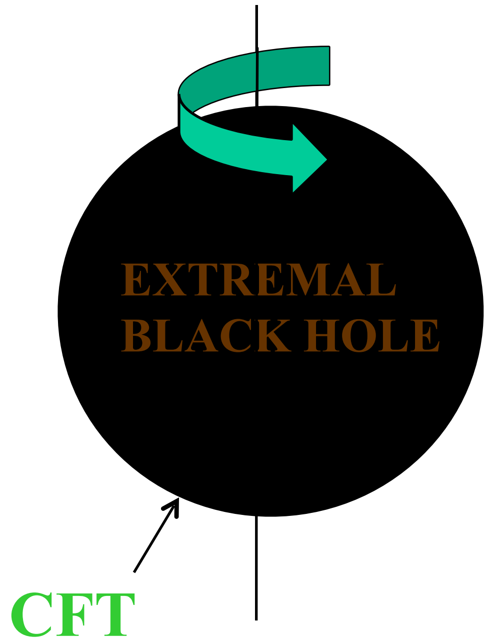
Holography for extended theories of gravity

Masoud Ghezelbash

Department of Physics and Engineering Physics
University of Saskatchewan
Canada

KERR/CFT CORRESPONDENCE
(Black Hole Holography)



The near-horizon states and physical quantities of an extremal rotating BH (4D or higher) could be identified with and obtained from a certain chiral CFT living on the boundary of BH.
Generalizing the idea to non-extremal rotating black holes also is possible by finding a local conformal invariance (known as hidden conformal symmetry) in the solution space of the wave equation for the propagating fields in the background of rotating BH.

KERR-SEN BLACK HOLE

$$
\begin{aligned}
ds^2 &= -(1-\frac{2M\tilde{r}}{\rho^2})dt^2+\rho^2(\frac{d\tilde{r}^2}{\Delta}+d\theta^2) \\
&- \frac{4M\tilde{r}a}{\rho^2}\sin^2\theta d\tilde{t}d\tilde{\phi}+\{\tilde{r}(\tilde{r}+\varrho)+a^2+\frac{2M\tilde{r}a^2\sin^2\theta}{\rho^2}\}\sin^2\theta d\tilde{\phi}^2
\end{aligned}
$$

$$
\begin{aligned}
\Phi &= -\ln\frac{r^2+a^2\cos^2(\theta)+2mr\sinh^2(\alpha/2)}{r^2+a^2\cos^2(\theta)} \\
A_t &= \frac{2mr\sinh(\alpha)}{r^2+a^2\cos^2(\theta)+2mr\sinh^2(\alpha/2)} \\
A_\phi &= \frac{2mra\sinh(\alpha)\sin^2(\theta)}{r^2+a^2\cos^2(\theta)+2mr\sinh^2(\alpha/2)}
\end{aligned}
$$

$$B_{t\phi} = \frac{2mra\sinh^2(\alpha/2)\sin^2(\theta)}{r^2+a^2\cos^2(\theta)+2mr\sinh^2(\alpha/2)}$$

Mass of BH

$$M = m\cosh^2(\alpha/2)$$

Charge of BH

$$Q = \frac{m}{\sqrt{2}}\sinh\alpha$$

Angular Velocity of Horizon

$$\Omega_H = \frac{a}{m(m+\sqrt{m^2-a^2})(1+\cosh(\alpha))}$$

$$T_H = \frac{\sqrt{m^2-a^2}}{2\pi m(m+\sqrt{m^2-a^2})(1+\cosh(\alpha))}$$

Hawking Temperature

Ang. Mom. of BH

$$J = ma\cosh^2(\alpha/2)$$

Horizon

$$r_H = M-\frac{\varrho}{2}+\frac{1}{2}\sqrt{(2M-\varrho)^2-4a^2}$$

$$\varrho = 2m\sinh^2(\alpha/2) = Q^2/M$$

To avoid any naked singularities $\longrightarrow$ $\mid J \mid \leq M^2 - \frac{1}{2}Q^2$

$$T_H = \frac{\sqrt{(2M^2 - Q^2)^2 - 4J^2}}{4\pi M(2M^2 - Q^2 + \sqrt{(2M^2 - Q^2)^2 - 4J^2})} \qquad \Omega_H = \frac{J}{M(2M^2 - Q^2 + \sqrt{(2M^2 - Q^2)^2 - 4J^2})}$$

Entropy of BH $\rightarrow$ $S = 2\pi M(M - \frac{Q^2}{2M} + \sqrt{(M - \frac{Q^2}{2M})^2 - \frac{J^2}{M^2}})$

Extremal Black Hole $J = M^2 - \frac{1}{2}Q^2$ $\Rightarrow$ $S = 2\pi J$

Could we obtain any of these results (especially entropy) from quantum theory of gravity? YES

Kerr, Kerr-Newman, Kerr-Bolt, Kerr-Bolt-(A)dS, Kerr-Sen, five and higher dimensional rotating black holes such as BMPV black hole in 5D N = 2 supergravity, ….

