}$$

L-loop generalization:

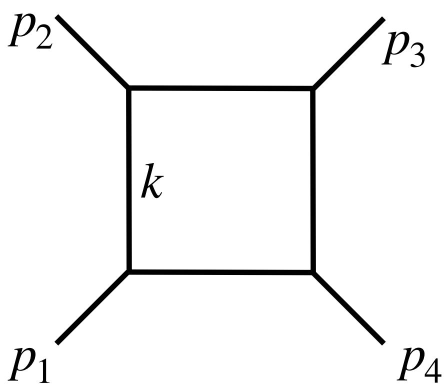
$$W_n^{(L)} \sim \int d^D y_1 \ldots d^D y_L \underbrace{\langle W_n \mathscr{L}(y_1) \ldots \mathscr{L}(y_L) \rangle_{\rm Born}}_{L\text{–loop integrand}}$$

Duality at integrand level

$$\mathcal{I}_n^{(L)}(x_1, \ldots x_n; y_1, \ldots y_L) = \langle W_n(x_1, \ldots x_n)\mathcal{L}(y_1)\ldots\mathcal{L}(y_L)\rangle_{\text{Born}}$$

[Eden, Korchemsky, Sokatchev 2010; Mason, Skinner 2010]

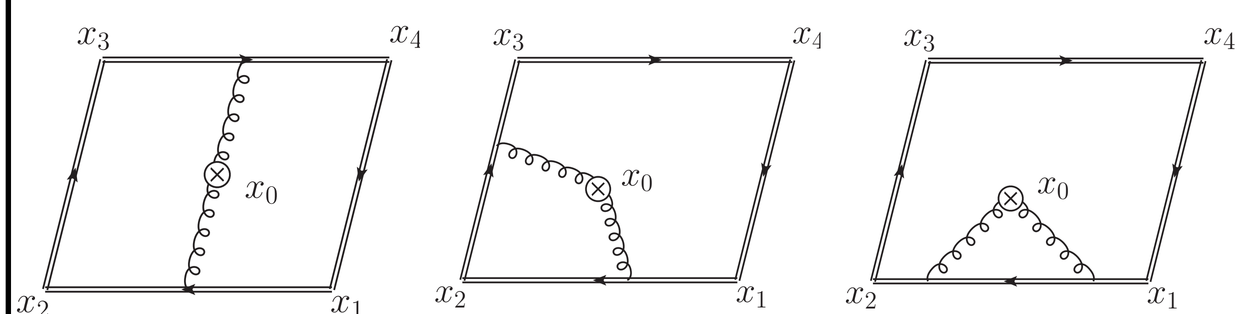
One-loop example:

$-st$ 

$$\mathcal{I}_n^{(L)}(x_1, \ldots x_4; x_0) = \frac{-st}{k^2(k-p_1)^2(k+p_2)^2(k+p_2+p_3)^2}$$

$$= \frac{-(x_1-x_3)^2(x_2-x_4)^2}{(x_0-x_2)^2(x_0-x_1)^2(x_0-x_3)^2(x_0-x_4)^2}$$

$k = x_2 - x_0\,,\quad p_i = x_{i+1} - x_i$



$$\langle W_4(x_1, \ldots x_4)\mathcal{L}(x_0)\rangle = \frac{-x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$$

$x_{ij} = x_i - x_j$

Recap part I

$$\mathcal{I}_n^{(L)}(x_1, \ldots x_n; y_1, \ldots y_L) = \langle W_n(x_1, \ldots x_n)\mathcal{L}(y_1)\ldots\mathcal{L}(y_L)\rangle_{\text{Born}}$$

Amplitude integrand in dual coordinates

Wilson loop with Lagrangian insertions

Duality between amplitudes and Wilson loops in planar N=4 sYM can be formulated at the level of rational loop integrands. Integrands are known explicitly to high loop orders, and in principle at any loop order.

Performing the loop integrations leads to divergences. We investigate a closely related, infrared and ultraviolet finite quantity.

Part 2 : Finite Wilson loop with Lagrangian insertion: definition; properties; overview results.

Finite Wilson loop with Lagrangian insertion

Wilson loop divergences exponentiate - this means $\log\langle W_n(x_1, \ldots, x_n)\rangle$ is free of subdivergences. [Korchemsky et al.]

