Building Integrand Bases at Two Loops

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Roadmap

- **Features and Challenges of *Doing* Perturbation Theory**
  - what mathematical form do observables take?
  - why is perturbation theory **so hard**?

    —and how can we make it *easier*?
- **Building Better Master Integrand Bases**
  - what makes for a *good* **basis** of Feynman integrals?
  - *prescriptive* integrand bases and their *stratification*
- **Applications and Illustrations**

Features and Challenges of Perturbation Theory

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coefficients: weight-0, on-shell (gauge-invariant, etc.)

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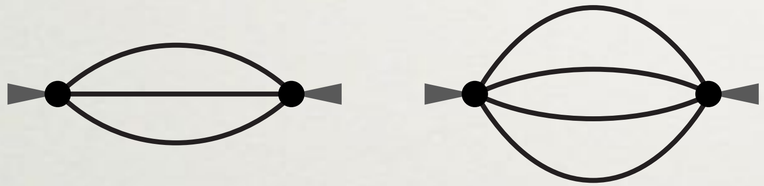

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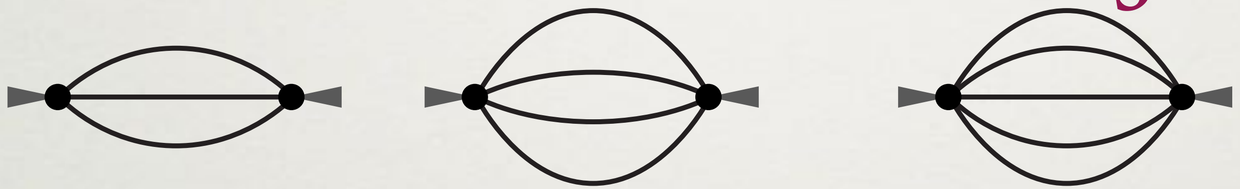

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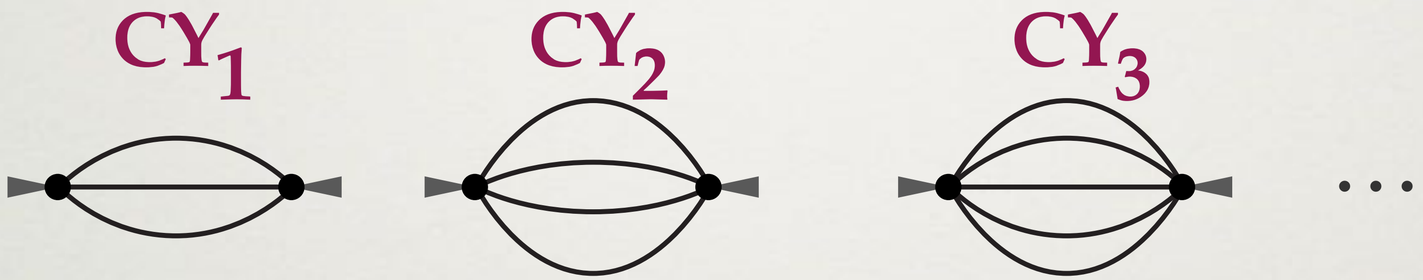

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contributes to electron (g-2)

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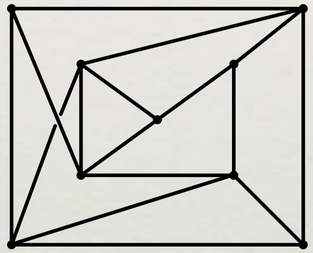

an 8-loop vacuum graph
evaluating to a **K3 period**

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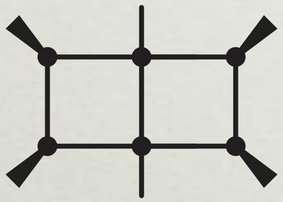

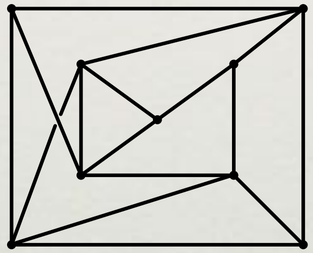

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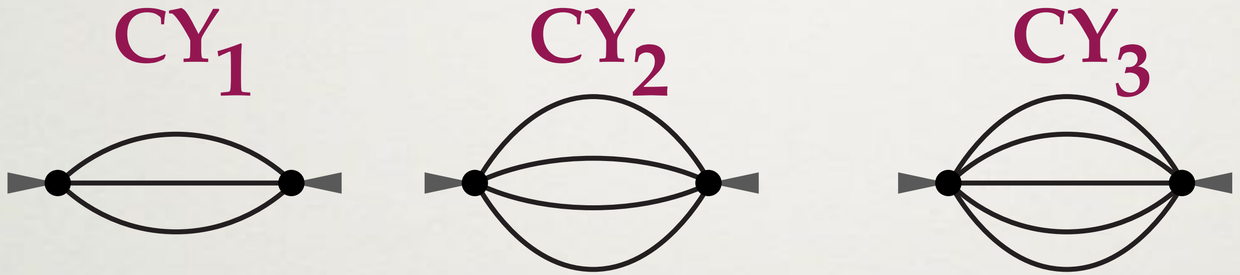

