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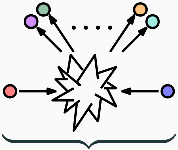
The duals of Feynman Integrals

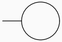
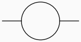
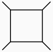
Andrzej Pokraka (based on 2104.06898 and 2112.00055 with Simon Caron-Huot)

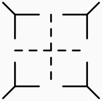
Loop Fest XX (University of Pittsburg 2022)

Integral reduction



messy numerators

$\Longrightarrow$ $\mathcal{A}_n^{\text{1-loop}} = \mathcal{R}_n + c_{\text{tad}}$  $+\, c_{\text{bub}}$  $+\, c_{\text{tri}}$  $+\, c_{\text{box}}$ 

$c_{\text{box}} =$  $= \prod (\text{trees})$ [Britto, Cachazo, Feng 2004]

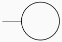
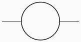
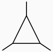
Generalized unitarity $\Longrightarrow$ Analytic structure of Feynman integrals (FIs) determined by its cuts!

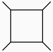
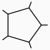
- Residue formulas for sub-topologies hard to find
- Harder when $d = 4 - 2\varepsilon$

Goal: Systematic formalism for extracting integral coeff in any dimension and for all topologies that maintains the spirit of GU

Integral reduction in generic dimension

Perform integral reduction in generic dimension

$\mathcal{A}_n^{\text{1-loop}} = c_{\text{tad}}(\varepsilon)$  $+\, c_{\text{bub}}(\varepsilon)$  $+\, c_{\text{tri}}(\varepsilon)$ 

$+\, c_{\text{box}}(\varepsilon)$  $+\, c_{\text{pent}}(\varepsilon)$ 

Want residue formulas for $c_\bullet$ that keep intuition from unitary cuts

Traditionally, done via integration by parts (IBP) identities

- Black box
- Squared propagators**
- Cuts are not utilized**

**These problems can be addressed in the IBP framework

Alternative: Intersection theory

Intersection theory $\implies$ extraction of $c_\bullet$ without IBP identities

[Mastrolia, Mizera 2018; Frellesvig, et. al. 2019, 2020; Mizera, AP 2019; Caron-Huot, AP 2021; ...]

FIs not unique: shifts by total derivative (IBPs) $\implies$ cohomology

$$|\varphi\rangle \in H^\bullet = \frac{\nabla \text{ - closed forms (FIs)}}{\nabla \text{ - exact forms (IBPs)}}$$

$\exists$ Poincaré dual cohomology $\langle\varphi^\vee| \in (H^\bullet)^\vee$

Inner product $\langle\varphi^\vee|\varphi\rangle$ integrate the wedge product (intersection number)

$$\langle\varphi_i^\vee|\varphi_j\rangle = \delta_{ij} \quad \implies \quad |\Phi\rangle = \sum_i |\varphi_i\rangle \underbrace{\langle\varphi_i^\vee|\Phi\rangle}_{\text{e.g., } c_{\text{box}}}$$

Computation of $\langle\bullet|\bullet\rangle$ is algebraic (residues) $\implies$ right direction

What is $(H^\bullet)^\vee$?

An unsatisfactory answer

$(H^\bullet)^\vee$ well known in math lit for a deformed problem

[Aomoto, Kita, Matsumoto, Yoshida 80's-present]

All propagators raised to non-integer powers ($\varepsilon\alpha \notin \mathbb{Z}$)

$\implies$ propagators (poles) $\to$ branch pts

$$\frac{1}{\ell^2 + m^2} \to \frac{1}{(\ell^2 + m^2)^{1+\varepsilon\alpha}} \qquad (\alpha \to 0 \text{ before } \varepsilon \to 0 \text{ at end})$$

Still missing the spirit of generalized unitarity:

- propagators (poles) $\to$ branch pts obscures connection to GU cuts

The answer for the undeformed dual space

[Caron-Huot, AP: 2104.06898, 2112.00055]

The undeformed dual space

Dual forms localized to GU cuts:

FIs: $\|\varphi\rangle$		dual forms: $\langle\varphi^\vee\|$
propagators/poles	$\leftrightarrow$	$\underbrace{\text{boundaries/zeros (on-shell)}}_{\text{compact support}}$

The dual cohomology spanned by holomorphic forms times:

- $\theta(|\text{propagators}| > \epsilon)$:
- $\mathrm{d}\theta(|\text{propagators}| > \epsilon)$:

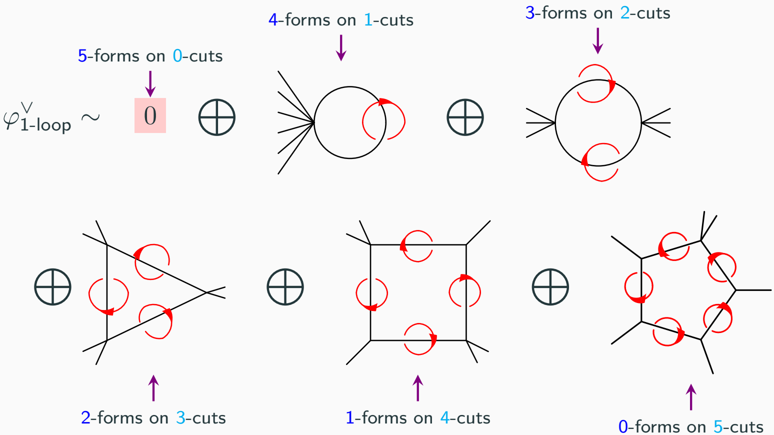
θ or $\mathrm{d}\theta$ for each propagator

The undeformed dual space

twist $(\mathrm{d} \to \nabla)$

$$\mathrm{d}^d\ell \propto \overbrace{(\ell_\perp^2)^{-\varepsilon}}^{\text{branch pts}} \overbrace{\frac{\mathrm{d}\ell_\perp^2 \wedge \mathrm{d}^4\ell}{\ell_\perp^2}}^{\text{5-form}}$$

 $\sim \mathrm{d}\theta(\text{propagator})$



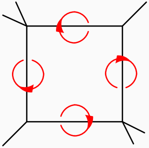
Streamlining the intersection number

$$\sim (\mathrm{d}\theta)^4 \wedge \left[\frac{\mathrm{d}^1 \ell_{\parallel}}{(\ell_\perp^2|_{\text{box}})^1} \right]_{\text{c-map}}$$

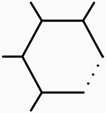
Localized/on-shell dual form

Any (1-loop) process

$c_{\text{box}}(\varepsilon) = \Big\langle$

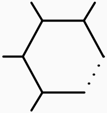


$\Big|$



$\Big\rangle$

$= \Big\langle \varphi^\vee_{\text{left-over}}(\ell_\perp^2|_{\text{box}}) \Big| \text{Res}_{\text{box}} \Big[$



$\Big] \Big\rangle$

Factorizes

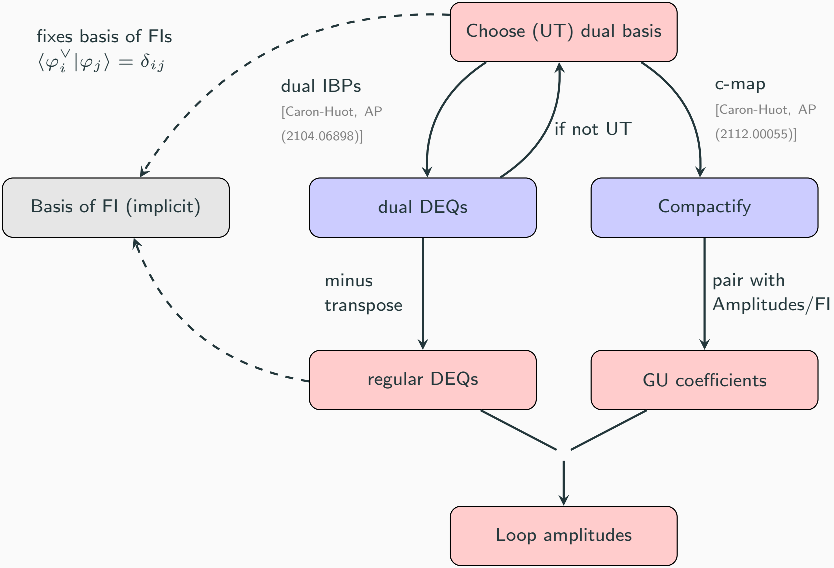
$=$ rational fn. of Mandelstams and ε

Left-over $\implies$ $\mathrm{d}\theta$'s at branch points ($\ell_\perp^2|_{\text{box}} = 0, \infty$) from c-map (essential for higher order terms in ε)

