Renormalized ϵ-finite master integrals and their virtues: the three-loop self energy case

Stephen P. Martin
Northern Illinois University
spmartin@niu.edu

Loopfest 2022
Pittsburgh
May 10, 2022

Based on: arXiv:2112.07694 and arXiv:2203.05042

Three frontiers in perturbative radiative corrections:

- Number of external legs
- Number of loops
- Number of independent scales (internal masses, external kinematic invariants)

Because I am a coward, I concentrate on problems where the number of external legs is 0 or 2.

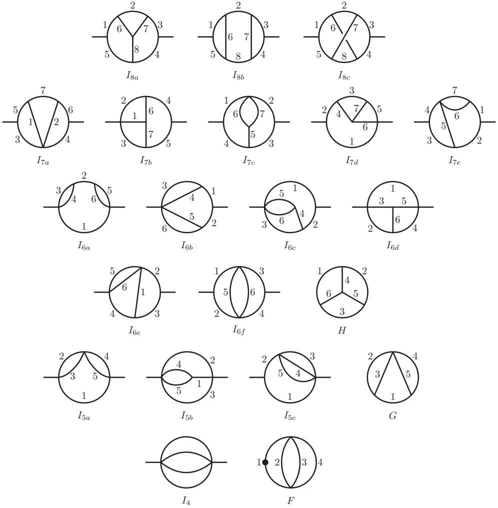
Motivation for the present work: 3-loop self-energy integrals for QCD, mixed QCD/EW and pure EW contributions to the pole masses of the W, Z, and Higgs bosons. [See also work of Lisong Chen and Ayres Freitas here at Pitt, Ina Hönemann (with Chaubey and Weinzierl) talk Friday.]

In extensions to BSM theories like SUSY, the number of independent masses can be large. Challenge: develop a push-button approach that works equally well for arbitrary internal masses.

For 3-loop vacuum (0 external legs) integrals, the problem of fast numerical evaluation with arbitrary masses is solved:

- 3VIL 1610.07720, by D. Robertson and SPM (differential equations method)
- TVID 1908.09887, 1702.02996, Bauberger, Freitas, Wiegand (dispersion relations)

3-loop self-energy (2 external legs) integrals have 1 external and up to 8 internal masses. The topologies are shown at right. Candidate masters include also these with doubled propagators.



Need:

• Reduction of observables to master integrals

• Numerical evaluation of masters

How to choose master integrals?

Many possible choices, appropriate for different goals:

- Simplify reduction of Green functions to masters
- Maximize flexibility (works for arbitrary masses)
- Simplify evaluation of masters (analytic when possible, accuracy, speed)
- Simplify presentation of results

These criteria do not always coincide!

I will advocate for an approach in which:

- numerical evaluation is straightforward but analytic evaluation is often not,
- avoid unnecessary functions arising in expansions of the dimensional regularization parameter ϵ

The problem of "spurious poles"

A master integral $\mathbf{I}_k$ in dimensional regularization has an expansion with poles in ϵ of order up to the loop order L:

$$\mathbf{I}_k = \sum_{n=-L}^{\infty} \mathbf{I}_k^{(n)} \epsilon^n,$$

and a Green function needed for some observable will be of the form

$$G = \sum_k c_k \mathbf{I}_k.$$

However, in general the coefficients c_k may themselves have poles in ϵ.

Fortunately, Chetyrkin, Faisst, Sturm, Tentuykov hep-ph/0601165 showed that one can always choose an ϵ-**finite basis** for masters, such that the coefficients c_k are finite as $\epsilon \to 0$. (The proof is constructive; the choice is not unique, but is independent of the observable.)

If one chooses an ϵ-finite basis, then all of the poles in ϵ are contained within the master integrals $\mathbf{I}_k$ themselves, not in the coefficients c_k.

Choosing an ϵ-finite basis is not always completely trivial.

