FOUR-LOOP FORM FACTORS

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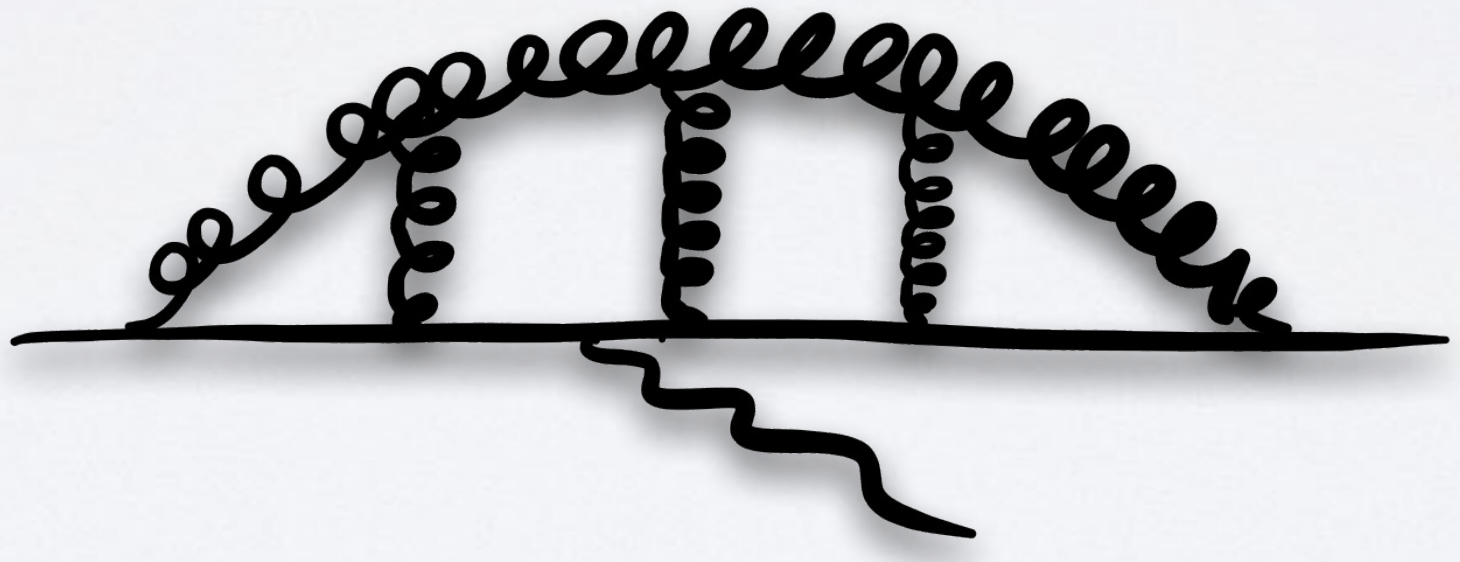
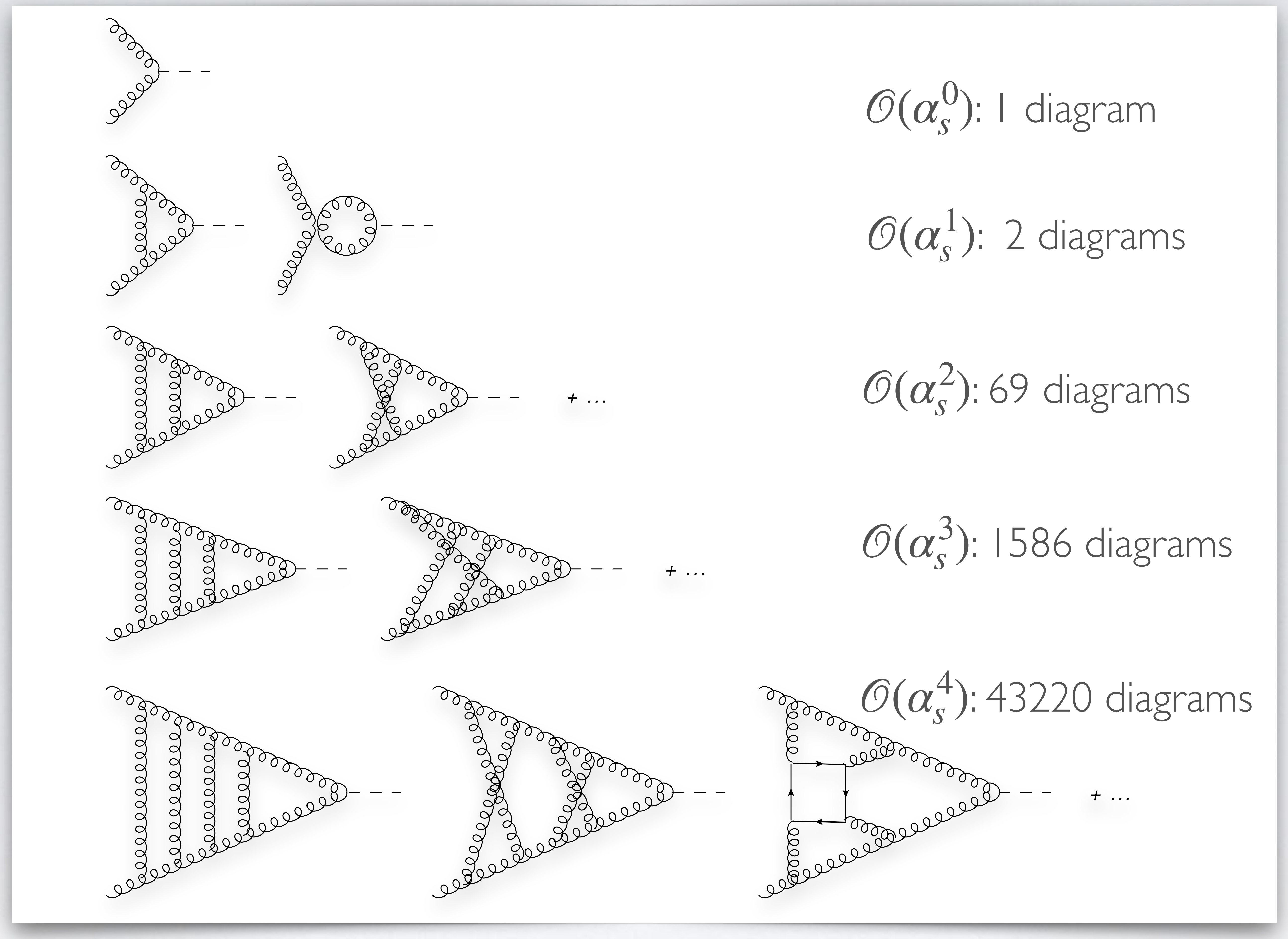


Loopfest XX

May 12-14, 2022, University of Pittsburgh

PERTURBATIVE EXPANSION OF FORM FACTORS



- Consider $q\bar{q}\gamma^*$, ggH, $b\bar{b}H$ form factors:
  - Virtual N4LO for Drell-Yan, Higgs prod./decay
  - Universal IR features of amplitudes

IR SUBTRACTION

- *[Catani '98, Aybat, Dixon, Sterman '06, Becher, Neubert '08, Gardi, Magnea '09, …]:*
  IR poles of renormalized amplitude may be minimally subtracted through multiplicative procedure:

$$\mathcal{M}^{\text{fin}} = \mathbf{Z}^{-1}\mathcal{M}^{\text{ren}}$$

  with Z matrix in color space, where anomalous dimension

$$\mathbf{\Gamma}(\mu, a) = -\mathbf{Z}^{-1}\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}\ln\mu}$$

  has simple process-independent features.

- Solution for Z matrix

$$\ln\mathbf{Z} = -\frac{1}{2}\int_0^a \frac{\mathrm{d}a'}{\beta(a') - \epsilon a'}\left(\mathbf{\Gamma}(\mu, a') - \frac{1}{2}\int_0^{a'}\frac{\mathrm{d}a''\ \Gamma'(a'')}{\beta(a'') - \epsilon a''}\right)$$

$$\mathbf{\Gamma}(\mu, a) = \sum_{n=1}^{\infty} a^n \mathbf{\Gamma}_n(\mu)\,,$$
$$\Gamma'(a) = \frac{\mathrm{d}\mathbf{\Gamma}(\mu, a)}{\mathrm{d}\ln(\mu)} = \sum_{n=1}^{\infty} a^n \Gamma'_n$$

- Anomalous matrix @ 2-loops: only color dipoles
  [Catani '98; Aybat, Dixon, Sterman '06; Becher, Neubert '08; Gardi, Magnea '09]

- Anomalous matrix @ 3-loops: also quadrupoles
  [Almelid, Duhr, Gardi '15; Henn, Mistlberger '16]
  recently confirmed for partonic scattering in full QCD
  [Caola, Chakraborty, Gambuti, AvM, Tancredi '21, '21]

talk: *Amlan Chakraborty*

- Anomalous matrix @ 4-loops: partial information
  [Becher, Neubert '19; Agarwal, Danish, Magnea, Pal, Tripathi '20, Agarwal, Magnea, Pal, Tripathi '21; Falcioni, Gardi, Maher, Milloy, Vernazza '21]

CUSP AND COLLINEAR ANOMALOUS DIMENSIONS

- For our form factors: Z is proportional to color unit matrix

$$\mathbf{Z} = Z_r \,,$$
$$\mathbf{\Gamma}_n = -\Gamma^r_n \ln\left(\frac{\mu^2}{-q^2 - i0}\right) - \gamma^r_n \,,$$
$$\Gamma'_n = -2\Gamma^r_n \,,$$

- We can extract cusp and collinear anomalous dimensions

$$\Gamma^r(a) = \sum_{n=1}^{\infty} a^n \Gamma^r_n \,,$$
$$\gamma^r(a) = \sum_{n=1}^{\infty} a^n \gamma^r_n$$

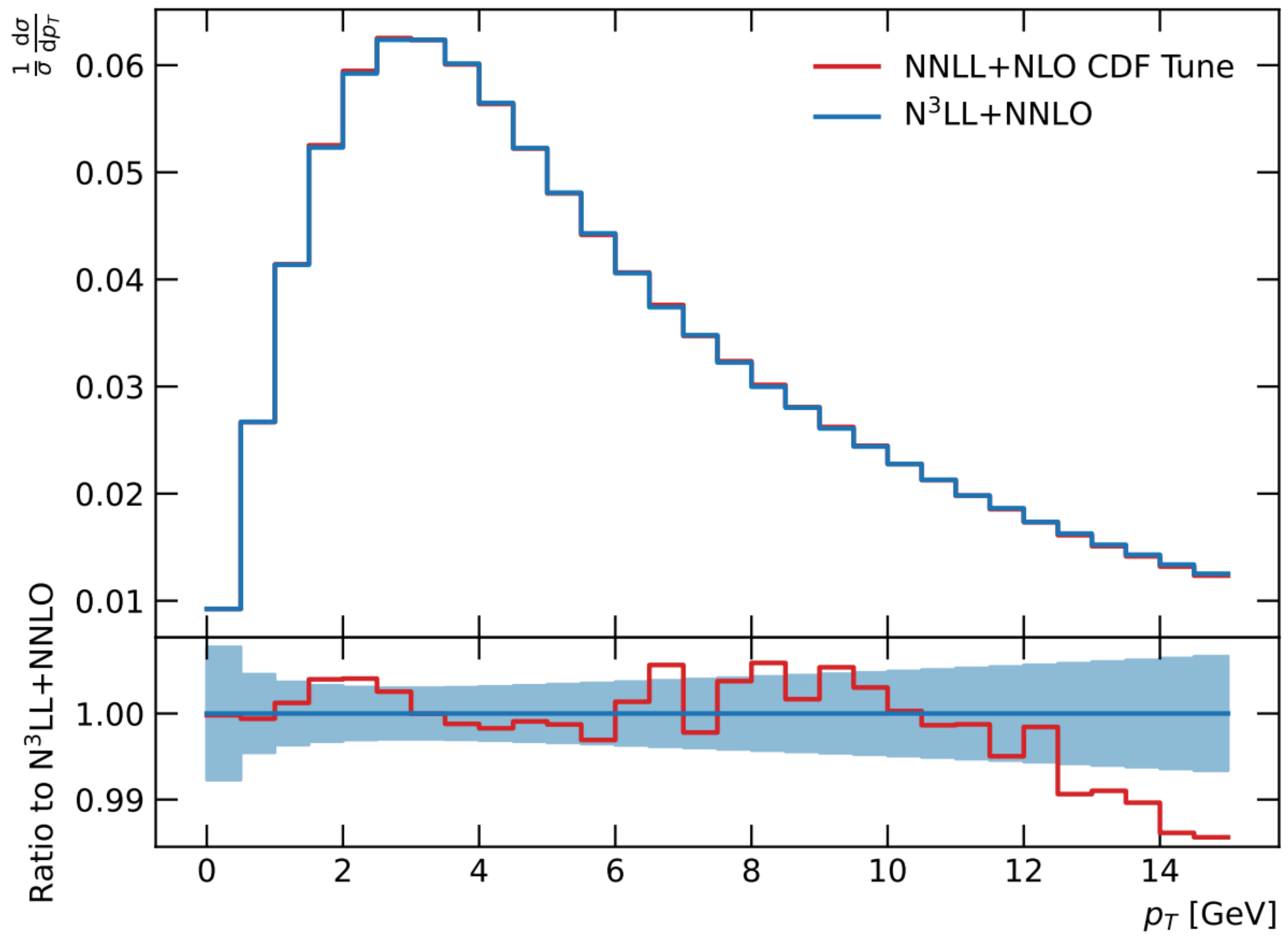
from poles of form factors ($1/\epsilon^2$ cusp, $1/\epsilon$ collinear)

- cusp@3 loops: only quadratic Casimir $\Gamma^r = T(r)T(r)\ \gamma^{cusp}$
- cusp@4 loops: also quartic Casimirs

