Asymptotic properties of null geodesics near null infinity

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Mar. 21st, 2022

Based on [2106.03150] and [2110.10917]

Collaborators：Keisuke Izumi, Tetsuya Shiromizu, Yoshimune Tomikawa, Hirotaka Yoshino

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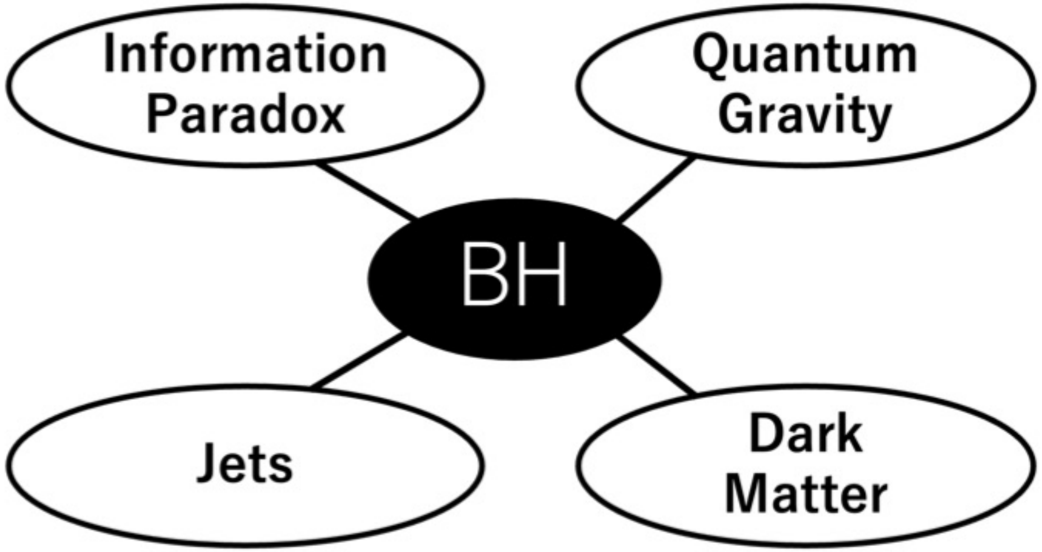
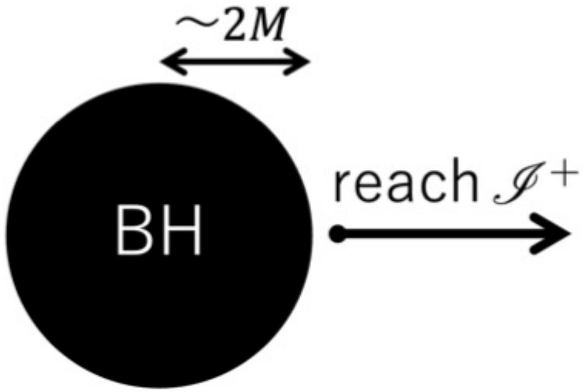
Black holes : null geodesics never reach $\mathscr{I}^+$

Black holes (BHs) provide us plentiful knowledge and discussion.

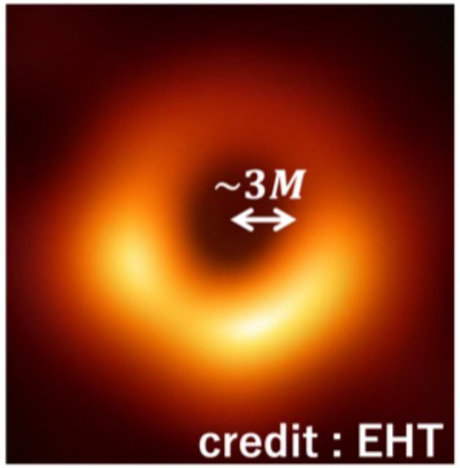


BH : Region from which null geodesics cannot reach $\mathscr{I}^+$

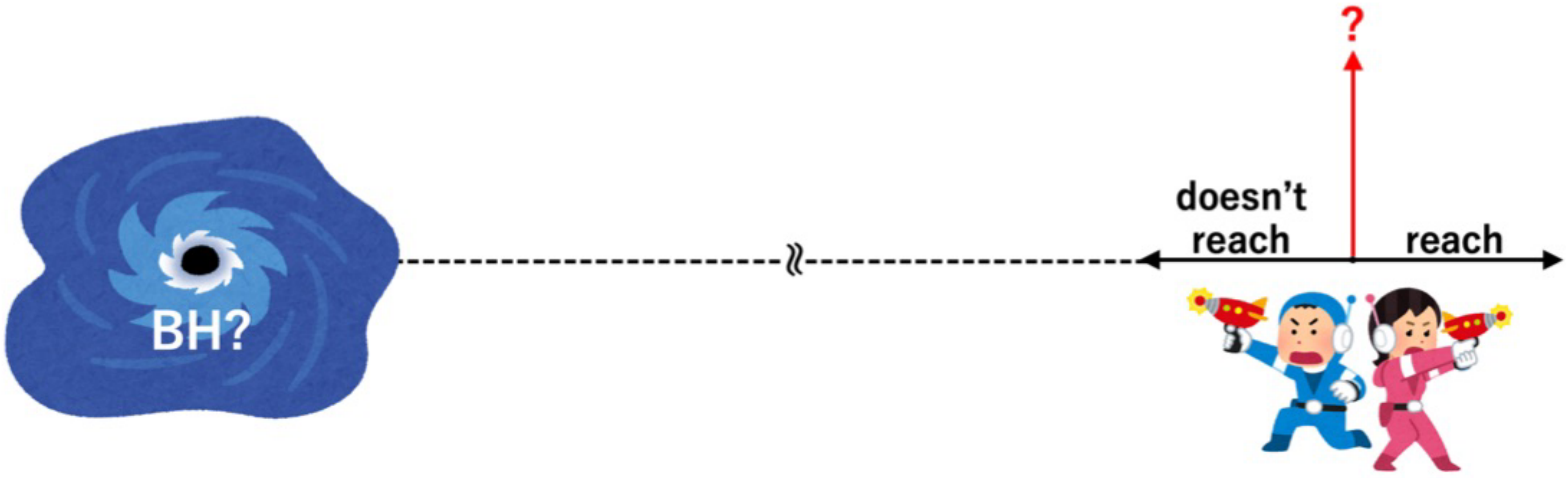
Observable region



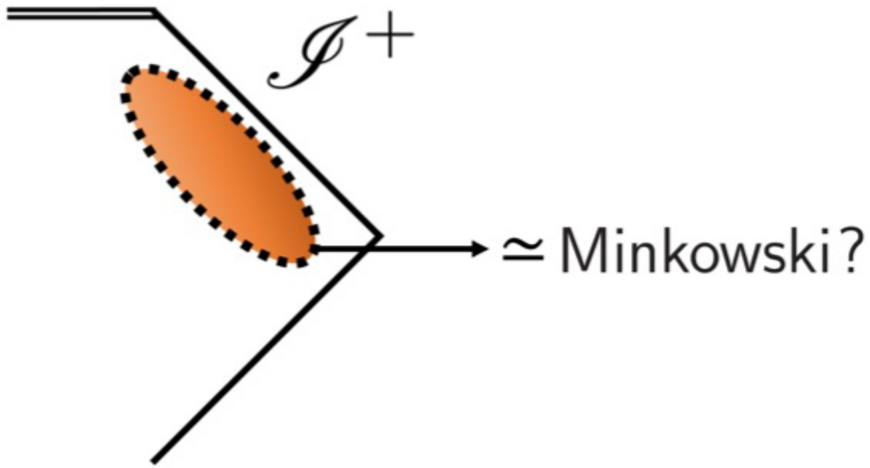
Alternative definition to BHs should be discussed.



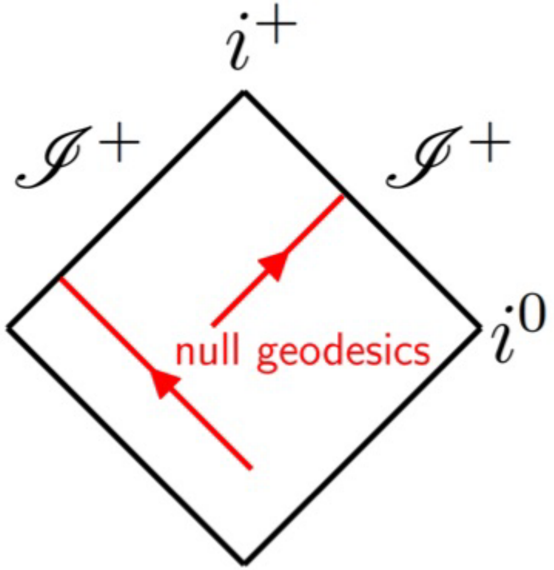
As the first step, we consider null geodesics in the direction of red arrow from near $\mathscr{I}^+$.