Suppose one has an integration-by-parts (IBP) identity of the form:

$$p_A \mathbf{I}_A + p_B \mathbf{I}_B + p_C \mathbf{I}_C + \ldots = 0$$

where the $\mathbf{I}_k$ are scalar integrals, some of which will be our chosen masters, and p_k are polynomial functions of the masses and ϵ.

Which integral $\mathbf{I}_k$ should one solve for?

- Solve for whichever $\mathbf{I}_k$ has the smallest numbers of momenta in numerators, propagators
- Solve for whichever $\mathbf{I}_k$ has the "simplest" coefficient p_k, since it will become a denominator in future expressions.
- Solve for one of the $\mathbf{I}_k$ such that p_k is non-zero as $\epsilon \to 0$. (This provides an ϵ-finite basis.)

These criteria are often in direct conflict. The straightforward construction of the ϵ-finite basis can be ugly in intermediate states.

One can also construct the ϵ-finite basis after doing the IBP relations, by trading some of masters for some of the solved-for integrals. Here, it is easier if we kept track of which integrals were solved for that would have been ϵ-finite.

Even after choosing an ϵ-finite basis, there is a more general problem. For a calculation at L-loop order, need masters at every loop order K, with $1 \leq K \leq L$.

Each K-loop master could be multiplied by an $L - K$ loop counterterm, or by other masters with UV poles as severe as $1/\epsilon^{L-K}$.

Naively, one needs expansions of K-loop masters to positive powers ϵ^{L-K}.

Solution: **renormalized ϵ-finite master integrals**.

Define by subtracting all UV divergences in a specific way (described on next slide), then take the limit $\epsilon \to 0$. Features of renormalized ϵ-finite masters:

- Masters are completely independent of ϵ, by construction
- Straightforward path to fast numerical evaluation
- Never necessary to expand to positive powers in ϵ, even if the calculation is later extended to even higher loop order.

Verified examples: effective potential and self-energies through 3-loop order.

Renormalized ϵ-finite master integrals

For a given L-loop ϵ-finite master $\mathbf{I}$, define

$$I \quad = \quad \lim_{\epsilon\to 0}\left[\mathbf{I} - \sum_{K=0}^{L} \mathbf{I}^{K,\,\mathrm{div}}\right],$$

where the K-loop UV sub-divergences being subtracted are

$$\mathbf{I}^{K,\,\mathrm{div}} \quad = \quad \sum_{\mathbf{J}_K} \mathbf{J}_K \sum_{n=1}^{K} \frac{1}{\epsilon^n} c^{(n)}_{\mathbf{J}_K}.$$

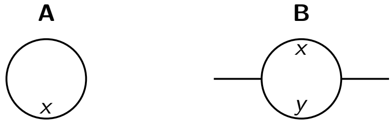
The $\mathbf{J}_K$ are the $(L-K)$-loop integrals obtained from $\mathbf{I}$ by collapsing UV-divergent K-loop sub-diagrams to a point. The counterterm coefficients $c^{(n)}_{\mathbf{J}_k}$ do not depend on ϵ, and are polynomials in the masses and kinematic invariants, chosen so that I is free of UV divergences.

Note that beyond 1-loop order, I is *not* the same as $I^{(0)}$, the coefficient of ϵ^0 in the expansion

$$\mathbf{I} = \sum_{n=-L}^{\infty} I^{(n)} \epsilon^n.$$

Also, the renormalized ϵ-finite master I does not depend on the positive power terms $I^{(1)}$, $I^{(2)}$, etc.