Near Horizon Geometry of Extremal BH with Horizon at $M - \frac{\varrho}{2}$

Scaling parameter

$$
\begin{aligned}
\tilde{r} &= (M - \frac{\varrho}{2})(1 + \frac{\lambda}{y}) \\
\tilde{t} &= \frac{2M}{\lambda} t \\
\tilde{\phi} &= \phi + t/\lambda
\end{aligned}
$$

$$
\begin{aligned}
ds^2 &= \frac{(2M - \varrho)\{\frac{1}{2}\varrho \sin^2\theta + M(1+\cos^2\theta)\}^2}{2M(1+\cos^2\theta) + \varrho \sin^2\theta}(\frac{-dt^2 + dy^2}{y^2}) \\
&+ \{M^2(1+\cos^2\theta) + \frac{1}{4}(-\varrho^2 \sin^2\theta - 4\varrho M \cos^2\theta)\} d\theta^2 \\
&+ \frac{4(2M-\varrho)M^2 \sin^2\theta}{2M(1+\cos^2\theta) + \varrho \sin^2\theta}(d\phi + \frac{dt}{y})^2
\end{aligned}
$$

or

AdS$_2$

$$
\begin{aligned}
ds^2 &= \{M^2(1+\cos^2\theta) + \frac{1}{4}(-\varrho^2 \sin^2\theta - 4\varrho M \cos^2\theta)\}\{\frac{-dt^2 + dy^2}{y^2} + d\theta^2 + \\
&+ \frac{4M^2 \sin^2\theta}{(\frac{1}{2}\varrho \sin^2\theta + M(1+\cos^2\theta))^2}(d\phi + \frac{dt}{y})^2\}
\end{aligned}
$$

Near-horizon Dilaton (in local coordinates)

$$\Phi = \ln \frac{(2M^2 - Q^2)(1+\cos^2\theta)}{Q^2 \sin^2\theta + 2M^2(1+\cos^2\theta)}$$

Near Horizon Gauge Field

$$A = -\frac{2\sqrt{2}Q(2M^2 - Q^2)\sin^2\theta}{(Q^2\sin^2\theta + 2M^2(1+\cos^2\theta))}\left(d\phi + \frac{dt}{y}\right)$$

Near Horizon 3-Form Field Strength

$$H = \{\mathcal{H}\frac{dy}{y^2} - \frac{1}{y}\mathcal{H}' d\theta\} \wedge dt \wedge d\phi$$

$$\mathcal{H} = \frac{2(2M^2 - Q^2)^2 Q^2 \sin^4\theta}{\{Q^2\sin^2\theta + 2M^2(1+\cos^2\theta)\}^2}$$

$$\Longrightarrow$$

$$H = d\mathcal{B}.$$

$$\mathcal{B} = -\frac{\mathcal{H}(\theta)}{y} dt \wedge d\phi$$

Global Coordinates

$$
\begin{aligned}
y &= \frac{1}{\cos\tau\sqrt{1+r^2}+r} \\
t &= y\sin\tau\sqrt{1+r^2} \\
\phi &= \varphi + \ln\left(\frac{\cos\tau + r\sin\tau}{1+\sin\tau\sqrt{1+r^2}}\right)
\end{aligned}
$$

The Global Near Horizon Metric

$$
\begin{aligned}
ds^2 &= \{M^2(1+\cos^2\theta) + \frac{1}{4}(-\varrho^2\sin^2\theta - 4\varrho M\cos^2\theta)\}\{-(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + d\theta^2 + \\
&+ \frac{4M^2\sin^2\theta}{(\frac{1}{2}\varrho\sin^2\theta + M(1+\cos^2\theta))^2}(d\varphi + r d\tau)^2\}
\end{aligned}
$$

AdS_2

a S^1 bundle over AdS_2

This geometry has a SL(2,R) isometry as well as a rotational U(1) isometry generated by the Killing vector ∂_φ

The near horizon geometry of rotating extremal black holes consists of a copy of AdS