RESUMMATION

Order	Anomalous Dimension γ_i (B)	Γ_{cusp} (A)	Fixed Order Matching (Y)
LL	-	1-loop	-
NLL	1-loop	2-loop	-
NLL' (+ NLO)	1-loop	2-loop	α_s
NNLL (+ NLO)	2-loop	3-loop	α_s
NNLL' (+ NNLO)	2-loop	3-loop	α_s^2
N^3LL (+ NNLO)	3-loop	4-loop	α_s^2
N^3LL' (+ N^3LO)	3-loop	4-loop	α_s^3
N^4LL (+ N^3LO)	4-loop	5-loop	α_s^3

[Isaacson, Fu, Yuan '21]



- W at small pT: important for mass measurement
- Fixed order breaks in this regime, requires resummation
- N3LL or higher needs four-loop cusp anomalous dim., some further works:
  - Hbb @ N3LL *[Ajjath, Chakraborty, Das, Mukherjee, Ravindran '19]* + many more
  - Energy-energy correlation @ N4LL *[Duhr, Mistlberger, Vita '21; Moult, Zhu, Zhu '21]*

talks: *Johannes Michel, Stefano Forte, …*

CALCULATIONAL SETUP

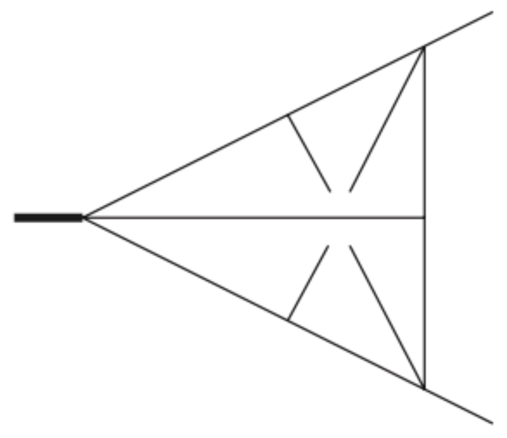
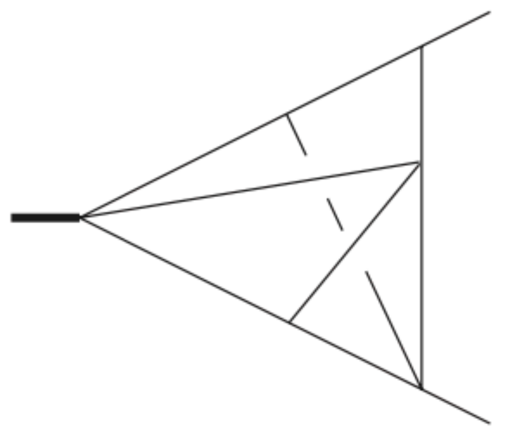
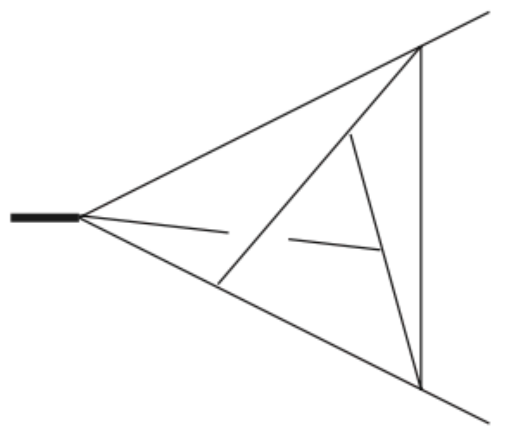
- Project started with E. Panzer, R. Schabinger several years ago
- 6k / 43k diagrams for $qq\gamma^*$ / ggH
- 100 top-level topologies (trivalent graphs)
- 10 integral families (sets of denominators)
- R_ξ gauge for matter content
- O(10^9) integrals in diagrams
- 5 / 6 ISPs for $qq\gamma^*$ / ggH
- IBP reductions with Finred based on finite field arithmetic + rational reconstruction *[AvM, Schabinger '14; Peraro '16; …]*
- 294 master integrals
- Choose finite master integrals
- Analytical integration with Hyperint [Panzer '14]
- Completion of weight 8 results: later

	A	B	C	D	E
D_1	k_1^2	$(k_1+k_2-k_3-k_4-p_1)^2$	$(k_1-k_3-p_1)^2$	$(k_1+k_2-k_3-k_4-p_1)^2$	$(k_1-k_2+k_3-k_4+p_1)^2$
D_2	k_2^2	$(k_2-k_4-p_1)^2$	$(k_3-k_4+p_1)^2$	$(k_2-k_3-p_1)^2$	$(k_1-k_2+k_3+p_1)^2$
D_3	k_3^2	$(k_4+p_1)^2$	$(k_1-k_3+p_2)^2$	$(k_2-p_1)^2$	$(k_1-k_2+p_1)^2$
D_4	k_4^2	$(k_1+k_2-k_3-k_4+p_2)^2$	$(k_1-k_2+p_2)^2$	$(k_1+k_2-k_3-k_4+p_2)^2$	$(k_1+p_1)^2$
D_5	$(k_1-p_1)^2$	$(k_1-k_4+p_2)^2$	k_1^2	$(k_1-k_4+p_2)^2$	$(k_1-k_2+k_3-k_4-p_2)^2$
D_6	$(k_1-k_2-p_1)^2$	$(k_4-p_2)^2$	k_2^2	$(k_1+p_2)^2$	$(k_1-k_2+k_3-p_2)^2$
D_7	$(k_1-k_2+k_3-p_1)^2$	k_1^2	k_3^2	k_1^2	$(k_2-k_3+p_2)^2$
D_8	$(k_1-k_2+k_3-k_4-p_1)^2$	k_2^2	k_4^2	k_2^2	k_1^2
D_9	$(k_1+p_2)^2$	k_3^2	$(k_2-k_3)^2$	k_3^2	k_2^2
D_{10}	$(k_1-k_2+p_2)^2$	k_4^2	$(k_1-k_2)^2$	k_4^2	k_3^2
D_{11}	$(k_1-k_2+k_3+p_2)^2$	$(k_2-k_3)^2$	$(k_3-k_4)^2$	$(k_2-k_3)^2$	k_4^2
D_{12}	$(k_1-k_2+k_3-k_4+p_2)^2$	$(k_1-k_3)^2$	$(k_1-k_4)^2$	$(k_1-k_4)^2$	$(k_2-k_3)^2$
D_{13}	$(k_1-k_2)^2$	$(k_1-k_4-p_1)^2$	$(k_1-k_2-p_1)^2$	$(k_2-k_4-p_1)^2$	$(k_3-p_2)^2$
D_{14}	$(k_2-k_3)^2$	$(k_2-k_4+p_2)^2$	$(k_3-k_4-p_2)^2$	$(k_1-k_3+p_2)^2$	$(k_1-k_2)^2$
D_{15}	$(k_3-k_4)^2$	$(k_2-k_4)^2$	$(k_1-p_1)^2$	$(k_2-k_4)^2$	$(k_1-k_3)^2$
D_{16}	$(k_1-k_2+k_3)^2$	$(k_1-k_2)^2$	$(k_1+p_2)^2$	$(k_1-k_3)^2$	$(k_1-k_4)^2$
D_{17}	$(k_2-k_3+k_4)^2$	$(k_1-k_4)^2$	$(k_1-k_3)^2$	$(k_3-k_4)^2$	$(k_2-k_4)^2$
D_{18}	$(k_1-k_2+k_3-k_4)^2$	$(k_1+k_2-k_3-k_4)^2$	$(k_2-k_4)^2$	$(k_1-k_2)^2$	$(k_3-k_4)^2$

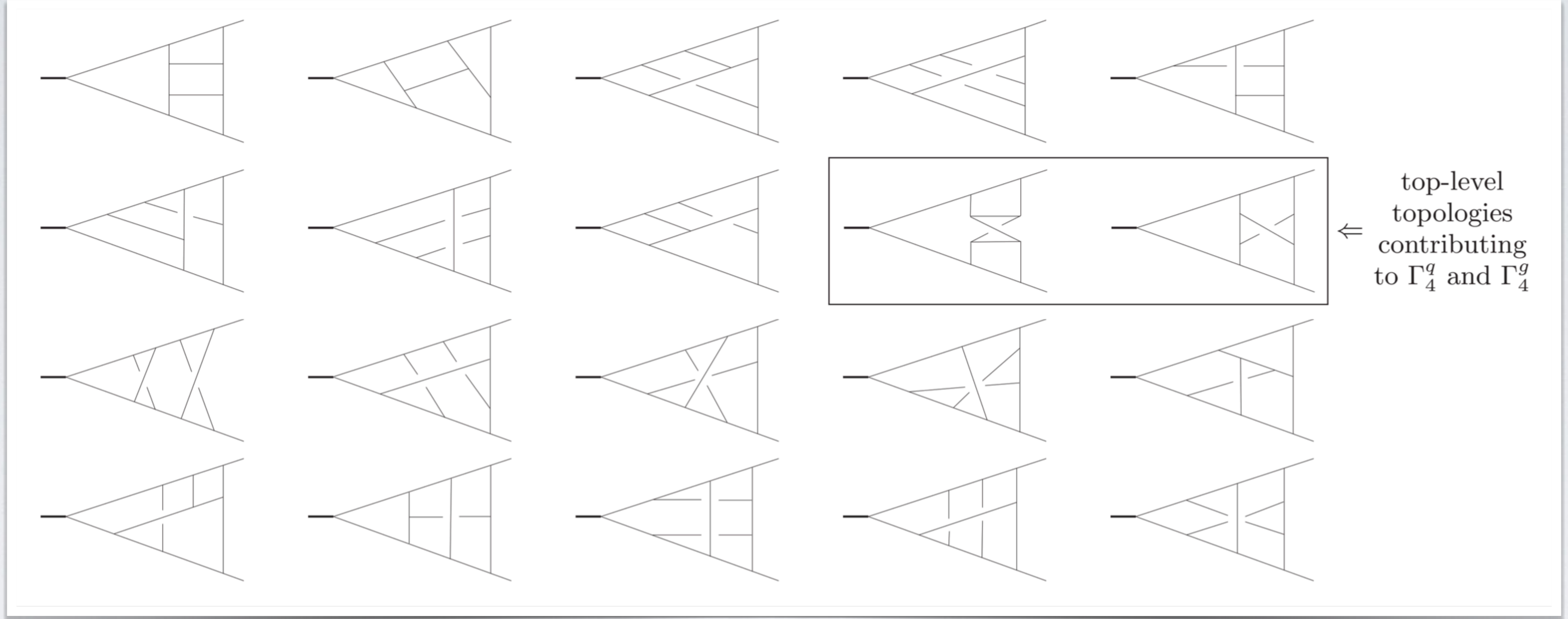
	F	G	H	I	J
D_1	$(k_1+k_2-k_3-k_4-p_1)^2$	$(k_1-k_2-k_3+k_4-p_1)^2$	$(k_1-p_1)^2$	$(k_1+k_3-k_4-p_1)^2$	k_1^2
D_2	$(k_1+k_2-k_4-p_1)^2$	$(k_1-k_2+k_4-p_1)^2$	$(k_1+k_2-p_1)^2$	$(k_3-k_4-p_1)^2$	k_2^2
D_3	$(k_2-p_1)^2$	$(k_1-k_2-p_1)^2$	$(k_1+k_2-k_3-p_1)^2$	$(k_4+p_1)^2$	k_3^2
D_4	$(k_1+k_2-k_3-k_4+p_2)^2$	$(k_1-k_2-k_3+k_4+p_2)^2$	$(k_1+k_2-k_3-k_4-p_1)^2$	$(k_2-k_4-p_1)^2$	k_4^2
D_5	$(k_1-k_3+p_2)^2$	$(k_2-k_4-p_2)^2$	$(k_2+p_2)^2$	$(k_1+k_3-k_4+p_2)^2$	$(k_1+p_1)^2$
D_6	$(k_1+p_2)^2$	k_3^2	$(k_1+k_2+p_2)^2$	$(k_1-k_4+p_2)^2$	$(k_1-k_3+p_1)^2$
D_7	k_1^2	k_4^2	$(k_1+k_2-k_3+p_2)^2$	$(k_4-p_2)^2$	$(k_1+k_2-k_3+p_1)^2$
D_8	k_2^2	$(k_1-k_2)^2$	$(k_1+k_2-k_3-k_4+p_2)^2$	k_1^2	$(k_1+k_2-k_3-k_4+p_1)^2$
D_9	k_3^2	$(k_2-k_3)^2$	k_1^2	k_2^2	$(k_3+p_2)^2$
D_{10}	$(k_1-k_2)^2$	$(k_2-k_4)^2$	k_2^2	k_3^2	$(k_1-k_3-p_2)^2$
D_{11}	$(k_2-k_4)^2$	$(k_3-k_4)^2$	k_3^2	k_4^2	$(k_1-k_3-k_4-p_2)^2$
D_{12}	$(k_1-k_4)^2$	$(k_1-k_3)^2$	k_4^2	$(k_2-k_4)^2$	$(k_1+k_2-k_3-k_4-p_2)^2$
D_{13}	$(k_2-k_4-p_1)^2$	$(k_2+p_1)^2$	$(k_1-k_2)^2$	$(k_2-k_4+p_2)^2$	$(k_1-k_2)^2$
D_{14}	$(k_1+k_2-k_3+p_2)^2$	$(k_1-k_2+k_4+p_2)^2$	$(k_1-k_3)^2$	$(k_1-k_2)^2$	$(k_1-k_3)^2$
D_{15}	k_4^2	$(k_2-p_2)^2$	$(k_1-k_4)^2$	$(k_1-k_3)^2$	$(k_1-k_4)^2$
D_{16}	$(k_3-k_4)^2$	k_1^2	$(k_2-k_3)^2$	$(k_1-k_4)^2$	$(k_2-k_3)^2$
D_{17}	$(k_1-k_3)^2$	k_2^2	$(k_2-k_4)^2$	$(k_2-k_3)^2$	$(k_2-k_4)^2$
D_{18}	$(k_2-k_3)^2$	$(k_1-k_4)^2$	$(k_3-k_4)^2$	$(k_3-k_4)^2$	$(k_3-k_4)^2$