For self-energy integrals at 1-loop order, with squared masses x, y:



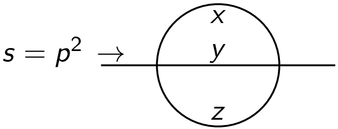
$$\begin{aligned}
\mathbf{A}(x) &= -\frac{x}{\epsilon} + A(x) + \epsilon A^{(1)}(x) + \epsilon^2 A^{(2)}(x) + \ldots \\
\mathbf{B}(x,y) &= \frac{1}{\epsilon} + B(x,y) + \epsilon B^{(1)}(x,y) + \epsilon^2 B^{(2)}(x,y) + \ldots,
\end{aligned}$$

and the renormalized master integrals are just

$$\begin{aligned}
A(x) &= \lim_{\epsilon \to 0} \left[\mathbf{A}(x) + \frac{x}{\epsilon} \right], \\
B(x,y) &= \lim_{\epsilon \to 0} \left[\mathbf{B}(x,y) - \frac{1}{\epsilon} \right].
\end{aligned}$$

A key point: if one uses renormalized ϵ-finite master integrals in any calculation at L loops, the functions $A^{(1)}(x)$, $A^{(2)}(x)$, …and $B^{(1)}(x,y)$, $B^{(2)}(x,y)$, … are **never** needed. They always cancel completely from any renormalized observable. This provides a powerful check!

Example at 2-loop order: the sunrise integral



The renormalized masters include:

$$S(x,y,z) \;=\; \lim_{\epsilon\to 0}\Big[\mathbf{S}(x,y,z) - \mathbf{S}^{1,\,\mathrm{div}}(x,y,z) - \mathbf{S}^{2,\,\mathrm{div}}(x,y,z)\Big],$$

with contributions from one-loop and two-loop UV sub-divergences:

$$\mathbf{S}^{1,\,\mathrm{div}}(x,y,z) \;=\; \frac{1}{\epsilon}\left[\mathbf{A}(x)+\mathbf{A}(y)+\mathbf{A}(z)\right],$$

$$\mathbf{S}^{2,\,\mathrm{div}}(x,y,z) \;=\; \frac{1}{2\epsilon^2}(x+y+z) + \frac{1}{2\epsilon}(s/2-x-y-z).$$

If we write

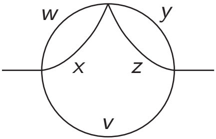
$$\mathbf{S}(x,y,z) = \frac{1}{\epsilon^2}S^{(-2)}(x,y,z) + \frac{1}{\epsilon}S^{(-1)}(x,y,z) + S^{(0)}(x,y,z) + \epsilon S^{(1)}(x,y,z) + \ldots,$$

then the corresponding renormalized master is

$$S(x,y,z) \;=\; S^{(0)}(x,y,z) - A^{(1)}(x) - A^{(1)}(y) - A^{(1)}(z).$$

Only this combination, and its derivatives with respect to x, y, z, are needed for renormalized quantities, not the individual integrals on the right side, and not $S^{(1)}$, etc.

In arXiv:2112.07694, obtained the renormalized master integrals for general 3-loop self-energy integrals, and found the renormalized ϵ-finite basis for various special cases with massless gauge bosons needed in Yang-Mills theories. For example:



$\mathbf{I}_{5a}$

$$I_{5a} \quad = \quad \lim_{\epsilon\to 0}\left[\mathbf{I}_{5a} - \mathbf{I}_{5a}^{1,\,\mathrm{div}} - \mathbf{I}_{5a}^{2,\,\mathrm{div}} - \mathbf{I}_{5a}^{3,\,\mathrm{div}}\right],$$

where

$$\mathbf{I}_{5a}^{1,\mathrm{div}}(v,w,x,y,z) \quad = \quad \frac{1}{\epsilon}\big[\mathbf{S}(v,w,x) + \mathbf{S}(v,y,z)\big],$$

$$\mathbf{I}_{5a}^{2,\mathrm{div}}(v,w,x,y,z) \quad = \quad -\frac{1}{\epsilon^2}\mathbf{A}(v) + \left(\frac{1}{2\epsilon} - \frac{1}{2\epsilon^2}\right)\big[\mathbf{A}(w) + \mathbf{A}(x) + \mathbf{A}(y) + \mathbf{A}(z)\big],$$

$$\mathbf{I}_{5a}^{3,\mathrm{div}}(v,w,x,y,z) \quad = \quad \left(-\frac{1}{6\epsilon^2} + \frac{1}{12\epsilon}\right)s + \left(-\frac{1}{6\epsilon^3} + \frac{1}{2\epsilon^2} - \frac{2}{3\epsilon}\right)(w+x+y+z) + \left(-\frac{1}{3\epsilon^3} + \frac{1}{3\epsilon^2} + \frac{1}{3\epsilon}\right)v.$$