Use Lagrangian insertion trick:

$$g^2 \partial_{g^2} \log\langle W_n\rangle \sim \int d^D x_0 \underbrace{\frac{\langle W_n \mathscr{L}(x_0)\rangle}{\langle W_n\rangle}}_{\equiv F_n}$$

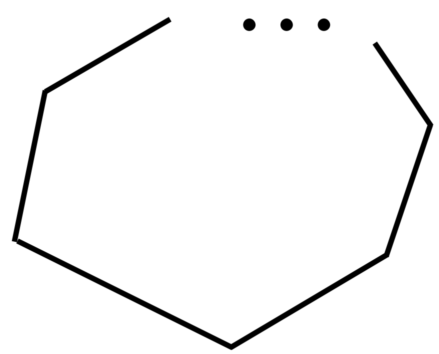
F_n can be computed in four dimensions. Divergences occur only upon integration over x_0 .

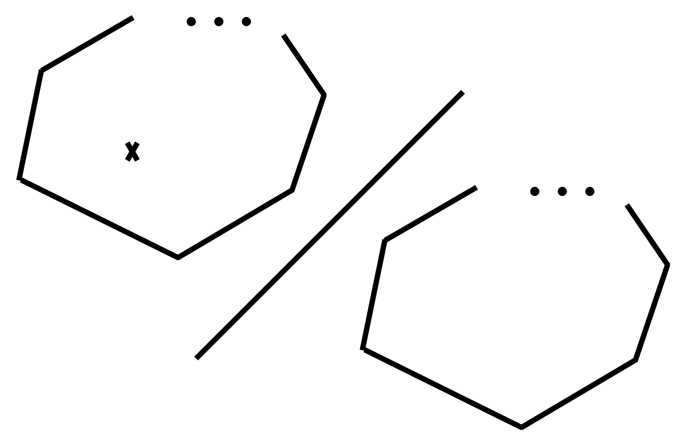
Remark: the full-color four-loop cusp anomalous dimension was first computed using this approach from the three-loop result for F. (In the present talk however we consider the planar limit.)

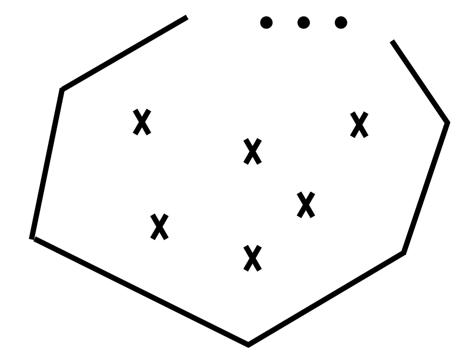
$$\log\langle W_n\rangle_{g^{2L}} \sim \frac{1}{L^2\epsilon^2}\Gamma^{(L)}_{\text{cusp}} + \ldots$$

[JMH, Korchemsky, Mistlberger, 2019]

Overview different Wilson loop correlators







$\langle W_n \rangle$	$\dfrac{\langle W_n \mathscr{L}(x_0) \rangle}{\langle W_n \rangle}$	$\langle W_n \mathscr{L}(y_1) \ldots \mathscr{L}(y_L) \rangle_{\text{Born}}$
Divergent	Finite	Finite
Anomalous dual conformal symmetry	Exact dual conformal symmetry	Exact dual conformal symmetry
Transcendental functions	Transcendental functions	Rational

F depends on the same kinematics and functions as planar massless QCD scattering amplitudes

Can use dual conformal symmetry to send x_0 to infinity:

$$f_n(p_1, \ldots, p_n) = \lim_{x_0 \to \infty} (x_0^2)^4 F_n(p_1, \ldots, p_n; x_0)$$

F (equivalently, f) depends on (3n-10) dimensionful variables.

n	Number of variables	Variables	Alphabet letters	Function space
4	2	s, t	$\{s, t, s+t\}$	'Harmonic polylogarithms' [Gehrmann, Remiddi; Maître]
5	5	$s_{i,i+1}$	20 parity-even letters, 5 parity-odd letters	'Planar pentagon functions' [Gehrmann, JMH; LoPresti; Chicherin, Sotnikov]
6	8	$s_{i,i+1}; s_{i,i+1,i+2}$ (one Gram condition)	Under investigation	Under investigation

Known results and general structure

Expected form: $f_n^{(L)} = \sum_{i,j} c_{i,j}\, r_{n,i}\, g_j^{(2L)}$

Constants

Leading singularities (rational, algebraic)

