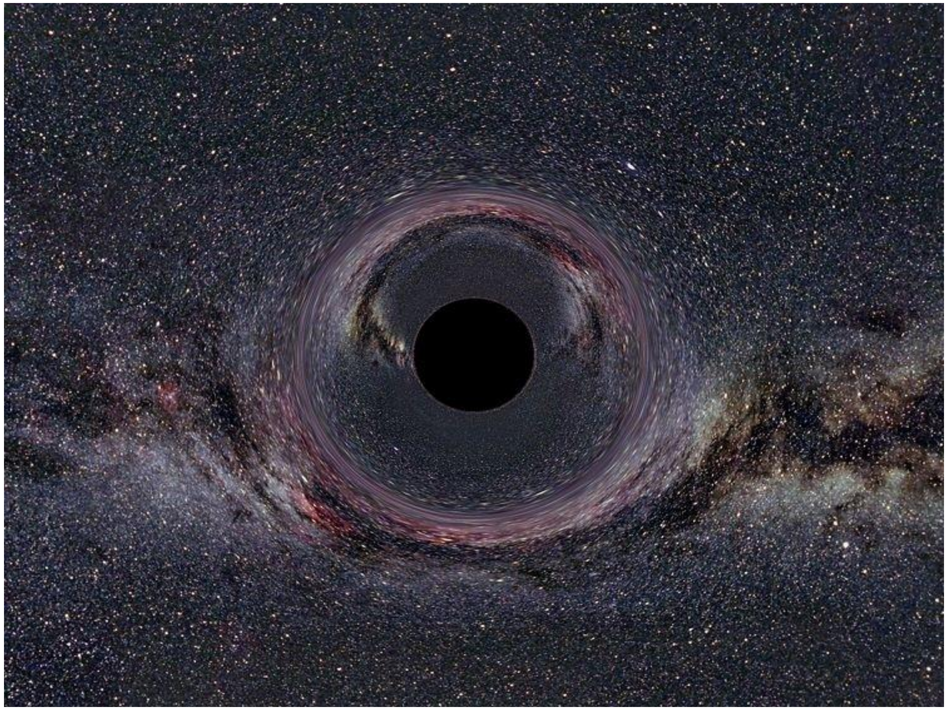
Lecture 2: Rotating black holes

David Kubizňák
(Charles University/Perimeter Institute)



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Rotating black holes

(plan for lecture 2)

a) Spherical black holes

b) Adding slow rotation: Lense-Thirring spacetimes

c) Kerr geometry and its miraculous properties

d) Black holes in higher dimensions

I) Spherical black holes

Schwarzschild solution (1916)

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 \underbrace{(\sin^2\theta d\phi^2 + d\theta^2)}_{d\Omega^2},$$

$$f = 1 - \frac{2M}{r}$$

Birkhoff's theorem: This is the *most* general *spherically symmetric* solution of vacuum Einstein equations.

Proof: start with a general spherical ansatz:

$$ds^2 = -e^{2\psi} f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2, \quad f = 1 - \frac{2m(r,t)}{r}, \quad \psi = \psi(r,t)$$

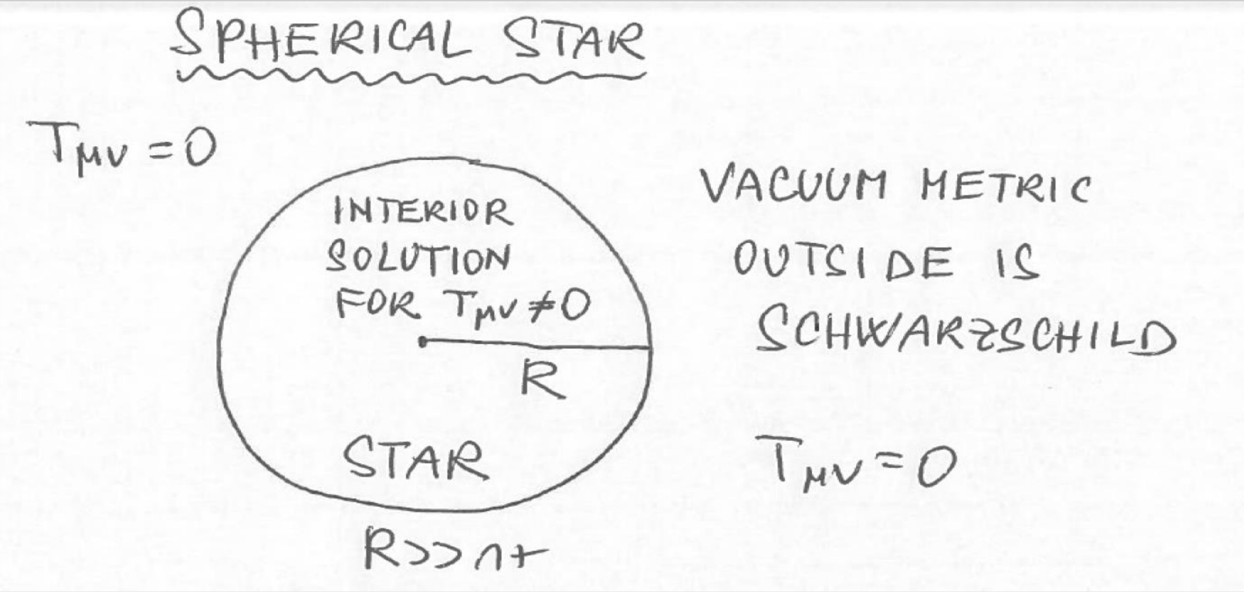
- Einstein equations

$$\frac{\partial m}{\partial r} = 4\pi r^2(-T_t^t), \quad \frac{\partial m}{\partial t} = -4\pi r^2(-T_r^t), \quad \frac{\partial \psi}{\partial r} = \frac{4\pi r}{f}(-T_t^t + T_r^r)$$

- By redefining time, we eliminate $\psi = \psi(t)$

Schwarzschild solution (1916)