Example: near horizon geometry of Kerr-Sen black hole

$$\begin{aligned} ds^2 &= \{M^2(1+\cos^2\theta) + \frac{1}{4}(-\varrho^2\sin^2\theta - 4\varrho M\cos^2\theta)\}\{\frac{-dt^2+dy^2}{y^2} + d\theta^2 + \\ &+ \frac{4M^2\sin^2\theta}{(\frac{1}{2}\varrho\sin^2\theta + M(1+\cos^2\theta))^2}(d\phi + \frac{dt}{y})^2\} \end{aligned}$$

AdS_2

a S^1 bundle over AdS_2

This geometry has a SL(2,R) isometry as well as a rotational U(1) isometry generated by the Killing vector ∂_φ

The u(1) rotational isometry can be enhanced to a Virasoro algebra with a non-trivial central charge!

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{J}{\hbar}m(m^2-1)\delta_{m+n,0} \longrightarrow c = \frac{12J}{\hbar}$$

The Cardy formula gives the entropy of the two dimensional CFT

$$S = 2\pi\sqrt{\frac{cL}{6}}$$

Energy

Central charge

First Law of Thermodynamics

$$dL = TdS \rightarrow dS = \pi\sqrt{\frac{c}{6L}}TdS \rightarrow \sqrt{L} = \pi\sqrt{\frac{c}{6}}T \Rightarrow S = \frac{\pi^2}{3}cT$$

Frolov-Thorne temperature of the near horizon region ~Temp. of left moving CFT

$$T_{F.T.} \equiv T_L = \frac{1}{2\pi}$$

$$\Rightarrow S_{microscopic} = 2\pi J \quad \checkmark$$

This is exactly equal to the macroscopic Bekenstein-Hawking entropy

$$S_{BH} = 2\pi J$$

Beside the perfect match of the macroscopic Bekenstein-Hawking entropy of Black hole with the Cardy entropy for CFT, the other supports for the correspondence is:

- **Super-radiant scattering off the black hole: The bulk scattering amplitudes are in precise agreement with the CFT results**

- **Real-time correlators of various perturbations in even near-extremal black hole could be computed directly from the bulk**

Extremal Kerr/CFT Conclusions:

The near-horizon states of an extremal black hole could be identified with a certain chiral CFT.
The corresponding Virasoro algebra is generated with a class of diffeomorphisms that preserves an appropriate boundary condition on the near-horizon geometry.
The black hole near-horizon geometry consists of a certain AdS structure; the central charges of dual CFT can be obtained by analyzing the asymptotic symmetry method

How about generic non-extremal Black Holes?

If Kerr/CFT correspondence is correct, then energy excitations of CFT should correspond to generic non-extremal black hole.

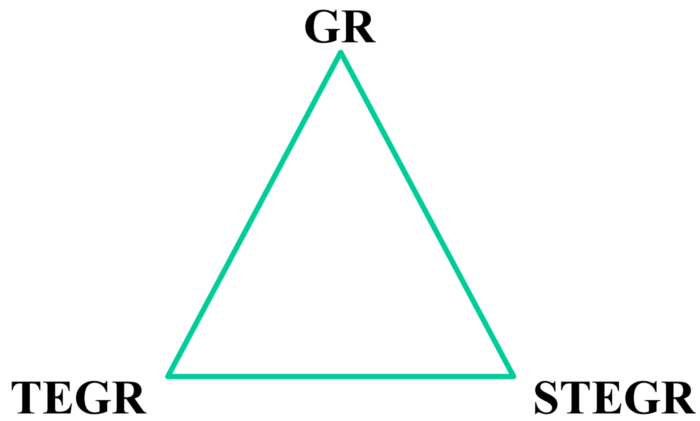
Problem:
Away from the extremality, there is no AdS structure for the near horizon geometry.
In fact the near horizon geometry is Rindler space with no known associated CFT.

Solution: Existence of conformal invariance in a near-horizon geometry is not a necessary condition for the interactions to exhibit conformal invariance.

Instead the existence of a local conformal invariance (known as hidden conformal symmetry)
in the solution space of the wave equation for the propagating field is sufficient to ensure a dual CFT description.