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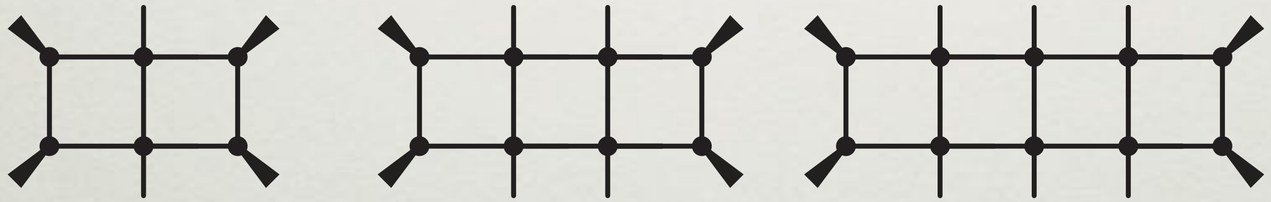

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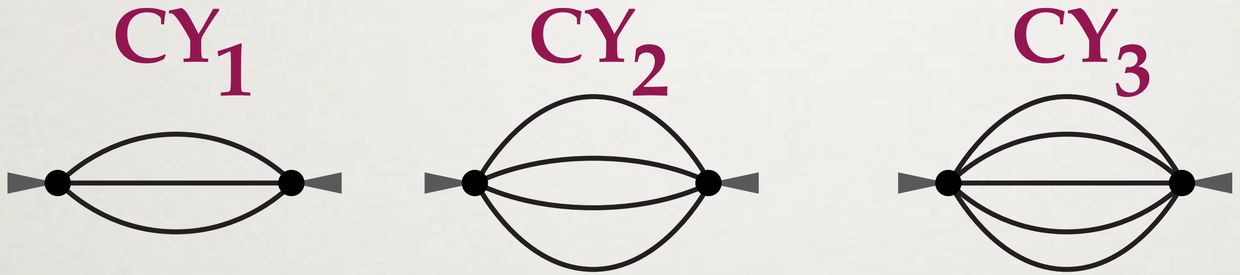

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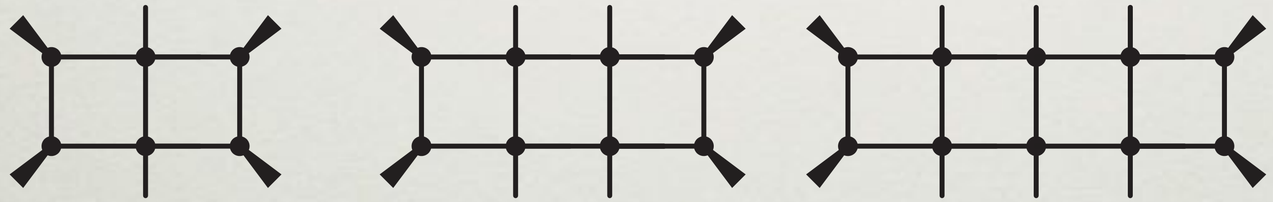

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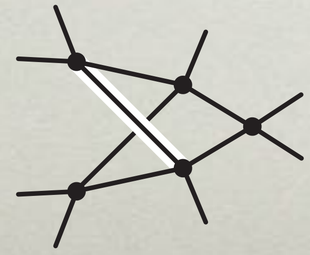

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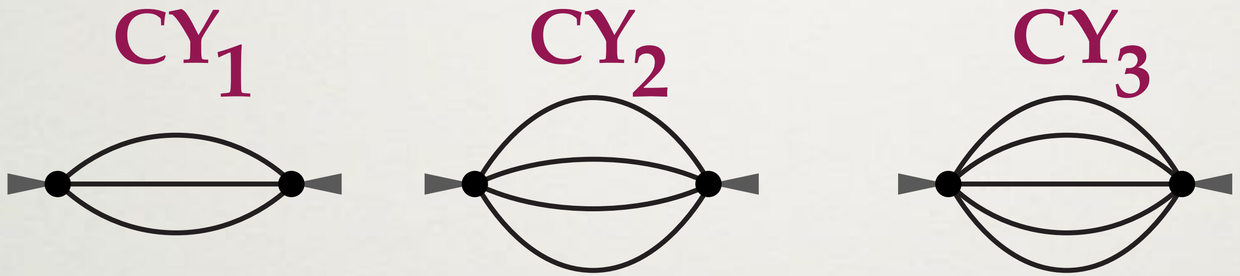