Transcendental functions of weight 2L

Loop order	Number of points	References
Tree-level	Any n	[JMH, Chicherin, 2022]
One loop	Any n	[Alday, Heslop, Sikorowski, 2012] [JMH, Chicherin, 2022]
Two loops	n=4,5	[Alday, JMH, Sikorowski, 2013] [JMH, Chicherin, 2022]
Three loops	n=4	[JMH, Korchemsky, Mistlberger, 2019]
All loops	n=4 'tree geometries'	[Arkani-Hamed, JMH, Trnka, 2021]
Strong coupling	n=4	[Alday, Buchbinder, Tseytlin, 2011]

Part 3 : Surprising new features

Leading singularities are conformally invariant

$$\mathbb{K}^{\alpha\dot{\alpha}}(PT)_n r_{n,i} = 0\,, \qquad \mathbb{K}^{\alpha\dot{\alpha}} = \sum_j \frac{\partial}{\partial \lambda_j^{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_j^{\dot{\alpha}}}$$

Momentum-space conformal generator, cf. [Witten, 2003]

$$(PT)_n = \frac{1}{\langle 12\rangle \ldots \langle (n-1)n\rangle}$$

We checked this in all known cases. [JMH, Chicherin, 2022]

Example: $$f_4^{(0)} = \lim_{x_0 \to \infty} (x_0^2)^4 F_4^{(0)} = -\, x_{13}^2 x_{24}^2 = -\, st$$

$$(PT)_4 f_4^{(0)} \sim \frac{[34]^2}{\langle 12\rangle^2} \qquad \mathbb{K}^{\alpha\dot{\alpha}} \frac{[34]^2}{\langle 12\rangle^2} = 0\,.$$

We found a Grassmannian formula that gives all leading singularities that have appeared so far. Is it the unique answer to the conformal invariance?

[JMH, Chicherin, 2022]

F has definite sign in the Amplituhedron region

The four-point Amplituhedron implies $s < 0, t < 0$.
In that region, we have:

$$(-1)^{L+1} f_4^{(L)} > 0 \qquad L = 0,1,2,3$$

[Arkani-Hamed, JMH, Trnka, 2021]

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At five points, the Amplituhedron 'lives' in a subspace of the Euclidean region defined by:

$$s_{i,i+1} < 0\,, \qquad \epsilon_5 = 4i\epsilon_{\mu\nu\rho\sigma} p_1^{\mu} p_2^{\mu} p_3^{\mu} p_4^{\mu} > 0$$

In this region, we find the positivity property:

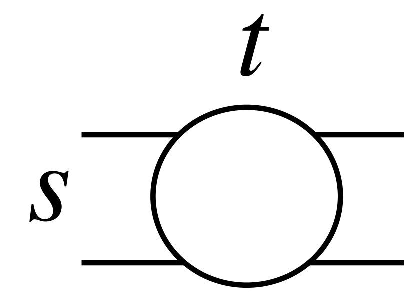
$$(-1)^{L+1} f_5^{(L)} > 0 \qquad L = 0,1,2$$ [Chicherin, JMH, 2022]

Nontrivial, individual terms can have other signs.

Is not true for $\epsilon_5 < 0$!

Four- and five-point (multi-)Regge limit are governed by the same formula

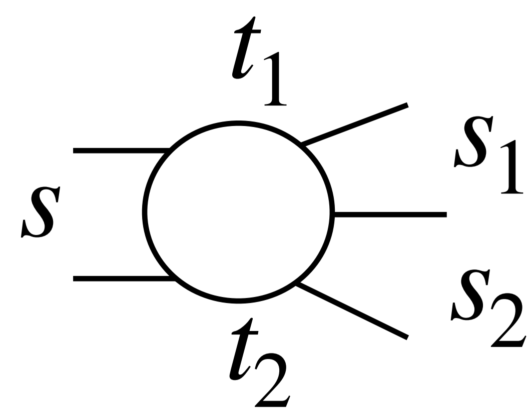
n=4 Euclidean Regge limit:



$s \gg t$

$L = \log \frac{s}{t} \gg 1$

n=5 Multi-Regge limit:



$s \gg s_1, s_2 \gg t_1, t_2$

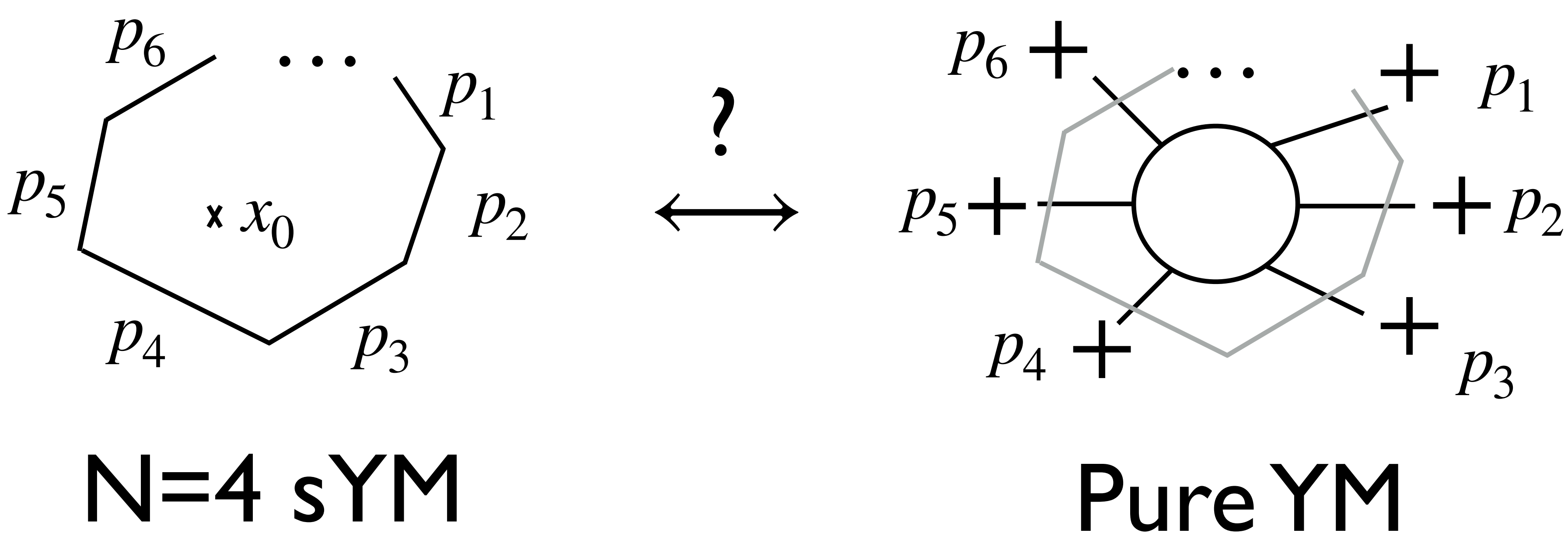
$L = \log \frac{s}{\sqrt{t_1 t_2}} \gg 1$

We find in both cases:

$$\log\left(\frac{f_n}{g^2 f_n^{(0)}}\right) = \left(-g^2 + \pi^2 g^4 + \mathcal{O}(g^6)\right) L^2 + \left(4\zeta_3 g^4 + \mathcal{O}(g^6)\right) L + \ldots$$

Can one understand this formula from first principles?

Duality with Yang-Mills all-plus amplitude

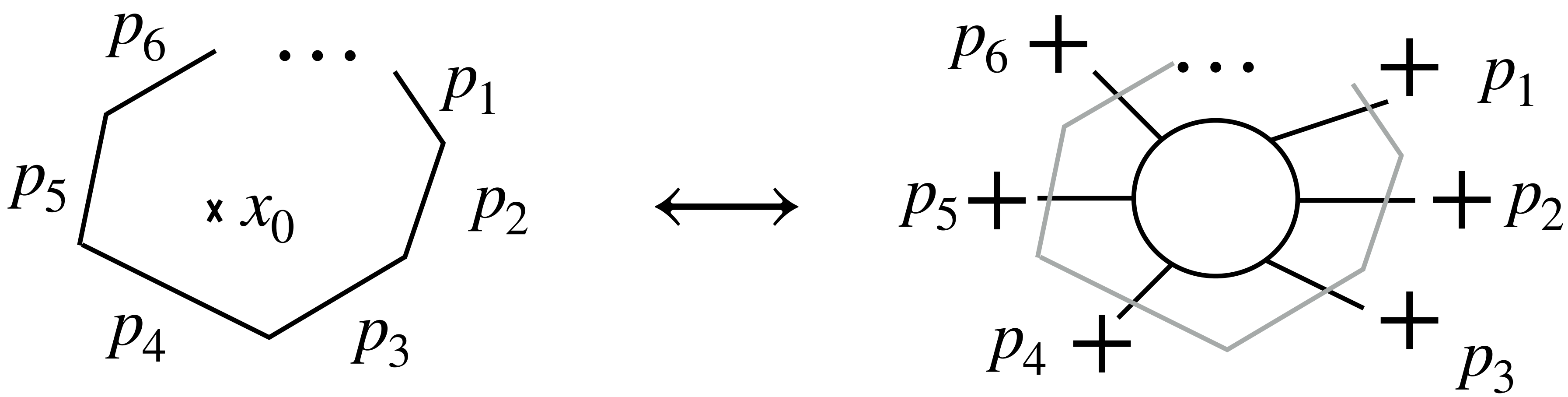


- Have the same cyclic symmetry
- Depend on the same kinematics (as $x_0 \to \infty$)
- One-loop all-plus amplitude is conformally invariant