This hidden conformal symmetry is a sufficient condition the scattering amplitudes exhibit conformal invariance though the space on which the field propagates doesn't have the conformal symmetry

Gravitational Trinity



$$\mathcal{S}_{GR} \equiv \frac{\kappa}{2} \int d^4x \, \sqrt{-g} \, R \qquad \mathcal{S}_{TEGR} \equiv \frac{\kappa}{2} \int d^4x \, \sqrt{-g} \, T \qquad \mathcal{S}_{STEGR} \equiv \frac{\kappa}{2} \int d^4x \, \sqrt{-g} \, Q$$

No Trinity

$$f(R)$$

$$f(T) \qquad f(Q)$$

$f(R)$ **Extension of GR-Dark energy and matter addressed as curvature effects on Astrophysical and cosmological scales - Explain well the acceleration of the universe - Explain galaxy rotation curves without dark matter/energy**

$f(T)$ **Extension of TEGR (torsion as a result of Weitzenbock connection, instead of Levi-Civita connection)**

$f(Q)$ **Extension of STEGR (non metricity which implies the covariant derivative of the metric does not vanish)**

f(T) Gravity

$$\mathcal{S} = \frac{1}{2\mathfrak{K}} \int d^4x \, |e| \left(f\left(T\right) - 2\Lambda - F \wedge {}^* F \right)$$

$$W^{\alpha}{}_{\mu\nu} = e_a{}^{\alpha} \partial_\nu e^a{}_\mu = -e^a{}_\mu \partial_\nu e_a{}^\alpha$$

Weitzenbock connection, which is curvature free, but has a non zero torsion

$$T^{\alpha}{}_{\mu\nu} = W^{\alpha}{}_{\nu\mu} - W^{\alpha}{}_{\mu\nu} = e_i{}^{\alpha} \left(\partial_\mu e^i{}_\nu - \partial_\nu e^i{}_\mu \right)$$

Scalar torsion

$$T = T^{\alpha}{}_{\mu\nu} S_{\alpha}{}^{\mu\nu}$$

$$S_{\alpha}{}^{\mu\nu} = \frac{1}{2} \left(K^{\mu\nu}{}_{\alpha} + \delta^{\mu}_{\alpha} T^{\beta\nu}{}_{\beta} - \delta^{\nu}_{\alpha} T^{\beta\mu}{}_{\beta} \right)$$

Contortion tensor

$$K_{\alpha\mu\nu} = \frac{1}{2} \left(T_{\nu\alpha\mu} + T_{\alpha\mu\nu} - T_{\mu\alpha\nu} \right)$$

Rotating Charged AdS Black Holes in $f(T) = T + \alpha T^2$ **Gravity**

Gravity field equations

$$S_\mu{}^{\rho\nu}\partial_\rho T f''(T) + \left[e^{-1}e^a{}_\mu \partial_\rho\left(ee_\alpha{}^\alpha S_\alpha{}^{\rho\nu}\right) - T^\alpha{}_{\lambda\mu}S_\alpha{}^{\nu\lambda}\right] f'(T) - \frac{\delta_\mu^\nu}{4}\left(f(T) + \frac{6}{l^2}\right)$$

$$= -\frac{\mathfrak{K}}{2}\mathcal{T}_{\mathrm{em}}{}^\nu{}_\mu$$

Maxwell's field equations $\quad \partial_\nu\left(\sqrt{-g}F^{\mu\nu}\right) = 0$

Black hole solution

$$ds^2 = -A(r)(\Xi dt - \Omega d\phi)^2 + \frac{dr^2}{B(r)} + \frac{r^2}{l^4}\left(\Omega dt - \Xi l^2 d\phi\right)^2 + \frac{r^2}{l^2}dz^2$$

$$ds^2 = -A(r)(\Xi dt - \Omega d\phi)^2 + \frac{dr^2}{B(r)} + \frac{r^2}{l^4}\left(\Omega dt - \Xi l^2 d\phi\right)^2 + \frac{r^2}{l^2}dz^2$$

Charge parameter

$$A(r) = r^2\Lambda_{eff} - \frac{M}{r} + \frac{3Q^2}{2r^2} + \frac{2Q^3\sqrt{6\,|\alpha|}}{6r^4}$$

The black hole may have 6 horizons!