The masters include A, S, I_{5a} (and first derivatives of the latter two with respect to squared masses). One never needs other terms in the ϵ expansions of $\mathbf{A}$, $\mathbf{S}$, $\mathbf{I}_{5a}$, even if one later extends to 4-loop order or beyond!

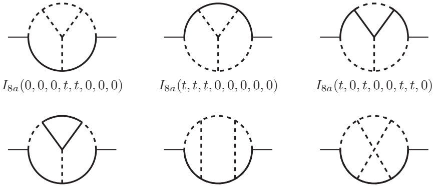
Again, note the constrast with the usual ϵ expansion. We could also expand

$$\mathbf{I}_{5a} = \frac{1}{\epsilon^3} I_{5a}^{(-3)} + \frac{1}{\epsilon^2} I_{5a}^{(-2)} + \frac{1}{\epsilon} I_{5a}^{(-1)} + I_{5a}^{(0)} + \epsilon I_{5a}^{(1)} + \ldots$$

However, it would be a tactical mistake to present results in terms of $I_{5a}^{(0)}$. This would necessitate also including $A^{(1)}$, $A^{(2)}$, $S^{(1)}$, etc. By writing renormalized quantities instead in terms of I_{5a}, we avoid having to include, and compute, these unnecessary integrals.

Note that in contrast to $A^{(1)}(x)$ and $A^{(2)}(x)$, the function $S^{(1)}(x, y, z)$ is more non-trivial to compute!

Application: single-mass self-energy integrals with odd thresholds (needed for 3-loop QCD corrections to W boson mass, with $t = M_t^2$).



A renormalized ϵ-finite basis is:

$$\{H(0,0,t,0,t,t),\ H(0,t,t,t,0,t),\ I_4(0,t,t,t),\ I_{5a}(t,0,0,t,t),\ I_{5b1}(t,t,t,0,t),\ I_{6a}(t,0,0,0,t,t),$$
$$I_{6b1}(t,0,0,t,t,t),\ I_{6d}(0,0,t,0,t,0),\ I_{6d}(0,0,t,t,0,t),\ I_{6d}(t,0,0,0,0,0),\ I_{6d}(t,0,0,t,t,t),$$
$$I_{6d}(t,t,t,t,t,0),\ I_{6d1}(t,t,t,t,t,0),\ I_{6e}(0,0,t,t,t,t),\ I_{6e}(0,t,0,0,0,t),\ I_{6e}(t,0,t,0,0,t),$$
$$I_{6e}(t,t,0,t,t,t),\ I_{6f}(0,t,t,0,0,t),\ I_{6f1}(t,0,0,t,0,t),\ I_{7a}(0,0,0,0,t,t,t),\ I_{7a}(0,0,t,t,0,0,0),$$
$$I_{7a}(0,t,t,0,0,t,0),\ I_{7a}(t,t,0,0,t,t,0),\ I_{7a}(t,t,t,t,0,0,t),\ I_{7a5}(t,t,0,0,t,t,0),\ I_{7b}(0,0,t,0,t,0,0),$$
$$I_{7b}(t,0,0,0,t,0,t),\ I_{7b}(t,t,t,0,t,t,0),\ I_{7c}(0,0,t,t,0,0,0),\ I_{7d}(0,t,0,t,0,t,0),$$
$$I_{7d}(0,t,0,t,t,0,t),\ I_{7e}(0,t,t,0,0,0,0),\ I_{7e}(0,t,t,0,0,t,t),\ I_{8a}(0,t,0,t,t,t,t,0),$$
$$I_{8b}(0,0,0,t,t,0,0,t),\ I_{8c}(0,0,0,t,t,0,0,t),\ I_{8c}^{pk}(t,t,t,0,0,0,0,0)\}.$$

Note: this is **not** the basis you would arrive at if you were motivated by simple denominators in IBP relations at intermediate steps!