IBP DETAILS

- Reduction of dots: “no-numerator syzygies” in Lee-Pomeransky rep.
  [Lee ‘14; Bitoun, Bogner, Klausen, Panzer ’17]
  - Need **higher-order annihilators**.
- Reduction of numerators: “no-dot syzygies” in Baikov rep. (some sectors)
  [Gluza, Kajda, Kosower ‘11; Schabinger ’11; Its ’15; Larsen, Zhang ’15; Böhm, Georgoudis, Larsen Schulze, Zhang ’18; …]
  - Used **linear algebra approach** *[Agarwal, Jones, AvM ’20]*.
- O(25k) sectors, up to O(10^8) eqs. per sector, up to O(40) finite fields, up to O(600) samples for variable
- Reduction tables: several TB compressed (checksums!)
- Inter-sector relation:

 $= \frac{4(2d-7)}{3d-11}$  $+ \frac{5(5-d)}{3d-11}$  $+$ subsectors,

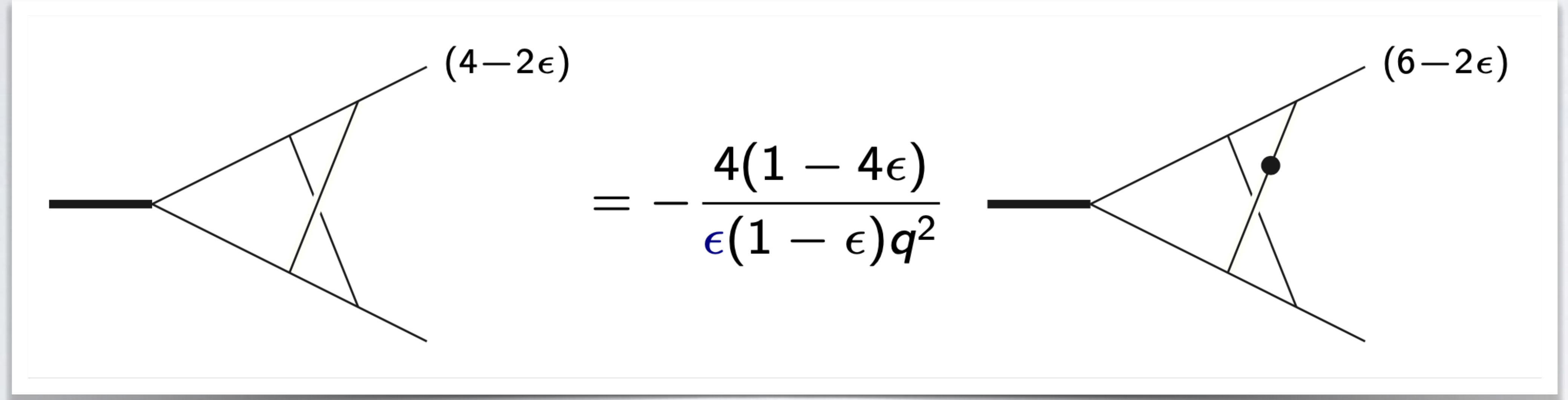
IRREDUCIBLE TOP-LEVEL TOPOLOGIES



- Last two rows: not linearly reducible out of the box
- Use variable transformations to render linearly reducible (known to work for all but last two)
- Hardest topologies contribute late in ϵ expansion

METHOD OF FINITE INTEGRALS

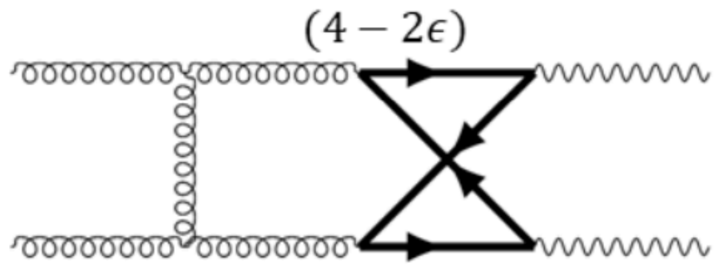
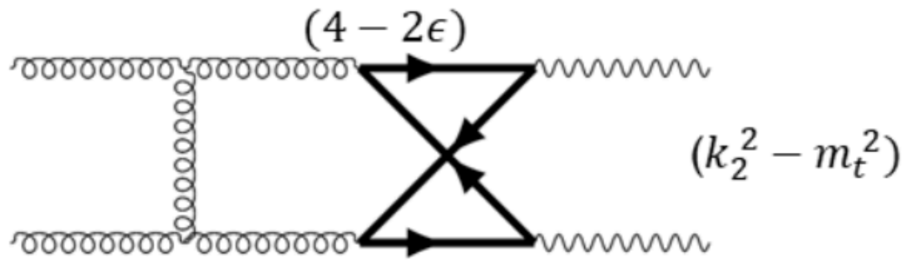
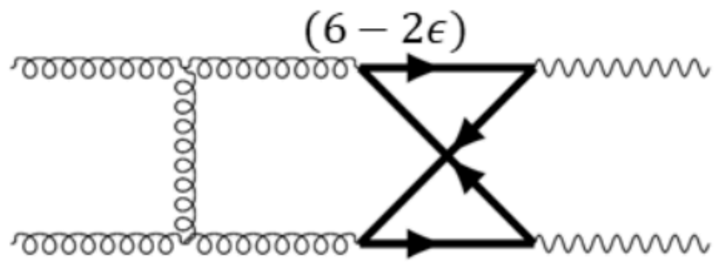
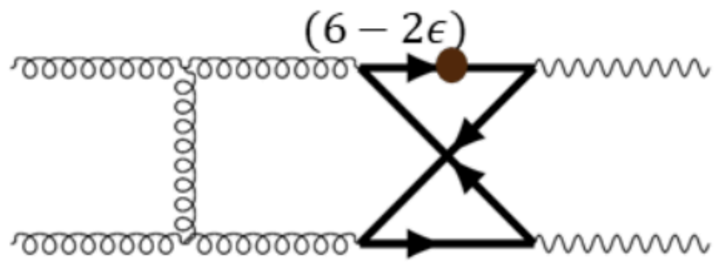
- Observation *[Panzer 2014; AvM, Panzer, Schabinger 2014]*:
  - any ***divergent*** loop integral expressible in terms of **finite** basis integrals

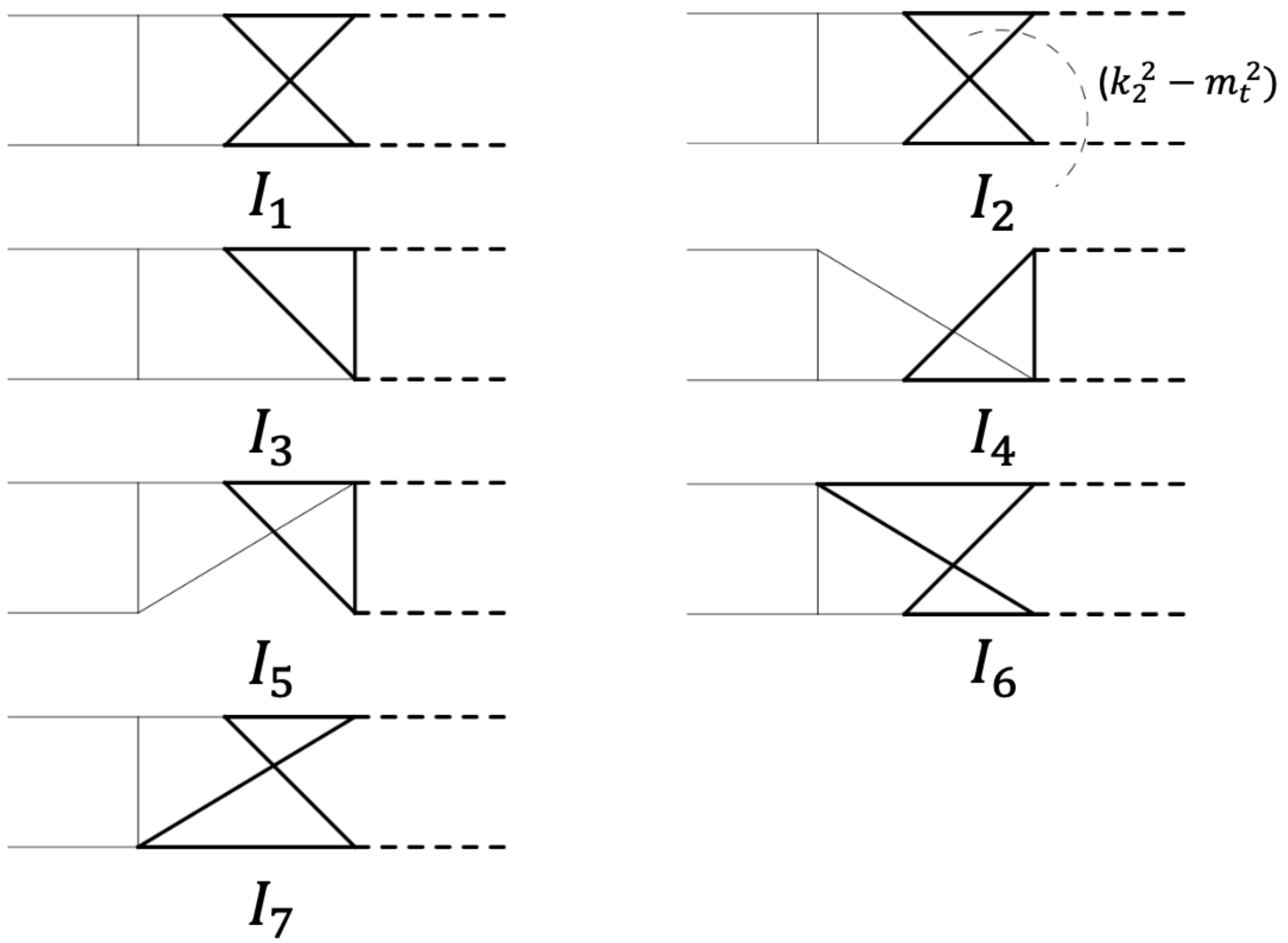


$+\ldots$

- Expand integrands of ***finite*** integrals around $\epsilon = (4-d)/2 \approx 0$
  - If linearly reducible: integrate ***analytically*** with HyperInt *[Panzer 2014]*
  - Improved ***numerical*** evaluations, used for HH *[Borowka, Greiner, Heinrich, Jones, Kerner '16]*, Hj *[Jones, Kerner, Lusioni '18]*, ZH *[Chen, Davies, Heinrich, Jones, Kerner, Mishima, Schlenk, Steinhauser '22]* …

GENERALIZED FINITE INTEGRALS

Integral	Rel.Err.	Timing(s)
	~2*10^-3	45
	~4*10^-2	63
	~8*10^-6	55
	~8*10^-4	60
Linear combination	~1*10^-4	18



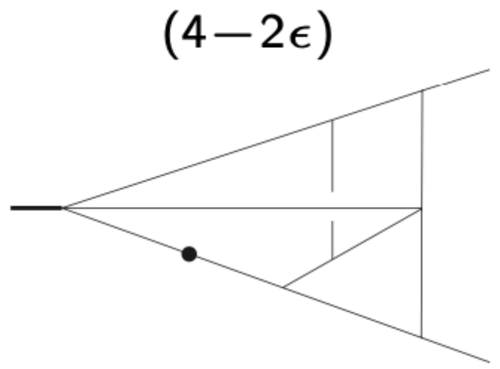
$$I = (m_z^2 - s - t)(sI_1 - I_6) + s(I_2 + I_3 - I_4 - I_5) - (m_z^2 - t)I_7$$

$$I(\nu_1, \ldots, \nu_N) = (-1)^{r+\Delta t}\,\Gamma(\nu - L\,d/2) \int \left(\prod_{j\in\mathcal{N}_T} \mathrm{d}x_j\right)\left(\prod_{j\in\mathcal{N}_t} \frac{x^{\nu_j - 1}}{\Gamma(\nu_j)}\right) \delta\left(1 - \sum_{j\in\mathcal{N}_T} x_j\right)$$
$$\left[\left(\prod_{j\in\mathcal{N}_{\setminus T}} \frac{\partial^{|\nu_j|}}{\partial x_j^{|\nu_j|}}\right)\left(\prod_{j\in\mathcal{N}_{\Delta t}} \frac{\partial^{|\nu_j|+1}}{\partial x_j^{|\nu_j|+1}}\right) \frac{\mathcal{U}^{\nu-(L+1)d/2}}{\mathcal{F}^{\nu - L\,d/2}}\right]_{x_j = 0\ \forall\ j\in\mathcal{N}_{\setminus T}} \qquad (\nu_j \in \mathbb{Z}).$$

[Agarwal, AvM, Jones 2020]