- **Metric outside** of spherical objects



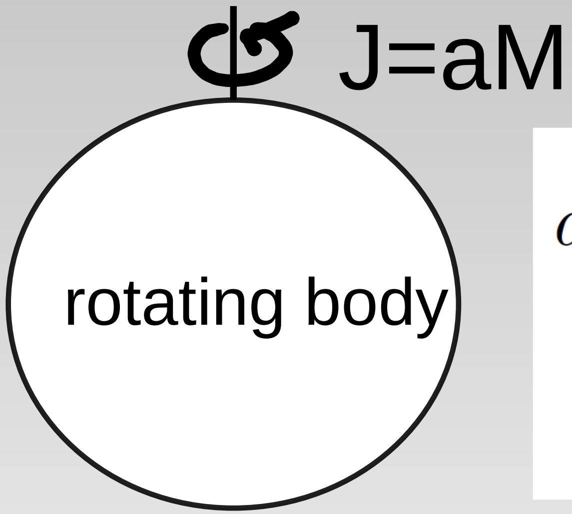
- **Black hole solution** (two singularities)
  - Horizon $r = r_+ = 2M$ (coordinate singularity)
  - Curvature singularity $r = 0$

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6}$$

(resolved in QG?)

II) Adding slow rotation: Lense-Thirring spacetimes

Lense-Thirring spacetime (1918)



$$ds^2 = -f dt^2 + \frac{dr^2}{f} + 2a(f-1)\sin^2\theta dt d\phi$$
$$+ r^2(\sin^2\theta d\phi^2 + d\theta^2)\,, \quad f = 1 - \frac{2M}{r}$$

- Spacetime outside a **slowly rotating body**
- **Approximate** (linear in a) vacuum solution of EE
- Linear in a approximation to **Kerr** (1963)
- After proper “modification” admits **hidden symmetries** (see Finn’s talk)
- Encodes **gravitomagnetic effects**

Frame dragging (gravitomagnetism)

= general-relativistic effect due to the **motion** (in particular rotation) **of matter** and gravitational waves, analogous in a way to electromagnetic induction.

- Already in the **weak field approximation**

$$T_{\mu\nu} = \begin{pmatrix} \rho & \vec{j} \\ \vec{j} & 0 \end{pmatrix}, \quad j_\mu = T_{0\mu} = (\rho, \vec{j})$$

$$g_{00} = -1 - 2\phi\,, \quad g_{0i} = -4A_i\,, \quad g_{ij} = (1 - 2\phi)\delta_{ij}$$

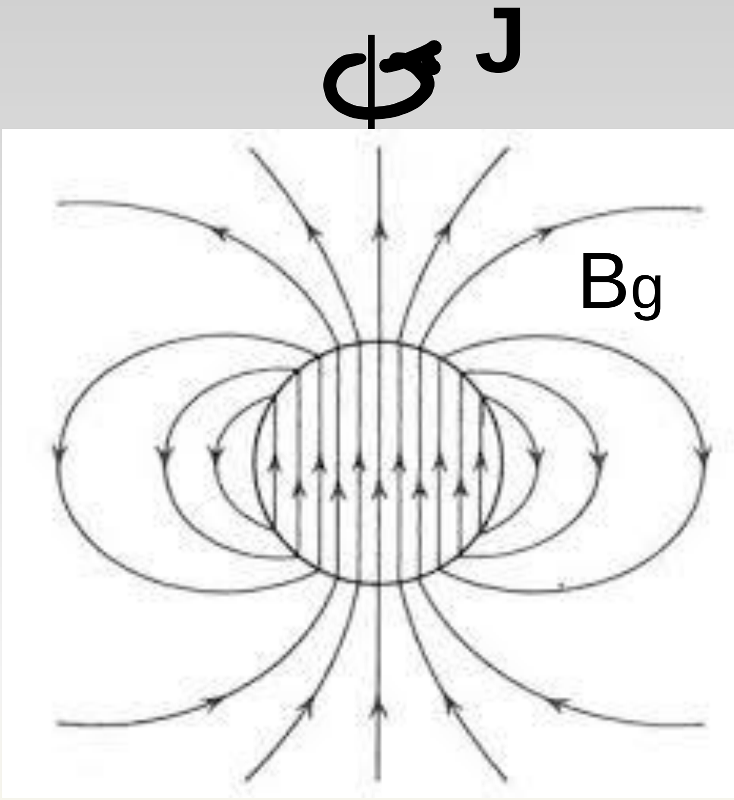
- Einstein & geodesic equations then yield

$$\Box A^\mu = -4\pi j^\mu\,, \quad \partial_\mu A^\mu = 0$$

$$\frac{d^2x^i}{dt^2} = E^i - 4F^i{}_j v^j$$

Frame dragging (gravitomagnetism)

- **Lens-Thirring (1918)**



- “radially infalling geodesic” experiences “**Coriolis type force**”

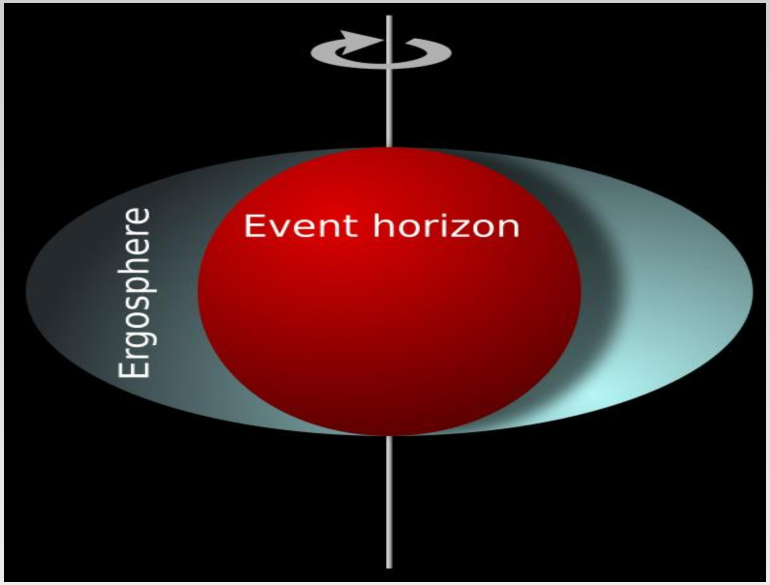
$$r\frac{d^2\varphi}{dt^2} = -\underbrace{\frac{2J}{r^3}}_{2\omega(r)}\frac{dr}{dt}$$