Null geodesics near $\mathscr{I}^+$: Main conclusion



Asymptotically flat spacetime



Minkowski spacetime

Q. Behavior of null geodesic near $\mathscr{I}^+$ is simple as in Minkowski? or not?

A. It is not simple in four-dim. spacetimes, and simple in higher-dim. spacetimes.

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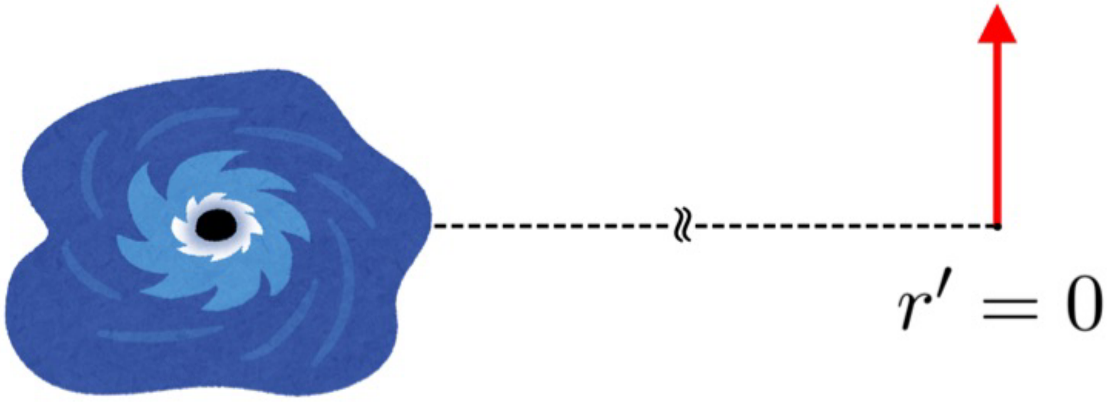
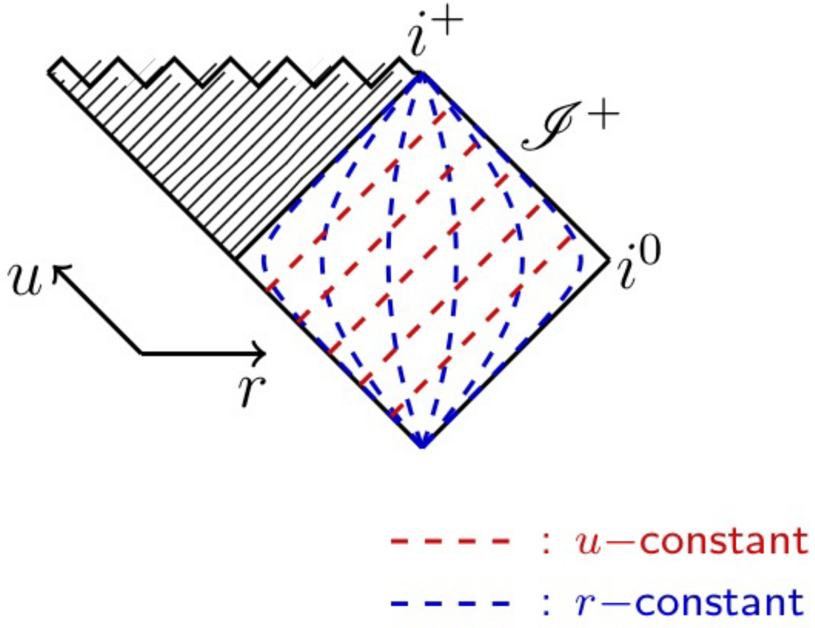
Bondi coordinates (n (≥ 4)-dimensional asym. flat spacetimes)

[Bondi *et al.* (1962), Sachs (1962), Tanabe *et al.* (2011)]

$$ds^2 = -Ae^B du^2 - 2e^B du dr + h_{IJ} r^2 (dx^I + C^I du)(dx^J + C^J du)$$

$(\det h_{IJ} = \det \omega_{IJ}, \quad \omega_{IJ}$: metric of unit $(n-2)$-sphere$)$





Behavior right after emission with $r' = 0$

[$'$: derivative with respect to affine parameter λ, $\dot{}$: u-derivative.]

Geodesic equation: $r'' = -\Gamma^r_{\mu\nu}(x^\mu)'(x^\nu)' = \Omega_{IJ} r (x^I)'(x^J)' + \mathcal{O}\left(r^{-(n-5)/2}\right)\left|(x^I)'\right|^2$

$$\Omega_{IJ} := \omega_{IJ} - \frac{1}{2}\dot{h}^{(1)}_{IJ} r^{-(n/2-2)} + \frac{1}{2}\dot{\mu}\,\omega_{IJ} r^{-(n/2-2)}$$

$n(\geq 5)$ dim.

At large r, $\Omega_{IJ} \simeq \omega_{IJ}$ is positive definite.

r is large enough $\Rightarrow r'' > 0$

four dim.

Ω_{IJ} may not be positive definite.

Ω_{IJ} is positive definite and r is large enough $\Rightarrow r'' > 0$

Def. of $h^{(1)}_{IJ}$, m in n dim.

$$h_{IJ} = \omega_{IJ} + h^{(1)}_{IJ}(u, x^I) r^{-(n/2-1)} + \mathcal{O}\left(r^{-(n-1)/2}\right), \quad A = 1 - \mu(u, x^I) r^{-(n/2-1)} + \mathcal{O}\left(r^{-(n-1)/2}\right)$$

$$ds^2 = -Ae^B du^2 - 2e^B du dr + h_{IJ} r^2 (dx^I + C^I du)(dx^J + C^J du)$$

ω_{IJ} : metric for unit $(n-2)$-sphere

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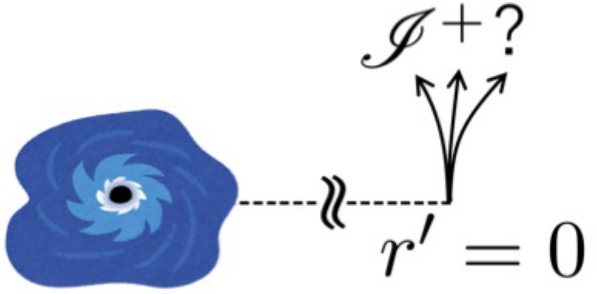
ω_{IJ} : metric for unit $(n-2)$-sphere

Global analysis of null geodesics

Amo-Izumi-Tomikawa-Yoshino-Shiromizu, *Phys. Rev. D* (2021) [2106.03150]



Technical analysis of global behavior yields:

- $n \geq 5$: Always reach $\mathscr{I}^+$
- $n = 4$: If $\Omega_{IJ} := \omega_{IJ} - \frac{1}{2}\dot{h}^{(1)}_{IJ} + \frac{1}{2}\dot{\mu}\omega_{IJ}$ is positive definite and $\dot{\mu} \leq 0$, reach $\mathscr{I}^+$.

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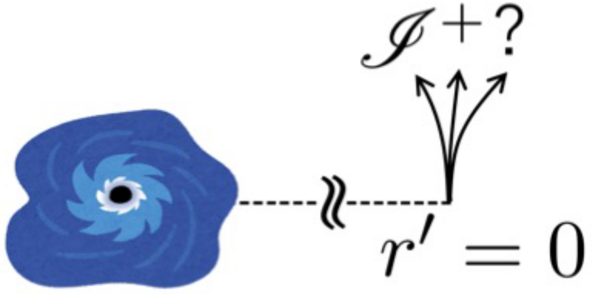

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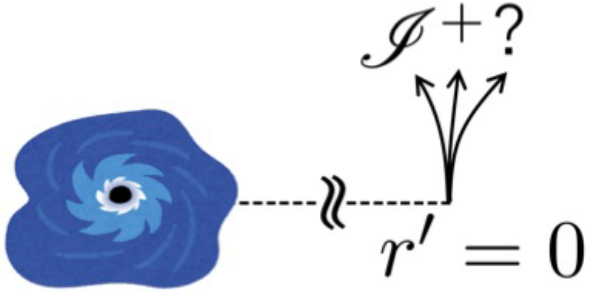