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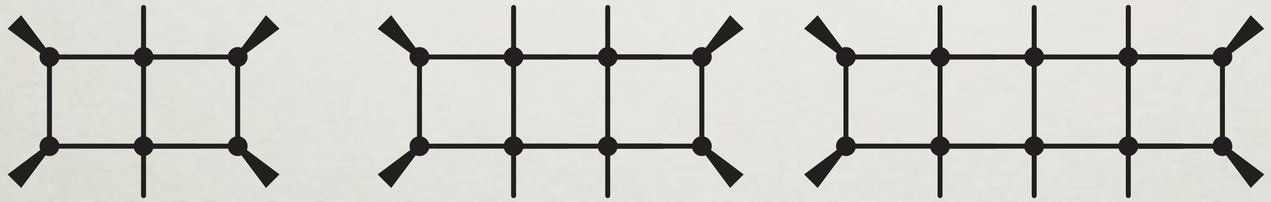

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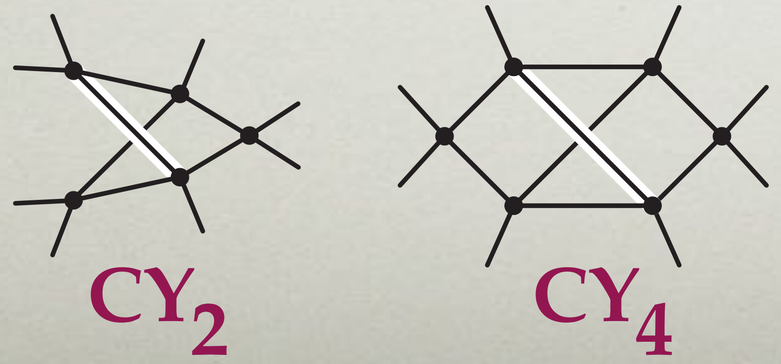

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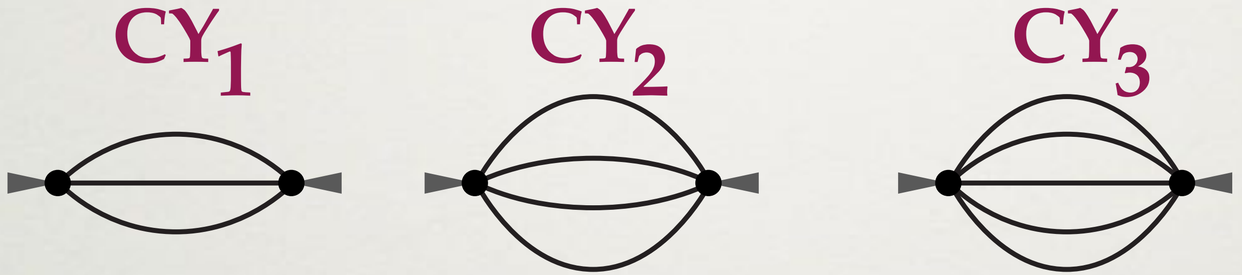

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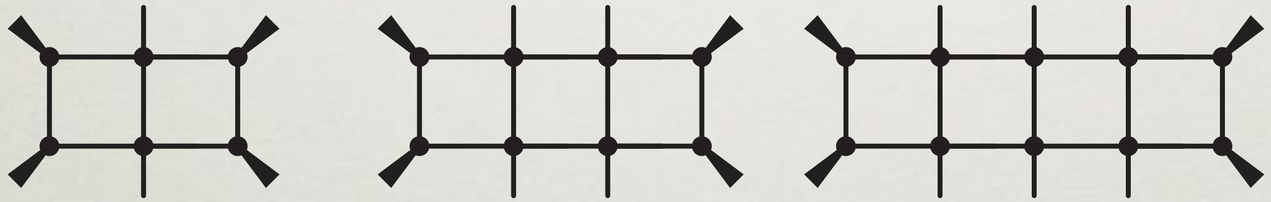

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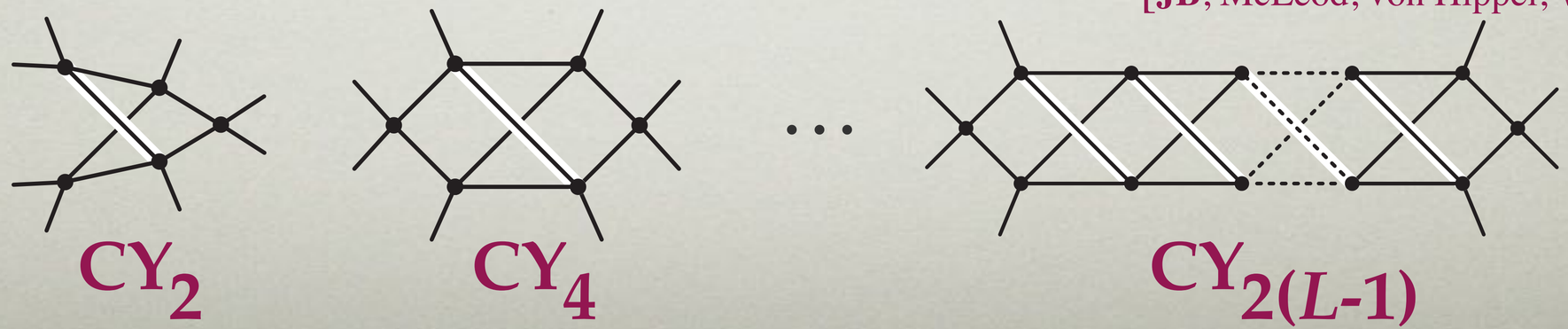