(Exercise)

- **Gyroscope precession** (“Larmor precession” due to gravitomagnetic field)

Frame dragging (gravitomagnetism)

- **Existence of a BH ergosphere**



Example of extremal frame dragging: particle inside the ergosphere has to corotate with the black hole

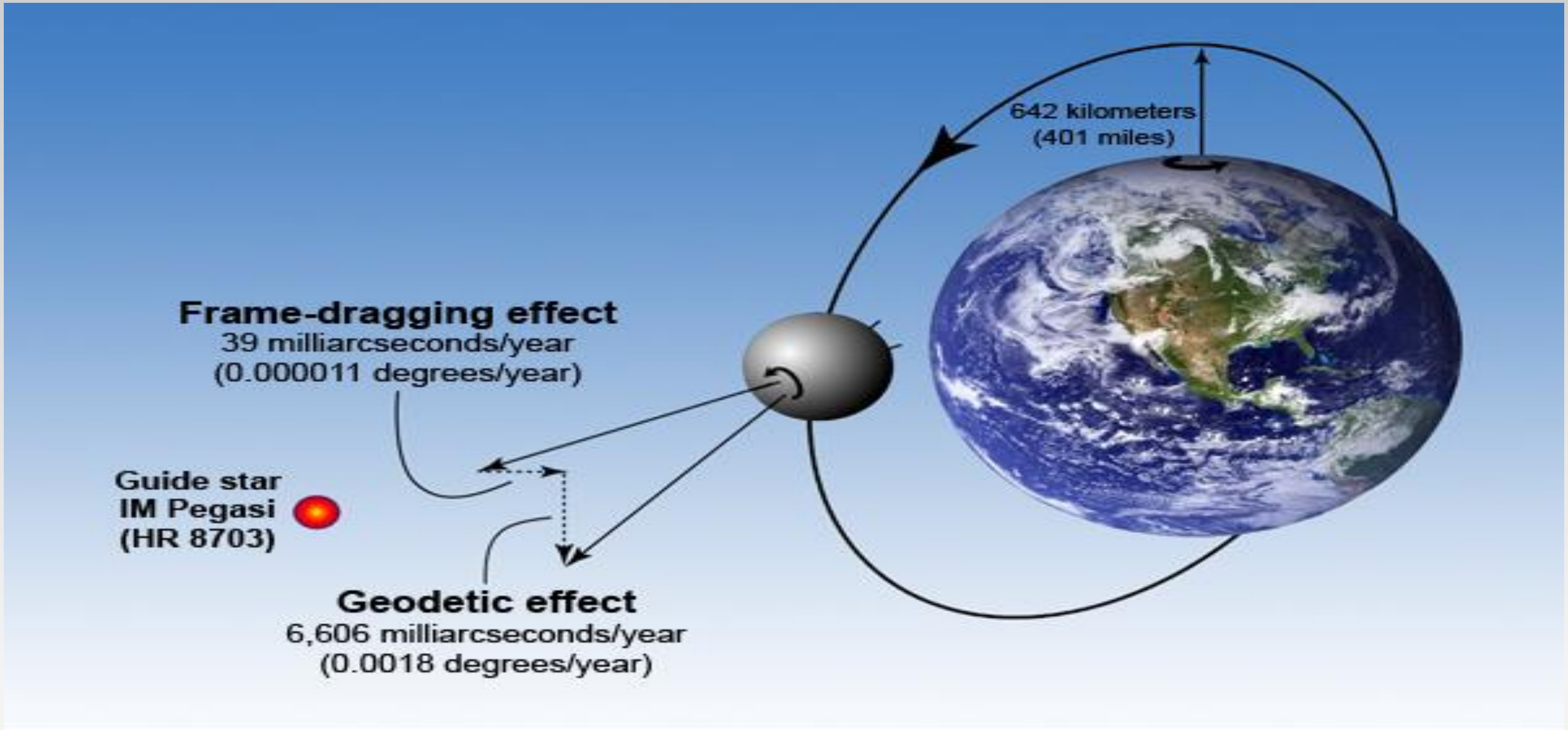
- **Astrophysical applications**

  - **Bardeen-Petterson effect** – aligning of the accretion disc along the black hole spin axis

  - **Precession** of orbits of stars near a supermassive black hole (back reaction on BH spin)

The Gravity Probe B Experiment

Everitt; et al. *"Gravity Probe B: Final Results of a Space Experiment to Test General Relativity".* Phys. Rev. Lett. **106** (22): 221101 (2011)



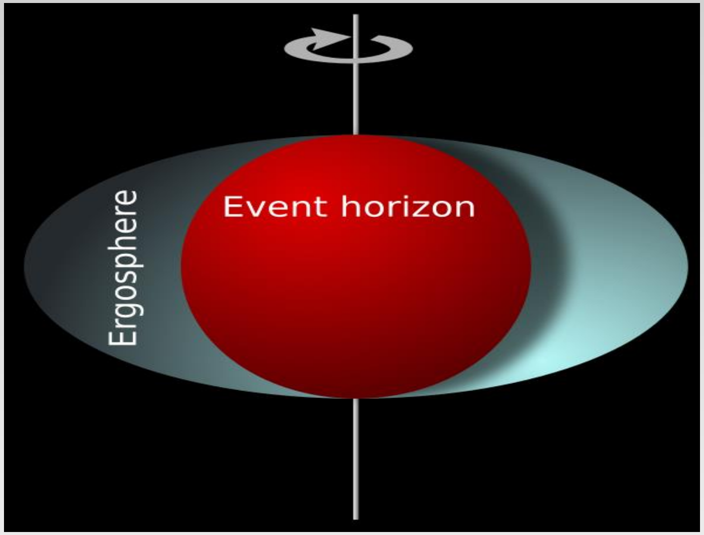
	Measured	Predicted
Geodetic precession (mas)	6602 ± 18	6606
Frame-dragging (mas)	37.2 ± 7.2	39.2

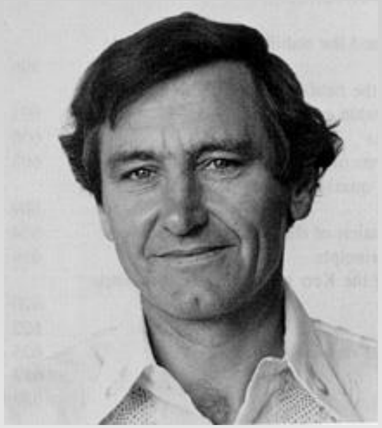
https://physics.aps.org/articles/v4/43

III) Kerr geometry and its miraculous properties

Kerr geometry

- **Unique vacuum** solution of Einstein equations describing a rotating black hole in 4d





Roy Patrick Kerr

- Discovered in 1963 by Kerr (4 years before Wheeler coins the term “black hole”).