Technical analysis of global behavior yields:

$$
\begin{aligned}
r'' &= -\left[\dot{m} r^{-1} + \mathcal{O}\left(r^{-2}\right)\right] u' r' + \left[\Omega_{IJ} r + \mathcal{O}\left(r^{0}\right)\right] \left(x^I\right)' \left(x^J\right)' \\
&\quad - \left[2 C^{(1)}{}_I r^{-1} + \mathcal{O}\left(r^{-2}\right)\right] r' \left(x^I\right)' + \left[2 B^{(1)} r^{-3} + \mathcal{O}\left(r^{-4}\right)\right] r'^2 \\
&\quad + \left[\left(C^{(1)}{}_I \dot{m} + m_{,I} - C^{(1)J} \dot{h}^{(1)}_{IJ}\right) r^{-1} + \mathcal{O}\left(r^{-2}\right)\right] \left(x^I\right)' u' \\
&= -\dot{m} r^{-1} u' r' + \left[\Omega_{IJ} r + \mathcal{O}\left(r^{0}\right)\right] \left(x^I\right)' \left(x^J\right)' + \mathcal{O}\left(r^{-1}\right) r' \left|\left(x^I\right)'\right| \\
&\quad + \left[2 B^{(1)} r^{-3} + \mathcal{O}\left(r^{-4}\right)\right] r'^2 \\
&> -\dot{m} r^{-1} u' r' + \left[\Omega_{IJ} r + \mathcal{O}\left(r^{0}\right)\right] \left(x^I\right)' \left(x^J\right)' - \tilde{C}_1 r^{-1} r' \left|\left(x^I\right)'\right| + \left[2 B^{(1)} r^{-3} + \mathcal{O}\left(r^{-4}\right)\right] r'^2 \\
&\geq -\dot{m} r^{-1} u' r' + \left[\Omega_{IJ} r + \mathcal{O}\left(r^{0}\right)\right] \left(x^I\right)' \left(x^J\right)' - \frac{1}{2} \tilde{C}_1 \left[r^{-2} r'^2 + \left|\left(x^I\right)'\right|^2\right] + \left[2 B^{(1)} r^{-3} + \mathcal{O}\left(r^{-4}\right)\right] r'^2 \\
&> -\dot{m} r^{-1} u' r' + \left[\Omega_{IJ} r + \mathcal{O}\left(r^{0}\right)\right] \left(x^I\right)' \left(x^J\right)' - \tilde{C}_2 r^{-2} r'^2
\end{aligned}
$$

ive definite and $\dot{\mu} \leq 0$, reach $\mathscr{I}^+$.

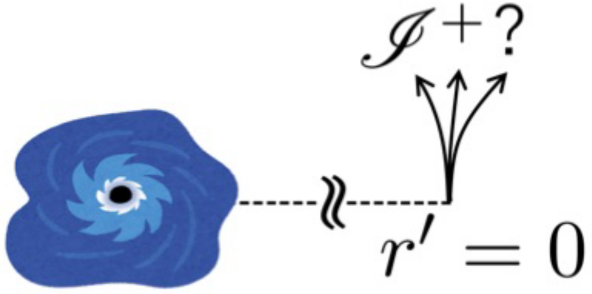
$A = 1 - \mu(u, x^I) r^{-1} + \mathcal{O}\left(r^{-2}\right)$

${}^2(dx^I + C^I du)(dx^J + C^J du)$

$(n-2)$-sphere

Global analysis of null geodesics

Amo-Izumi-Tomikawa-Yoshino-Shiromizu, *Phys. Rev. D* (2021) [2106.03150]



Technical analysis of global behavior yields:

$r'' = -[\dot{m} r^{-1} + \mathcal{O}(r^{-2})] u'r' + [\Omega$

$- [2C^{(1)}{}_I r^{-1} + \mathcal{O}(r^{-2}$

$+ [(C^{(1)}{}_I \dot{m} + m_{,I} - C$

$= -\dot{m} r^{-1} u'r' + [\Omega_{IJ} r + \mathcal{O}(r^0)$

$+ [2B^{(1)} r^{-3} + \mathcal{O}(r^{-4}$

$> -\dot{m} r^{-1} u'r' + [\Omega_{IJ} r + \mathcal{O}(r^0)$

$\geq -\dot{m} r^{-1} u'r' + [\Omega_{IJ} r + \mathcal{O}(r^0)$

$> -\dot{m} r^{-1} u'r' + [\Omega_{IJ} r + \mathcal{O}(r^0)$

$$\frac{r''}{r'} > -\frac{\tilde{C}_2}{r^2} r'.$$

By integrating out this inequality,

$$\log r' > \frac{\tilde{C}_2}{r} + \tilde{C}_3$$

is obtained, where $\tilde{C}_3$ is the integral constant. Thus, we have

$$r' > \exp\left(\frac{\tilde{C}_2}{r} + \tilde{C}_3\right) > \tilde{C}_4,$$

where $\tilde{C}_4 := e^{\tilde{C}_3} > 0$. Integrating this inequality again, we obtain

$$r > \tilde{C}_4 \lambda + \tilde{C}_5,$$

ve definite and $\dot{\mu} \leq 0$, reach $\mathscr{I}^+$.

$A = 1 - \mu(u, x^I) r^{-1} + \mathcal{O}(r^{-2})$

$(dx^I + C^I du)(dx^J + C^J du)$

$n - 2)$-sphere

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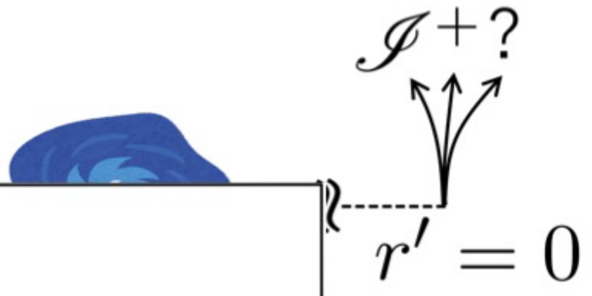

Technical analysis of global behavior yields:

By integrat

is obtained

where $\tilde{C}_4$:

$$\left[\omega_{IJ} + \left(h^{(1)}_{IJ} + m\omega_{IJ}\right) r^{-1} + \mathcal{O}\left(r^{-2}\right)\right] \left(x^I\right)' \left(x^J\right)'$$
$$= r^{-2}u'^2 + 2\left[r^{-2} + mr^{-3} + \mathcal{O}\left(r^{-4}\right)\right]u'r' - \left[2C^{(1)}{}_I r^{-2} + \mathcal{O}\left(r^{-3}\right)\right] \left(x^I\right)' u'$$
$$= \left[r^{-2} + \mathcal{O}\left(r^{-3}\right)\right]u'^2 + 2\left[r^{-2} + mr^{-3} + \mathcal{O}\left(r^{-4}\right)\right]u'r' + \mathcal{O}\left(r^{-1}\right)\left|\left(x^I\right)'\right|^2,$$

$$0 \le r^{-2}u'\left|\left(x^I\right)'\right| = \left(r^{-3}u'^2\right)^{1/2}\left[r^{-1}\left|\left(x^I\right)'\right|^2\right]^{1/2} \le \frac{1}{2}\left[r^{-3}u'^2 + r^{-1}\left|\left(x^I\right)'\right|^2\right]$$

which implies

$$r^{-2}u'\left(x^I\right)' = \mathcal{O}\left(r^{-3}\right)u'^2 + \mathcal{O}\left(r^{-1}\right)\left|\left(x^I\right)'\right|^2$$

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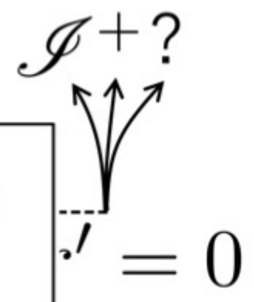


Technical analysis of global be

$$
\begin{aligned}
r'' &= -\left[\dot m r^{-1} + \mathcal{O}\left(r^{-2}\right)\right] u' r' + \left[\Omega \ldots \right.\\
&\quad - \left[2C^{(1)}{}_I r^{-1} + \mathcal{O}\left(r^{-2}\right) \ldots \right.\\
&\quad + \left[\left(C^{(1)}{}_I \dot m + m_{,I} - C \ldots \right.\right.\\
&= -\dot m r^{-1} u' r' + \left[\Omega_{IJ} r + \mathcal{O}\left(r^0\right) \ldots \right.\\
&\quad + \left[2B^{(1)} r^{-3} + \mathcal{O}\left(r^{-4}\right) \ldots \right.\\
&> -\dot m r^{-1} u' r' + \left[\Omega_{IJ} r + \mathcal{O}\left(r^0\right) \ldots \right.\\
&\geq -\dot m r^{-1} u' r' + \left[\Omega_{IJ} r + \mathcal{O}\left(r^0\right) \ldots \right.\\
&> -\dot m r^{-1} u' r' + \left[\Omega_{IJ} r + \mathcal{O}\left(r^0\right) \ldots \right.
\end{aligned}
$$

By integrat

is obtained

where $\tilde{C}_4$:

$\left[\omega_{IJ} + \ldots\right.$

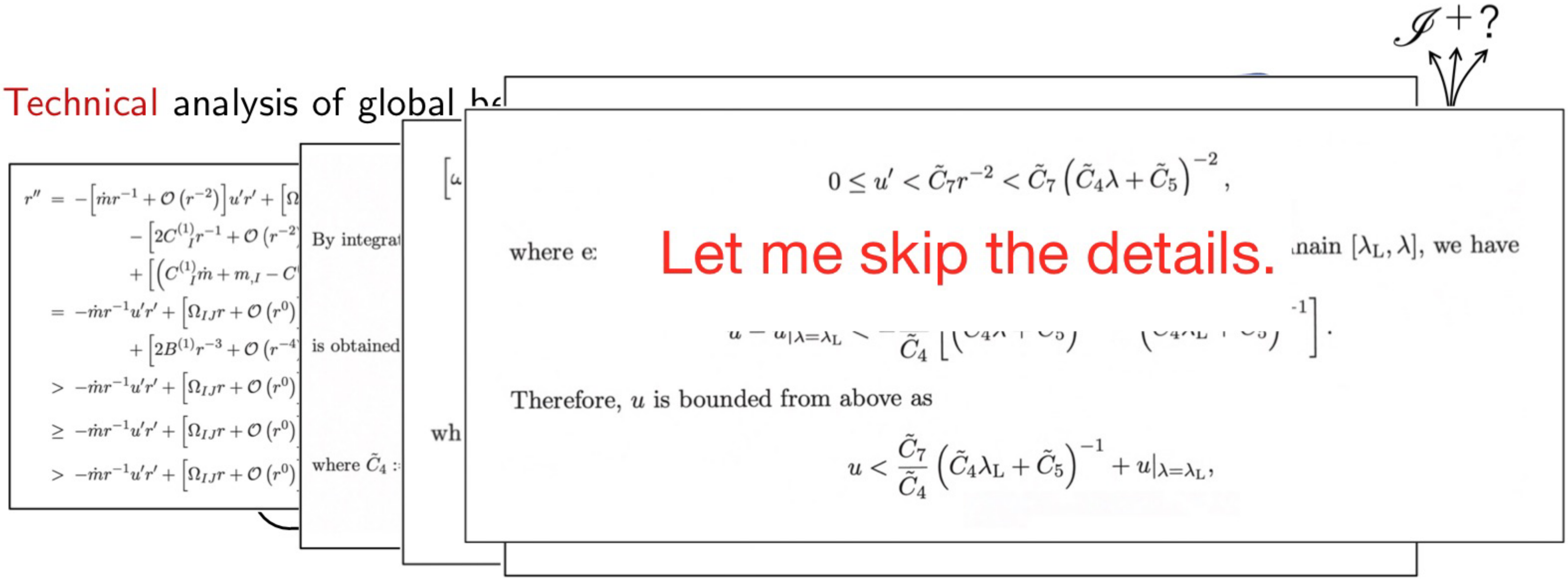
which

$$
\begin{aligned}
u'' &= \left[\left(-\dot{B}^{(1)} + \frac{1}{2}m\right) r^{-2} + \mathcal{O}\left(r^{-3}\right)\right] u'^2 + \mathcal{O}\left(r^{-3}\right) u'^2 + \mathcal{O}\left(r^{-1}\right) \left|\left(x^I\right)'\right|^2 \\
&\qquad - \left[\omega_{IJ} r + \frac{1}{2} h^{(1)}_{IJ} + \mathcal{O}\left(r^{-1}\right)\right] \left(x^I\right)' \left(x^J\right)' \\
&= \mathcal{O}\left(r^{-2}\right) u'^2 - \left[\omega_{IJ} + \mathcal{O}\left(r^{-1}\right)\right] r \left(x^I\right)' \left(x^J\right)' \\
&= \mathcal{O}\left(r^{-2}\right) u'^2 - \left[\frac{1}{r} + \mathcal{O}\left(r^{-2}\right)\right] u'^2 - \left[\frac{2}{r} + \mathcal{O}\left(r^{-2}\right)\right] u' r' \\
&= -\left[\frac{1}{r} + \mathcal{O}\left(r^{-2}\right)\right] u'^2 - \left[\frac{2}{r} + \mathcal{O}\left(r^{-2}\right)\right] u' r' \\
&< -\left(\frac{2}{r} - \frac{\tilde{C}_6}{r^2}\right) u' r'
\end{aligned}
$$

Global analysis of null geodesics

Amo-Izumi-Tomikawa-Yoshino-Shiromizu, *Phys. Rev. D* (2021) [2106.03150]

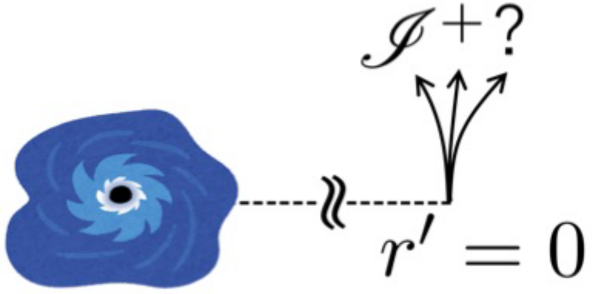
Technical analysis of global be



Let me skip the details.

Global analysis of null geodesics

Amo-Izumi-Tomikawa-Yoshino-Shiromizu, *Phys. Rev. D* (2021) [2106.03150]



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Def. of $h^{(1)}_{IJ}$, m in four dim.

$$h_{IJ} = \omega_{IJ} + h^{(1)}_{IJ}(u, x^I)r^{-1} + \mathcal{O}\left(r^{-2}\right), \quad A = 1 - \mu(u, x^I)r^{-1} + \mathcal{O}\left(r^{-2}\right)$$

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ω_{IJ} : metric for unit $(n-2)$-sphere

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Traceless part of extrinsic curvature σ_{ab} of $r =$const.

[σ_{ab}: Traceless part of extrinsic curvature]

<u>$n(\geq 5)$ dim.</u>

$$\sigma_{uu} = \frac{n-2}{n-1} r^{-1} + \mathcal{O}\left(r^{-(n/2-1)}\right)$$

$$\sigma_{uI} = \sigma_{Iu} = \mathcal{O}\left(r^{-(n/2-1)}\right)$$

$$\sigma_{IJ} = \frac{1}{n-1}\omega_{IJ} r + \mathcal{O}\left(r^{-(n/2-3)}\right)$$

→ (Leading part of σ_{ab}) $\neq 0$

<u>four dim.</u>

$$\sigma_{uu} = \left(\frac{2}{3} + \frac{1}{3}\dot{\mu}\right) r^{-1} + \mathcal{O}\left(r^{-2}\right)$$

$$\sigma_{uI} = \sigma_{Iu} = \mathcal{O}\left(r^{-1}\right)$$

$$\sigma_{IJ} = \left(\frac{1}{3}\omega_{IJ} - \frac{1}{2}\dot{h}^{(1)}_{IJ} + \frac{1}{6}\dot{\mu}\omega_{IJ}\right) r + \mathcal{O}\left(r^{0}\right)$$

(Leading part of σ_{ab}) is 0

$$\Leftrightarrow \Omega_{IJ} := \omega_{IJ} - \tfrac{1}{2}\dot{h}^{(1)}_{IJ} + \tfrac{1}{2}\dot{\mu}\omega_{IJ} = 0$$

key quantity for null geodesics!

Approximate photon surface

[σ_{ab}: Traceless part of extrinsic curvature]

Photon surface [Claudel et al. (2001), Perlick (2005)]

The photon surface is a hypersurface where all initially tangent null geodesics remain tangent.

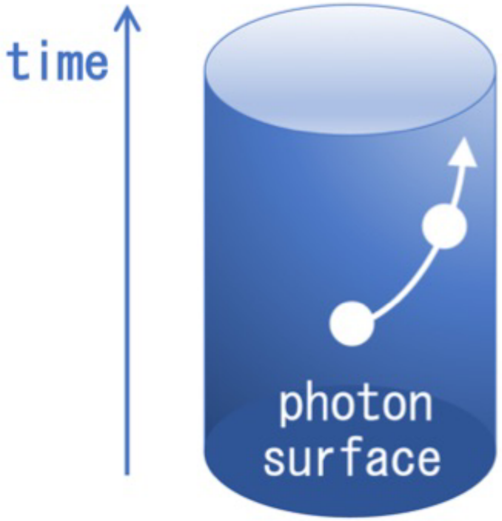
S is a photon surface $\Leftrightarrow \sigma_{ab} = 0$ on S

$r =$const. is a photon surface

$$\Leftrightarrow \sigma_{ab} = 0 \Leftrightarrow \Omega_{IJ} = 0 \Leftrightarrow r'' = 0$$

→ We call $r =$const. "approximate photon surfaces"

when $\sigma_{ab} = 0$ at the leading order in r^{-1} expansion.



Condition for approximate photon surfaces

Amo-Shiromizu-Izumi-Yoshino-Tomikawa *Phys. Rev. D* (2022) [2110.10917]

$n(\geq 5)$ dim.

$$\sigma_{uu} = \frac{n-2}{n-1} r^{-1} + \mathcal{O}\left(r^{-(n/2-1)}\right)$$
$$\sigma_{uI} = \sigma_{Iu} = \mathcal{O}\left(r^{-(n/2-1)}\right)$$
$$\sigma_{IJ} = \frac{1}{n-1}\omega_{IJ} r + \mathcal{O}\left(r^{-(n/2-3)}\right)$$

(Leading part of σ_{ab}) $\neq 0$

$r =$const. cannot be approximate photon surfaces.

four dim.

$$\sigma_{uu} = \left(\frac{2}{3} + \frac{1}{3}\dot{\mu}\right) r^{-1} + \mathcal{O}\left(r^{-2}\right)$$
$$\sigma_{uI} = \sigma_{Iu} = \mathcal{O}\left(r^{-1}\right)$$
$$\sigma_{IJ} = \left(\frac{1}{3}\omega_{IJ} - \frac{1}{2}\dot{h}^{(1)}_{IJ} + \frac{1}{6}\dot{\mu}\omega_{IJ}\right) r + \mathcal{O}\left(r^{0}\right)$$

(Leading part of σ_{ab}) is 0

$$\Leftrightarrow \Omega_{IJ} := \omega_{IJ} - \tfrac{1}{2}\dot{h}^{(1)}_{IJ} + \tfrac{1}{2}\dot{\mu}\omega_{IJ} = 0$$

$r =$const. are approximate photon surfaces if and only if $\Omega_{IJ} = 0 \quad (\Leftrightarrow r'' = 0)$.