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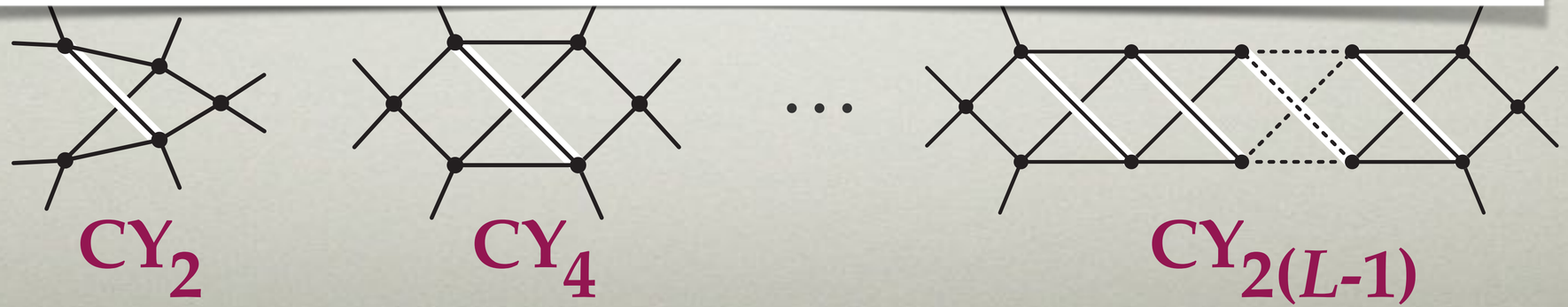

Outside a *small* list of **extremely** limited/simple cases (which are widely expected* to be exceptions):

almost all observables in ***almost all*** QFTs are expected to be non-polylogarithmic

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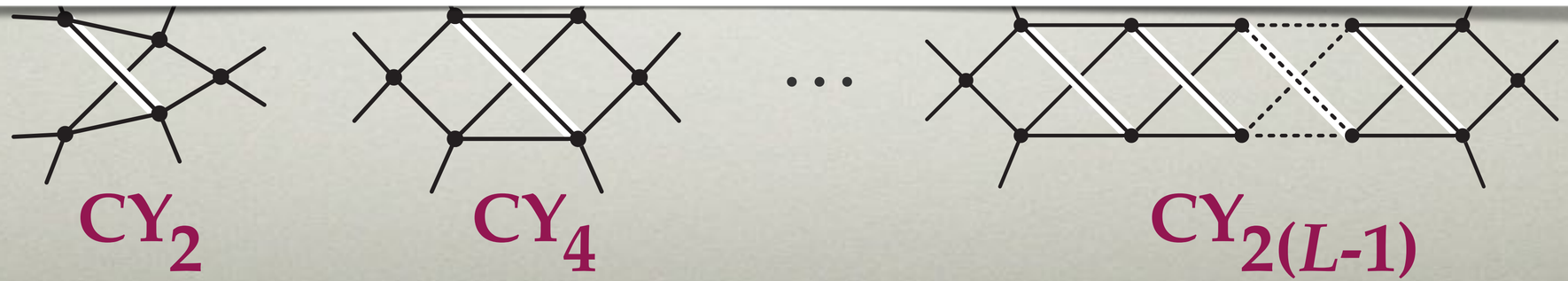

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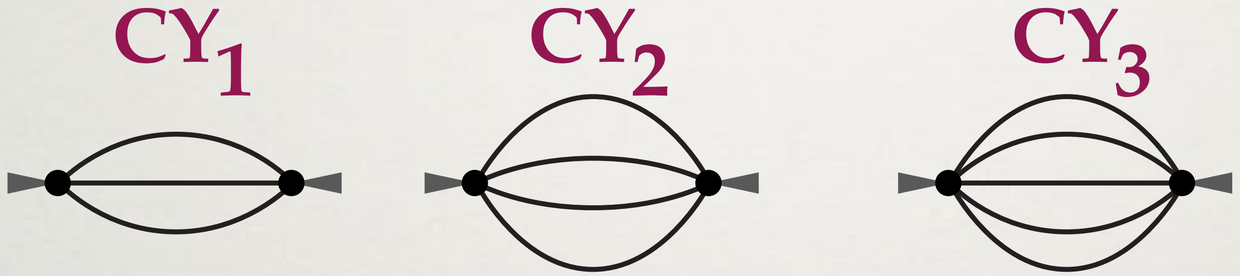