- Provided cosmic censorship, Kerr solution is a final configuration of gravitational collapse – generic in our Universe.

Kerr geometry

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - a dt\right]^2$$

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad f = 1 - \frac{2M}{r}$$

$$\Delta = r^2 f + a^2$$

- Possesses **two parameters**: mass and rotation (no hair theorem)

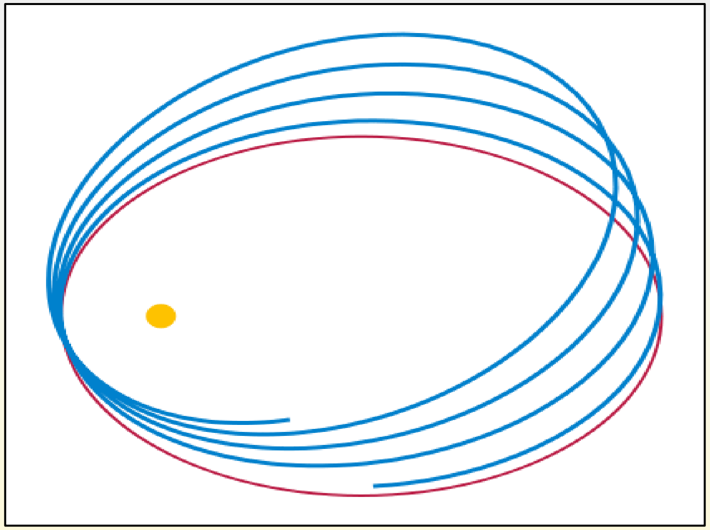
- no (physical) interior solutions are known – interpretation as field outside rotating objects is questionable

Remarkable properties

- **Geodesic motion is completely integrable**
  - The metric is stationary and axisymmetric: $\boldsymbol{k} = \partial_t \,, \ \boldsymbol{m} = \partial_\varphi$

$$g_{ab} u^a u^b = -1 \,, \quad k_a u^a = -E \,, \quad m_a u^a = L$$

  - **1968 Carter** discovers a mysterious constant of motion which results in a complete integrability of geodesic motion



$$K_{ab} u^a u^b = K$$

Carter's constant derivation (Exercise)

- Consider **Hamilton-Jacobi equation**

$$\frac{\partial S}{\partial \lambda} + g^{\alpha\beta}\frac{\partial S}{\partial x^\alpha}\frac{\partial S}{\partial x^\beta} = 0$$

- Try the following separation ansatz:

$$S = -\kappa\lambda + \mathcal{E}t - h\varphi - R(r) - \Lambda(\theta)$$

$$\kappa = \frac{1}{\Delta\Sigma}\left((r^2+a^2)\partial_t S + a\partial_\varphi S\right)^2 - \frac{\Delta}{\Sigma}(\partial_r S)^2 - \frac{1}{\Sigma}(\partial_\theta S)^2 - \frac{1}{\Sigma\sin^2\theta}\left(a\sin^2\theta\partial_t S + \partial_\varphi S\right)^2$$

- Re-shuffling:

$$\begin{aligned} C &= -\kappa r^2 + \frac{1}{\Delta}\left((r^2+a^2)\mathcal{E} - ah\right)^2 - \Delta R'^2 \\ &= \Lambda'^2 + \frac{1}{\sin^2\theta}\left(a\sin^2\theta\mathcal{E} - h\right)^2 + \kappa a^2\cos^2\theta \end{aligned}$$

- Since

$$\partial_\alpha S = p_\alpha \qquad C = K^{\alpha\beta}p_\alpha p_\beta \qquad \nabla_{(\gamma}K_{\alpha\beta)} = 0.$$

$$K = -r^2 g^{-1} + \frac{1}{\Delta}\left((r^2+a^2)\partial_t + a\partial_\varphi\right)^2 - \Delta\partial_r^2$$

- **Field equations decouple and separate**

  Scalar field, Dirac, electromagnetic, and gravitational perturbations **decouple and separate variables** (Carter 1968, Teukolsky 1972, Chandrasekhar & Page 1976, Wald 1978)

Enables to study:

- black hole shadow
- plasma accretion
- black hole stability
- Hawking evaporation
- …

- **Kerr-Schild form**: the metric can be written as a linear in mass deformation of the flat space

$$\boldsymbol{g} = \mathring{\boldsymbol{g}} + \frac{2Mr}{\Sigma}\,\boldsymbol{l}\,\boldsymbol{l}$$

- **Special algebraic type** of the Weyl tensor

Principal tensor

"All" the above properties can be attributed to the existence of a single object called:

Principal tensor = a (non-degenerate) closed conformal Killing-Yano 2-form

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a$$

For example: Carter's constant corresponds to the "square" of principal tensor

$$K_{ab} = h_{ac}h_b{}^c + \frac{1}{2}g_{ab}h^2$$

Special algebraic type: follows from integrability conditions of the above object

Canonical Carter's form

- Starting from the Boyer-Lindquist form, let's perform the following **coordinate transformation**:

$$y = a\cos\theta\,, \quad \psi = \phi/a\,, \quad \tau = t - a\phi$$

and for convience: $x = ir$ $\quad b_x = iM$

$$\boldsymbol{g} = \frac{\Delta_y}{y^2 - x^2}(\boldsymbol{d}\tau + x^2\boldsymbol{d}\psi)^2 + \frac{\Delta_x}{x^2 - y^2}(\boldsymbol{d}\tau + y^2\boldsymbol{d}\psi)^2$$

$$+ \frac{y^2 - x^2}{\Delta_y}\boldsymbol{d}y^2 + \frac{x^2 - y^2}{\Delta_x}\boldsymbol{d}x^2$$

$$\Delta_x = (a^2 - x^2) + 2b_x x\,,$$

$$\Delta_y = (a^2 - y^2)$$

Canonical Carter's form

$$\boldsymbol{g} = \frac{\Delta_y}{y^2 - x^2}(\boldsymbol{d}\tau + x^2\boldsymbol{d}\psi)^2 + \frac{\Delta_x}{x^2 - y^2}(\boldsymbol{d}\tau + y^2\boldsymbol{d}\psi)^2 + \frac{y^2 - x^2}{\Delta_y}\boldsymbol{d}y^2 + \frac{x^2 - y^2}{\Delta_x}\boldsymbol{d}x^2$$