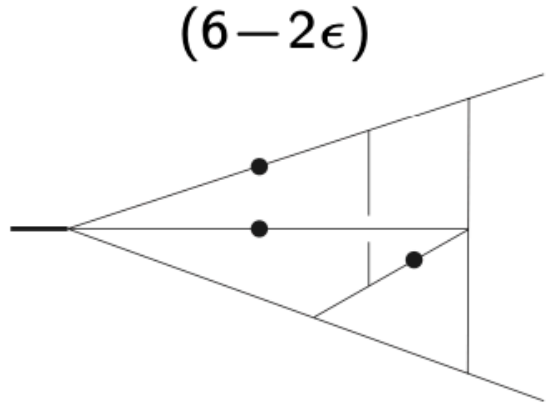
"NICE" FINITE INTEGRALS

- Example: 10 terms in ϵ for weight 6 in conventional basis:



$$
\begin{aligned}
&= \frac{1}{\epsilon^8}\left(-\frac{1}{144}\right) + \frac{1}{\epsilon^7}\left(-\frac{1}{12}\right) + \frac{1}{\epsilon^6}\left(\frac{1}{24}\zeta_2 - \frac{7}{36}\right) + \frac{1}{\epsilon^5}\left(\frac{29}{24}\zeta_3 + \frac{1}{2}\zeta_2 - \frac{1}{72}\right) \\
&+ \frac{1}{\epsilon^4}\left(\frac{71}{16}\zeta_2^2 + \frac{29}{2}\zeta_3 + \frac{39}{16}\zeta_2 + \frac{335}{144}\right) + \frac{1}{\epsilon^3}\left(\frac{1819}{24}\zeta_5 - \frac{23}{6}\zeta_2\zeta_3 + \frac{213}{4}\zeta_2^2 + \frac{1211}{24}\zeta_3 + \frac{431}{48}\zeta_2\right. \\
&\left.+ \frac{47}{18}\right) + \frac{1}{\epsilon^2}\left(-\frac{1285}{24}\zeta_3^2 + \frac{80579}{1008}\zeta_2^3 + \frac{1819}{2}\zeta_5 - 46\zeta_2\zeta_3 + \frac{25787}{160}\zeta_2^2 + \frac{417}{8}\zeta_3 - \frac{1175}{48}\zeta_2 - \frac{7277}{72}\right) \\
&+ \frac{1}{\epsilon}\left(\frac{434203}{192}\zeta_7 - \frac{7139}{24}\zeta_2\zeta_5 - \frac{54139}{120}\zeta_2^2\zeta_3 - \frac{1285}{2}\zeta_3^2 + \frac{80579}{84}\zeta_2^3 + \frac{5571}{2}\zeta_5 - \frac{9005}{24}\zeta_2\zeta_3 + \frac{967}{480}\zeta_2^2\right. \\
&\left.- \frac{4045}{8}\zeta_3 - \frac{733}{24}\zeta_2 + \frac{57635}{72}\right) - \frac{2023}{12}\zeta_{5,3} - \frac{30581}{4}\zeta_3\zeta_5 - \frac{6829}{24}\zeta_2\zeta_3^2 + \frac{45893321}{100800}\zeta_2^4 + \frac{434203}{16}\zeta_7 \\
&- \frac{7139}{2}\zeta_2\zeta_5 - \frac{54139}{10}\zeta_2^2\zeta_3 - \frac{10706}{3}\zeta_3^2 + \frac{7987951}{3360}\zeta_2^3 + \frac{1309}{12}\zeta_5 - \frac{30317}{24}\zeta_2\zeta_3 - \frac{43847}{96}\zeta_2^2 + \frac{32335}{24}\zeta_3 \\
&+ \frac{2553}{4}\zeta_2 - \frac{334727}{72} + \mathcal{O}(\epsilon)\,.
\end{aligned}
$$

- Only 1 term for weight 6 for a nice finite integral:



$$= -\frac{3}{2}\zeta_3^2 - \frac{4}{3}\zeta_2^3 + 10\zeta_5 + 2\zeta_2\zeta_3 - \frac{1}{5}\zeta_2^2 - 6\zeta_3 + \mathcal{O}(\epsilon)$$

ANALYTICAL CUSP ANOMALOUS DIMENSION

$$\Gamma_4^r = N_f^3 C_R \left(\frac{64}{27}\zeta_3 - \frac{32}{81}\right)$$
$$+ N_f^2 C_A C_R \left(-\frac{224}{15}\zeta_2^2 + \frac{2240}{27}\zeta_3 - \frac{608}{81}\zeta_2 + \frac{923}{81}\right)$$
$$+ N_f^2 C_F C_R \left(\frac{64}{5}\zeta_2^2 - \frac{640}{9}\zeta_3 + \frac{2392}{81}\right)$$
$$+ N_f C_A^2 C_R \left(\frac{2096}{9}\zeta_5 + \frac{448}{3}\zeta_3\zeta_2 - \frac{352}{15}\zeta_2^2 - \frac{23104}{27}\zeta_3 + \frac{20320}{81}\zeta_2 - \frac{24137}{81}\right)$$
$$+ N_f C_A C_F C_R \left(160\zeta_5 - 128\zeta_3\zeta_2 - \frac{352}{5}\zeta_2^2 + \frac{3712}{9}\zeta_3 + \frac{440}{3}\zeta_2 - \frac{34066}{81}\right)$$
$$+ N_f C_F^2 C_R \left(-320\zeta_5 + \frac{592}{3}\zeta_3 + \frac{572}{9}\right)$$
$$+ N_f \frac{d_F^{abcd} d_R^{abcd}}{N_R} \left(-\frac{1280}{3}\zeta_5 - \frac{256}{3}\zeta_3 + 256\zeta_2\right)$$
$$+ \frac{d_A^{abcd} d_R^{abcd}}{N_R} \left(-384\zeta_3^2 - \frac{7936}{35}\zeta_2^3 + \frac{3520}{3}\zeta_5 + \frac{128}{3}\zeta_3 - 128\zeta_2\right)$$
$$+ C_A^3 C_R \left(-16\zeta_3^2 - \frac{20032}{105}\zeta_2^3 - \frac{3608}{9}\zeta_5 - \frac{352}{3}\zeta_3\zeta_2 + \frac{3608}{5}\zeta_2^2 + \frac{20944}{27}\zeta_3 - \frac{88400}{81}\zeta_2 + \frac{84278}{81}\right)$$

where $R = F, A$ for $r = q, g$. Note the quartic Casimir (dd) contributions.

$$\Gamma_4^{\mathcal{N}=4} = \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left(-384\zeta_3^2 - \frac{7936}{35}\zeta_2^3\right) + C_A^4 \left(-16\zeta_3^2 - \frac{20032}{105}\zeta_2^3\right),$$

$$\ln(F) = \sum_{L=1}^{\infty} a^L z^{L\epsilon} \left(-\frac{\Gamma_L}{2(L\epsilon)^2} - \frac{\mathcal{G}_L}{2L\epsilon} + L_L^{\text{fin}}\right)$$

N=4: *[Henn, Mistlberger, Korchemsky '19; Huber, AvM, Panzer, Schabinger, Yang '19]*

- Wilson line method (with a conjecture): *[Brüser, Grozin, Henn, Stahlhofen '19; Henn, Mistlberger, Korchemsky '19]*
- Quartics from form factors: *[Lee, Smirnov, Smirnov, Steinhauser '19]*
- Full calculation from QCD form factors: *[AvM, Panzer, Schabinger '20]*

ANALYTICAL COLLINEAR ANOMALOUS DIMENSIONS

$$
\begin{aligned}
\gamma_4^q = {} & C_F^4\Big(11760\,\zeta_7 - 768\,\zeta_5\zeta_2 + \frac{256}{5}\zeta_3\zeta_2^2 - 2304\,\zeta_3^2 - \frac{33776}{35}\zeta_2^3 - 5040\,\zeta_5 - 240\,\zeta_3\zeta_2 - \frac{1368}{5}\zeta_2^2 + 4008\,\zeta_3 - 900\,\zeta_2 + \frac{4873}{12}\Big) \\
& + C_F^3 C_A\Big(-21840\,\zeta_7 + 4128\,\zeta_5\zeta_2 + \frac{512}{5}\zeta_3\zeta_2^2 + 6440\,\zeta_3^2 + \frac{634376}{315}\zeta_2^3 - 1952\,\zeta_5 - \frac{3976}{3}\zeta_3\zeta_2 + \frac{8668}{5}\zeta_2^2 - 6520\,\zeta_3 + 2334\,\zeta_2 - \frac{2085}{2}\Big) \\
& + C_F^2 C_A^2\Big(17220\,\zeta_7 - 4208\,\zeta_5\zeta_2 - \frac{128}{5}\zeta_3\zeta_2^2 - \frac{14204}{3}\zeta_3^2 - \frac{43976}{35}\zeta_2^3 + \frac{10708}{9}\zeta_5 + \frac{4192}{9}\zeta_3\zeta_2 - \frac{48680}{27}\zeta_2^2 + \frac{259324}{27}\zeta_3 - \frac{93542}{27}\zeta_2 + \frac{29639}{18}\Big) \\
& + C_F C_A^3\Big(-\frac{45511}{6}\zeta_7 + \frac{1648}{3}\zeta_5\zeta_2 - \frac{4132}{15}\zeta_3\zeta_2^2 + \frac{5126}{9}\zeta_3^2 - \frac{77152}{315}\zeta_2^3 + \frac{175166}{27}\zeta_5 + \frac{15400}{9}\zeta_3\zeta_2 + \frac{186742}{135}\zeta_2^2 - \frac{1751224}{243}\zeta_3 + \frac{1062149}{729}\zeta_2 + \frac{7179083}{26244}\Big) \\
& + \frac{d_F^{abcd}d_A^{abcd}}{N_F}\Big(3484\,\zeta_7 + 1024\,\zeta_5\zeta_2 - \frac{736}{5}\zeta_3\zeta_2^2 - \frac{3344}{3}\zeta_3^2 + \frac{27808}{315}\zeta_2^3 - \frac{1840}{9}\zeta_5 - 1792\,\zeta_3\zeta_2 + \frac{224}{15}\zeta_2^2 - \frac{7808}{9}\zeta_3 - \frac{2176}{3}\zeta_2 + 192\Big) \\
& + n_f C_F^3\Big(368\,\zeta_3^2 - \frac{117344}{315}\zeta_2^3 + \frac{3872}{3}\zeta_5 - \frac{512}{3}\zeta_3\zeta_2 - \frac{668}{5}\zeta_2^2 - \frac{1120}{9}\zeta_3 + 322\,\zeta_2 + \frac{27949}{108}\Big) \\
& + n_f C_F^2 C_A\Big(-\frac{3400}{3}\zeta_3^2 + \frac{5744}{35}\zeta_2^3 - \frac{4472}{3}\zeta_5 + \frac{3904}{9}\zeta_3\zeta_2 + \frac{105488}{135}\zeta_2^2 - \frac{23518}{81}\zeta_3 + \frac{673}{27}\zeta_2 - \frac{1092511}{972}\Big) \\
& + n_f C_F C_A^2\Big(\frac{6916}{9}\zeta_3^2 + \frac{24184}{315}\zeta_2^3 + \frac{6088}{27}\zeta_5 - \frac{3584}{9}\zeta_3\zeta_2 - \frac{17164}{45}\zeta_2^2 + \frac{140632}{243}\zeta_3 - \frac{445117}{729}\zeta_2 + \frac{326863}{1944}\Big) \\
& + n_f\frac{d_F^{abcd}d_F^{abcd}}{N_F}\Big(\frac{1216}{3}\zeta_3^2 + \frac{9472}{315}\zeta_2^3 - \frac{21760}{9}\zeta_5 + 128\,\zeta_3\zeta_2 - \frac{320}{3}\zeta_2^2 - \frac{5312}{9}\zeta_3 + \frac{4544}{3}\zeta_2 - 384\Big) \\
& + n_f^2 C_F^2\Big(\frac{1040}{9}\zeta_5 - \frac{224}{9}\zeta_3\zeta_2 - \frac{8032}{135}\zeta_2^2 - \frac{4232}{81}\zeta_3 + \frac{1972}{27}\zeta_2 + \frac{9965}{486}\Big) \\
& + n_f^2 C_F C_A\Big(-\frac{1184}{9}\zeta_5 + \frac{256}{9}\zeta_3\zeta_2 + \frac{152}{15}\zeta_2^2 + \frac{14872}{243}\zeta_3 + \frac{41579}{729}\zeta_2 - \frac{97189}{17496}\Big) \\
& + n_f^3 C_F\Big(\frac{128}{135}\zeta_2^2 + \frac{1424}{243}\zeta_3 + \frac{16}{27}\zeta_2 - \frac{37382}{6561}\Big)
\end{aligned}
$$

[Agarwal, AvM, Panzer, Schabinger '21]

$$
\gamma_4^{\mathcal{N}=4} = -300\,\zeta_7 - 256\,\zeta_5\zeta_2 - 384\,\zeta_4\zeta_3 + \frac{1}{N_c^2}\Big[5226\,\zeta_7 + 1536\,\zeta_5\zeta_2 - 552\,\zeta_4\zeta_3\Big]
$$

(N=4 planar color: *[Dixon '17]*)

$$
\begin{aligned}
\gamma_4^g = {} & C_A^4\Big(-\frac{2671}{6}\zeta_7 - \frac{896}{3}\zeta_5\zeta_2 - \frac{2212}{15}\zeta_3\zeta_2^2 - \frac{286}{9}\zeta_3^2 - \frac{674696}{945}\zeta_2^3 + \frac{19232}{27}\zeta_5 + \frac{1588}{3}\zeta_3\zeta_2 + \frac{249448}{135}\zeta_2^2 + \frac{36380}{243}\zeta_3 - \frac{1051411}{729}\zeta_2 + \frac{10672040}{6561}\Big) \\
& + \frac{d^A_{abcd}d^A_{abcd}}{N_A}\Big(3484\,\zeta_7 + 1024\,\zeta_5\zeta_2 - \frac{736}{5}\zeta_3\zeta_2^2 - \frac{3344}{3}\zeta_3^2 + \frac{39776}{315}\zeta_2^3 + \frac{2720}{9}\zeta_5 - 2336\,\zeta_3\zeta_2 - \frac{1808}{15}\zeta_2^2 - \frac{12512}{9}\zeta_3 + 64\,\zeta_2 + \frac{128}{9}\Big) \\
& + n_f C_A^3\Big(-\frac{596}{9}\zeta_3^2 + \frac{148976}{945}\zeta_2^3 + \frac{16066}{27}\zeta_5 + 148\,\zeta_3\zeta_2 - \frac{69502}{135}\zeta_2^2 - \frac{260822}{243}\zeta_3 + \frac{155273}{729}\zeta_2 - \frac{421325}{1944}\Big) \\
& + n_f C_A^2 C_F\Big(152\,\zeta_3^2 + \frac{5632}{315}\zeta_2^3 + \frac{8}{9}\zeta_5 - 176\,\zeta_3\zeta_2 - \frac{1196}{45}\zeta_2^2 + \frac{29606}{81}\zeta_3 + \frac{3023}{9}\zeta_2 - \frac{903983}{972}\Big) \\
& + n_f C_A C_F^2\Big(-80\,\zeta_3^2 - \frac{320}{7}\zeta_2^3 - \frac{1600}{3}\zeta_5 + \frac{148}{5}\zeta_2^2 + \frac{1592}{3}\zeta_3 - 2\,\zeta_2 + \frac{685}{12}\Big) + n_f C_F^3\Big(46\Big) \\
& + n_f\frac{d^A_{abcd}d^F_{abcd}}{N_A}\Big(\frac{1216}{3}\zeta_3^2 - \frac{14464}{315}\zeta_2^3 - \frac{30880}{9}\zeta_5 + 1216\,\zeta_3\zeta_2 + \frac{2464}{15}\zeta_2^2 + \frac{2560}{9}\zeta_3 - 64\,\zeta_2 + \frac{448}{9}\Big) \\
& + n_f^2 C_A^2\Big(-\frac{1024}{9}\zeta_5 - 32\,\zeta_3\zeta_2 + \frac{3128}{135}\zeta_2^2 + \frac{37354}{243}\zeta_3 - \frac{13483}{729}\zeta_2 + \frac{611939}{17496}\Big) \\
& + n_f^2 C_A C_F\Big(\frac{304}{9}\zeta_5 + \frac{32}{3}\zeta_3\zeta_2 + \frac{128}{45}\zeta_2^2 - \frac{1688}{81}\zeta_3 - \frac{172}{9}\zeta_2 + \frac{1199}{18}\Big) + n_f^2 C_F^2\Big(-\frac{352}{9}\zeta_3 + \frac{676}{27}\Big) \\
& + n_f^2\frac{d^F_{abcd}d^F_{abcd}}{N_A}\Big(\frac{1024}{3}\zeta_3 - \frac{1408}{9}\Big) + n_f^3 C_A\Big(\frac{256}{135}\zeta_2^2 - \frac{400}{243}\zeta_3 - \frac{16}{81}\zeta_2 - \frac{15890}{6561}\Big) + n_f^3 C_F\Big(\frac{308}{243}\Big)
\end{aligned}
$$