$$B(r) = A(r)\beta(r)$$

Mass parameter

$$\beta(r) = \left(1 + Q\sqrt{6\,|\alpha|}/r^2\right)^{-2}$$

Rotation parameter

$$\Xi = \sqrt{1 + \frac{\Omega^2}{l^2}}$$

Cosmological constant parameter

$$\Lambda_{eff} = \frac{1}{36|\alpha|}$$

Gauge potential $$\tilde{\Phi}(r) = -\Phi(r)\left(\Omega d\phi - \Xi dt\right)$$

$$\Phi(r) \;=\; \frac{Q}{r} + \frac{Q^2\sqrt{6|\alpha|}}{3r^3}$$

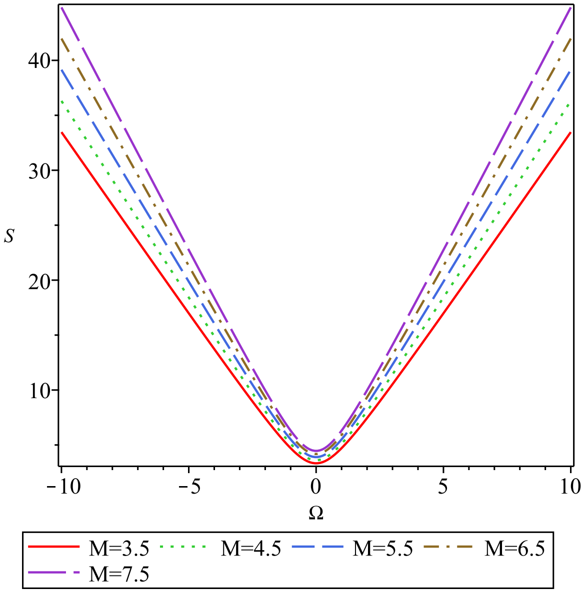
Black hole torsion $$T(r) = \frac{4A'(r)B(r)}{rA(r)} + \frac{2B(r)}{r^2}$$

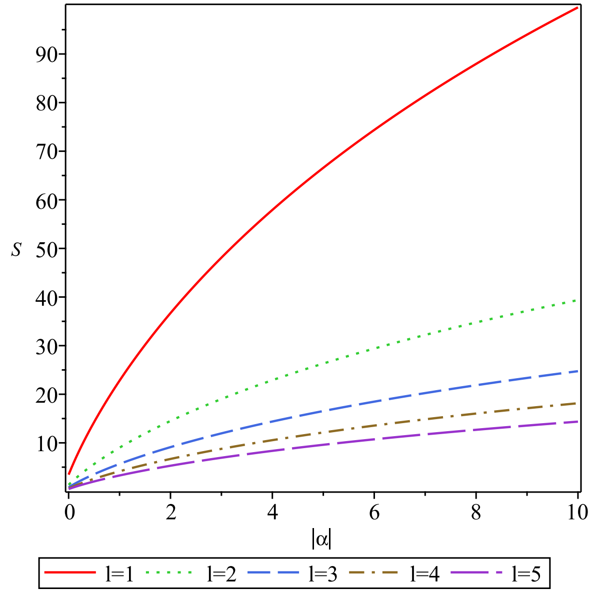
Entropy of black hole

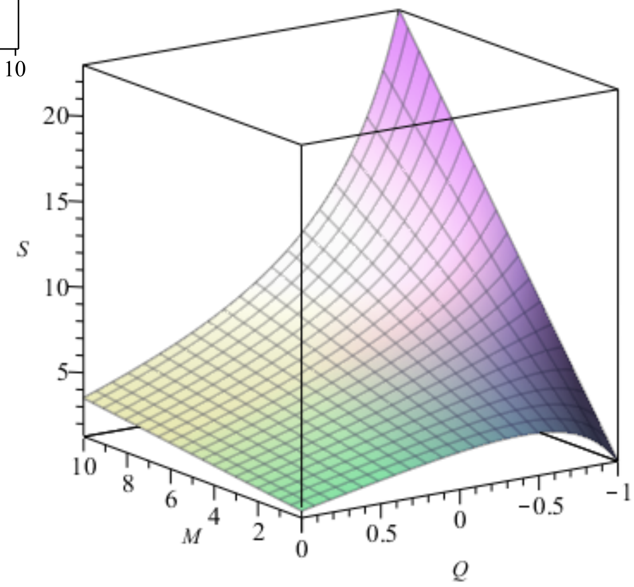
$$S = \frac{f'\left(T\right)\mathscr{A}}{4}$$

Outer event horizon area $$\mathscr{A} = \int_0^{2\pi} d\phi \int_0^{L} dz \sqrt{-g|_{dt=dr=0}} = \frac{2\pi r_+^2 \Xi L}{l}$$

$$S = \frac{\pi \Xi L \left(7{r_+}^6 + 9\sqrt{6\,|\alpha|}Q{r_+}^4 + 18M\,|\alpha|\,{r_+}^3 - 54Q^2\,|\alpha|\,{r_+}^2 - 42\sqrt{6|\alpha|^3}Q^3\right)}{9l\left(Q\sqrt{6\,|\alpha|} + {r_+}^2\right)^2}$$