These are free of IR singularities, and can be evaluated for arbitrary s using the differential equations method, either in series form or numerically.

Evaluation using differential equations method

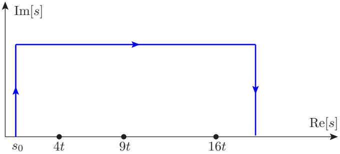
The renormalized ϵ-finite masters are algebraically closed under differentiation with respect to kinematic quantities, including the external momentum invariant s. So:

$$\frac{d}{ds} I_j = \sum_k c_{jk} I_k$$

In this approach, the coefficients c_{jk} are rational functions of s and the masses, and do not depend on ϵ, by construction.

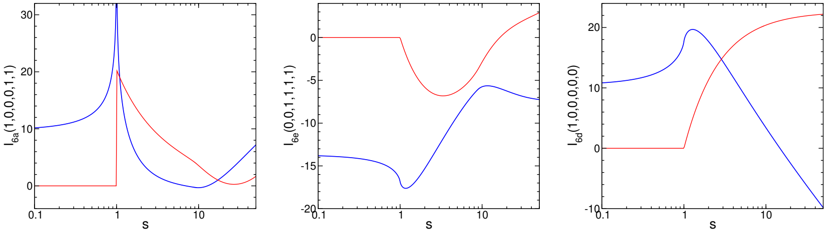
Coupled first-order differential equations are easy to solve numerically!

Integrate along a contour in the upper-half complex s plane, to avoid threshold singularities and get branch cuts correct:



As initial boundary conditions, use expansions around $s = 0$ in terms of the 3-loop vacuum integrals, known analytically.

Sample results with $t = 1$ (blue = real part, red = imaginary part):



Series solutions converge for $s < t$, can be matched to series expansions around threshold points $s = t$ and $s = 9t$.

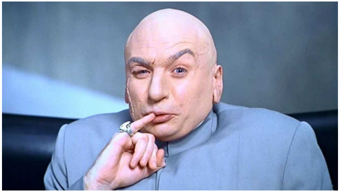
Although there are no IR divergences in the masters, threshold singularities can occur, as in $I_{6a}(1, 0, 0, 0, 1, 1)$. This gives an opportunity for checks; must cancel in physical observables.

Advantages of the numerical differential equations method

- Fast and accurate
- Solve for all masters for a given topology, simultaneously with a single Runge-Kutta calculation
- Easy error estimates (change contour, vary step size)

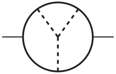
The same strategy has already been used in the programs TSIL (Two-loop Self-energy Integral Library) and 3VIL (Three-loop Vacuum Integral Library), written with Dave Robertson, each with running time a fraction of a second. (See also TVID by Bauberger, Freitas, Wiegand.)

Looking ahead to the 3-loop self-energy case with arbitrary masses, 3VIL can be used to find the initial boundary conditions for the Runge-Kutta integration near $s = 0$.

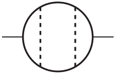


Other sets of renormalized ϵ-finite masters found in arXiv:2112.07694:

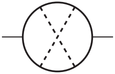
- All massless
- All same massive
- Integrals needed for QCD corrections to Z boson and Higgs self-energies:



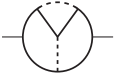
$I_{8a}(t,t,t,t,t,0,0,0)$



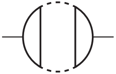
$I_{8b}(t,t,t,t,t,0,0,t)$



$I_{8c}(t,t,t,t,t,0,0,t)$



$I_{8a}(t,0,t,t,t,t,t,0)$



$I_{8b}(t,0,t,t,t,t,t,0)$

$I_{8b}(0,0,t,t,0,0,t,0)$

$I_{8a}(0,t,0,0,0,t,t,0)$

and descendants. The renormalized ϵ-finite masters are:

$$\{H(0,0,t,0,t,t),\ H(0,t,t,t,0,t),\ I_4(t,t,t,t),\ I_{5a}(t,0,t,0,t),\ I_{5b}(0,t,t,t,t),\ I_{5c}(t,t,t,t,t),$$
$$I_{6c}(t,t,t,0,t,t),\ I_{6c2}(t,t,t,0,0,0),\ I_{6d}(0,0,0,t,t,t),\ I_{6d}(0,t,t,t,t,0),\ I_{6d}(t,0,t,0,t,0),$$
$$I_{6d}(t,0,t,t,0,t),\ I_{6e}(0,0,0,0,t,t),\ I_{6e}(0,t,t,t,0,t),\ I_{6e}(t,t,t,0,t,t),\ I_{6e1}(t,t,t,0,0,0),$$
$$I_{6f}(0,0,0,0,t,t),\ I_{6f5}(0,0,0,0,t,t),\ I_{7a}(0,0,t,t,t,t,t),\ I_{7a}(t,t,0,0,0,0,t),\ I_{7a}(0,t,0,t,0,t,0),$$
$$I_{7a3}(t,0,t,0,t,0,0),\ I_{7b}(0,t,t,t,t,0,0),\ I_{7b}(t,0,t,0,0,0,t),\ I_{7b}(t,0,t,t,t,t,0),$$
$$I_{7b4}(t,0,t,t,t,t,0),\ I_{7b4}(t,t,0,t,t,0,t),\ I_{7c}(t,t,t,t,0,0,0),\ I_{7d}(t,t,0,t,0,t,0),$$
$$I_{7d}(t,t,0,t,t,0,t),\ I_{7e}(0,0,0,0,0,t,t),\ I_{7e}(0,0,t,t,t,0,0),\ I_{8a}(t,0,t,t,t,t,t,0),$$
$$I_{8a}(t,t,t,t,t,0,0,0),\ I_{8b}(t,0,t,t,t,t,t,0),\ I_{8b}(0,0,t,t,0,0,t,0),\ I_{8b}(t,t,t,t,t,0,0,t),$$
$$I_{8c}(t,0,t,t,t,t,t,0),\ I_{8c}^{pk}(t,t,t,t,t,0,0,t)\}$$

Application: calculation of 3-loop QCD contributions to W, Z, Higgs boson masses in the pure $\overline{MS}$ scheme.

Inputs are $\overline{MS}$ Lagrangian parameters:

$$g_3,\ g,\ g',\ v,\ m_H^2,\ \lambda,\ y_t,\ y_b,\ \ldots$$

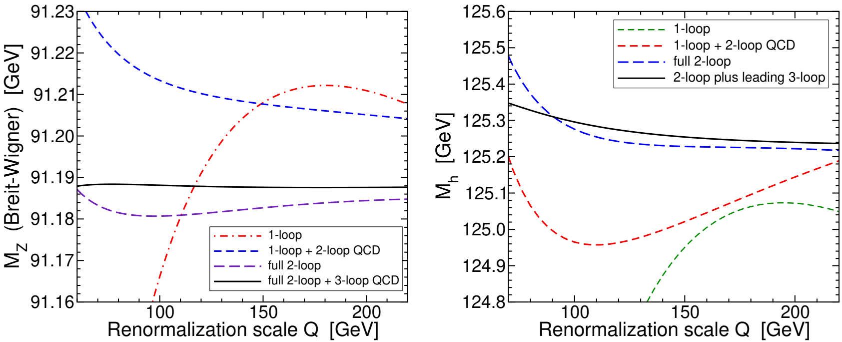
Outputs are G_F, α_0, M_Z, M_t, M_W, M_h, etc.

