Lecture 3: Kerr-NUT-AdS spacetimes: towers of symmetries

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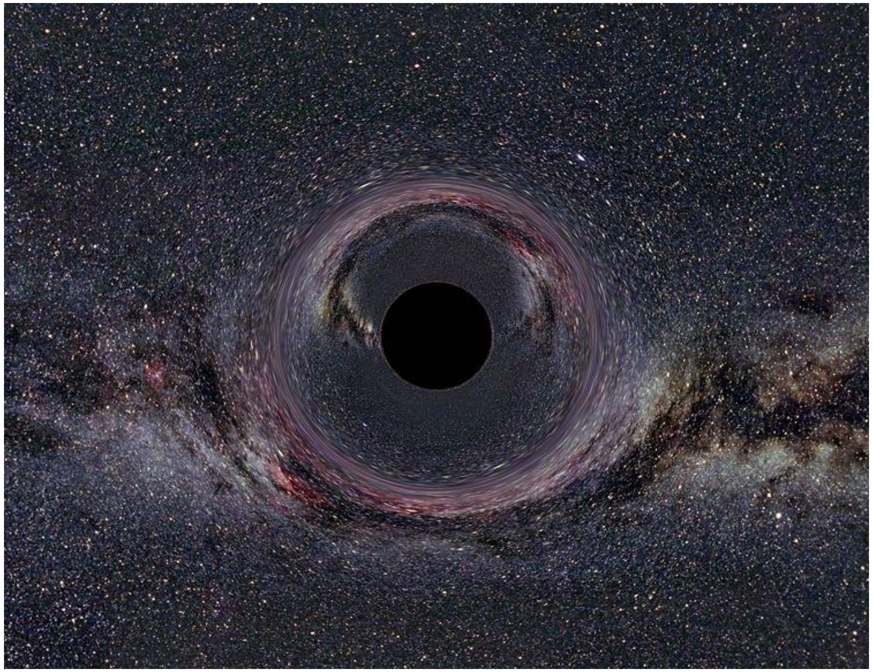

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Kerr-NUT-AdS spacetimes: towers of symmetries

(plan for lecture 3)

a) More on Killing tensors

b) Principal tensor

c) Liouville’s integrability

d) Geodesic integrability: towers of Carter’s constants

I) More on Killing tensors

Killing tensors

=totally symmetric tensors obeying

$$\nabla^{(a_1} K^{a_2 a_3 \ldots a_{p+1})} = 0.$$

Generate ***constants of geodesic motion*** of degree p

M. Walker and R. Penrose, Comm. Math. Phys. 18 , 265 (1970).

$$\mathcal{K}_p = K^{a_1 \ldots a_p} \, p_{a_1} \cdots p_{a_p}$$

Poisson commute with the Hamiltonian generating geodesic flow

$$\mathcal{H} = \tfrac{1}{2} g^{ab} p_a p_b$$

Reducibility

$$K_{(1)}^{(a} K_{(2)}^{bc)}, \quad \text{or} \quad K_{(3)}^{(a} K_{(4)}^{b} K_{(5)}^{c)},$$

Algebra of Killing tensors

Killing tensors form an algebra with respect to (symmetric) ***Schouten-Nijenhuis brackets***:

$$\{\mathcal{K}_p, \mathcal{K}_q\} = \frac{\partial \mathcal{K}_p}{\partial q^i}\frac{\partial \mathcal{K}_q}{\partial p_i} - \frac{\partial \mathcal{K}_q}{\partial q^i}\frac{\partial \mathcal{K}_p}{\partial p_i}$$

$$\equiv [K_p, K_q]^{a_1 a_2 \ldots a_{p+q-1}}_{\mathrm{SN}}\, p_{a_1} p_{a_2} \cdots p_{a_{p+q-1}} \,.$$

$$[K_p, K_q]^{a_1 \ldots a_{p+q-1}}_{\mathrm{SN}} = pK_p{}^{c(a_1 \ldots p-1}\nabla_c K_q{}^{a_p \ldots a_{p+q-1})} - qK_q{}^{c(a_1 \ldots a_{q-1}}\nabla_c K_p{}^{a_q \ldots a_{q+p-1})}$$