Similar discussion is valid for Dynamically Transversely Trapping Surface (See [1909.08420]).

Explicit example in Outgoing Vaidya metric

Amo-Shiromizu-Izumi-Yoshino-Tomikawa [*Phys. Rev. D* (2022) 2110.10917]

Outgoing Vaidya metric :

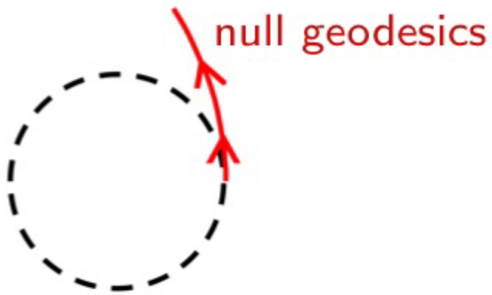
$$ds^2 = -\left(1 - \frac{2M(u)}{r}\right) du^2 - 2dudr + r^2 \omega_{IJ} dx^I dx^J,$$

with

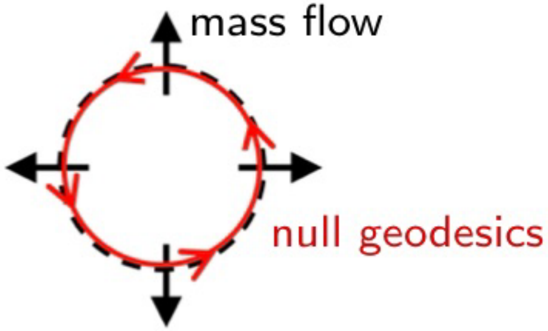
$$\dot{M}(u) = -\left(1 - \frac{2M(u)}{r_{\rm PS}}\right)\left(1 - \frac{3M(u)}{r_{\rm PS}}\right).$$

$r' = 0 \Rightarrow r'' = 0$ on $r = r_{\rm PS}$ (at full order).

→ $r = r_{\rm PS}$: exact photon surface (for arbitrarily large $r_{\rm PS}$)



Schwarzchild spacetime



Outgoing Vaidya spacetime

Table of Contents

Conclusion

[Asymptotic behavior of null geodesics]

- Investigated the condition to reach $\mathscr{I}^+$
- Focused on null geodesics emitted in $r' = 0$

[Asymptotic behavior of extrinsic curvature]

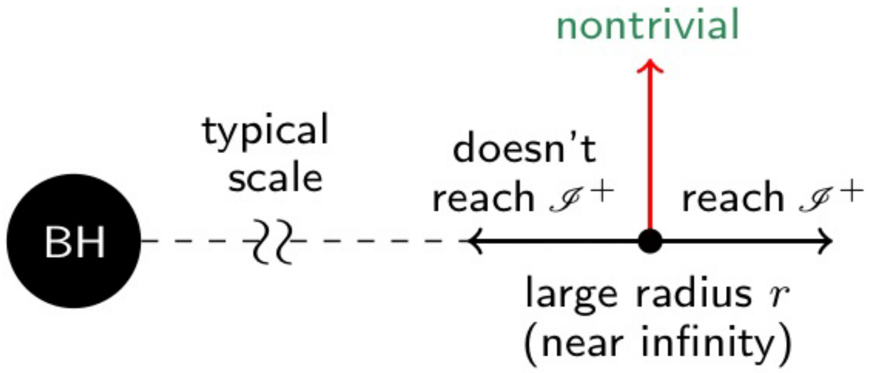
- Photon surface may exist for large r in approximate sense.
- Exact discussion is valid for specific spacetimes.

A key quantity $\Omega_{IJ} := \omega_{IJ} - \frac{1}{2}\dot{h}^{(1)}_{IJ} r^{-(n/2-2)} + \frac{1}{2}\dot{\mu}\omega_{IJ} r^{-(n/2-2)}$

This leads to non-trivial difference between four-dim. and higher-dim.

Overview

Naive sketch



Bondi coordinates (u, r, x^I) (n (≥ 4) dimensional asym. flat spacetimes)

[Bondi *et al.* (1962), Sachs (1962), Tanabe *et al.* (2011)]

$$g_{IJ} = \boxed{\omega_{IJ} r^2} + \boxed{h^{(1)}_{IJ}(u, x^I) r^{-(n/2-3)}} + \mathcal{O}\left(r^{-(n-5)/2}\right)$$

$$g_{uu} = \boxed{-1} + \boxed{m(u, x^I) r^{-(n-3)}} + O(r^{-(n/2-1)})$$

flat — relevant to the result in $n = 4$

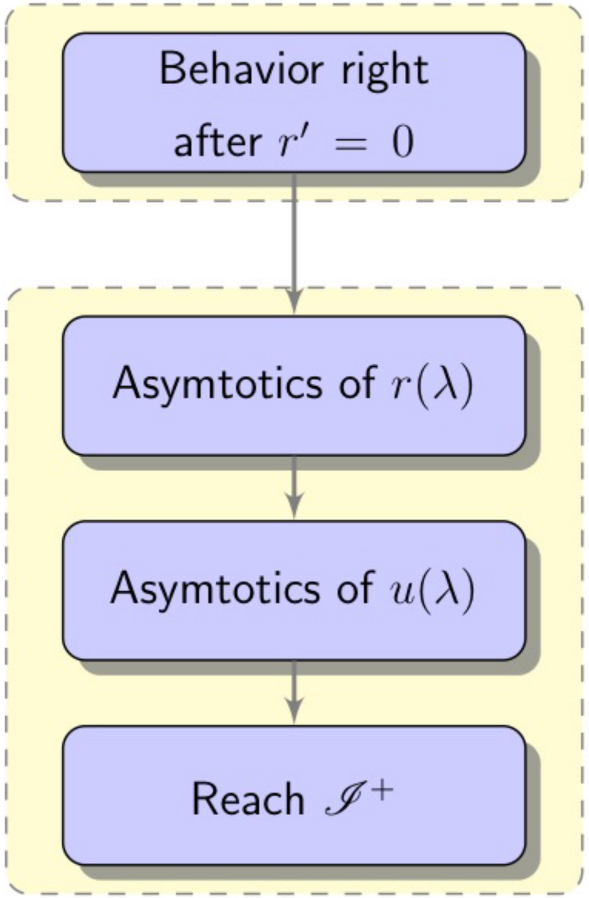
Conclusion for nontrivial null geodesics (in general asymptotically flat spacetimes)

$(n \geq 5)$ dimensions: Always reach $\mathscr{I}^+$

4 dimensions: $\Omega_{IJ} := \underset{\substack{\uparrow \\ \text{positive definite}}}{\omega_{IJ}} - \frac{1}{2}\underset{\substack{\uparrow \\ \text{time-derivative of GW}}}{\dot{h}^{(1)}_{IJ}} + \frac{1}{2}\underset{\substack{\uparrow \\ \text{time-derivative of mass}}}{\dot{\mu}}\omega_{IJ}$ is positive definite and $\dot{\mu} \leq 0$ $\Rightarrow$ reach $\mathscr{I}^+$

Nontrivial difference between 4 dim. and higher dim.!

Rough story of proof



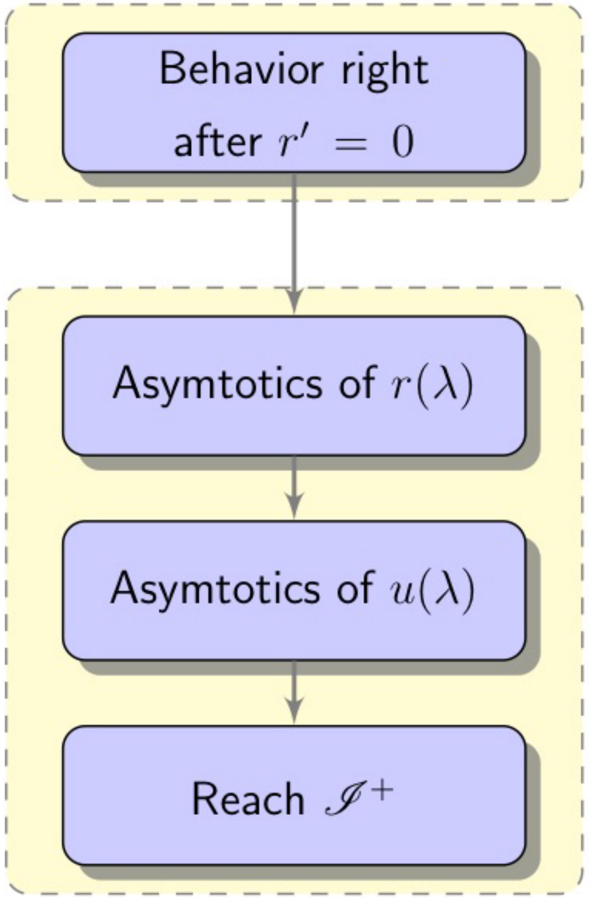
Impose assumptions in four dim.