- More generally: **off-shell metric** $\Delta_x = \Delta_x(x)\,, \quad \Delta_y = \Delta_y(y)$
- Form of this metric **uniquely determined** by the principal tensor

$$\xi^a_{(\tau)} = \frac{1}{3}\nabla_b h^{ba} = \partial^a_\tau\,, \quad \xi^a_{(\psi)} = -k^a{}_b\xi^b_{(\tau)} = \partial^a_\psi$$

$$h^a{}_b\hat{n}^b = -rn^a \qquad h^a{}_b\hat{e}^b = ye^a$$

- All separability/integrability properties **remain true** off-shell.

Canonical Carter's form

- Imposing the vacuum (with Lambda) Einstein equations:

$$R_{ab} = \Lambda g_{ab}$$

… and in particular

$$R = 4\Lambda$$

- gives

$$\ddot{\Delta}_x - \Delta_y'' = 4\Lambda(y^2 - x^2)$$

which can be solved by separation of variables and yields

$$\Delta_x = (a^2 - x^2)(1 + \Lambda x^2/3) + 2b_x x$$
$$\Delta_y = (a^2 - y^2)(1 + \Lambda y^2/3) + 2b_y y$$

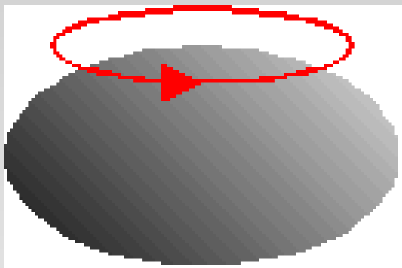
… on-shell **Kerr-NUT-AdS spacetime**

IV) Black holes in higher dimensions

What about black holes in higher dimensions?

(motivated by string theory, brane world scenarios, GR)

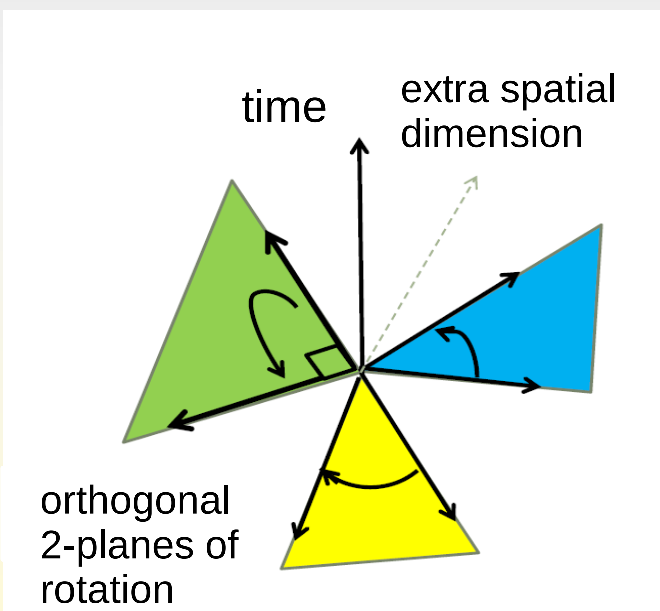
- **Myers-Perry** generalization of the Kerr metric (1986)



rotates in m=[(D-1)/2]
orthogonal planes

Robert C. Myers

Malcolm J. Perry



Rotation group SO(d-1) admits the Cartan's subgroup [U(1)]^m.

What about black holes in higher dimensions?

- **Myers-Perry** generalization of the Kerr metric (1986)

$$ds^2 = -dt^2 + \frac{U dr^2}{V - 2M} + \frac{2M}{U}\left(dt + \sum_{i=1}^{n} a_i \mu_i^2 d\phi_i\right)^2 + \sum_{i=1}^{n}(r^2 + a_i^2)(\mu_i^2 d\phi_i^2 + d\mu_i^2) + \varepsilon r^2 d\mu_{n+\varepsilon}^2 ,$$

$$V = r^{\varepsilon - 2} \prod_{i=1}^{n}(r^2 + a_i^2), \quad U = V\left(1 - \sum_{i=1}^{n} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}\right)$$

$$\sum_{i=1}^{n} \mu_i^2 + \varepsilon \mu_{n+\varepsilon}^2 = 1$$

- The constraint can be eliminated by the **Jacobi transformation**

$$\mu_i^2 = \frac{\prod_{\alpha=1}^{n-1}(a_i^2 - y_\alpha^2)}{\prod_{k=1}^{\prime n}(a_i^2 - a_k^2)}$$

(diagonalizes the mu-part of the metric)

What about black holes in higher dimensions?

- **Kerr-NUT-(A)dS spacetimes**: adding the cosmological constant and NUT parameters

Chen, Lü, Pope, Class. Quant. Grav. 23 , 5323 (2006).

$$\boldsymbol{g} = \sum_{\mu=1}^{n} \left[\frac{U_\mu}{X_\mu}\, \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{n-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right] + \varepsilon \frac{c}{A^{(n)}} \Big(\sum_{k=0}^{n} A^{(k)} \boldsymbol{d}\psi_k \Big)^2$$