For example:

$$[K_{(i)}, K_{(j)}]^{abc}_{\mathrm{SN}} \equiv K^{e(a}_{(i)}\nabla_e K^{bc)}_{(j)} - K^{e(a}_{(j)}\nabla_e K^{bc)}_{(i)}$$

$$[\xi, K_p]^{a_1 \ldots a_p}_{\mathrm{SN}} = \mathcal{L}_\xi K_p^{a_1 \ldots a_p} \,.$$

Algebra of Killing tensors

- Spec: **metric** g is a (trivial) Killing tensor

$$[\xi, g]_{\mathrm{SN}}^{ab} = \mathcal{L}_\xi g^{ab} = -2\nabla^{(a}\xi^{b)}$$

$$[K_p, g]_{\mathrm{SN}}^{a_1\ldots a_p} = -p\nabla^{(a_1}K_p^{a_2\ldots a_p)}$$

- In other words: **Killing vector** and **Killing tensor** equations can be conveniently expressed as

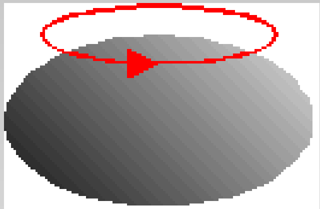
$$[\xi, g]_{\mathrm{SN}} = 0 \qquad [K_p, g]_{\mathrm{SN}} = 0$$

- Note also that in principle one can generate higher-rank Killing tensors by employing SN brackets.

$$[K_p, K_q]_{\mathrm{SN}}^{a_1\ldots a_{p+q-1}}$$

Examples of spacetimes with rank-2 KTs

1. Kerr geometry (in all dimensions)



   P. Krtouš, D. Kubizňák, D. N. Page, and V. P. Frolov, Killing-Yano Tensors, Rank-2 Killing Tensors, and Conserved Quantities in Higher Dimensions, JHEP 0702 (2007) 004

2. Taub-NUT space: generalization of Runge-Lenz vector

   G. W. Gibbons and P. J. Ruback, "The Hidden Symmetries Of Taub-NUT And Monopole Scattering," Phys. Lett. B 188 (1987) 226.

3. Various SUGRA black holes

   D.D. Chow, Symmetries of supergravity black holes, Class. Quant. Grav. 27, 205009 (2010) , arXiv:0811:1264.

Not known spacetimes with irreducible higher-rank Killing-Stackel tensors!

See, however:

- **Finn's talk**
- G. Gibbons, T. Houri, DK, C. Warnick, *Some spacetimes with higher-rank Killing tensors*, PLB700 (2011), 68.

II) Principal Killing-Yano tensor

Families of Killing-Yano tensors

for a general differential p-form

$$\nabla \boldsymbol{\omega} = (\text{exterior} + \text{divergence} + \text{harmonic})\ \text{parts}$$

Conformal Killing-Yano (CKY) tensor

$$\nabla_X \boldsymbol{k} = \frac{1}{p+1} \boldsymbol{X} \lrcorner\, d\boldsymbol{k} - \frac{1}{D-p+1} \boldsymbol{X}^\flat \wedge \delta \boldsymbol{k}\,.$$

Killing-Yano (KY) tensor: divergence part is missing

closed CKY tensor: exterior part is missing

Under Hodge duality divergence part transforms into exterior part and vice versa.

$$*(\text{closed CKY}) = \text{KY}$$

Principal Killing-Yano tensor

= (non-degenerate) closed CKY 2-form

$$\nabla_X \boldsymbol{h} = \boldsymbol{X}^\flat \wedge \boldsymbol{\xi}\,.$$

$$\nabla_X h_{ab} = 2X_{[a}\,\xi_{b]}$$

It follows

$$\boldsymbol{dh} = 0$$

$$\xi_b = \frac{1}{D-1}\nabla_a h^a{}_b$$

non-degenerate: full matrix rank, eigenvalues are functionally independent (can be used as coordinates)

Eigenvalues of $-\boldsymbol{h}^2$:

$$\{\underbrace{x_1^2,\ldots,x_1^2}_{2l_1},\ldots,\underbrace{x_n^2,\ldots,x_n^2}_{2l_n},\underbrace{\xi_1^2,\ldots,\xi_1^2}_{2m_1},\ldots,\underbrace{\xi_N^2,\ldots,\xi_N^2}_{2m_N},\underbrace{0,\ldots,0}_{K}\}\,.$$