Holography for the Rotating Charged AdS Black Holes in $f\left(T\right) = T + \alpha T^2$ Gravity

Consider a massless scalar probe in the background of black hole

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right) = 0$$

Thanks to three Killing vectors $\psi\left(t, r, z, \phi\right) = e^{-i\omega t + ikz + im\phi}R\left(r\right)$

$$B(r)\frac{d^2R\left(r\right)}{dr^2} + \left(rB(r)\frac{dA(r)}{dr} + rA(r)\frac{dB(r)}{dr} + 4A(r)B(r)\right)\frac{dR(r)}{dr} + V(r)R\left(r\right) = 0$$

$$V(r) = \frac{r^2\left(\Xi l^2\omega - \Omega m\right)^2 - A(r)l^2\left\{k^2l^4\Xi^4 + k^2\Omega^4 + l^2\left[m^2\Xi^2 - 2m\Xi\Omega\omega + \Omega^2\left(\omega^2 - 2\Xi^2k^2\right)\right]\right\}}{A(r)r^2\left(\Xi^2l^2 - \Omega^2\right)^2}$$

In near horizon region: $A(r) \simeq K\left(r-r_{+}\right)\left(r-r_{*}\right)$

$$K=15{r_{+}}^{4}\Lambda_{eff}-3Mr_{+}+\frac{3Q^{2}}{2} \qquad r_{*}=r_{+}-\frac{2r_{+}\left(2{r_{+}}^{4}\Lambda_{eff}-Mr_{+}+Q^{2}\right)}{10{r_{+}}^{4}\Lambda_{eff}-2Mr_{+}+Q^{2}}$$

Considering low energy scalar probe and closeness of r_{+} **to** r_{*}

$$\frac{d}{dr}\left\{\left(r-r_{+}\right)\left(r-r_{*}\right)\frac{d}{dr}R\left(r\right)\right\}+\left[\left(\frac{r_{+}-r_{*}}{r-r_{+}}\right)\mathcal{A}+\left(\frac{r_{+}-r_{*}}{r-r_{*}}\right)\mathcal{B}+\mathcal{C}\right]R\left(r\right)=0$$

$$\mathcal{A}=\frac{\mathcal{D}m^{2}+\mathcal{E}m\omega}{K^{2}{r_{+}}^{2}{r_{*}}^{3}(\Xi^{2}l^{2}-\Omega^{2})^{2}\left(r_{+}-r_{*}\right)^{2}\beta}+\frac{\mathcal{F}\omega^{2}}{K{r_{+}}^{2}(\Xi^{2}l^{2}-\Omega^{2})^{2}\left(r_{+}-r_{*}\right)^{2}\beta}-C_{1}$$

$$\mathcal{B}=\frac{\mathcal{G}m^{2}+\mathcal{I}m\omega}{K^{2}{r_{+}}^{3}r_{*}(\Xi^{2}l^{2}-\Omega^{2})^{2}\left(r_{+}-r_{*}\right)^{2}\beta}+\frac{\mathcal{J}\omega^{2}}{K{r_{+}}^{2}(\Xi^{2}l^{2}-\Omega^{2})^{2}\left(r_{+}-r_{*}\right)^{2}\beta}+C_{2}$$

Generators of CFT

$$H_1 = i\partial_+ \qquad \bar{H}_1 = i\partial_-$$

$$H_0 = i\left(\omega^+\partial_+ + \frac{1}{2}y\partial_y\right) \qquad \bar{H}_0 = i\left(\omega^-\partial_- + \frac{1}{2}y\partial_y\right)$$

$$H_{-1} = i(\omega^{+2}\partial_+ + \omega^+ y\partial_y - y^2\partial_-) \qquad \bar{H}_{-1} = i(\omega^{-2}\partial_- + \omega^- y\partial_y - y^2\partial_+)$$

$$SL(2,R)_L \ \times \ SL(2,R)_R$$

$$[H_0, H_{\pm 1}] = \mp i H_{\pm 1} \qquad [H_{-1}, H_1] = -2iH_0$$

$$[\bar{H}_0, \bar{H}_{\pm 1}] = \mp i \bar{H}_{\pm 1} \qquad [\bar{H}_{-1}, \bar{H}_1] = -2i\bar{H}_0$$