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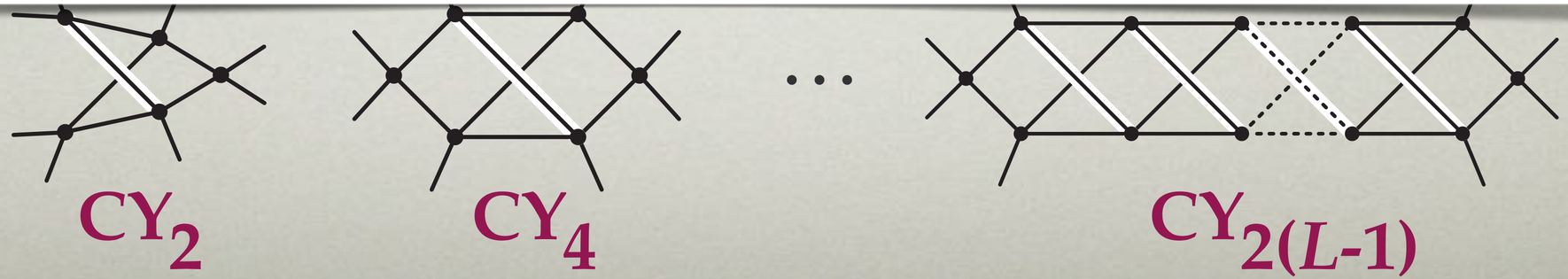

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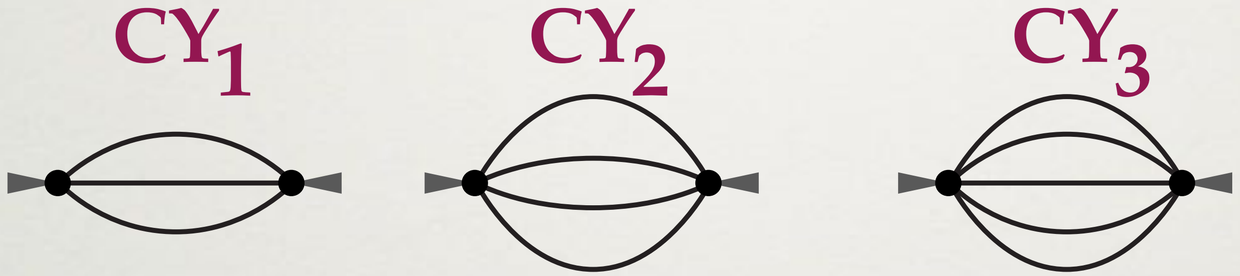

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- kinematic data (momenta/masses) control the moduli
  - often **extremely** *singular*

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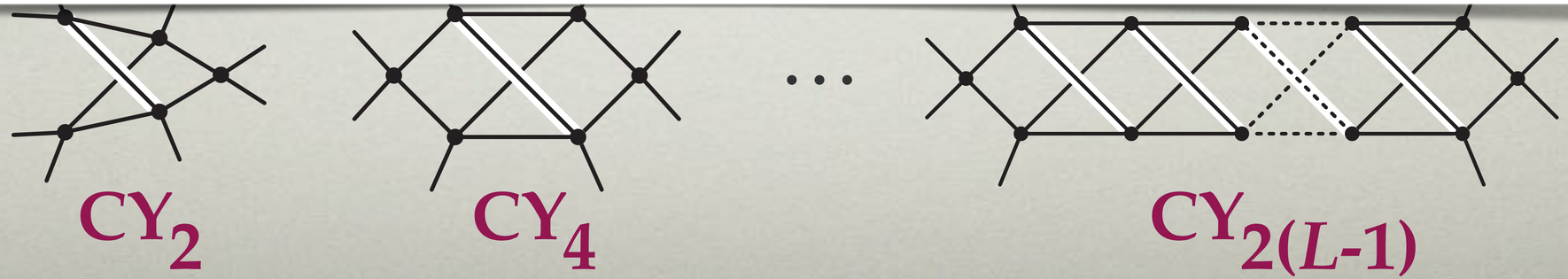

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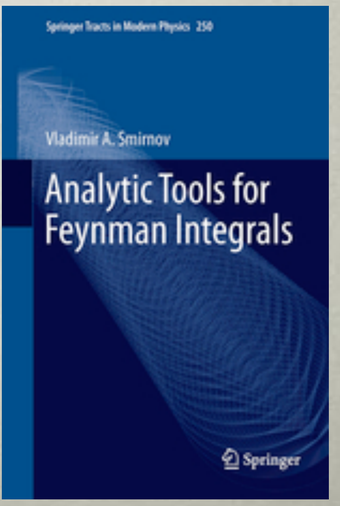

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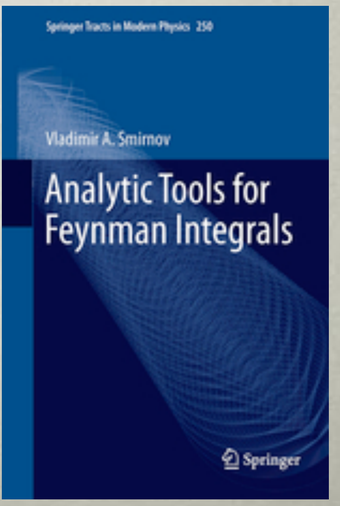
  - don't satisfy nice/canonical differential equations
  - contain ***multiple*** **elliptic**(+worse(!)) geometries,
  - …

$$\mathrm{I}_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9} = e^{2\gamma_E \epsilon} \iint \frac{d^D k_1 d^D k_2}{(i\pi^{d/2})^2} \frac{P_8^{-a_8}\, P_9^{-a_9}}{P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7}}$$