Outline

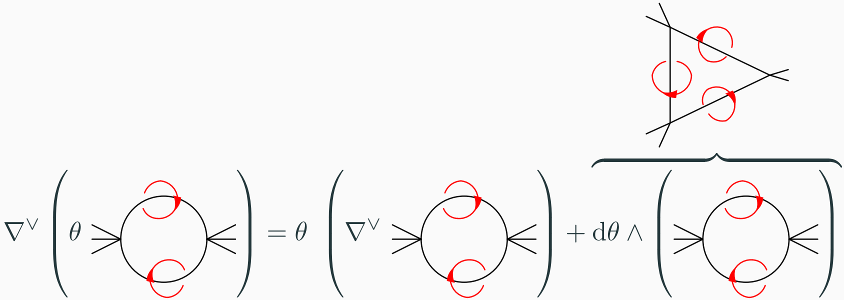
1. Workflow and key steps
    - DEQs
    - Compute intersection number (c-map)
2. New technology
    - Where are we in the development phase?
    - Roadblock: Multi-variate intersections are difficult
3. Conclusions and future directions

Workflow



Choose (UT) dual basis
fixes basis of FIs
$\langle \varphi_i^\vee | \varphi_j \rangle = \delta_{ij}$
dual IBPs
[Caron-Huot, AP (2104.06898)]
if not UT
c-map
[Caron-Huot, AP (2112.00055)]
Basis of FI (implicit)
dual DEQs
Compactify
minus transpose
pair with Amplitudes/FI
regular DEQs
GU coefficients
Loop amplitudes

Dual DEQs are transposed



Dual IBP-forms are simple: supported on cuts and no squared propagators!

Obtained simple form for canonical differential equations in **any** dimension for **both** 1-loop dual and Feynman forms

[Caron-Huot, AP (2104.06898); Bourjaily, Gardi, McLeod, Vergu (2020); Abreu, Britto, Duhr, Gardi (2017) Volovich, Spradlin (2011); Schläfli (1860)]

Compactifying (c-map)

$\varphi_c^\vee$ must have compact support for well defined intersection number

$$\langle \varphi^\vee | \varphi \rangle \propto \int_{\mathbb{C}^5} \varphi_c^\vee \wedge \varphi$$

Compact support regulates the singularities of φ

Given $\varphi^\vee$ with partial compact support $\exists\ \varphi_c^\vee$ such that $\varphi^\vee - \varphi_c^\vee = \nabla^\vee(\bullet)$

Construct local primitives to remove support

$$\varphi_c^\vee = \varphi^\vee - \mathrm{d}\theta(\text{other propagators})\psi^\vee - \mathrm{d}\theta(\text{branch pts})\psi^\vee$$

$$\nabla^\vee \psi^\vee \equiv \varphi^\vee \text{ near other propagators and branch pts}$$

as a Laurent series

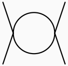
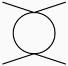
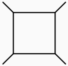
Trivial for 1-forms (integrate order by order), harder for $(p > 1)$-forms

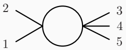
c-map in practice

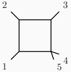
For $(p > 1)$-dual forms, there are 2 options:

- Fibration [Mizera 2019]: 1-variable at a time
  - ☺ Reuse simplicity of 1-form c-map
  - ☹ Matrix/vector valued
  - ☹ Construct $\varphi_c^\vee$ for dual basis all at once
  - ☹ Singularities at ∞ (from numerators) get progressively worse
  - 😐 Choice: coordinates and order of fibration
- Combinatoric [Matsumoto 80's]: deal with the multi-variate nature
  - ☺ Construct $\varphi_c^\vee$ individual dual forms
  - ☹ Need primitives for all ways of approaching a singularity
  - ☹ Need primitives for the primitives
  - 😐 Doable for $p = 2$
  - 😐 Choice: coordinates and which primitives are used where