ANALYTICAL FORM FACTORS @ 4-LOOP QCD

- Partial results for finite parts of form factors @ 4-loop QCD:
  [Henn, Smirnov, Smirnov, Steinhauser '16; Henn, Smirnov, Smirnov, Steinhauser, Lee '16; Lee, Smirnov, Smirnov, Steinhauser '17, '19]

- Partial results for finite parts of for factors @ 4-loop QCD:
  [AvM, Schabinger '16, '19, '19]

- Complete form factors @ 4-loop QCD:
  [Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21, '22; Chakraborty, Huber, Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '22]

- See also:

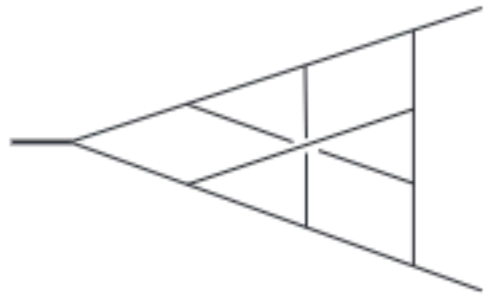
  talks: *Fabian Lange, Marco Niggetiedt*

  - Recent results for form factors with masses + singlet contrib. @ 3-loop QCD:
    [Fael, Lange, Schönwald, Steinhauser '22; Czakon, Niggetiedt '20; Chen, Czakon, Niggetiedt '21; Gehrmann, Primo '21]

  - First steps towards inclusive H cross section at 4th order (soft-collinear contributions):
    [Moch, Ruijl, Ueda, Vermaseren, Vogt '17, '18; Das, Moch, Vogt '19, '20]

METHOD OF DIFFERENTIAL EQUATIONS

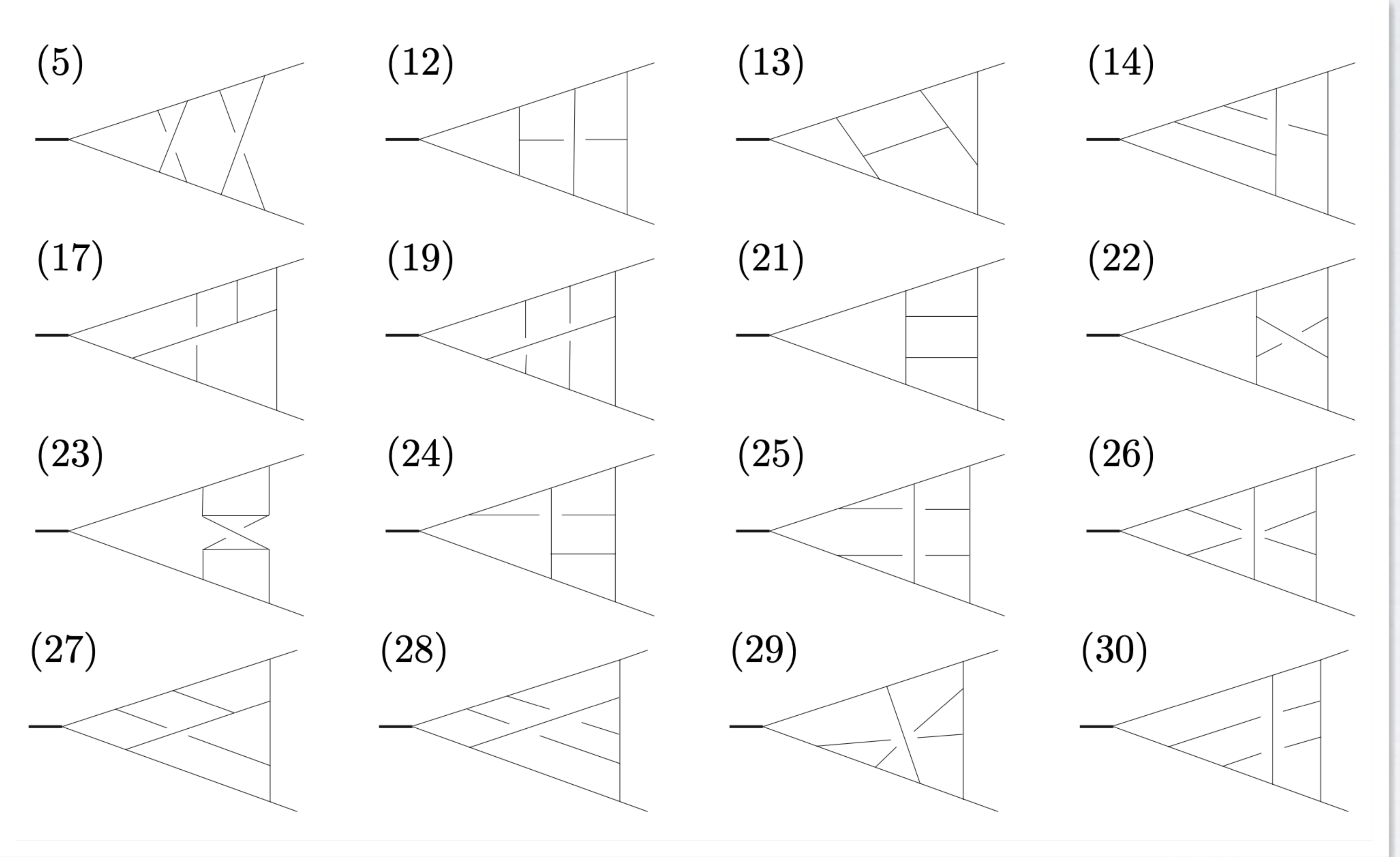
- Take a second leg off-shell, $x = q_1^2/q^2$,
  transport from x=1 (propagator) to x=0 (one-scale FF) *[Henn, Smirnov, Smirnov '13]*
- Reductions with Fire 6 *[A.V. Smirnov, Chukharev '19]*, canonical form *[Henn '13]* with Libra *[Lee '20]*
- Example topology with singularities at x = 0, 1, -1,1/4, 4:
  - 2-scale letters: $\frac{1}{x-1}, \quad \frac{1}{x+1}, \quad \frac{1}{x-4}, \quad \frac{1}{x-1/4}, \quad \frac{1}{(1-x)\sqrt{x}}, \quad \frac{1}{x\sqrt{x-1/4}}, \quad \frac{1}{x\sqrt{1/x-1/4}}$
  - 1-scale G(…, 1) with weights $0, \quad \pm 1, \quad \pm i\sqrt{3}, \quad e^{\pm i\pi/3}, \quad e^{\pm 2i\pi/3}, \quad e^{\pm i\pi/3}/2$ mapped to MZVs



$$
\begin{aligned}
= {} & \frac{1}{\epsilon^8}\left(\frac{7}{18}\right) + \frac{1}{\epsilon^7}\left(\frac{55}{24}\right) + \frac{1}{\epsilon^6}\left(-\frac{67}{9}\zeta_2 - \frac{797}{144}\right) + \frac{1}{\epsilon^5}\left(-\frac{442}{9}\zeta_3 - \frac{643}{18}\zeta_2 + \frac{1193}{144}\right) + \frac{1}{\epsilon^4}\left(-\frac{9199}{360}\zeta_2^2 - \frac{3547}{18}\zeta_3\right.\\
& \left. + \frac{7793}{72}\zeta_2 + \frac{1013}{48}\right) + \frac{1}{\epsilon^3}\left(-\frac{2858}{3}\zeta_5 + \frac{27617}{36}\zeta_3\zeta_2 - \frac{3439}{180}\zeta_2^2 + \frac{60893}{72}\zeta_3 - \frac{1897}{8}\zeta_2 - \frac{43895}{144}\right) + \frac{1}{\epsilon^2}\left(\frac{179927}{72}\zeta_3^2 - \frac{40853}{252}\zeta_2^3\right.\\
& \left. - 2780\zeta_5 + \frac{23467}{9}\zeta_3\zeta_2 + \frac{132359}{180}\zeta_2^2 - \frac{66607}{24}\zeta_3 - \frac{5423}{72}\zeta_2 + \frac{311383}{144}\right) + \frac{1}{\epsilon}\left(-\frac{1015395}{32}\zeta_7 + \frac{30493}{2}\zeta_5\zeta_2 + \frac{274199}{90}\zeta_3\zeta_2^2\right.\\
& \left. + \frac{44984}{9}\zeta_3^2 - \frac{540823}{420}\zeta_2^3 + \frac{477281}{24}\zeta_5 - \frac{412181}{36}\zeta_3\zeta_2 - \frac{117101}{30}\zeta_2^2 + \frac{410629}{72}\zeta_3 + \frac{400999}{72}\zeta_2 - \frac{622069}{48}\right) + \frac{122261}{15}\zeta_{5,3}\\
& + \frac{1298525}{12}\zeta_5\zeta_3 - \frac{942899}{36}\zeta_3^2\zeta_2 - \frac{121150681}{9000}\zeta_2^4 - \frac{2558101}{16}\zeta_7 + \frac{360793}{6}\zeta_5\zeta_2 - \frac{53821}{18}\zeta_3\zeta_2^2 - \frac{1428953}{72}\zeta_3^2 + \frac{2037031}{168}\zeta_2^3\\
& - \frac{1989461}{24}\zeta_5 + \frac{526387}{12}\zeta_3\zeta_2 + \frac{245017}{18}\zeta_2^2 + \frac{738547}{72}\zeta_3 - \frac{1198061}{24}\zeta_2 + \frac{10519199}{144} + \mathcal{O}(\epsilon),
\end{aligned}
\tag{6}
$$

N=4 SYM SUDAKOV FORM FACTOR @ 4 LOOPS

$$F = \frac{1}{N}\int \mathrm{d}^4x\, e^{-iq\cdot x}\,\langle \phi^a_{12}(p_1)\phi^b_{12}(p_2)|\,(\phi^c_{34}\phi^c_{34})(x)\,|0\rangle,$$



$$\begin{aligned}F^{(4)} = 2\Big[&8I^{(1)}_{\mathrm{p},1} + 2I^{(2)}_{\mathrm{p},2} - 2I^{(3)}_{\mathrm{p},3} + 2I^{(4)}_{\mathrm{p},4} + \frac{1}{2}I^{(5)}_{\mathrm{p},5} + 2I^{(6)}_{\mathrm{p},6} + 4I^{(7)}_{\mathrm{p},7} + 2I^{(9)}_{\mathrm{p},8} - 2I^{(10)}_{\mathrm{p},9} + I^{(12)}_{\mathrm{p},10}\\
&+ I^{(12)}_{\mathrm{p},11} + 2I^{(13)}_{\mathrm{p},12} + 2I^{(14)}_{\mathrm{p},13} - 2I^{(17)}_{\mathrm{p},14} + 2I^{(17)}_{\mathrm{p},15} - 2I^{(19)}_{\mathrm{p},16} + I^{(19)}_{\mathrm{p},17} + I^{(21)}_{\mathrm{p},18} + \frac{1}{2}I^{(25)}_{\mathrm{p},19} + 2I^{(30)}_{\mathrm{p},20} + 2I^{(13)}_{\mathrm{p},21}\\
&+ 4I^{(14)}_{\mathrm{p},22} - 2I^{(14)}_{\mathrm{p},23} - I^{(14)}_{\mathrm{p},24} + 4I^{(17)}_{\mathrm{p},25} - I^{(17)}_{\mathrm{p},26} - 2I^{(17)}_{\mathrm{p},27} - 2I^{(17)}_{\mathrm{p},28} - I^{(19)}_{\mathrm{p},29} - I^{(19)}_{\mathrm{p},30} + I^{(19)}_{\mathrm{p},31} - \frac{1}{2}I^{(30)}_{\mathrm{p},32}\Big]\\
&+ \frac{48}{N_c^2}\Big[\frac{1}{2}I^{(21)}_1 + \frac{1}{2}I^{(22)}_2 + \frac{1}{2}I^{(23)}_3 - I^{(24)}_4 + \frac{1}{4}I^{(25)}_5 - \frac{1}{4}I^{(26)}_6 - \frac{1}{4}I^{(26)}_7 + 2I^{(27)}_8 + I^{(28)}_9\\
&+ 4I^{(29)}_{10} + I^{(30)}_{11} + I^{(27)}_{12} - \frac{1}{2}I^{(28)}_{13} + I^{(29)}_{14} + I^{(29)}_{15} + I^{(30)}_{16} + I^{(30)}_{17} + I^{(30)}_{18} + I^{(22)}_{19} + I^{(22)}_{20}\\
&- I^{(24)}_{21} + \frac{1}{4}I^{(24)}_{22} + \frac{1}{2}I^{(28)}_{23}\Big].\end{aligned}$$