Canonical metric element

a) Darboux basis:

$$\boldsymbol{g} = \delta_{ab}\boldsymbol{\omega}^{\hat{a}}\boldsymbol{\omega}^{\hat{b}} = \sum_{\mu=1}^{n}(\boldsymbol{\omega}^{\hat{\mu}}\boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}}\tilde{\boldsymbol{\omega}}^{\hat{\mu}}) + \varepsilon\boldsymbol{\omega}^{\hat{0}}\boldsymbol{\omega}^{\hat{0}},$$

$$\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu \boldsymbol{\omega}^{\hat{\mu}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{\mu}}.$$

$$\boxed{D = 2n + \varepsilon}$$

(PKY is non-degenerate, Euclidean signature)

b) Towers of symmetries:

construction based on the following Lemma:

Lemma ([Krtouš *et al.*, 2007b]). *Let $\boldsymbol{k}^{(1)}$ and $\boldsymbol{k}^{(2)}$ be two closed CKY tensors. Then their exterior product $\boldsymbol{k} \equiv \boldsymbol{k}^{(1)} \wedge \boldsymbol{k}^{(2)}$ is also a closed CKY tensor.*

P. Krtouš, DK, D.N. Page, V.P. Frolov, Killing-Yano Tensors, Rank-2 Killing Tensors, and Conserved Quantities in Higher Dimensions, JHEP 0702 (2007) 004.

Towers of hidden symmetries

closed CKY tensors:

$$h^{(j)} \equiv h^{\wedge j} = \underbrace{h \wedge \ldots \wedge h}_{\text{total of } j \text{ factors}} .$$

Killing-Yano tensors:

$$f^{(j)} \equiv *h^{(j)} .$$

$$\nabla_{(\alpha_1} f_{\alpha_2)\alpha_3 \ldots \alpha_{p+1}} = 0 .$$

Killing tensors:

$$K^{(j)}_{ab} \equiv \frac{1}{(D-2j-1)!(j!)^2} f^{(j)}{}_{ac_1 \ldots c_{D-2j-1}} f^{(j)}{}_b{}^{c_1 \ldots c_{D-2j-1}} .$$

$$K^{(j)} = \sum_{\mu=1}^{n} A^{(j)}_{\mu} (\omega^{\hat{\mu}} \omega^{\hat{\mu}} + \tilde{\omega}^{\hat{\mu}} \tilde{\omega}^{\hat{\mu}}) + \varepsilon A^{(j)} \omega^{\hat{0}} \omega^{\hat{0}} .$$

$$\nabla_{(a} K^{(j)}_{bc)} = 0$$

where

$$A^{(j)} = \sum_{\nu_1 < \cdots < \nu_j} x^2_{\nu_1} \ldots x^2_{\nu_j} , \quad A^{(j)}_{\mu} = \sum_{\substack{\nu_1 < \cdots < \nu_j \\ \nu_i \neq \mu}} x^2_{\nu_1} \ldots x^2_{\nu_j} ,$$

Tower of explicit symmetries:

Primary Killing vector:

$$\boldsymbol{\xi} = \boldsymbol{l}_{(0)} = \frac{1}{D-1} \nabla \cdot \boldsymbol{h}$$

Secondary Killing vectors:

$$\boldsymbol{l}_{(j)} = \boldsymbol{K}_{(j)} \cdot \boldsymbol{\xi}$$

Since all symmetries generated from a single object h, they all **mutually** (Schouten-Nijenhuis) **commute**:

$$[\boldsymbol{l}_{(i)}, \boldsymbol{K}_{(j)}] = 0\,, \quad [\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}] = 0\,.$$

$$[K^{(j)}, K^{(l)}]_{abc} \equiv K^{(j)}_{e(a} \nabla^e K^{(l)}_{bc)} - K^{(l)}_{e(a} \nabla^e K^{(j)}_{bc)} = 0\,.$$

c) Canonical coordinates:

- The n “eigenvalues” x_μ are natural coordinates $\{x_\mu, \psi_j\}$
- These can be “upgraded” by adding (n+ε) new Killing coordinates: $\boldsymbol{l}_{(k)} = \boldsymbol{\partial}_{\psi_k}$

d) Canonical metric:

$$\boldsymbol{g} \;=\; \delta_{ab}\boldsymbol{\omega}^{\hat a}\boldsymbol{\omega}^{\hat b} = \sum_{\mu=1}^{n}(\boldsymbol{\omega}^{\hat\mu}\boldsymbol{\omega}^{\hat\mu} + \tilde{\boldsymbol{\omega}}^{\hat\mu}\tilde{\boldsymbol{\omega}}^{\hat\mu}) + \varepsilon\boldsymbol{\omega}^{\hat 0}\boldsymbol{\omega}^{\hat 0}\,,$$