Do not impose assumptions in higher dim.

We use

- Geodesic equations
- Geodesic is null
- Initially $r' = 0$

Rough story of the proof



Impose assumptions in four dim.

Do not impose assumptions in higher dim.

We use

- Geodesic equations
- Geodesic is null 6 slides
- Initially $r' = 0$

Preparation - Bondi coordinates (1)

$$ds^2 = -Ae^B du^2 - 2e^B dudr + h_{IJ} r^2 (dx^I + C^I du)(dx^J + C^J du),$$

where A, B, C^I and h_{IJ} are functions of u, r, and x^I.

We expand h_{IJ} near future null infinity as

$$h_{IJ} = \omega_{IJ} + \sum_{k \geq 0} h_{IJ}^{(k+1)} r^{-(n/2+k-1)},$$

where ω_{IJ} is the metric for the unit $(n-2)$-sphere, $k \in \mathbb{Z}$ for even dimensions, and $2k \in \mathbb{Z}$ for odd dimensions.

Preparation - Bondi coordinates (2)

By using the vacuum Einstein equations $R_{\mu\nu} = 0$, the falloff behavior of A, B, and C^I can be given as [Tanabe *et al.* (2011)]

$$
\begin{aligned}
A &= 1 + \sum_{k=0}^{k<n/2-2} A^{(k+1)} r^{-(n/2+k-1)} - m(u, x^I) r^{-(n-3)} + \mathcal{O}\left(r^{-(n-5/2)}\right), \\
B &= B^{(1)} r^{-(n-2)} + \mathcal{O}\left(r^{-(n-3/2)}\right), \\
C^I &= \sum_{k=0}^{k<n/2-1} C^{(k+1)I} r^{-(n/2+k)} + J^I(u, x^I) r^{-(n-1)} + \mathcal{O}\left(r^{-(n-1/2)}\right),
\end{aligned}
$$

where $A^{(k+1)}$, $B^{(1)}$, $C^{(k+1)I}$, m, and J^I are functions of u and x^I.

Preparation - Bondi coordinates (3)

The nonzero components of the metric and of the inverse metric behave as

$$\begin{aligned} g_{uu} &= -Ae^{B} + h_{IJ}C^{I}C^{J}r^{2} = -1 - A^{(1)}r^{-(n/2-1)} + mr^{-(n-3)} + \mathcal{O}\left(r^{-(n-1)/2}\right), \\ g_{ur} &= -e^{B} = -1 - B^{(1)}r^{-(n-2)} + \mathcal{O}\left(r^{-(n-3/2)}\right), \\ g_{IJ} &= h_{IJ}r^{2} = \omega_{IJ}r^{2} + h^{(1)}_{IJ}r^{-(n/2-3)} + \mathcal{O}\left(r^{-(n-5)/2}\right), \\ g_{uI} &= h_{IJ}C^{J}r^{2} = C^{(1)}{}_{I}r^{-(n/2-2)} + \mathcal{O}\left(r^{-(n-3)/2}\right), \end{aligned}$$

Preparation - Bondi coordinates (4)

In particular, in four dimensions, the behavior of the metric components is written as

$$
\begin{aligned}
g_{uu} &= -1 + mr^{-1} + \mathcal{O}\left(r^{-2}\right), & g_{ur} = -1 - B^{(1)} r^{-2} + \mathcal{O}\left(r^{-3}\right),\\
g_{IJ} &= \omega_{IJ} r^2 + h^{(1)}_{IJ} r + \mathcal{O}\left(r^{0}\right), & g_{uI} = C^{(1)}_{\ \ I} + \mathcal{O}\left(r^{-1}\right)\\
g^{ur} &= -1 + B^{(1)} r^{-2} + \mathcal{O}\left(r^{-3}\right), & g^{rr} = 1 - mr^{-1} + \mathcal{O}\left(r^{-2}\right),\\
g^{rI} &= C^{(1)I} r^{-2} + \mathcal{O}\left(r^{-3}\right), & g^{IJ} = \omega^{IJ} r^{-2} - h^{(1)IJ} r^{-3} + \mathcal{O}\left(r^{-4}\right)
\end{aligned}
$$

Preparation - Geodesic equations for r and u

$$
\begin{aligned}
r'' &= \left[-\frac{1}{2}\dot{A}^{(1)}r^{-(n/2-1)}+\mathcal{O}\left(r^{-(n-1)/2}\right)\right]u'^2+\left[\frac{n-2}{2}A^{(1)}r^{-n/2}+\mathcal{O}\left(r^{-(n+1)/2}\right)\right]u'r' \\
&\quad +\left[(n-2)B^{(1)}r^{-(n-1)}+\mathcal{O}\left(r^{-(n-1/2)}\right)\right]r'^2 \\
&\quad +\left[\left(-\frac{n-4}{2}C^{(1)}_{I}-A^{(1)}_{,I}\right)r^{-(n/2-1)}+\mathcal{O}\left(r^{-(n-1)/2}\right)\right]u'\left(x^I\right)' \\
&\quad -\left[\frac{n}{2}C^{(1)}_{I}r^{-(n/2-1)}+\mathcal{O}\left(r^{-(n-1)/2}\right)\right]r'\left(x^I\right)' \\
&\quad +\left[\omega_{IJ}r-\frac{1}{2}\dot{h}^{(1)}_{IJ}r^{-(n/2-3)}+\mathcal{O}\left(r^{-(n-5)/2}\right)\right]\left(x^I\right)'\left(x^J\right)', \\
u'' &= -\left[\frac{n-2}{4}A^{(1)}r^{-n/2}+\mathcal{O}\left(r^{-(n+1)/2}\right)\right]u'^2-2\left[-\frac{n-4}{4}C^{(1)}_{I}r^{-(n/2-1)}+\mathcal{O}\left(r^{-(n-1)/2}\right)\right]u'\left(x^I\right)' \\
&\quad -\left[\omega_{IJ}r-\frac{n-6}{4}h^{(1)}_{IJ}r^{-(n/2-2)}+\mathcal{O}\left(r^{-(n-3)/2}\right)\right]\left(x^I\right)'\left(x^J\right)',
\end{aligned}
$$

Preparation - The null condition

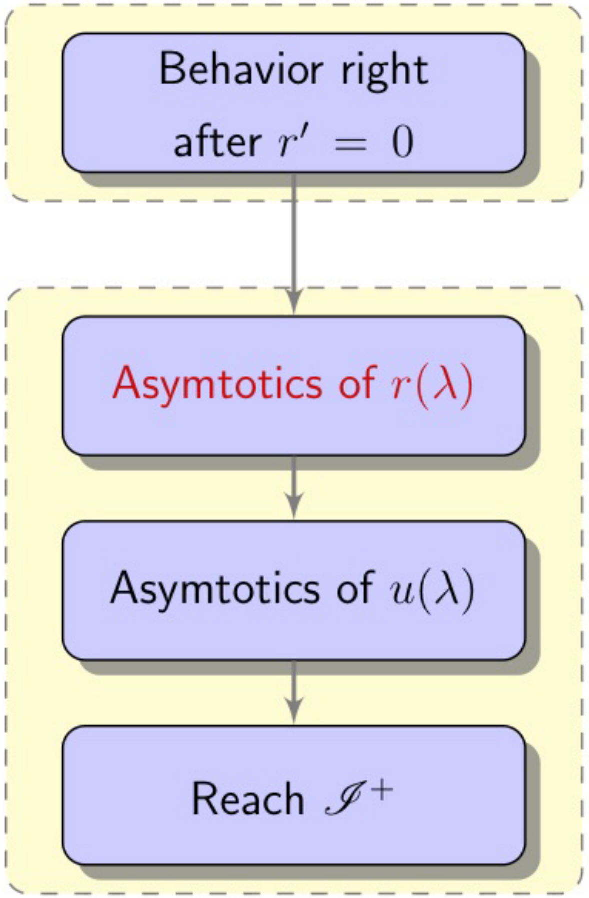
The null condition for the tangent vector of null geodesics gives us

$$\begin{aligned} u'^2 \;=\; & -2\left[1+mr^{-1}+\mathcal{O}\left(r^{-2}\right)\right]u'r' + \left[\omega_{IJ}r^2+\left(h^{(1)}_{IJ}+m\omega_{IJ}\right)r+\mathcal{O}\left(r^0\right)\right]\left(x^I\right)'\left(x^J\right)' \\ & +\left[2C^{(1)}{}_I+\mathcal{O}\left(r^{-1}\right)\right]\left(x^I\right)'u'. \end{aligned}$$

This is algebraically solved as

$$u' \;=\; \frac{-e^B r' + h_{IJ}C^J r^2\left(x^I\right)' + \sqrt{\left[e^B r' - h_{IJ}C^J r^2 (x^I)'\right]^2 + (Ae^B - h_{IJ}C^I C^J r^2) h_{KL} r^2 (x^K)'(x^L)'}}{Ae^B - h_{MN}C^M C^N r^2}$$

Rough story of the proof



4 slides

Impose assumptions in four dim.