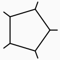
Putting things together: 1-Loop amplitudes

$$\mathcal{A}_4^{h_1 h_2 h_3 h_4} = c_{\mathsf{bub}_s} \Big($$

$$\Big) + c_{\mathsf{bub}_t} \Big($$

$$\Big) + c_{\mathsf{box}} \Big($$

$$\Big)$$

$$\mathcal{A}_5^{h_1 h_2 h_3 h_4 h_5} = \overbrace{c_{\mathsf{bub}_{s_{12}}}}^{\times 5} \Big($$


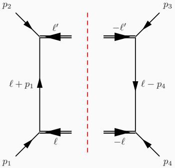
$$\Big) + \cdots + \overbrace{c_{\mathsf{box}_{45}}}^{\times 5} \Big($$


$$\Big)$$

$$+ \cdots + \underbrace{c_{\mathsf{pent}}}_{\times 1} \Big($$

$$\Big)$$

$$c_\bullet = \Big\langle \mathrm{d}\theta_{\color{red}{\mathsf{cut}}}(\mathsf{dual\ form}) \Big| \mathsf{Feynman\ form} \Big\rangle = \Big\langle \mathsf{dual\ form} \Big| \underbrace{\mathsf{Res}_{\color{red}{\mathsf{cut}}}(\mathsf{Feynman\ form})}_{} \Big\rangle$$

from cuts by glueing trees



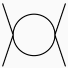
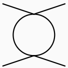
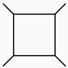
Massless 1-Loop amplitudes and 4-dimensional limits

Proof of concept: reproduced 4- and 5-point gluon amplitudes at 1-loop

[Bern, Dixon, Kosower (90's), see 2112.00055 for details]

More interesting, systematic study of 4d-limit

$$\frac{c_s^{(\mathrm{tri})}}{\varepsilon^2} + \frac{c_s^{(\mathrm{bub})}}{\varepsilon} + c_s^{(\mathrm{rat})} \qquad\qquad \frac{c^{(\mathrm{box})}}{\varepsilon^2}$$

$\mathcal{A}_4 = c_{\mathrm{bub}_s}(\varepsilon) \Big($  $\Big) + c_{\mathrm{bub}_t}(\varepsilon) \Big($  $\Big) + c_{\mathrm{box}}(\varepsilon) \Big($  $\Big)$

$$-2 + 2\varepsilon \log(-s) + \varepsilon^2(\zeta_2 - \log^2(-s)) \qquad\qquad -\varepsilon^2 \left(\pi^2 + \log^2(t/s)\right)$$

Massless 1-Loop amplitudes

$$c_{\mathsf{bub}_s}[\varphi](\varepsilon) = \sum_{(\ell_\perp^2, \ell_-, \ell_+)} \mathsf{Res}_{\ell_-} \mathsf{Res}_{\ell_+} \mathsf{Res}_{\ell_\perp^2} \left[\underbrace{\psi^\vee_{(\ell_\perp^2, \ell_+, \ell_-)}}_{\text{triple primitive of bub-dual}} (\mathsf{Res}_{s\text{-channel}} \varphi) \right]$$

		$(0,\infty)$	(∞,∞)	$(D_{\mathsf{tri}}, \mathsf{soft})$	$(D_{\mathsf{tri}}, \infty)$
ε^{-2}	$\ell_\perp^2 = 0$			✓	
	$\ell_\perp^2 = \infty$				
ε^{-1}	$\ell_\perp^2 = 0$		✓		
	$\ell_\perp^2 = \infty$				
ε^{0}	$\ell_\perp^2 = 0$		✓		✓
	$\ell_\perp^2 = \infty$		✓		✓

Better coordinates? [Badger 2008]

Next goal: 2-loop rational coefficients

Conclusions and Future Directions

Recovered generalized unitarity in arbitrary dimension from twisted relative cohomology

1-loop warm up : [S. Caron-Huot, AP (2021); S. Caron-Huot, AP (2021)]

- 1-loop uniform transcendental basis of dual forms
- Obtained DEQ for this basis in any dimension
- Constructed 1-loop 4- and 5-point gluon amplitudes from their cuts (coeffs all orders in ε)
- 4-dimensional limit extract rational terms (ε/ε)
- Robust checks at intermediate steps

Next goals:

- 2-loops (loop-by-loop) [M. Giroux, AP (ongoing)]
- Improve c-map efficiency (what are good choices?)
- Rational terms at 2-loops
- Dual IBP identities are simpler?