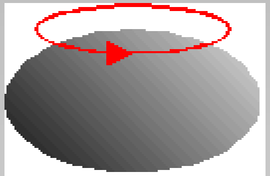
$$\boldsymbol{\omega}^{\hat\mu} \;=\; \frac{\boldsymbol{d}x_\mu}{\sqrt{Q_\mu}}\,,\quad \tilde{\boldsymbol{\omega}}^{\hat\mu} = \sqrt{Q_\mu}\sum_{j=0}^{n-1} A_\mu^{(j)}\boldsymbol{d}\psi_j\,,\quad \boldsymbol{\omega}^{\hat 0} = \sqrt{\frac{-c}{A^{(n)}}}\sum_{j=0}^{n} A^{(j)}\boldsymbol{d}\psi_j\,.$$

$$Q_\mu \;=\; \frac{X_\mu}{U_\mu}\,,\quad U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(x_\nu^2 - x_\mu^2)\,.$$

$$X_\mu \;=\; X_\mu(x_\mu).$$

e) Kerr-NUT-(A)dS spacetime:

$$X_\mu = X_\mu(x_\mu).$$



- **Einstein space** condition: $$R_{ab} = (-1)^n (D-1) c_n \, g_{ab} \, ,$$

  implies the specific form of metric functions:

$$X_\mu = \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} + \frac{\varepsilon c}{x_\mu^2} \, .$$

W. Chen, H. Lü and C. N. Pope, Class. Quant. Grav. 23 , 5323 (2006).

- Constants are related to mass, NUT parameters, rotations, and cosmological constant

<u>Uniqueness Theorem</u>: The most general solution of vacuum Einstein equations that admits the principal Killing-Yano tensor is the Kerr-NUT-(A)dS spacetime.

Krtouš, Frolov, DK, *Hidden Symmetries of Higher Dimensional Black Holes and Uniqueness of the Kerr-NUT-(A)dS spacetime*, Phys. Rev. D 78 (2008) 064022.

III) Liouville's integrability

Integrable systems

- **Conserved quantity** (constant/integral of motion)