Do not impose assumptions in higher dim.

We use

- Geodesic equations
- Geodesic is null
- Initially $r' = 0$

Null condition (four dimensions)

Let us prove the existence of the lower bound of r'. The null condition:

$$u' = \frac{-e^{B}r' + h_{IJ}C^{J}r^{2}\left(x^{I}\right)' + \sqrt{\left[e^{B}r' - h_{IJ}C^{J}r^{2}(x^{I})'\right]^{2} + (Ae^{B} - h_{IJ}C^{I}C^{J}r^{2})h_{KL}r^{2}(x^{K})'(x^{L})'}}{Ae^{B} - h_{MN}C^{M}C^{N}r^{2}}.$$

gives

$$u' = \left[r + \mathcal{O}\left(r^{0}\right)\right]\left|\left(x^{I}\right)'\right|.$$

Geodesic equation for r (four dimensions)

$\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ are positive constants.

$$\begin{aligned} r'' &= -\dot{\mu}r^{-1}u'r' + \left[\Omega_{IJ}r + \mathcal{O}\left(r^0\right)\right]\left(x^I\right)'\left(x^J\right)' + \mathcal{O}\left(r^{-1}\right)r'\left|\left(x^I\right)'\right| + \left[2B^{(1)}r^{-3} + \mathcal{O}\left(r^{-4}\right)\right]r'^2 \\ &> -\dot{\mu}r^{-1}u'r' + \left[\Omega_{IJ}r + \mathcal{O}\left(r^0\right)\right]\left(x^I\right)'\left(x^J\right)' - \tilde{C}_1 r^{-2} r'^2 \end{aligned}$$

Assuming $\Omega_{IJ} := \omega_{IJ} - \frac{1}{2}\dot{h}^{(1)}_{IJ} + \frac{1}{2}\dot{\mu}\omega_{IJ}$ is positive definite and $\dot{\mu} \leq 0$,

$$\begin{aligned} r'' &> -\tilde{C}_1 r^{-2} r'^2 \\ \therefore \quad r' &> \exp\left(\frac{\tilde{C}_1}{r} + \tilde{C}_2\right) > e^{\tilde{C}_2} \\ \therefore \quad r &> e^{\tilde{C}_2}\lambda + \tilde{C}_3 \end{aligned}$$

Null condition (higher dimensions)

Let us prove the existence of the lower bound of r'.

The null condition:

$$u' = \frac{-e^B r' + h_{IJ} C^J r^2 \left(x^I\right)' + \sqrt{\left[e^B r' - h_{IJ} C^J r^2 (x^I)'\right]^2 + (Ae^B - h_{IJ} C^I C^J r^2) h_{KL} r^2 (x^K)'(x^L)'}}{Ae^B - h_{MN} C^M C^N r^2}.$$

gives

$$u' = \left[r + \mathcal{O}\left(r^{-(n/2-2)}\right)\right] \left|\left(x^I\right)'\right|.$$

Geodesic equation for r (higher dimensions)

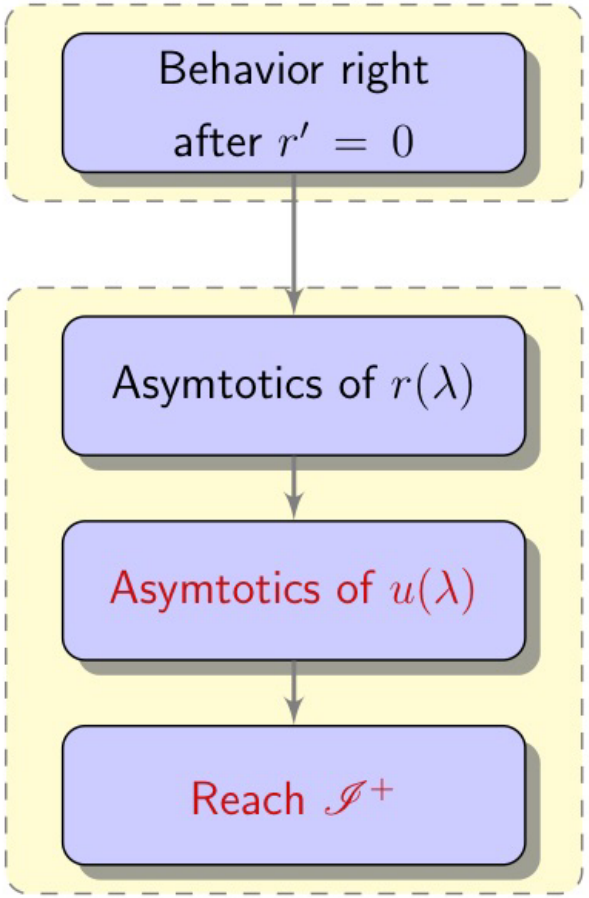
$0<\alpha<1$, $\hat{C}_1$, $\hat{C}_2$, and $\hat{C}_3$ are positive constants.

$$\begin{aligned} r'' &= \left[\omega_{IJ} r - \frac{1}{2}\left(\dot{A}^{(1)}\omega_{IJ} + \dot{h}^{(1)}_{IJ}\right) r^{-(n/2-3)} + \mathcal{O}\left(r^{-(n-5)/2}\right)\right]\left(x^I\right)'\left(x^J\right)' \\ &\quad + \mathcal{O}\left(r^{-(n/2-2)}\right) r' \left|\left(x^I\right)'\right| + \left[(n-2)B^{(1)} r^{-(n-1)} + \mathcal{O}\left(r^{-(n-1/2)}\right)\right] r'^2 \\ &> \left(\omega_{IJ} r + \mathcal{O}\left(r^{\alpha}\right)\right)\left(x^I\right)'\left(x^J\right)' - \hat{C}_1 r^{-(n+\alpha-4)} r'^2 \\ &\geq -\hat{C}_1 r^{-(n+\alpha-4)} r'^2 \end{aligned}$$

Without assuming any conditions,

$$r > \hat{C}_2\lambda + \hat{C}_3$$

Rough story of the proof



2 slides

Impose assumptions in four dim.

Do not impose assumptions in higher dim.

We use

- Geodesic equations
- Geodesic is null
- Initially $r' = 0$

Geodesic equation for u

The procedure is quite similar, thus we focus on four dimensions.

We define $\mathscr{I}^+$ as the region where $r=\infty$ and u is finite.

With the arithmetic-geometric mean inequality, the null condition is

$$\left[\omega_{IJ}+\mathcal{O}\left(r^{-1}\right)\right]\left(x^I\right)'\left(x^J\right)'=\left[r^{-2}+\mathcal{O}\left(r^{-3}\right)\right]u'^2+\left[2r^{-2}+\mathcal{O}\left(r^{-3}\right)\right]u'r'.$$

Substituting this to the geodesic equation for u,

$$\begin{aligned} u'' &= \left[\left(-\dot{B}^{(1)}+\tfrac{1}{2}m\right)r^{-2}+\mathcal{O}\left(r^{-3}\right)\right]u'^2+\mathcal{O}\left(r^{-3}\right)u'^2+\mathcal{O}\left(r^{-1}\right)\left|\left(x^I\right)'\right|^2 \\ &< -\left(\tfrac{2}{r}-\tfrac{\tilde{C}_6}{r^2}\right)u'r' \end{aligned}$$

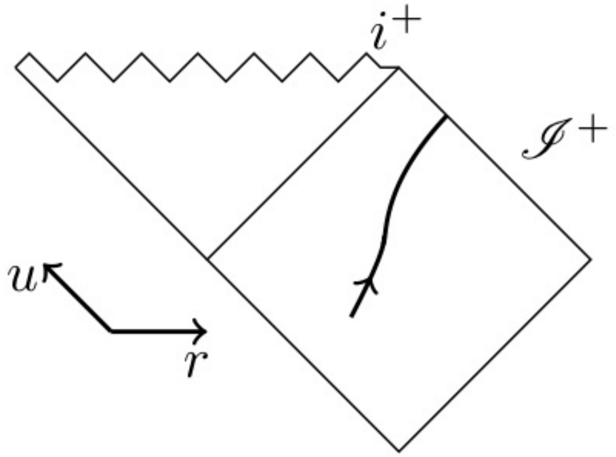
$$\therefore \quad \left(r^2\exp\left(\frac{\tilde{C}_6}{r}\right)u'\right)'<0$$

Upper bound of u

By $\left(r^2 \exp\left(\frac{\tilde{C}_6}{r}\right) u'\right)' < 0$, u is bounded from above as

$$u < \frac{\tilde{C}_7}{\tilde{C}_4}\left(\tilde{C}_4\lambda_{\mathrm{L}} + \tilde{C}_5\right)^{-1} + u|_{\lambda=\lambda_{\mathrm{L}}},$$

$\therefore$ r goes to infinity while u remains finite.