The Casimir operators of the $SL(2,R)_L \ \times \ SL(2,R)_R$

$$\mathcal{H}^2 = \bar{\mathcal{H}}^2 = -H_0^2 + \frac{1}{2}(H_1 H_{-1} + H_{-1}H_1) = \frac{1}{4}(y^2\partial_y^2 - y\partial_y) + y^2\partial_+\partial_-$$

Choosing the conformal coordinates

$$\omega^+ = \sqrt{\frac{r-r_+}{r-r_*}}e^{2\pi T_R\phi+2n_R t}$$

$$\omega^- = \sqrt{\frac{r-r_+}{r-r_*}}e^{2\pi T_L\phi+2n_L t}$$

$$y = \sqrt{\frac{r_+-r_*}{r-r_*}}e^{\pi(T_L+T_R)\phi+(n_L+n_R)t}$$

We find

$$\mathcal{H}^2 = (r-r_+)(r-r_*)\frac{\partial^2}{\partial r^2} + (2r-r_+-r_*)\frac{\partial}{\partial r} + \left(\frac{r_+-r_*}{r-r_*}\right)\left[\left(\frac{n_L-n_R}{4\pi G}\partial_\phi - \frac{T_L-T_R}{4G}\partial_t\right)^2 + C_2\right]$$
$$- \left(\frac{r_+-r_*}{r-r_+}\right)\left[\left(\frac{n_L+n_R}{4\pi G}\partial_\phi - \frac{T_L+T_R}{4G}\partial_t\right)^2 - C_1\right]$$

Temperatures of CFT

$$T_R = \frac{r_+ K\left(r_+ - r_*\right)\left(\Xi^2 l^2 - \Omega^2\right)\sqrt{\beta r_+ r_* \delta}}{4\pi\delta}$$

$$T_L = \frac{r_+ K\left(\Xi^2 l^2 - \Omega^2\right)\left[{r_+}^4 + 2{r_+}^3 r_* + 6{r_+}^2 {r_*}^2 - 2{r_*}^3 r_+\left(K l^2 - 1\right) + {r_*}^4\right]\sqrt{\beta r_+ r_* \delta}}{4\pi(r_+ + r_*)^3\delta}$$