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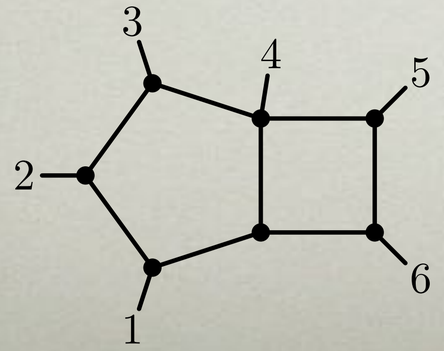

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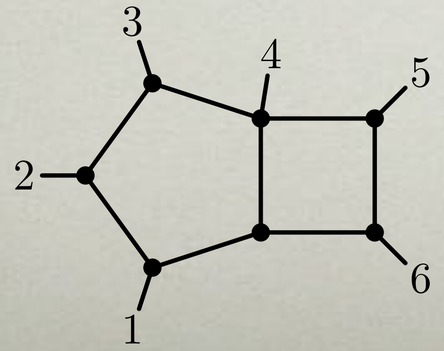

$$\Rightarrow \frac{s_{13}}{s_{12}s_{23}s_{56}}\left(\begin{array}{c}\mathrm{tr}_+[p_3,p_{12},p_6,p_1]\big(\mathrm{Li}_4(\cdots)\text{'s}+\ldots\big)\\ +\mathrm{tr}_+[p_{12},p_6,p_1,p_3]\big(\mathrm{Li}_4(\cdots)\text{'s}+\ldots\big)\end{array}\right)$$

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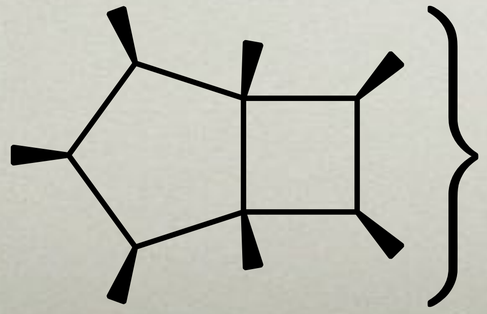

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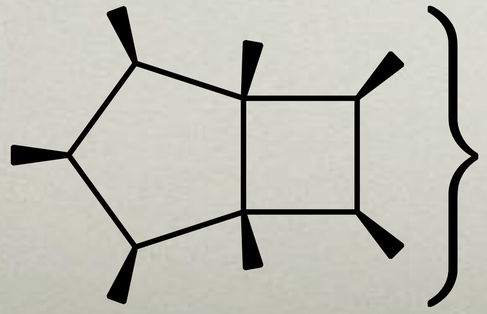

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- **Avoid regularization** when**ever** possible:
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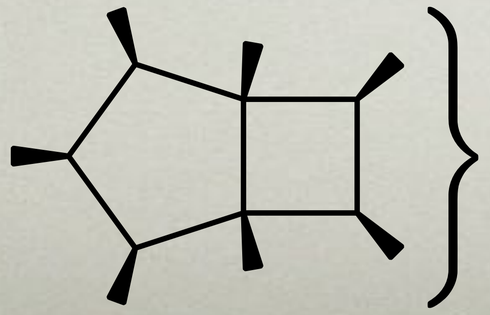
[**JB**, Langer, Patatoukos (2021); …]



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Improving Loop Integration by Building Better Bases

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- **Key observation:** viewed as a potential element of *some* basis, ***every*** **Feynman** integrand can be interesting!
  - Why not try to find the ***best/easiest*** integrands—and use these?

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$$\mathfrak{B}^{\text{SM}} \supset \mathfrak{B}^{\mathcal{N}=2} \supset \mathfrak{B}^{\mathcal{N}=4}$$

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  - recently, we gave an intrinsically graph-theoretic definition of power-counting for ***non-planar*** integrand bases

[**JB**, Herrmann, Langer, Trnka (2020)]

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- ¿Can we further stratify each part by ***transcendental*** *structure*?

$$\overbrace{\underbrace{\{\text{polylogs}\} \oplus \{\text{elliptic-polylogs}\} \oplus \{\text{K3-polylogs}\} \oplus \cdots}_{\{\text{max-weight}\} \oplus \{\text{next-to-max-weight}\} \oplus \cdots \oplus \{\text{rational}\}}}^{\{\text{finite}\}}$$

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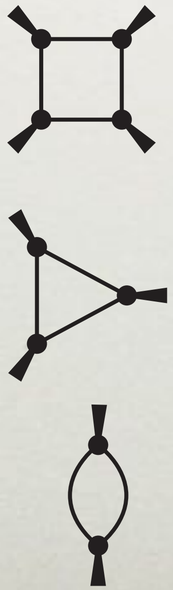