Asymptotic symmetry

ξ: generator of the asymptotic symmentry, $Q_\xi := (x^\mu)' \xi_\mu$, Q_ξ is asymptotic conserved quantity：

$$Q_\xi = \mathrm{constant} + \mathcal{O}\left(r^{-n/2}\right).$$

In particular,

$\xi = \partial_u$: asymptotic conservation of energy

$\xi = f^I \partial_I$: asymptotic conservation of angular momentum

Details of Bondi coordinates

[Bondi *et al.* (1962), Sachs (1962), Tanabe *et al.* (2011)]

Bondi coordinates which well describes $n\ (\geq 4)$ dim. asymptotically flat spacetimes :

$$ds^2 = -Ae^B du^2 - 2e^B dudr + h_{IJ} r^2 (dx^I + C^I du)(dx^J + C^J du)$$

ω_{IJ} is the metric of unit $(n-2)$-dim. sphere,

$$h_{IJ} = \omega_{IJ} + \sum_{k\geq 0} h_{IJ}^{(k+1)} r^{-(n/2+k-1)}, \qquad B = O(r^{-(n-2)}),$$

$$A = 1 - m(u, x^I) r^{-(n-3)} + O(r^{-(n/2-1)}), \qquad C^I = O(r^{-(n/2)})$$

$$\det h_{IJ} = \det \omega_{IJ}$$

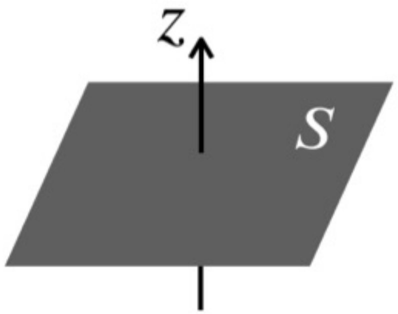
Photon surface

Def. [Claudel et al. (2001)]

The photon surface is a hypersurface where all initially tangent null geodesics remain tangent.

Trivial Example：

$z = 0$ in Minkowski spacetime $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$



Traceless part of extrinsic curvature σ_{ab} : $n(\geq 4)$-dim.

$$\sigma^a_b := \chi^a_b - (n-1)^{-1}\delta^a_b\chi^c_c,$$

$$\sigma_{uu} = \frac{n-2}{n-1}r^{-1} - \frac{n-2}{2(n-1)}\dot{A}^{(1)}r^{-(n/2-1)} + \frac{n-2}{2(n-1)}\dot{\mu}r^{-(n-3)} + \mathcal{O}\left(r^{-(n-1)/2}\right)$$

$$\sigma_{uI} = \sigma_{Iu} = -\left(\frac{n^2-n-4}{4(n-1)}C^{(1)}_I + \frac{1}{2}A^{(1)}_{,I}\right)r^{-(n/2-1)} + \left(\frac{1}{2}m_{,I} - \frac{1}{2}C^{(1)J}\dot{h}^{(1)}_{IJ} - \frac{1}{2(n-1)}\dot{A}^{(1)}C^{(1)}_I\right)r^{-(n-3)}$$
$$+ \frac{1}{2(n-1)}\dot{\mu}C^{(1)}_I r^{-(3n/2-5)} + \mathcal{O}\left(r^{-(n-1)/2}\right)$$

$$\sigma_{IJ} = \frac{1}{n-1}\omega_{IJ}r - \frac{1}{2}\dot{h}^{(1)}_{IJ}r^{-(n/2-3)} - \frac{1}{2(n-1)}\dot{A}^{(1)}\omega_{IJ}r^{-(n/2-3)} + \frac{1}{2(n-1)}\dot{\mu}\omega_{IJ}r^{-(n-5)} + \mathcal{O}\left(r^{-(n-5)/2}\right)$$

Traceless part of extrinsic curvature σ_{ab} : Four dimensions

$$\sigma_{uu} = \left(\frac{2}{3} + \frac{1}{3}\dot{\mu}\right) r^{-1} + \mathcal{O}\left(r^{-2}\right)$$
$$\sigma_{uI} = \sigma_{Iu} = \mathcal{O}\left(r^{-1}\right)$$
$$\sigma_{IJ} = \left(\frac{1}{3}\omega_{IJ} - \frac{1}{2}\dot{h}^{(1)}_{IJ} + \frac{1}{6}\dot{\mu}\omega_{IJ}\right) r + \mathcal{O}\left(r^{0}\right)$$

Leading part of σ_{ab} is 0

$\Leftrightarrow \dot{\mu} = -2$ and $\dot{h}^{(1)}_{IJ} = 0$

$\Leftrightarrow \Omega_{IJ} := \omega_{IJ} - \frac{1}{2}\dot{h}^{(1)}_{IJ} + \frac{1}{2}\dot{\mu}\omega_{IJ} = 0$

Bondi coordinates (n (≥ 4)-dimensional asym. flat spacetimes)

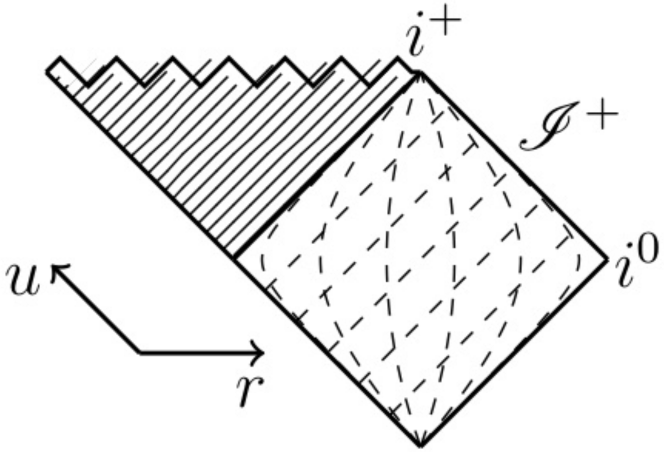
[Bondi *et al.* (1962), Sachs (1962), Tanabe *et al.* (2011)]

$$ds^2 = -Ae^B du^2 - 2e^B du dr + h_{IJ} r^2 (dx^I + C^I du)(dx^J + C^J du)$$

($\det h_{IJ} = \det \omega_{IJ}$, ω_{IJ} : the metric of the unit $(n-2)$-sphere)

$$h_{IJ} = \omega_{IJ} + h^{(1)}_{IJ}(u, x^I) r^{-(n/2-1)} + \mathcal{O}\left(r^{-(n-1)/2}\right),$$
$$A = 1 - m(u, x^I) r^{-(n-3)} + \mathcal{O}\left(r^{-(n/2-1)}\right).$$

flat; relevant to the result in $n = 4$



Behavior right after the emission with $r' = 0$

[$'$: derivative with respect to affine parameter λ , $\dot{}$: u-derivative.]

$n(\geq 5)$ dim.

$$r'' = -\Gamma^r_{\mu\nu}(x^\mu)'(x^\nu)'$$
$$= \omega_{IJ} r (x^I)'(x^J)'$$
$$+\mathcal{O}\left(r^{-(n/2-3)}\right)\left|(x^I)'\right|^2$$

$\therefore$ ~~ω_{IJ} is positive definite and~~ r is large enough

$\downarrow$

always true

$\Rightarrow r'' > 0$

four dim.

$$r'' = -\Gamma^r_{\mu\nu}(x^\mu)'(x^\nu)'$$
$$= \Omega_{IJ} r (x^I)'(x^J)' + O(r^0)\left|(x^I)'\right|^2$$

$$\Omega_{IJ} := \omega_{IJ} - \frac{1}{2}\dot{h}^{(1)}_{IJ} + \frac{1}{2}\dot{\mu}\omega_{IJ}$$

$\therefore$ Ω_{IJ} is positive definite and r is large enough

$\Rightarrow r'' > 0$

Def. of $h^{(1)}_{IJ}$, m in four dim.

$$h_{IJ} = \omega_{IJ} + h^{(1)}_{IJ}(u, x^I)r^{-1} + \mathcal{O}\left(r^{-2}\right), \quad A = 1 - m(u, x^I)r^{-1} + \mathcal{O}\left(r^{-2}\right),$$
$$ds^2 = -Ae^B du^2 - 2e^B dudr + h_{IJ}r^2(dx^I + C^I du)(dx^J + C^J du)$$