This is a complementary alternative to the usual on-shell scheme, with inputs

$$M_Z,\ G_\mu,\ \alpha,\ \Delta\alpha_{\rm hadronic},\ M_t,\ \alpha_S^{(5)}(M_Z),\ M_h,\ \ldots$$

For more details, see my PHENO 2022 talk.

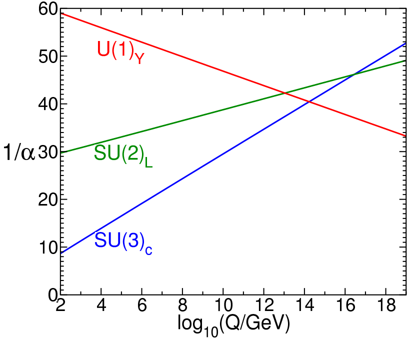
Renormalization scale (Q) dependence of calculated Z and Higgs pole masses:

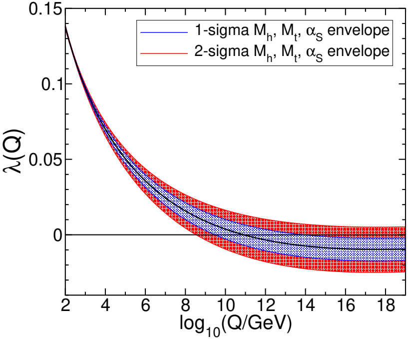


Scale dependence of M_Z is extremely small. (Almost certainly a lucky accident.)

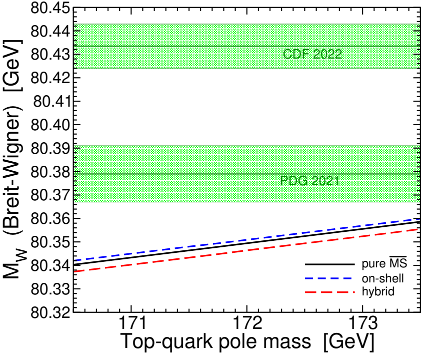
This M_h calculation is the state-of-the-art in the Standard Model (all 2-loop effect, 3-loop QCD and QCD/mixed effects). Scale dependence of M_h is much smaller than current experimental uncertainty, but this will change in the future.

The public code SMDR provides automated state-of-the-art evaluation of multiloop effects in the $\overline{MS}$ scheme, including: effective potential, renormalization group running, decoupling at thresholds, matching between running masses and pole masses, fits to data. Gives running renormalized Standard Model Lagrangian parameters $g_3, g, g', v, m_H^2, \lambda, y_t, y_b, \ldots$, appropriate for matching to your favorite UV completion model, or running to very high energy scales.





Comparison of the state-of-the-art computations of M_W in the three schemes:



on-shell = Awramik, Czakon, Freitas, Weiglein, hep-ph/0311148

hybrid = Degrassi, Gambino, Giardino arXiv:1411.7040

pure $\overline{MS}$ = arXiv:2203.05042, arXiv:1503.03782,
SMDR arXiv:1907.02500 (w/ D. Robertson)
available from github
from command line: `./calc_fit -int`

Input/fit data for M_Z, G_μ, α, $\Delta\alpha_{\rm had}$, α_S, . . . are from 2021 PDG.

Differences between schemes are consistent with ± 4 MeV theoretical error, excluding parameteric uncertainties.

4-loop QCD effects, neglected so far in pure $\overline{MS}$ scheme, are tiny.

Outlook

- Organizing results in terms of renormalized ϵ-finite master integrals allows both efficient evaluation and presentation of results. Number of renormalized masters is smaller than number of un-renormalized masters. Even better, never need expansions to positive powers of ϵ.
- Successful applications so far include
  - full 3-loop effective potential for a general theory arXiv:1709.02397
  - full 2-loop plus 3-loop QCD corrections to W, Z, Higgs boson self-energy functions and pole masses. See arXiv:2112.07694 and the public code SMDR.
- Extension to the problem of general 3-loop self-energy integrals with arbitrary masses is straightforward, in principle...