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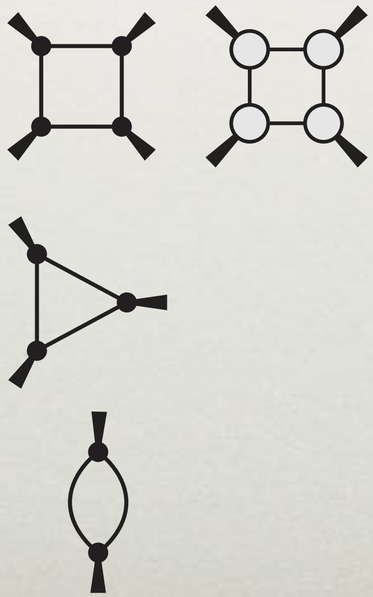

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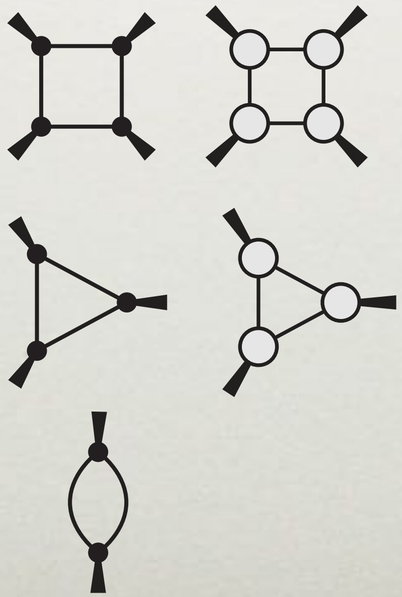

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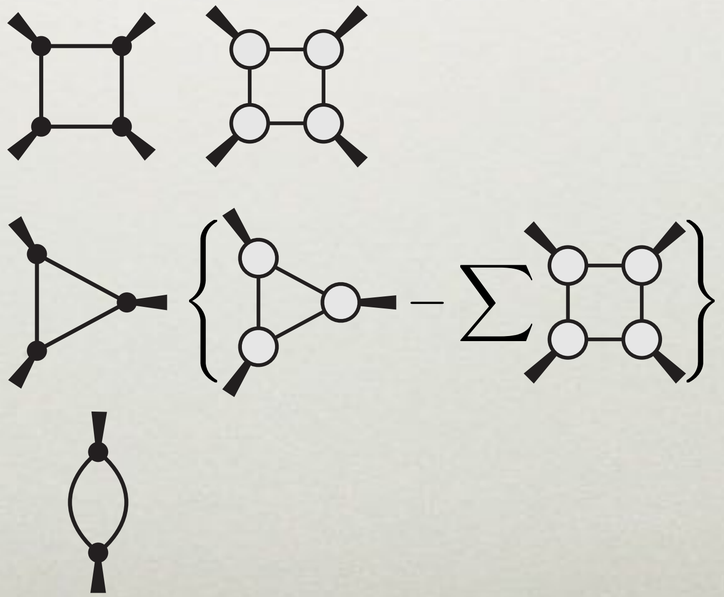

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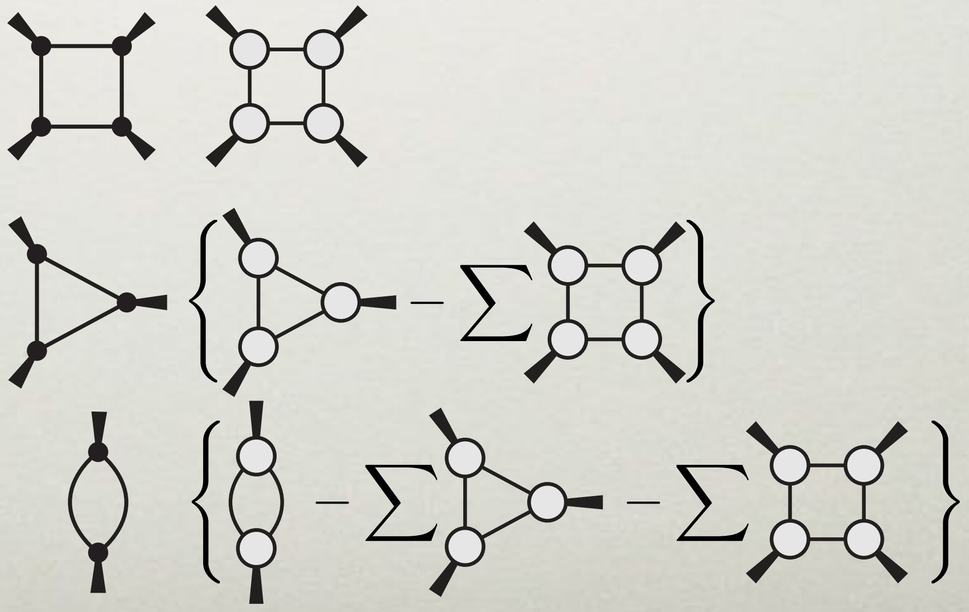