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1 \\ \nu_1<\dots<\nu_k}}^{n} x_{\nu_1}^2 \dots x_{\nu_k}^2$$

$$A_\mu^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j=1 \\ \nu_1<\dots<\nu_j \\ \nu_i\neq\mu}}^{n} x_{\nu_1}^2 \dots x_{\nu_j}^2$$

$$U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\nu^2 - x_\mu^2)$$

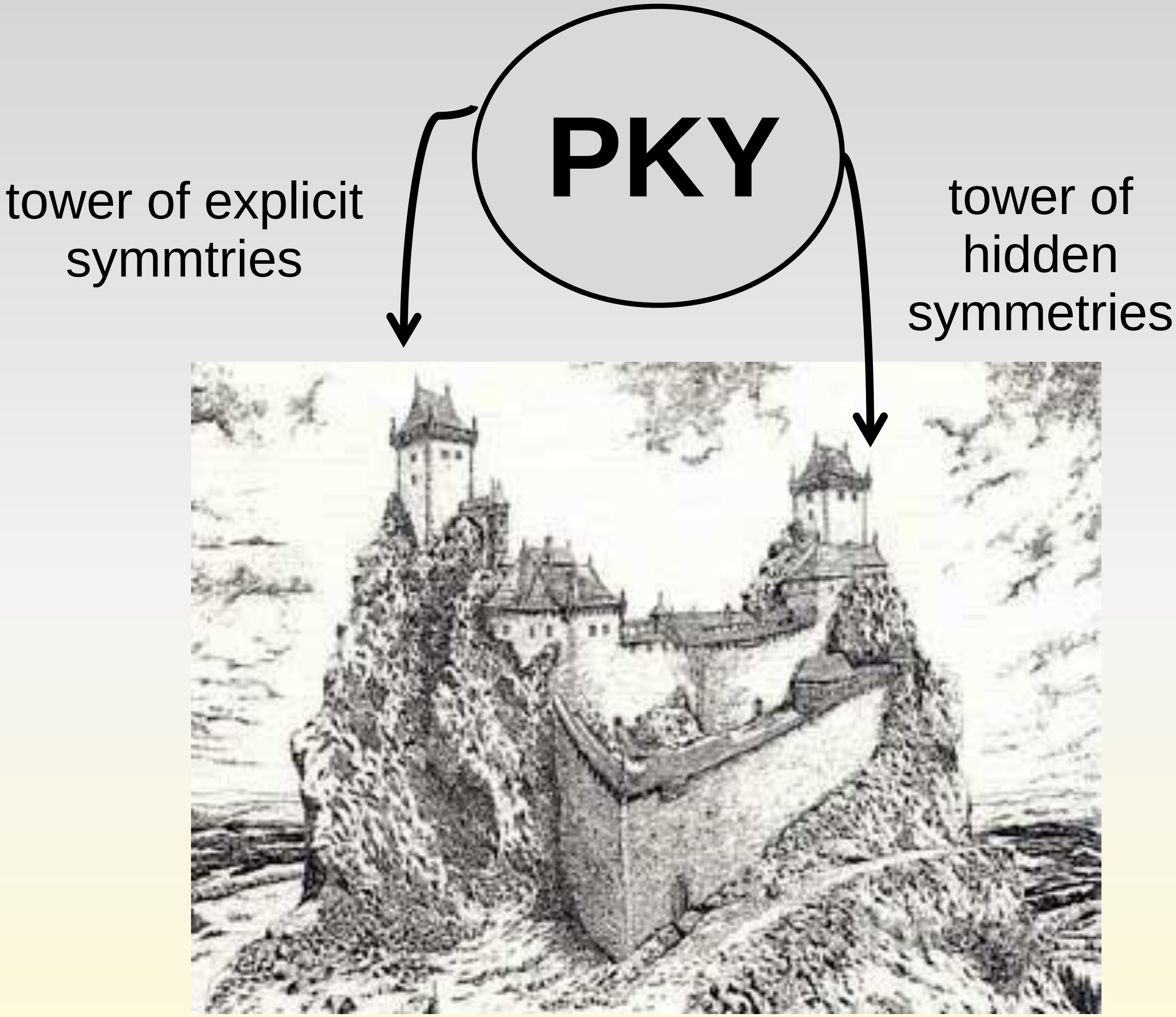
$$X_\mu = \begin{cases} -2b_\mu\, x_\mu + \sum_{k=0}^{n} c_k\, x_\mu^{2k} & \text{for } D \text{ even} , \\ -\dfrac{c}{x_\mu^2} - 2b_\mu + \sum_{k=1}^{n} c_k\, x_\mu^{2k} & \text{for } D \text{ odd} . \end{cases}$$

$$X_\mu = X_\mu(x_\mu)$$

- “direct” higher-dimensional generalization of **Carter’s canonical form**

Towers of hidden symmetries

(Off-shell) Kerr-NUT-AdS admits a **principal Killing-Yano tensor**. From this tensor one can generate **towers** of explicit and hidden symmetries, which guarantee its miraculous symmetries, similar in many aspects to those of the Kerr geometry**.**



- Integrability of geodesic motion
- Separation of test field equations
- Special algebraic type D
- SUSY for spinning particles
- Kerr-Schild form

Other black holes in higher dimensions

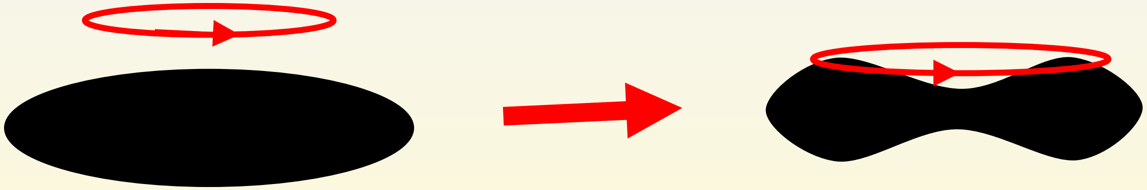
- **2001: Black** ring = black holes with S^2 x S^1 horizon topology