$$\{g, H\} = 0$$

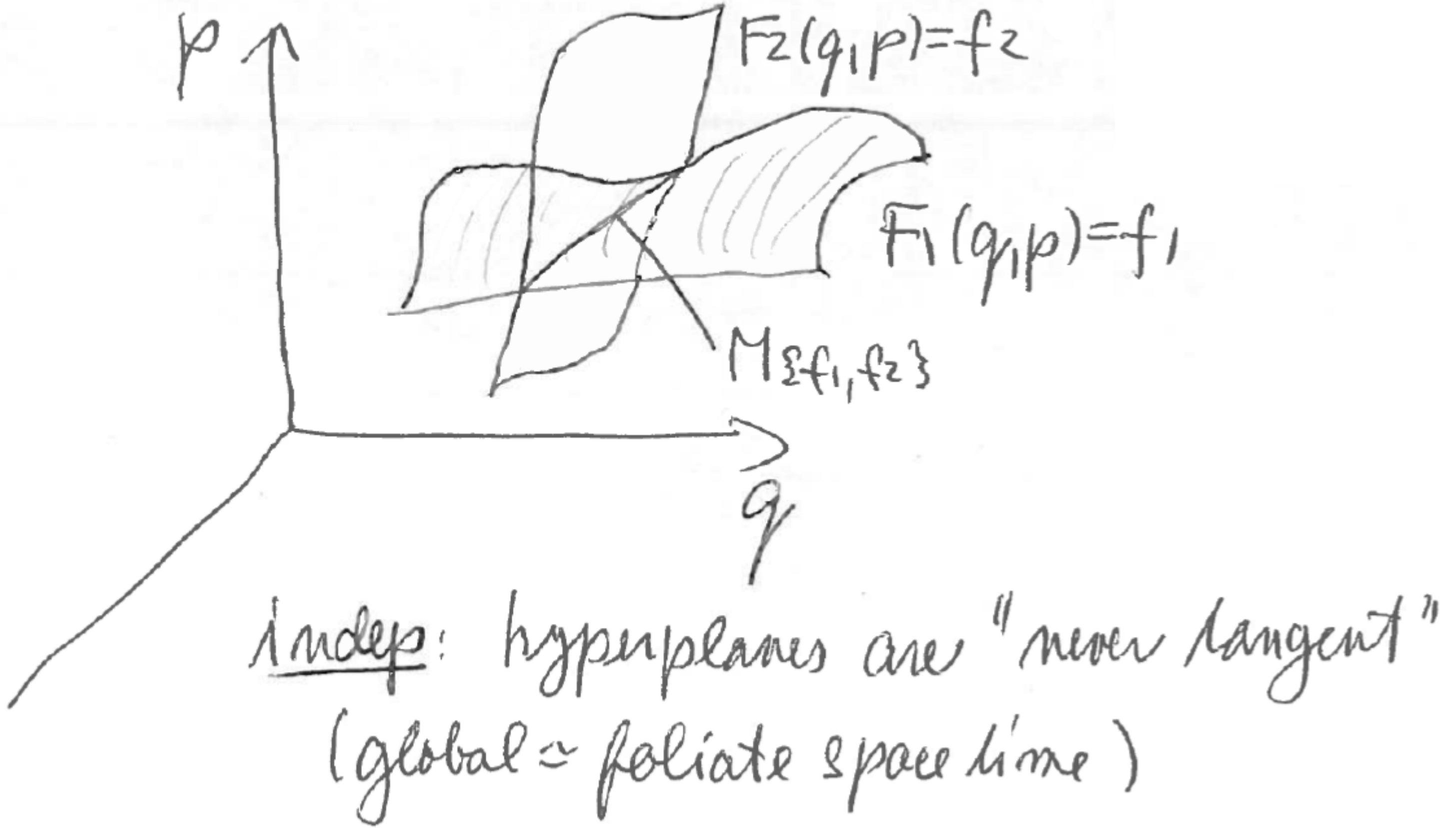
Definition. The dynamical system with n degrees of freedom ($2n$-dimensional phase space) is *completely (Liouville) integrable*, if it possess n independent conserved quantities $F_i(q,p) = f_i$, $\{H, F_i\} = 0$, that are in *involution*: $\{F_i, F_j\} = 0 \ \forall i, j$.

- **Nice piece of 19th century mathematics:**

Liouville's theorem. The solution of equations of motion of a completely integrable system can be obtained by "*quadrature*", that is by a finite number of algebraic operations and integrations.

Integrable systems

Independence. Each integral defines a hypersurface in phase space, dynamical trajectories must remain in this surface:



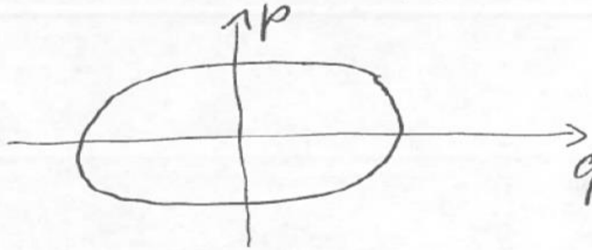
$M_{\{f\}}$ given by $\{F_i = f_i\}$ has dimension n.

One cannot have more than n independent integrals of motion that are in involution. Otherwise, the Poisson bracket would be degenerate. This implies that $H = H(F_i)$.

Integrable systems

Under a suitable global hypothesis, $M_{\{f\}}$ is an n-dimensional tori T_n.

Ex: harmonic oscillator $H = \frac{1}{2}(p^2 + \omega^2 q^2)$



phase space is fibred into ellipses $H = E$ .... T_1 (except $(0,0)$ .. stationary point)

- introduce $p = \rho\cos\theta$, $q = \frac{\rho}{\omega}\sin\theta$

"action-angle var."

$\Rightarrow$ motion $= \{\rho = \sqrt{2E},\ \theta = \omega t + \theta_0\}$

- generalization $H = \frac{1}{2}\sum_i (p_i^2 + \omega_i^2 q_i^2)$ (*)

F_i ... n conserved quantities in involution

$M_f \equiv \{F_i = f_i\}$ is T_n

"all integrable systems look like (*)"

One can show that whenever the Hamilton–Jacobi completely separates, the motion is completely integrable.