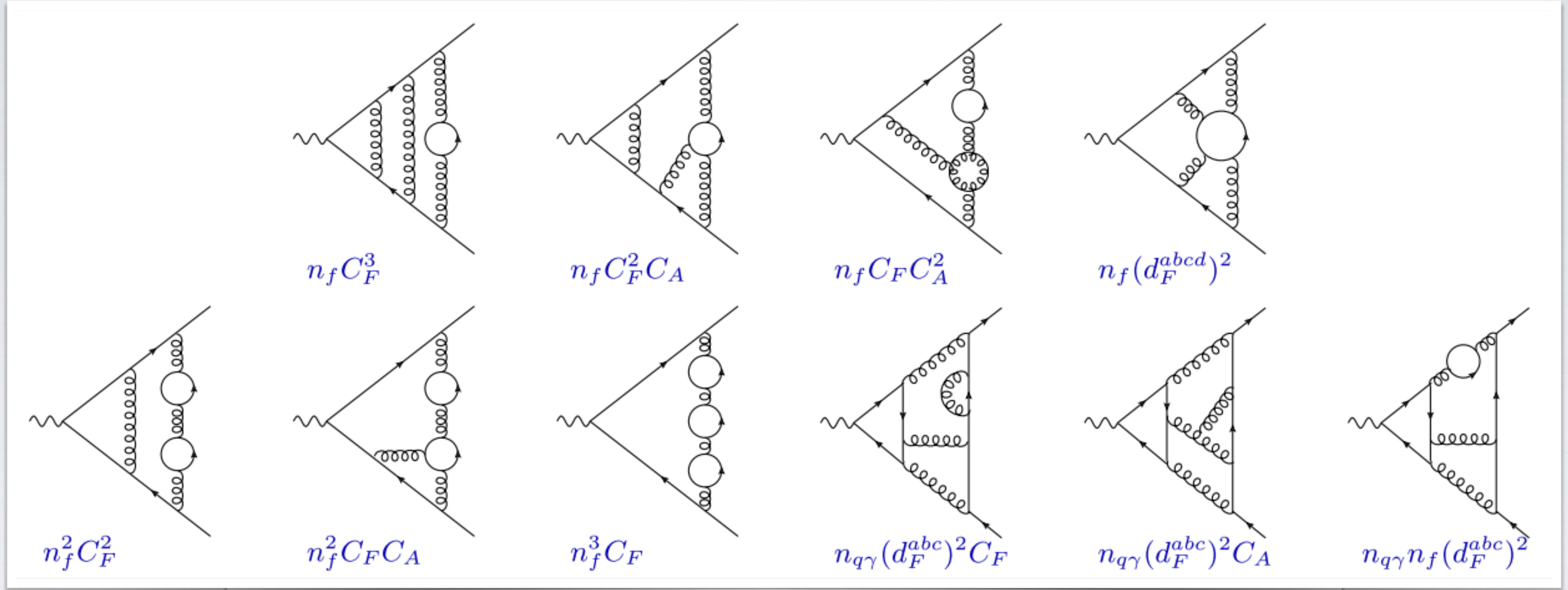
$$I^{(21)}_1 = I^{(21)}_{\mathrm{p},18}, \qquad I^{(25)}_5 = I^{(25)}_{\mathrm{p},19}, \qquad I^{(30)}_{11} = I^{(30)}_{\mathrm{p},20}.$$

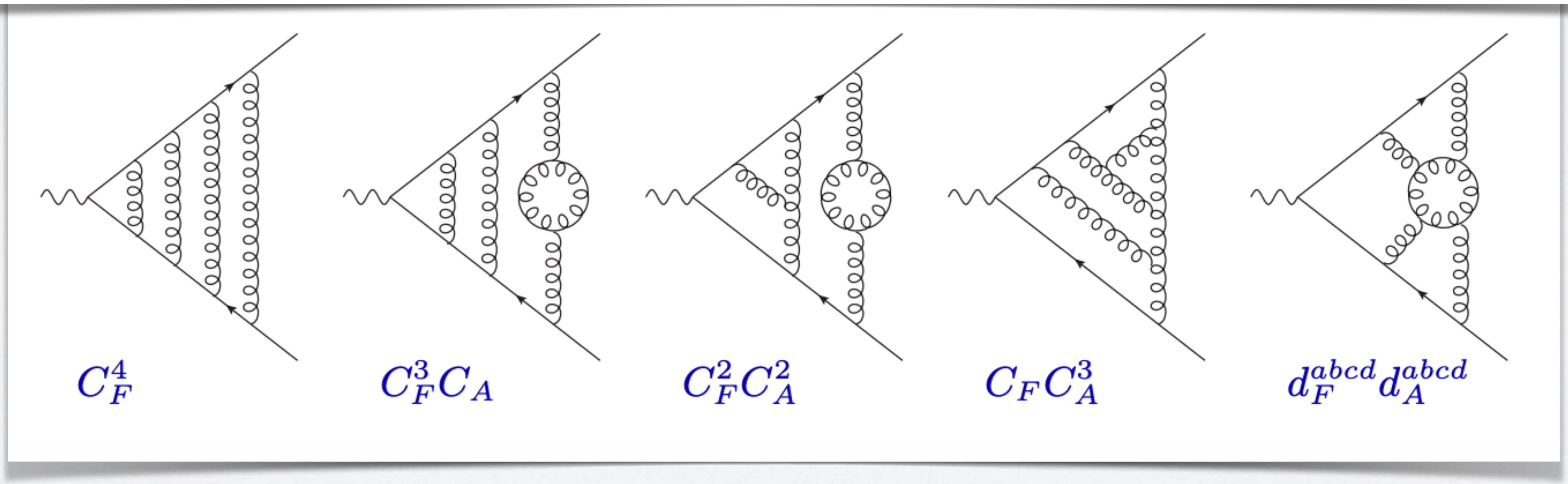
Integrand: *[Boels, Huber, Yang '17]*

$$\begin{aligned}F_4 = \Big[&\frac{1}{\epsilon^8}\left(\frac{2}{3}\right) + \frac{1}{\epsilon^6}\left(\frac{2}{3}\zeta_2\right) + \frac{1}{\epsilon^5}\left(-\frac{38}{9}\zeta_3\right) + \frac{1}{\epsilon^4}\left(\frac{5}{18}\zeta_2^2\right) + \frac{1}{\epsilon^3}\left(\frac{1082}{15}\zeta_5 + \frac{23}{3}\zeta_3\zeta_2\right) + \frac{1}{\epsilon^2}\left(\frac{10853}{54}\zeta_3^2 + \frac{95477}{945}\zeta_2^3\right)\\
&+ \frac{1}{\epsilon}\left(\frac{541619}{126}\zeta_7 - \frac{15529}{45}\zeta_5\zeta_2 + \frac{39067}{135}\zeta_3\zeta_2^2\right) + \left(-\frac{808}{45}\zeta_{5,3} + \frac{499927}{45}\zeta_5\zeta_3 - \frac{35707}{27}\zeta_3^2\zeta_2 + \frac{71888861}{31500}\zeta_2^4\right)\Big]\\
&+ \frac{1}{N_c^2}\Big[\frac{1}{\epsilon^2}\left(18\,\zeta_3^2 + \frac{372}{35}\zeta_2^3\right) + \frac{1}{\epsilon}\left(-\frac{2613}{4}\zeta_7 - 192\,\zeta_5\zeta_2 + \frac{138}{5}\zeta_3\zeta_2^2\right) + \left(390\,\zeta_{5,3} - 7638\,\zeta_5\zeta_3 - 24\,\zeta_3^2\zeta_2 - \frac{248383}{175}\zeta_2^4\right)\end{aligned}$$

[Lee, AvM, Schabinger, Steinhauser, Smirnov, Smirnov '21]

$q\bar{q}\gamma^*$ FORM FACTOR @ 4 LOOPS





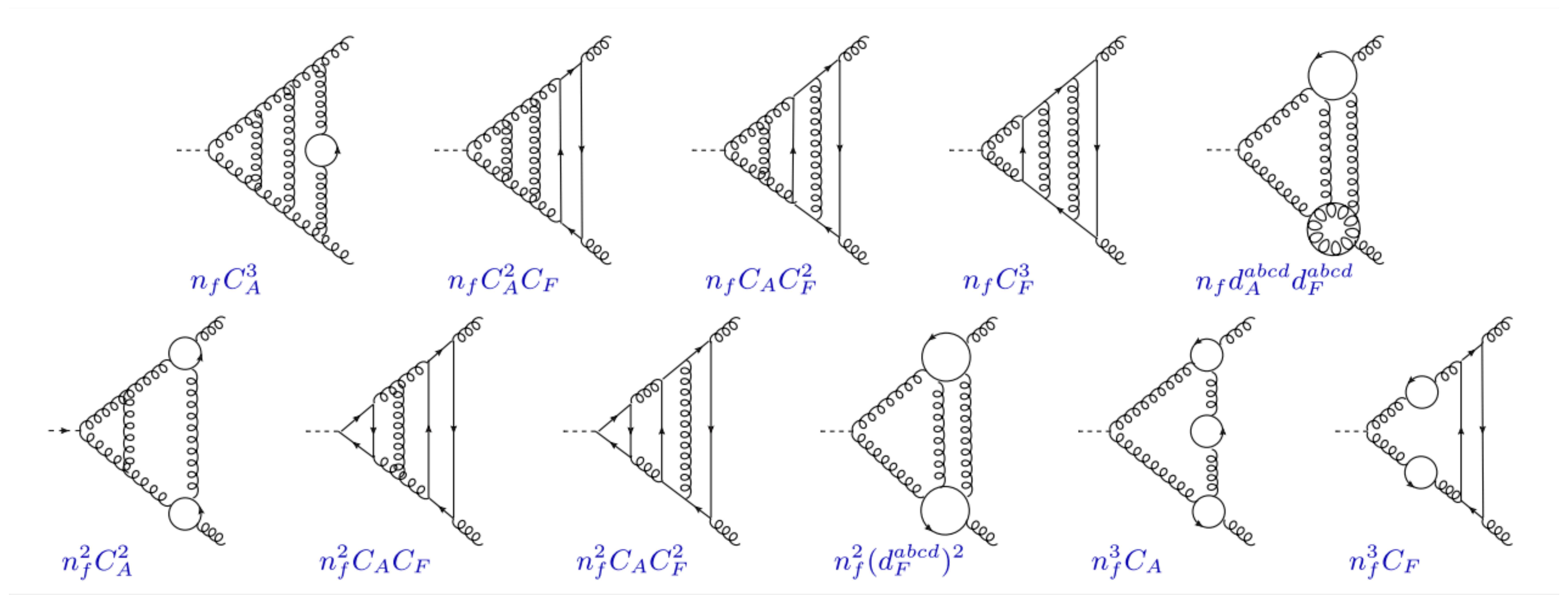
[Lee, AvM, Schabinger, Steinhauser, Smirnov, Smirnov ’21, ‘22]

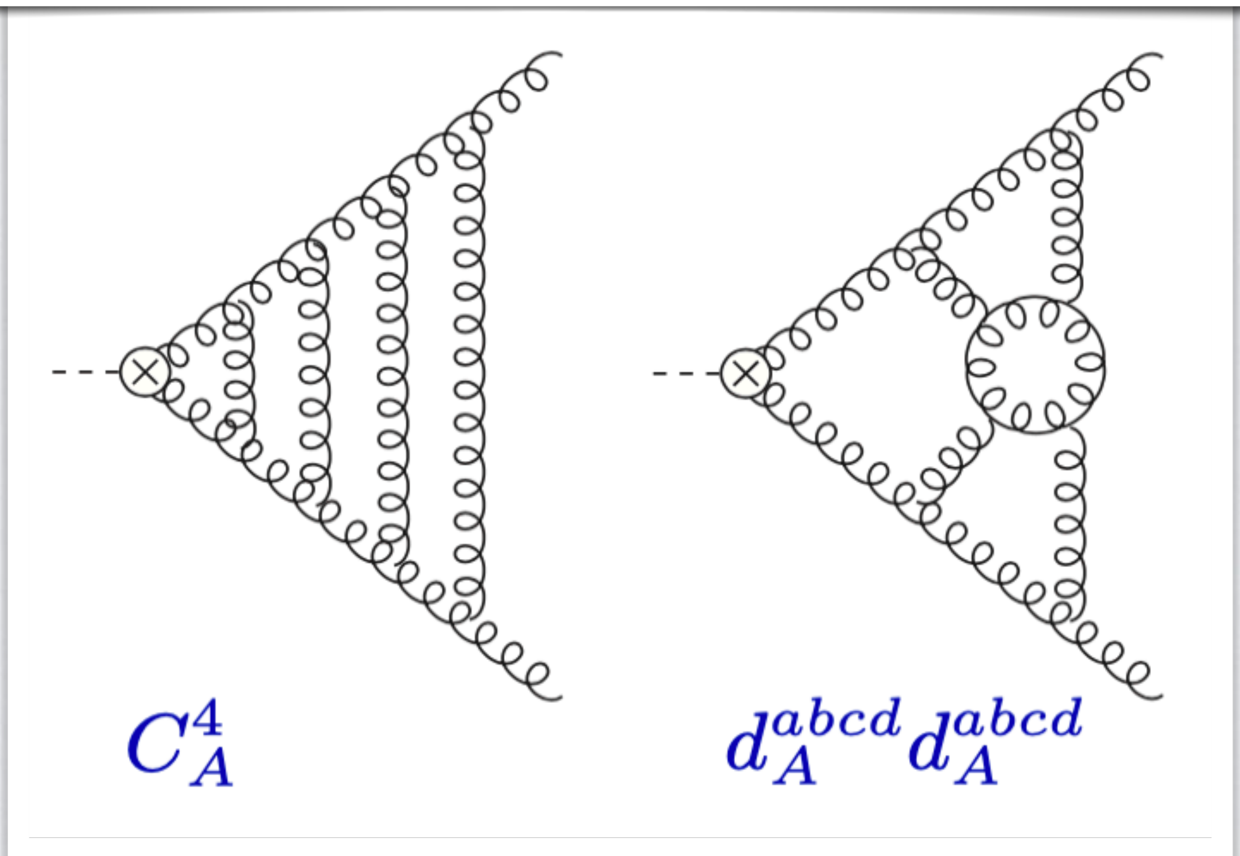
$q\bar{q}\gamma^*$ FORM FACTOR @ 4 LOOPS