Roberto Emparan

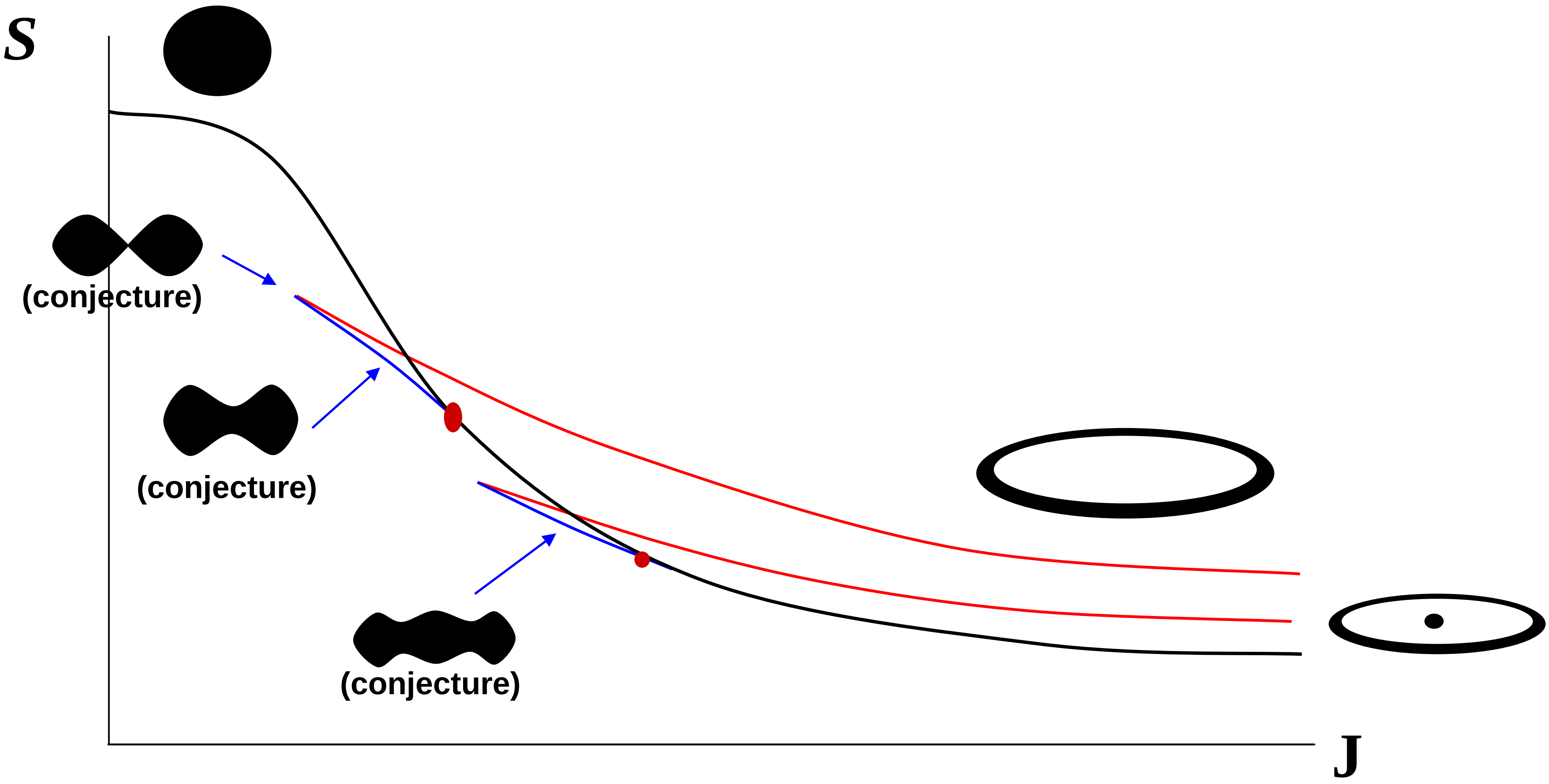
Harvey Reall

"Bended black string, whose gravity is compensated for by the centrifugal force."

- **"Bumpy"** black holes



Phase diagram of vacuum black holes



(Emparan, Harmark, Obers, Rodriguez, 2007)

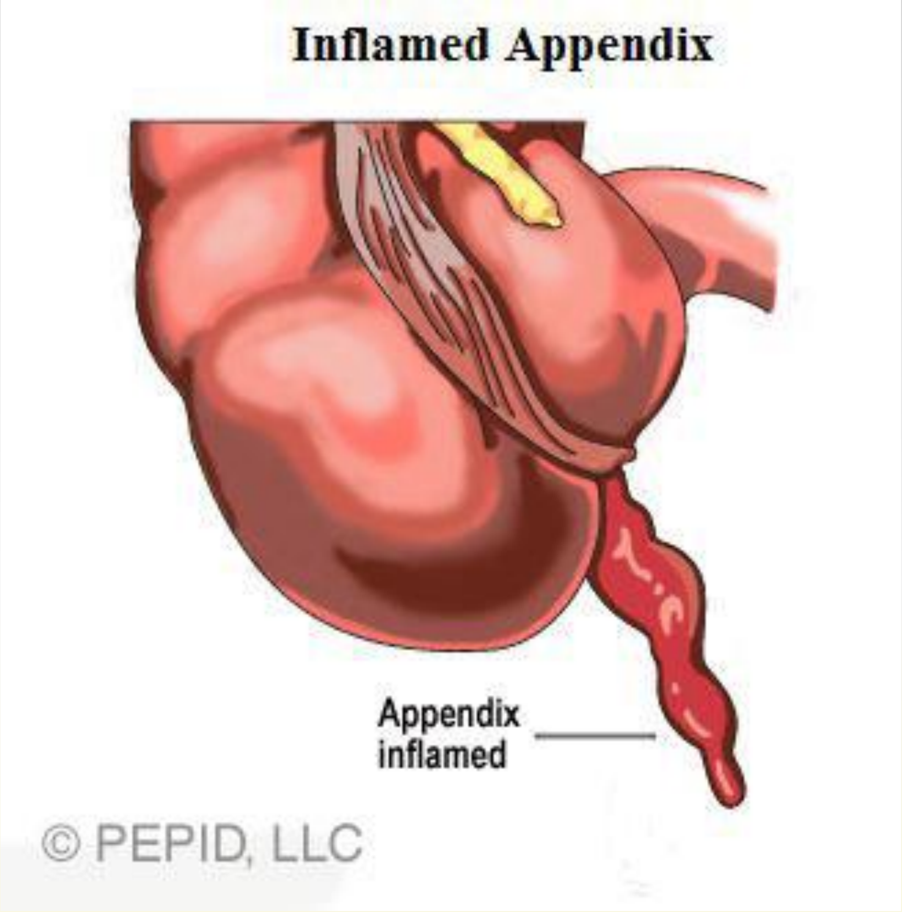
Summary of lecture 2

1) In 4d, **Kerr** metric describes a **unique** black hole solution. It possess a remarkable hidden symmetry of **principal Killing-Yano** (PKY) tensor which determines its miraculous properties.