Idea of Proof of Liouville's theorem

- **Proof is constructive**: shows how to integrate system by one integration and several algebraic operations.
- The idea is to construct a canonical transformation (by constructing the corresponding generating function) to coordinates where the EOM are trivial.

– We want to find a canonical transformation $(q^i, p_i) \to (F^i, \psi_i)$, where F^i are our conserved quantities. If we succeed, then

$$\begin{aligned} \dot{F}^i &= \{H, F^i\} = 0\,, \\ \dot{\psi}_i &= \{H, \psi_i\} = \frac{\partial H}{\partial F^i} = \Omega_i = \Omega_i(F_j) \quad \ldots \text{constant in time.} \end{aligned}$$

This then means that we get a solution

$$F^i(t) = F^i(0)\,, \quad \psi_i(t) = \psi_i(0) + \Omega_i t\,.$$

Canonical transformations: generating functions

- Strength of Hamiltonian dynamics derives from **canonical transformations** (gauge freedom)

$$Q^j = Q^j(q^i, p_i)\,, \qquad P_j = P_j(q^i, p_i)$$

$$\omega = dp_i \wedge dq^i = dP_i \wedge dQ^i$$

- Consider Cartan's 1-forms

$$\theta = p_j dq^j\,, \qquad \omega = d\theta \qquad\qquad \tilde{\theta} = P_j dQ^j\,, \qquad \omega = d\tilde{\theta}$$

$$d(\theta - \tilde{\theta}) = 0 \quad \Rightarrow \quad \theta - \tilde{\theta} = dF \quad \text{locally}$$

- So we have a generating function F:

$$\boxed{p_j dq^j - P_j dQ^j = dF}$$

Canonical transformations: generating functions

- In particular

$$S\ (q^j, Q^j) = F(q^j, p_i(Q^j, q^j)) \qquad \left[\det \frac{\partial(Q,q)}{\partial(q,p)} \neq 0\right]$$

- Then we have

$$p_j dq^j - P_j dQ^j = \frac{\partial S}{\partial q^j} dq^j + \frac{\partial S}{\partial Q^j} dQ^j$$

- Which yields the following relations for S:

$$p_j = \frac{\partial S}{\partial q^j}\,, \qquad P_j = -\frac{\partial S}{\partial Q^j}$$

Idea of Proof of Liouville's theorem

– We want to find a canonical transformation $(q^i, p_i) \to (F^i, \psi_i)$, where F^i are our conserved quantities. If we succeed, then

$$\begin{aligned} \dot{F}^i &= \{H, F^i\} = 0\,, \\ \dot{\psi}_i &= \{H, \psi_i\} = \frac{\partial H}{\partial F^i} = \Omega_i = \Omega_i(F_j) \quad \ldots \text{constant in time.} \end{aligned} \tag{2.74}$$

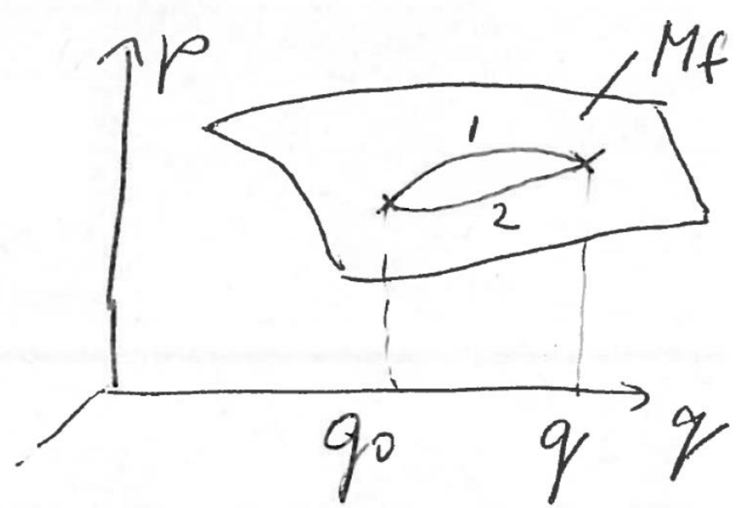
This then means that we get a solution

$$F^i(t) = F^i(0)\,, \quad \psi_i(t) = \psi_i(0) + \Omega_i t\,. \tag{2.75}$$

– To construct this let's find the corresponding generating function S. We have $M_{\{f\}} = \{F^i(p, q) = f^i\}$. In principle we can invert this, to get $p_i = p_i(f, q)$

on $M_{\{f\}}$ and can define

$$\boxed{S(q, F) = \int_{q_0}^{q} p_i dq^i\,.}$$



Idea of Proof of Liouville's theorem

If such an integral exists (see figure) then (c.f. (2.19))

$$\psi_j = -\frac{\partial S}{\partial F^j} \tag{2.77}$$

gives the desired canonical transformation. [Note that $\frac{\partial S}{\partial q^i} = p_i$ is automatically satisfied.]