$$
\begin{aligned}
F^{\mathrm{fin}}_{q,4} = {} & C_F^4\Big(-\tfrac{2208}{5}\zeta_{5,3} - 1792\,\zeta_5\zeta_3 + 840\,\zeta_3^2\zeta_2 - \tfrac{7508687}{63000}\zeta_2^4 - \tfrac{29919}{2}\zeta_7 - 2696\,\zeta_5\zeta_2 + \tfrac{2009}{5}\zeta_3\zeta_2^2 + 5072\,\zeta_3^2 + \tfrac{563503}{630}\zeta_2^3 + \tfrac{44977}{3}\zeta_5 - 1930\,\zeta_3\zeta_2 + \tfrac{19375}{16}\zeta_2^2 - \tfrac{129505}{12}\zeta_3 + \tfrac{26749}{8}\zeta_2 + \tfrac{153365}{384}\Big) \\
& + C_F^3 C_A\Big(-\tfrac{692}{5}\zeta_{5,3} + 3696\,\zeta_5\zeta_3 - \tfrac{8536}{3}\zeta_3^2\zeta_2 + \tfrac{506012}{1125}\zeta_2^4 + \tfrac{474205}{24}\zeta_7 + \tfrac{37975}{9}\zeta_5\zeta_2 - \tfrac{113287}{90}\zeta_3\zeta_2^2 - 8504\,\zeta_3^2 + \tfrac{2013857}{3780}\zeta_2^3 + \tfrac{325717}{36}\zeta_5 + \tfrac{787613}{54}\zeta_3\zeta_2 - \tfrac{32251333}{6480}\zeta_2^2 - \tfrac{288281}{72}\zeta_3 - \tfrac{6575143}{432}\zeta_2 - \tfrac{1147289}{192}\Big) \\
& + C_F^2 C_A^2\Big(1046\,\zeta_{5,3} - 5104\,\zeta_5\zeta_3 + \tfrac{24208}{9}\zeta_3^2\zeta_2 - \tfrac{3829877}{4725}\zeta_2^4 - \tfrac{248037}{16}\zeta_7 - \tfrac{6781}{18}\zeta_5\zeta_2 + \tfrac{64919}{45}\zeta_3\zeta_2^2 + \tfrac{1022996}{81}\zeta_3^2 - \tfrac{103553}{420}\zeta_2^3 - \tfrac{1113539}{216}\zeta_5 - \tfrac{20087587}{972}\zeta_3\zeta_2 + \tfrac{95100011}{29160}\zeta_2^2 - \tfrac{51597389}{2916}\zeta_3 + \tfrac{2779278167}{104976}\zeta_2 + \tfrac{9643400117}{839808}\Big) \\
& + C_F C_A^3\Big(-\tfrac{14161}{30}\zeta_{5,3} + \tfrac{21577}{6}\zeta_5\zeta_3 - \tfrac{1963}{3}\zeta_3^2\zeta_2 + \tfrac{10233079}{15750}\zeta_2^4 + \tfrac{616417}{144}\zeta_7 - 397\,\zeta_5\zeta_2 - \tfrac{19823}{45}\zeta_3\zeta_2^2 - \tfrac{845393}{108}\zeta_3^2 - \tfrac{8189719}{11340}\zeta_2^3 - \tfrac{8979437}{3240}\zeta_5 + \tfrac{720313}{108}\zeta_3\zeta_2 - \tfrac{283307}{1620}\zeta_2^2 + \tfrac{32942281}{1458}\zeta_3 - \tfrac{540427967}{34992}\zeta_2 - \tfrac{3289233097}{209952}\Big) \\
& + \frac{d_F^{abcd}d_A^{abcd}}{N_F}\Big(260\,\zeta_{5,3} - 5092\,\zeta_5\zeta_3 - 16\,\zeta_3^2\zeta_2 - \tfrac{496766}{525}\zeta_2^4 + 3518\,\zeta_7 - \tfrac{4744}{3}\zeta_5\zeta_2 + \tfrac{6584}{15}\zeta_3\zeta_2^2 + \tfrac{39986}{9}\zeta_3^2 + \tfrac{526496}{945}\zeta_2^3 - \tfrac{180566}{27}\zeta_5 + \tfrac{3020}{3}\zeta_3\zeta_2 + \tfrac{1220}{9}\zeta_2^2 + \tfrac{169532}{27}\zeta_3 + \tfrac{10570}{9}\zeta_2 - \tfrac{1580}{3}\Big) \\
& + n_f C_F^3\Big(\tfrac{2013}{2}\zeta_7 - \tfrac{1124}{9}\zeta_5\zeta_2 - \tfrac{7567}{45}\zeta_3\zeta_2^2 - \tfrac{3032}{3}\zeta_3^2 - \tfrac{20477}{378}\zeta_2^3 - \tfrac{105215}{18}\zeta_5 - \tfrac{113617}{81}\zeta_3\zeta_2 + \tfrac{3288893}{3240}\zeta_2^2 + \tfrac{802207}{162}\zeta_3 + \tfrac{1539611}{1944}\zeta_2 - \tfrac{1841095}{7776}\Big) \\
& + n_f C_F^2 C_A\Big(-\tfrac{1219}{4}\zeta_7 + 114\,\zeta_5\zeta_2 + \tfrac{15934}{45}\zeta_3\zeta_2^2 + \tfrac{10904}{81}\zeta_3^2 - \tfrac{808}{105}\zeta_2^3 + \tfrac{44981}{18}\zeta_5 + \tfrac{189565}{81}\zeta_3\zeta_2 - \tfrac{6376939}{3645}\zeta_2^2 + \tfrac{25114571}{5832}\zeta_3 - \tfrac{547858717}{104976}\zeta_2 + \tfrac{273777229}{419904}\Big) \\
& + n_f C_F C_A^2\Big(\tfrac{19141}{72}\zeta_7 - \tfrac{127}{3}\zeta_5\zeta_2 - \tfrac{6904}{45}\zeta_3\zeta_2^2 + \tfrac{4958}{9}\zeta_3^2 + \tfrac{345871}{11340}\zeta_2^3 - \tfrac{3862513}{3240}\zeta_5 - \tfrac{92201}{108}\zeta_3\zeta_2 + \tfrac{316999}{540}\zeta_2^2 - \tfrac{40209899}{5832}\zeta_3 + \tfrac{213890551}{34992}\zeta_2 + \tfrac{5309402065}{839808}\Big) \\
& + n_f\frac{d_F^{abcd}d_F^{abcd}}{N_F}\Big(-1240\,\zeta_7 + \tfrac{992}{3}\zeta_5\zeta_2 - \tfrac{3952}{15}\zeta_3\zeta_2^2 + \tfrac{680}{9}\zeta_3^2 + \tfrac{41620}{189}\zeta_2^3 + \tfrac{95098}{27}\zeta_5 + \tfrac{92}{3}\zeta_3\zeta_2 + \tfrac{7552}{45}\zeta_2^2 - \tfrac{13414}{27}\zeta_3 - \tfrac{21566}{9}\zeta_2 + \tfrac{3190}{3}\Big) \\
& + n_{q\gamma} C_F\frac{d_F^{abc}d_F^{abc}}{N_F}\Big(\tfrac{11536}{3}\zeta_7 + \tfrac{1280}{3}\zeta_5\zeta_2 + \tfrac{1408}{5}\zeta_3\zeta_2^2 - 672\,\zeta_3^2 - \tfrac{25808}{105}\zeta_2^3 + \tfrac{10160}{3}\zeta_5 - \tfrac{2672}{3}\zeta_3\zeta_2 - \tfrac{1392}{5}\zeta_2^2 - \tfrac{2752}{3}\zeta_3 - 1376\,\zeta_2 - \tfrac{7040}{9}\Big) \\
& + n_{q\gamma} C_A\frac{d_F^{abc}d_F^{abc}}{N_F}\Big(-\tfrac{13972}{3}\zeta_7 - 1840\,\zeta_5\zeta_2 - \tfrac{784}{5}\zeta_3\zeta_2^2 - \tfrac{8752}{3}\zeta_3^2 - \tfrac{523448}{945}\zeta_2^3 - \tfrac{11740}{9}\zeta_5 + \tfrac{7192}{3}\zeta_3\zeta_2 - \tfrac{43948}{45}\zeta_2^2 + \tfrac{12568}{3}\zeta_3 + \tfrac{39344}{9}\zeta_2 + \tfrac{20384}{9}\Big) \\
& + n_f^2 C_F^2\Big(\tfrac{4556}{81}\zeta_3^2 + \tfrac{3520}{189}\zeta_2^3 + \tfrac{3796}{27}\zeta_5 + \tfrac{18802}{243}\zeta_3\zeta_2 + \tfrac{107507}{810}\zeta_2^2 - \tfrac{514580}{729}\zeta_3 + \tfrac{5818805}{26244}\zeta_2 - \tfrac{73476853}{209952}\Big) \\
& + n_f^2 C_F C_A\Big(-\tfrac{622}{27}\zeta_3^2 + \tfrac{1654}{135}\zeta_2^3 + \tfrac{22874}{135}\zeta_5 - \tfrac{956}{27}\zeta_3\zeta_2 - \tfrac{18431}{135}\zeta_2^2 + \tfrac{719659}{1458}\zeta_3 - \tfrac{26318309}{34992}\zeta_2 - \tfrac{689230799}{839808}\Big) \\
& + n_{q\gamma} n_f\frac{d_F^{abc}d_F^{abc}}{N_F}\Big(\tfrac{1408}{3}\zeta_3^2 + \tfrac{11264}{135}\zeta_2^3 + \tfrac{3520}{9}\zeta_5 - \tfrac{448}{3}\zeta_3\zeta_2 + \tfrac{608}{9}\zeta_2^2 - 224\,\zeta_3 - \tfrac{4448}{9}\zeta_2 - \tfrac{3136}{9}\Big) \\
& + n_f^3 C_F\Big(-\tfrac{106}{135}\zeta_5 + \tfrac{4}{9}\zeta_3\zeta_2 + \tfrac{3044}{405}\zeta_2^2 + \tfrac{104}{243}\zeta_3 + \tfrac{19766}{729}\zeta_2 + \tfrac{1865531}{52488}\Big)
\end{aligned}
$$

[Lee, AvM, Schabinger, Steinhauser, Smirnov, Smirnov ‘21, ’22]

ggH FORM FACTOR @ 4 LOOPS





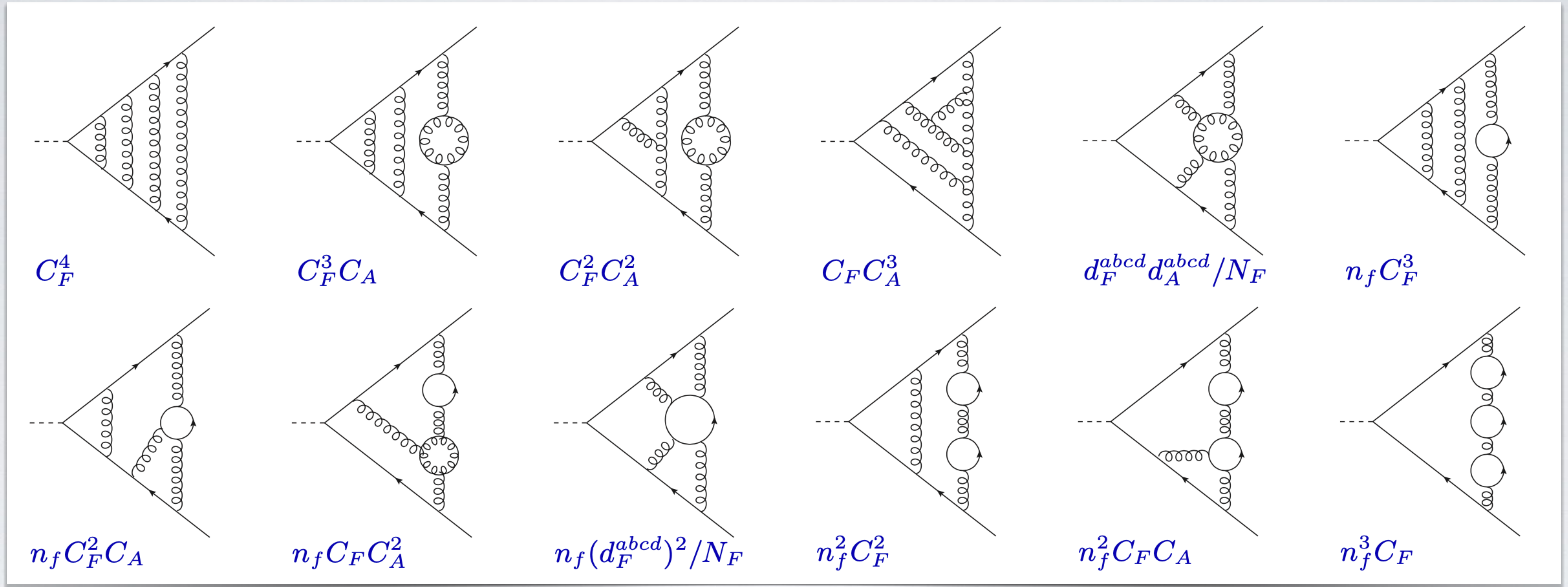
[Lee, AvM, Schabinger, Steinhauser, Smirnov, Smirnov ’21, ’22]