2) Slightly more generally, PKY exists for (off-shell) Carter's **canonical metric**, which on-shell yields Kerr-NUT-AdS spacetimes.

3) Generalizes to higher dimensions, for (off-shell) **Kerr-NUT-AdS spacetimes.** (Other black holes exist in higher dimensions.)

Appendices



Differential forms

- **Definition.** A <u>*differential p-form*</u> ω is a totally antisymmetric tensor of type $(0,p)$, that is,

$$\omega_{\alpha_1\ldots\alpha_p} = \omega_{[\alpha_1\ldots\alpha_p]} = \frac{1}{p!}\sum_{\mathrm{perm}\pi} \mathrm{sign}(\pi)\,\omega_{\alpha_{\pi(1)}\ldots\alpha_{\pi(p)}}\,.$$

Hence, a differential form is antisymmetric under exchange of any 2 indices. We shall denote Λ^p_x a vector space of p-forms at x. One can show that it has a dimensionality $dim\Lambda^p_x = \binom{n}{p}$.

- **Definition.** A <u>*wedge product*</u> $\wedge : \Lambda^p_x \times \Lambda^q_x \to \Lambda^{p+q}_x$:

$$(\omega\wedge\nu)_{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_q} = \frac{(p+q)!}{p!q!}\omega_{[\alpha_1\ldots\alpha_p}\nu_{\beta_1\ldots\beta_q]}\,.$$

That is, $\omega\wedge\nu$ is a $(p+q)$-form. It obeys

$$\omega\wedge\nu = (-1)^{pq}\nu\wedge\omega\,.$$

Since dx^α is a coordinate basis of 1-forms, general p-form can be written as

$$\boxed{\omega = \frac{1}{p!}\omega_{\alpha_1\ldots\alpha_p}dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_p}\,.}$$

Differential forms

- For any vector V, we define an *inner derivative* $i_V : \Lambda^p \to \Lambda^{p-1}$:

$$i_V\omega = V \lrcorner\, \omega = V \cdot \omega = \omega(V) : \qquad (V \lrcorner\, \omega)_{\alpha_1 \dot{\alpha}_{p-1}} = V^\beta \omega_{b\alpha_1 \dots \alpha_{p-1}} \, .$$

Properties of inner derivative:

i) i_V is linear

ii) i_V is linear in V : $i_{fV+gW} = f i_V + g i_W$.

iii) graded Leibnitz rule: For $\omega \in \Lambda^p$ we have

$$i_V(\omega \wedge \nu) = (i_V\omega) \wedge \nu + (-1)^p \omega \wedge i_V \nu \, .$$

iv)

$$i_v i_W + i_W i_V = 0 \quad \text{spec.} \quad i_V^2 = 0 \, .$$

- **Definition.** *Exterior derivative* $d : \Lambda^p \to \Lambda^{p+1}$ is defined as follows:

i) On a function f we have $d : f \to df = \frac{\partial f}{\partial x^\alpha} dx^\alpha$.

ii) On a p-form ω we then have

$$d : \omega \to d\omega = \frac{1}{p!} d\omega_{\alpha_1 \dots \alpha_p} \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p} \, .$$

That is $(d\omega)_{\alpha_1 \dots \alpha_{p+1}} = (p+1)\partial_{[\alpha_1} \omega_{\alpha_2 \dots \alpha_{p+1}]}$.
Note that we have $d^2 = 0$. Conversely, a p-form α is called *closed* when $d\alpha = 0$. It is called *exact* when $\alpha = d\beta$. Any closed form α can be *locally* written as $\alpha = d\beta$ but not *globally*.

Differential forms

- *Cartan's lemma.* For a vector field V and a p-form ω, we have the following identity:

$$\boxed{\mathcal{L}_V \omega = V \lrcorner\, d\omega + d(V \lrcorner\, \omega)\,.}$$

In particular, this implies that

$$\mathcal{L}_V df = d\mathcal{L}_V f\,.$$

- Having a metric, we can also define Hodge dual $* : \Lambda^p \to \Lambda^{n-p}$

$$\boxed{(*\alpha)_{\alpha_1\ldots\alpha_{n-p}} = \frac{1}{p!}\epsilon_{\alpha_1\ldots\alpha_{n-p}}{}^{\beta_1\ldots\beta_p}\alpha_{\beta_1\ldots\beta_p}}$$

Where Levi-Civita tensor $\epsilon_{\alpha_1\ldots\alpha_n} = \sqrt{|g|}e_{\alpha_1\ldots\alpha_n}$

in terms of permutation symbol:

$$e_{\alpha_1\ldots\alpha_n} = \begin{cases} +1 & \text{for } \alpha_1,\ldots,\alpha_n \text{ even permutations of } 1,\ldots,n \\ -1 & \text{for odd permutations} \end{cases}$$

Differential forms

- Having a metric, we can also define co-derivative δ:

$$\delta = \mp(-1)^{n(p+1)} * d* : \ \Lambda^p \to \Lambda^{p-1}\,.$$

Here n is the dimensions of the manifold and upper/lower sign applies to Riemannian/Lorentzian signature. We then find

$$(\delta\omega)_{\alpha_2\dots\alpha_p} = -\nabla^{\alpha_1}\omega_{\alpha_1\dots\alpha_p}\,.$$

- Integration. A p-form ω can be integrated over a p-dimensional (sub)manifold. Writing $\omega = f dx^1 \wedge \dots dx^p$ we then define

$$\int_{O_p}\omega = \int_{\psi(O_p)} f dx^1 \dots dx^p \quad \text{where r.h.s. is defined as Lebesgue integral.}$$

Note that this definition is independent of coordinates, as we have

$$\omega = f' dx'^1 \wedge \dots \wedge dx'^p\,, \quad f' = f \det\Big(\frac{\partial x^\mu}{\partial x'^\nu}\Big)\,.$$

Stokes theorem. The following identity is valid

$$\boxed{\int_\Omega d\omega = \int_{\partial\Omega}\omega\,.}$$