– One finally needs to show that S is well defined, that is integral (2.76) is independent of integration path. One can show that is is exactly equivalent to the requirement that F^i are in involution: $\{F^i, F^j\} = 0$.

– So we have obtained the solution of EOM by one integral (2.76) and some algebraic operations (needed to express p as function of q and F).

IV) Goedesic integrability: towers of Carter's constants

Complete integrability of geodesic motion

Definition. The dynamical system with n degrees of freedom ($2n$-dimensional phase space) is *completely (Liouville) integrable*, if it possess n independent conserved quantities $F_i(q,p) = f_i$, $\{H, F_i\} = 0$, that are in *involution*: $\{F_i, F_j\} = 0\ \forall i, j$.

D=2n+ε constants of motion:

- Killing vectors: $$\Psi_k = \boldsymbol{l}_{(k)} \cdot \boldsymbol{u}$$ … $n + \epsilon$
- Moreover we have Killing tensors:

$$\kappa_j = K^{(j)}{}_{ab} u^a u^b = \boldsymbol{u} \cdot \boldsymbol{K}^{(j)} \cdot \boldsymbol{u}\,.$$ … n

…tower of **Carter's constants**

Complete integrability of geodesic motion

- **Involution:** "trivial" – follows from SN bracket commutation

$$\{\kappa_i, \kappa_j\} = 0\,, \quad \{\kappa_i, \Psi_j\} = 0\,, \quad \{\Psi_i, \Psi_j\} = 0$$

- Indeed

$$\{\mathcal{K}_p, \mathcal{K}_q\} = \frac{\partial \mathcal{K}_p}{\partial q^i}\frac{\partial \mathcal{K}_q}{\partial p_i} - \frac{\partial \mathcal{K}_q}{\partial q^i}\frac{\partial \mathcal{K}_p}{\partial p_i}$$
$$\equiv [K_p, K_q]^{a_1 a_2 \ldots a_{p+q-1}} p_{a_1} p_{a_2} \cdots p_{a_{p+q-1}}\,.$$

and

$$[\boldsymbol{l}_{(i)}, \boldsymbol{K}_{(j)}] = 0\,, \quad [\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}] = 0\,.$$

$$[K^{(j)}, K^{(l)}]_{abc} \equiv K^{(j)}_{e(a} \nabla^e K^{(l)}_{bc)} - K^{(l)}_{e(a} \nabla^e K^{(j)}_{bc)} = 0\,.$$

Complete integrability of geodesic motion

- **Functional independence**: gradients on the phase space are linearly independent.

- Since all Killing vectors and Killing tensors are independent (in the x-direction), it is enough to show what happens in the "**momentum direction**".

$$J = \partial_p \kappa_0 \wedge \cdots \wedge \partial_p \kappa_{n-1} \wedge \partial_p \Psi_0 \wedge \cdots \wedge \partial_p \Psi_{d-n-1}$$

- One can show that we have

$$J \propto \partial_{x_1} \wedge \cdots \wedge \partial_{x_n} \wedge \partial_{\psi_0} \wedge \cdots \wedge \partial_{\psi_{d-n-1}} \neq 0$$

D.N. Page, DK, M. Vasudevan, P. Krtouš, *Complete Integrability of Geodesic Motion in General Higher-Dimensional Rotating Black-Hole Spacetimes*, PRL 98 (2007) 061102.

Summary of lecture 3

1) **Killing tensors** form an **algebra** w.r.t **Schouten-Nijenhuis** (SN) brackets. Spacetimes with higher-rank Killing tensors still yet to be explored (see Finn's talk).

2) **Kerr-NUT-AdS** is the **unique** spacetime admitting the **principal Killing-Yano tensor** in all dimensions. It generates towers of explicit and hidden symmetries.

3) These symmetries in particular guarantee **complete integrability of geodesic motion** in the Liouville sense (more predictable than weather on Earth).