(also coefficients of two-loop transcendental functions)

[JMH, Power, Zoia, 2019; Badger et al, 2019]

Duality with Yang-Mills all-plus amplitude



N=4 sYM   Pure YM

- Have the same cyclic symmetry
- Depend on the same kinematics (as $x_0 \to \infty$)
- One-loop all-plus amplitude is conformally invariant

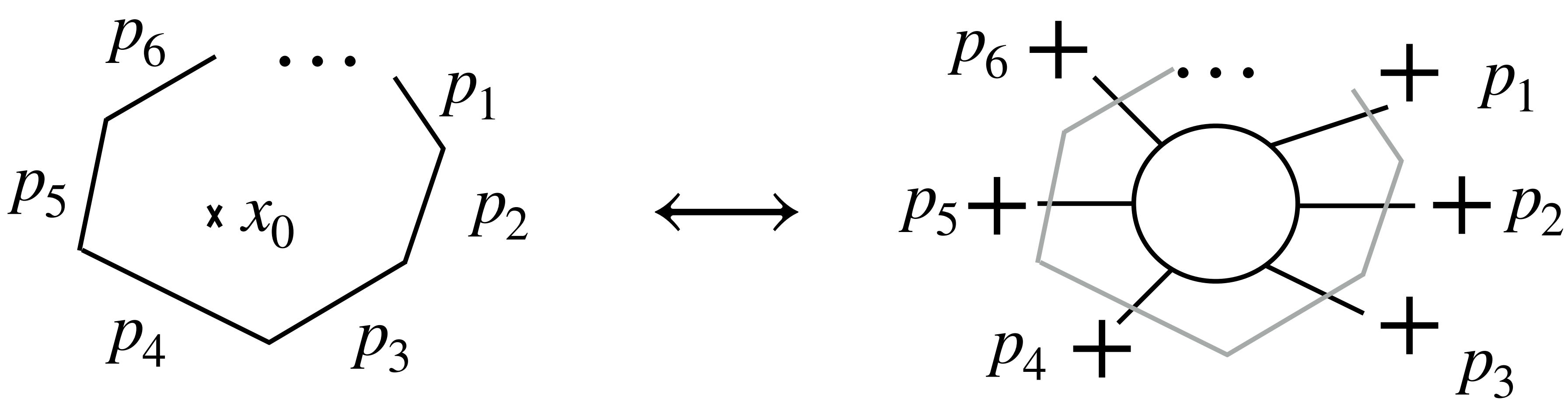
(also coefficients of two-loop transcendental functions)

[JMH, Power, Zoia, 2019; Badger et al, 2019]

At leading order, objects are equal! $A^{(1)}_{\mathrm{YM},n} = \mathrm{PT}_n\, f^{(0)}_n$

[Chicherin, JMH, 2022]

Duality with Yang-Mills all-plus amplitude



At higher loops, the maximal weight part of the infrared-renormalised amplitude $A_n^{\mathrm{YM}\,(L+1)}$ matches with the Wilson loop $f_n^{(L)}$. [Chicherin, JMH, 2022]

Checks: $f_n^{(1)} \longleftrightarrow A_n^{\mathrm{YM}\,(2)}$ [Dunbar et al., 2016]

$f_4^{(2)} \longleftrightarrow A_4^{\mathrm{YM}\,(3)}$ [Jin, Luo, 2019; Caola et al, 2021]

$f_4^{(3)}$ and $f_5^{(2)}$ predict the maximal weight part of the 4-loop 4-gluon and 3-loop 5-gluon all-plus amplitude!

Discussion

Wilson loop with Lagrangian insertion in N=4 sYM has similarities to finite parts of massless QCD amplitudes.

We found several remarkable properties:

- Conformal symmetry of leading singularities
- Positivity in Amplituhedron kinematics
- Simplicity in Regge limit
- Duality with all-plus Yang-Mills amplitudes