ggH FORM FACTOR @ 4 LOOPS

$$
\begin{aligned}
F^{\text{fin}}_{g,4} = {} & C_A^4\Big(-\frac{181}{30}\zeta_{5,3}+\frac{2377}{6}\zeta_5\zeta_3+\frac{271}{9}\zeta_3^2\zeta_2+\frac{4583689}{27000}\zeta_2^4-\frac{224939}{72}\zeta_7+\frac{5423}{6}\zeta_5\zeta_2+\frac{18931}{90}\zeta_3\zeta_2^2+\frac{418801}{162}\zeta_3^2+\frac{353093}{1620}\zeta_2^3+\frac{1203647}{135}\zeta_5-\frac{1806605}{486}\zeta_3\zeta_2-\frac{778313}{5832}\zeta_2^2-\frac{47586469}{1944}\zeta_3+\frac{32379341}{104976}\zeta_2+\frac{5165679667}{139968}\Big)\\
& +\frac{d_A^{abcd}d_A^{abcd}}{N_A}\Big(260\,\zeta_{5,3}-5092\,\zeta_5\zeta_3-16\,\zeta_3^2\zeta_2-\frac{496766}{525}\zeta_2^4-\frac{6776}{3}\zeta_7-5016\,\zeta_5\zeta_2+\frac{2992}{3}\zeta_3\zeta_2^2+\frac{31588}{3}\zeta_3^2+\frac{1073972}{945}\zeta_2^3-6460\,\zeta_5+\frac{6752}{9}\zeta_3\zeta_2+\frac{24616}{45}\zeta_2^2+\frac{68410}{9}\zeta_3-\frac{4682}{27}\zeta_2-\frac{1310}{9}\Big)\\
& +n_fC_A^3\Big(-\frac{8390}{9}\zeta_7+\frac{991}{9}\zeta_5\zeta_2-\frac{2129}{45}\zeta_3\zeta_2^2-\frac{32425}{324}\zeta_3^2-\frac{702253}{5670}\zeta_2^3+\frac{566977}{540}\zeta_5+\frac{67831}{162}\zeta_3\zeta_2-\frac{2333729}{29160}\zeta_2^2+\frac{9686917}{1944}\zeta_3+\frac{113944685}{104976}\zeta_2-\frac{20463665839}{839808}\Big)\\
& +n_fC_A^2C_F\Big(\frac{16003}{12}\zeta_7+\frac{230}{9}\zeta_5\zeta_2-\frac{44}{15}\zeta_3\zeta_2^2-\frac{1787}{3}\zeta_3^2+\frac{32254}{945}\zeta_2^3+\frac{143197}{36}\zeta_5+\frac{78590}{81}\zeta_3\zeta_2-\frac{44839}{540}\zeta_2^2+\frac{8317937}{1944}\zeta_3-\frac{293267}{3888}\zeta_2-\frac{573672965}{46656}\Big)\\
& +n_fC_AC_F^2\Big(-\frac{9580}{3}\zeta_7-300\,\zeta_5\zeta_2+12\,\zeta_3\zeta_2^2-368\,\zeta_3^2-\frac{39328}{945}\zeta_2^3-\frac{92317}{18}\zeta_5+\frac{193}{3}\zeta_3\zeta_2-5\,\zeta_2^2+\frac{700879}{108}\zeta_3-\frac{217}{36}\zeta_2+\frac{1156175}{1296}\Big)\\
& +n_fC_F^3\Big(3360\,\zeta_7-2940\,\zeta_5-156\,\zeta_3+\frac{169}{2}\Big)\\
& +n_f\frac{d_A^{abcd}d_F^{abcd}}{N_A}\Big(\frac{2464}{3}\zeta_7+1824\,\zeta_5\zeta_2-\frac{1088}{3}\zeta_3\zeta_2^2-\frac{15700}{3}\zeta_3^2-\frac{245536}{945}\zeta_2^3+\frac{108692}{9}\zeta_5+\frac{1544}{9}\zeta_3\zeta_2-\frac{35108}{45}\zeta_2^2-\frac{89932}{9}\zeta_3+\frac{9580}{27}\zeta_2+\frac{6944}{9}\Big)\\
& +n_f^2C_A^2\Big(\frac{9452}{81}\zeta_3^2+\frac{15044}{945}\zeta_2^3-\frac{38071}{135}\zeta_5+\frac{3113}{486}\zeta_3\zeta_2+\frac{78953}{3240}\zeta_2^2+\frac{1103621}{1944}\zeta_3-\frac{25105537}{104976}\zeta_2+\frac{3255482741}{839808}\Big)\\
& +n_f^2C_AC_F\Big(-270\,\zeta_3^2-\frac{10084}{945}\zeta_2^3-\frac{23572}{27}\zeta_5-\frac{944}{9}\zeta_3\zeta_2-\frac{764}{135}\zeta_2^2-\frac{724883}{486}\zeta_3-\frac{4790}{27}\zeta_2+\frac{48037931}{11664}\Big)\\
& +n_f^2C_F^2\Big(\frac{800}{3}\zeta_3^2+\frac{13696}{945}\zeta_2^3+\frac{3920}{3}\zeta_5+\frac{32}{3}\zeta_3\zeta_2-\frac{212}{15}\zeta_2^2-1592\,\zeta_3+\frac{58}{9}\zeta_2+\frac{32137}{216}\Big)\\
& +n_f^2\frac{d_F^{abcd}d_F^{abcd}}{N_A}\Big(512\,\zeta_3^2-960\,\zeta_5+\frac{384}{5}\zeta_2^2+1520\,\zeta_3-\frac{9008}{9}\Big)\\
& +n_f^3C_A\Big(-\frac{194}{15}\zeta_5+\frac{124}{27}\zeta_3\zeta_2-\frac{944}{405}\zeta_2^2-\frac{17818}{243}\zeta_3+\frac{9430}{729}\zeta_2-\frac{8399887}{52488}\Big)\\
& +n_f^3C_F\Big(\frac{640}{27}\zeta_5-\frac{64}{9}\zeta_3\zeta_2+\frac{112}{45}\zeta_2^2+\frac{4060}{27}\zeta_3+\frac{64}{3}\zeta_2-\frac{233953}{972}\Big)
\end{aligned}
$$

[Lee, AvM, Schabinger, Steinhauser, Smirnov, Smirnov ’21,’22]

$b\bar{b}H$ FORM FACTOR @ 4 LOOPS



[Chakraborty, Huber, Lee, AvM, Schabinger, Steinhauser, Smirnov, Smirnov '22]

$b\bar{b}H$ FORM FACTOR @ 4 LOOPS

$$
\begin{aligned}
F_{b,4}^{\mathrm{fin}} = {} & C_F^4\Big(-\frac{2208}{5}\zeta_{5,3}-1792\,\zeta_5\zeta_3+840\,\zeta_3^2\zeta_2-\frac{7508687}{63000}\zeta_2^4-\frac{12321}{2}\zeta_7-4448\,\zeta_5\zeta_2+\frac{2081}{5}\zeta_3\zeta_2^2+2940\,\zeta_3^2+\frac{31403}{45}\zeta_2^3+13323\,\zeta_5+292\,\zeta_3\zeta_2+\frac{972}{5}\zeta_2^2-9275\,\zeta_3+\frac{7029}{4}\zeta_2-\frac{22259}{12}\Big)\\
&+C_F^3C_A\Big(-\frac{692}{5}\zeta_{5,3}+3696\,\zeta_5\zeta_3-\frac{8536}{3}\zeta_3^2\zeta_2+\frac{506012}{1125}\zeta_2^4+\frac{178357}{24}\zeta_7+\frac{83443}{9}\zeta_5\zeta_2-\frac{107401}{90}\zeta_3\zeta_2^2-2880\,\zeta_3^2+\frac{1145267}{3780}\zeta_2^3-\frac{73607}{36}\zeta_5+\frac{320363}{54}\zeta_3\zeta_2-\frac{3878479}{1620}\zeta_2^2-\frac{526531}{36}\zeta_3-\frac{4901615}{648}\zeta_2+\frac{2888701}{216}\Big)\\
&+C_F^2C_A^2\Big(1046\,\zeta_{5,3}-5104\,\zeta_5\zeta_3+\frac{24208}{9}\zeta_3^2\zeta_2-\frac{3829877}{4725}\zeta_2^4-\frac{105405}{16}\zeta_7-\frac{91561}{18}\zeta_5\zeta_2+\frac{64541}{45}\zeta_3\zeta_2^2+\frac{697187}{81}\zeta_3^2+\frac{113683}{1260}\zeta_2^3-\frac{125555}{216}\zeta_5-\frac{12580021}{972}\zeta_3\zeta_2+\frac{52786259}{29160}\zeta_2^2+\frac{29217731}{5832}\zeta_3+\frac{279041783}{26244}\zeta_2-\frac{526960807}{52488}\Big)\\
&+C_FC_A^3\Big(-\frac{14161}{30}\zeta_{5,3}+\frac{21577}{6}\zeta_5\zeta_3-\frac{1963}{3}\zeta_3^2\zeta_2+\frac{10233079}{15750}\zeta_2^4+\frac{258199}{144}\zeta_7+1056\,\zeta_5\zeta_2-\frac{23288}{45}\zeta_3\zeta_2^2-\frac{702221}{108}\zeta_3^2-\frac{2000759}{2268}\zeta_2^3-\frac{9786737}{3240}\zeta_5+\frac{444085}{108}\zeta_3\zeta_2+\frac{184637}{810}\zeta_2^2+\frac{8121343}{1458}\zeta_3-\frac{146447531}{34992}\zeta_2+\frac{3966128773}{419904}\Big)\\
&+\frac{d_F^{abcd}d_A^{abcd}}{N_F}\Big(260\,\zeta_{5,3}-5092\,\zeta_5\zeta_3-16\,\zeta_3^2\zeta_2-\frac{496766}{525}\zeta_2^4-1228\,\zeta_7-\frac{12808}{3}\zeta_5\zeta_2+\frac{14216}{15}\zeta_3\zeta_2^2+\frac{72674}{9}\zeta_3^2+\frac{768632}{945}\zeta_2^3-\frac{65546}{27}\zeta_5+\frac{2516}{3}\zeta_3\zeta_2+\frac{8692}{45}\zeta_2^2+\frac{112346}{27}\zeta_3+\frac{8194}{9}\zeta_2-\frac{1588}{3}\Big)\\
&+n_fC_F^3\Big(\frac{2013}{2}\zeta_7-\frac{1124}{9}\zeta_5\zeta_2-\frac{7567}{45}\zeta_3\zeta_2^2-\frac{3764}{3}\zeta_3^2-\frac{107227}{1890}\zeta_2^3-\frac{70907}{18}\zeta_5-\frac{72811}{81}\zeta_3\zeta_2+\frac{432143}{810}\zeta_2^2+\frac{1934375}{324}\zeta_3+\frac{172627}{972}\zeta_2-\frac{6554087}{3888}\Big)\\
&+n_fC_F^2C_A\Big(-\frac{1219}{4}\zeta_7+114\,\zeta_5\zeta_2+\frac{15934}{45}\zeta_3\zeta_2^2+\frac{9446}{81}\zeta_3^2-\frac{1846}{21}\zeta_2^3+\frac{45995}{18}\zeta_5+\frac{155563}{81}\zeta_3\zeta_2-\frac{3347782}{3645}\zeta_2^2-\frac{1262017}{5832}\zeta_3-\frac{145213765}{104976}\zeta_2+\frac{756958495}{419904}\Big)\\
&+n_fC_FC_A^2\Big(\frac{19141}{72}\zeta_7-\frac{127}{3}\zeta_5\zeta_2-\frac{6904}{45}\zeta_3\zeta_2^2+\frac{5462}{9}\zeta_3^2+\frac{153371}{2268}\zeta_2^3-\frac{2343853}{3240}\zeta_5-\frac{73985}{108}\zeta_3\zeta_2+\frac{120913}{540}\zeta_2^2-\frac{16605365}{5832}\zeta_3+\frac{46423375}{34992}\zeta_2-\frac{2567430839}{839808}\Big)\\
&+n_f\frac{d_F^{abcd}d_F^{abcd}}{N_F}\Big(-1240\,\zeta_7+\frac{992}{3}\zeta_5\zeta_2-\frac{3952}{15}\zeta_3\zeta_2^2-\frac{4504}{9}\zeta_3^2+\frac{215876}{945}\zeta_2^3+\frac{101938}{27}\zeta_5+\frac{572}{3}\zeta_3\zeta_2-\frac{8}{45}\zeta_2^2-\frac{18202}{27}\zeta_3-\frac{18254}{9}\zeta_2+\frac{3488}{3}\Big)\\
&+n_f^2C_F^2\Big(\frac{4556}{81}\zeta_3^2+\frac{3520}{189}\zeta_2^3-\frac{1568}{27}\zeta_5+\frac{3358}{243}\zeta_3\zeta_2+\frac{45551}{810}\zeta_2^2-\frac{612127}{1458}\zeta_3+\frac{74333}{6561}\zeta_2-\frac{11290865}{104976}\Big)\\
&+n_f^2C_FC_A\Big(-\frac{622}{27}\zeta_3^2+\frac{1654}{135}\zeta_2^3+\frac{19094}{135}\zeta_5-\frac{20}{27}\zeta_3\zeta_2-\frac{1957}{27}\zeta_2^2+\frac{408781}{1458}\zeta_3-\frac{4264925}{34992}\zeta_2+\frac{176182813}{839808}\Big)\\
&+n_f^3C_F\Big(-\frac{106}{135}\zeta_5+\frac{4}{9}\zeta_3\zeta_2+\frac{328}{81}\zeta_2^2+\frac{14}{243}\zeta_3+\frac{1946}{729}\zeta_2+\frac{6460}{6561}\Big)
\end{aligned}
$$

[Chakraborty, Huber, Lee, AvM, Schabinger, Steinhauser, Smirnov, Smirnov ’22]

CHECKS AND FINDINGS

- Master integrals
  - many checked analytically
  - with Fiesta 5 *[A.V. Smirnov, Shapurov, Vysotsky '21]* to 10^(-4) relative error otherwise.
- IR subtraction works
  - Non-trivial test of IR prediction and quark collinear anom.dim.
  - Note: renormalization very different for $q\bar{q}\gamma*$, $b\bar{b}H$ due to Yukawa coupling and α_s
- Max. "transcendental weight" of form factors:
  - agree all with N=4 (after adjusting reps.)
  - for all poles and the finite parts
  - for leading and subleading color !

SUMMARY

- First 4-loop form factors in full-color QCD: $q\bar{q}\gamma^*$, ggH, $b\bar{b}H$
- From poles: cusp and collinear anomalous dimensions
- Confirmed prediction for IR structure