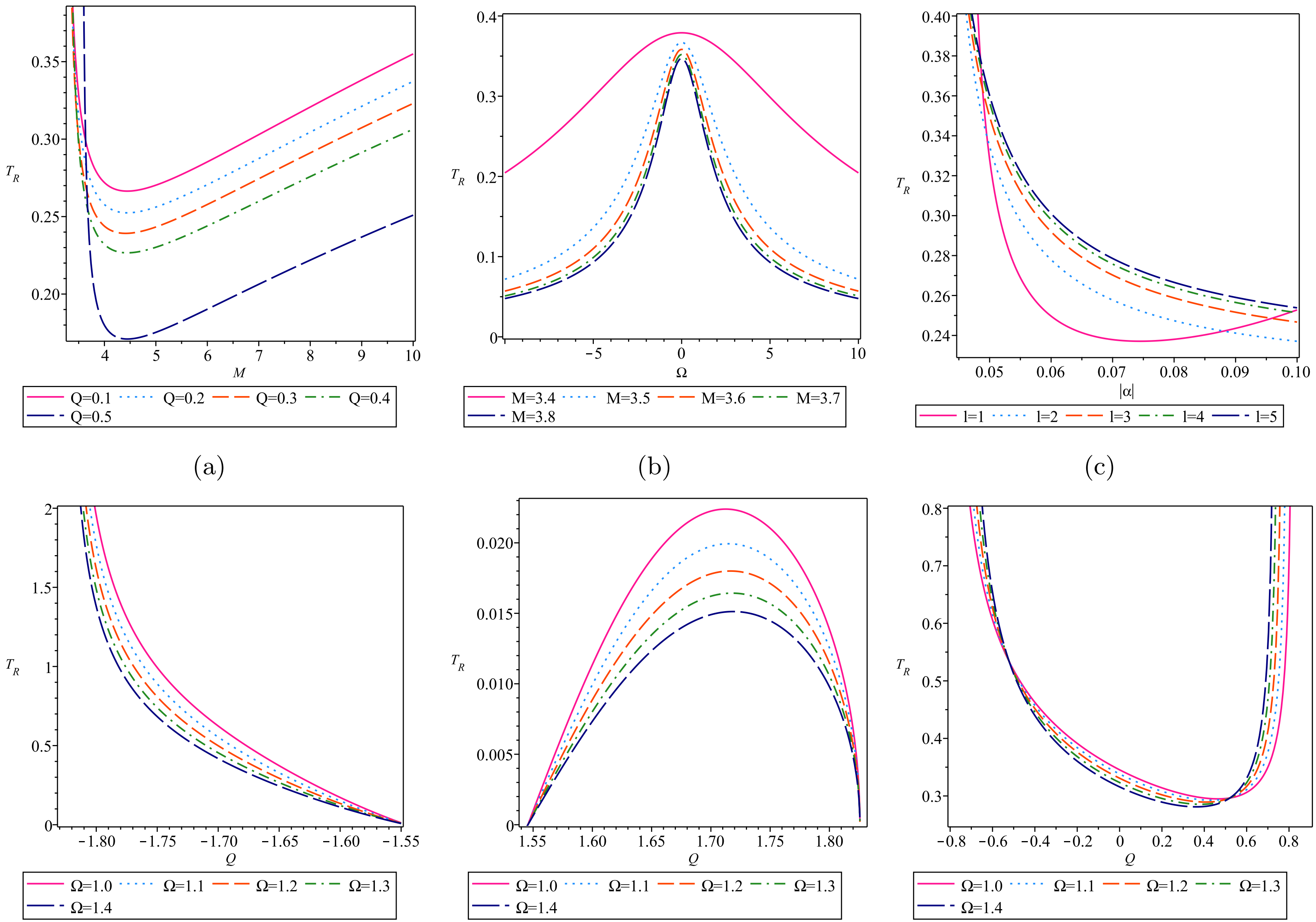
Mode numbers of CFT

$$n_R = 0$$

$$n_L = \frac{{r_*}^2 r_+ K\left(\Omega^2 - \Xi^2 l^2\right)\sqrt{\beta r_+ r_* \delta}}{2\Omega l^2 \Xi (r_+ + r_*)^3}$$

Cardy entropy for the CFT

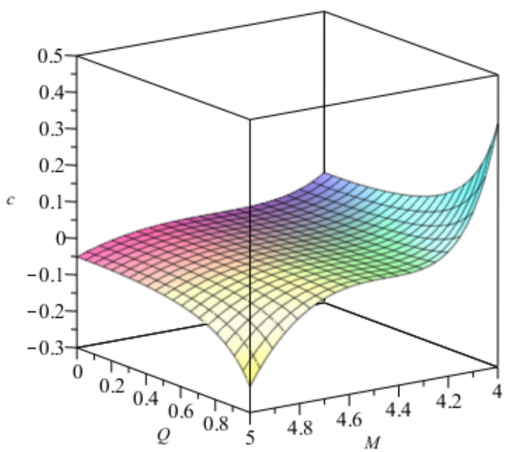
$$S_{CFT} = \frac{\pi^2}{3}\left(c_L T_L + c_R T_R\right)$$



T_R
M
Q=0.1 Q=0.2 Q=0.3 Q=0.4
Q=0.5
(a)
T_R
Ω
M=3.4 M=3.5 M=3.6 M=3.7
M=3.8
(b)
T_R
|α|
l=1 l=2 l=3 l=4 l=5
(c)
T_R
Q
Ω=1.0 Ω=1.1 Ω=1.2 Ω=1.3
Ω=1.4
T_R
Q
Ω=1.0 Ω=1.1 Ω=1.2 Ω=1.3
Ω=1.4
T_R
Q
Ω=1.0 Ω=1.1 Ω=1.2 Ω=1.3
Ω=1.4

Choosing the central charges for the CFT

$$c \equiv c_L = c_R = \frac{12\Xi\delta L\varpi(r_+ + r_*)^3}{lKr_+{}^2\left(\Xi^2l^2 - \Omega^2\right)\left(r_+{}^3 + 2r_+{}^2r_* + 3r_+r_*{}^2 - l^2Kr_*{}^3\right)\left(Q\sqrt{6\,|\alpha|} + r_+{}^2\right)^2\sqrt{\beta r_+ r_* \delta}}$$



We find the Cards entropy is exactly the same as the entropy of black hole

$$S = \frac{\pi\Xi L\left(7r_+{}^6 + 9\sqrt{6\,|\alpha|}Qr_+{}^4 + 18M\,|\alpha|\,r_+{}^3 - 54Q^2\,|\alpha|\,r_+{}^2 - 42\sqrt{6|\alpha|^3}Q^3\right)}{9l\left(Q\sqrt{6\,|\alpha|} + r_+{}^2\right)^2}$$

Thank you for your attention!

Questions ?