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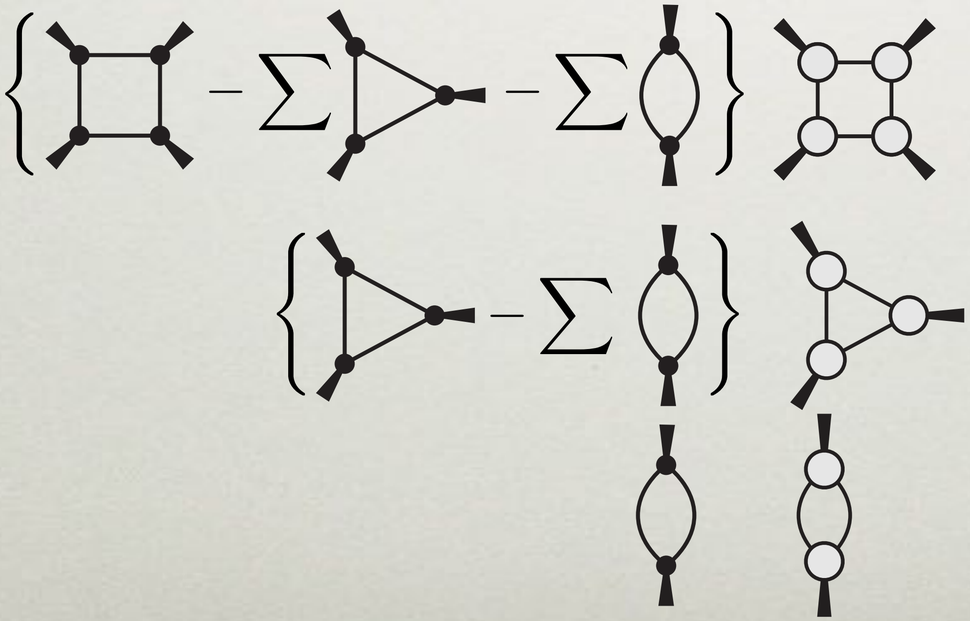

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- Choosing a maximal set of IR/UV-divergence-probing contours ensures(?) that the basis is split into finite/divergent subspaces

Thank you!

Stratifying Rigidity

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[**JB**, Kalyanapuram, (*in prep*.)]

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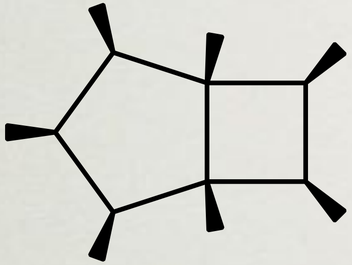
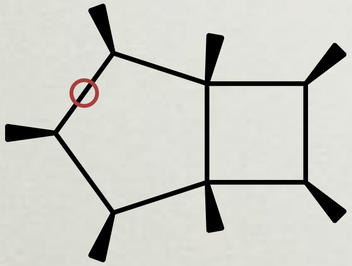
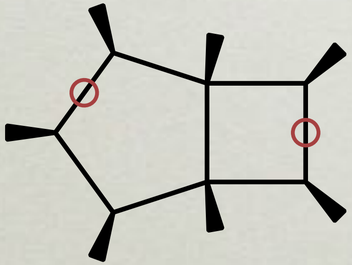


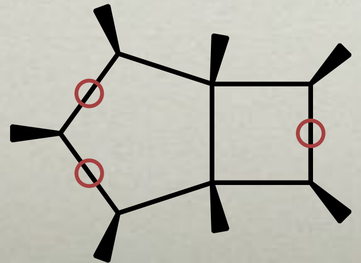
$$\Leftrightarrow \mathcal{I}_0 := \frac{(\ell_1|X)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$



$$\Leftrightarrow \mathcal{I}_1(N) := \frac{(\ell_1|N)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$



$$\Leftrightarrow \mathcal{I}_2(N,M) := \frac{(\ell_1|N)(\ell_2|M)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_2|X)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$



$$\Leftrightarrow \mathcal{I}_3(\vec{N},M) := \frac{(\ell_1|N_1)(\ell_1|N_2)(\ell_2|M)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_1|X)(\ell_2|X)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$

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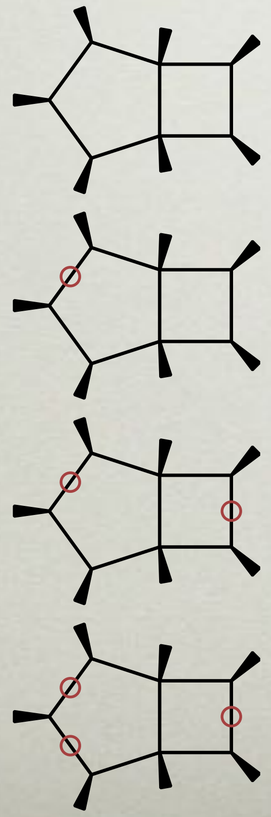

# ints	# polylogs	# elliptics	# impure/mixed
1	0	0	1
6	0	4 (4)	2
36	10	12 (12)	14
120	76	44 (22)	0