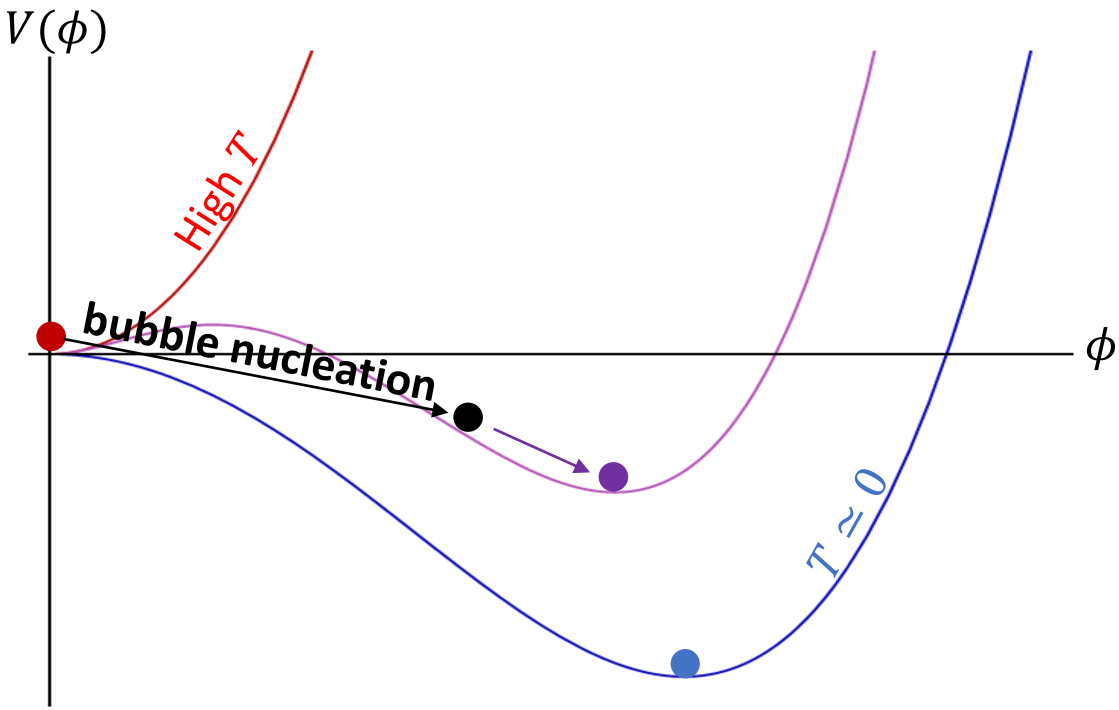
Primordial black holes from first-order phase transitions

Tae Hyun Jung

Florida State University

Cosmological first-order phase transitions



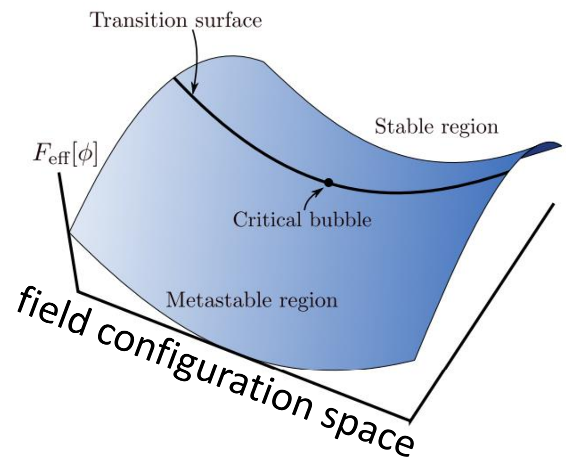
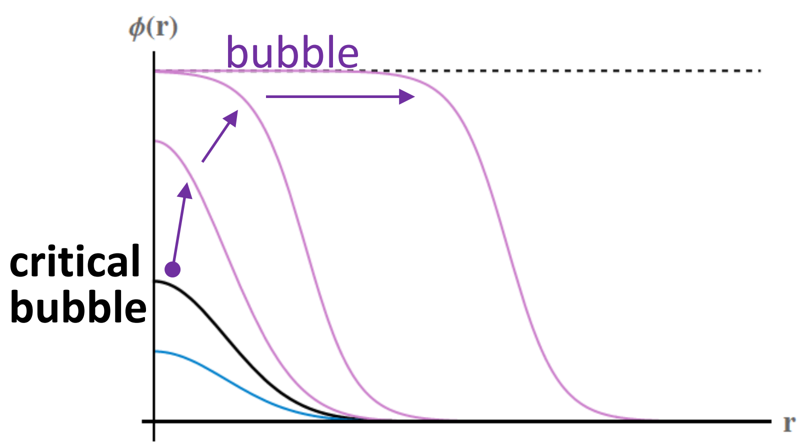
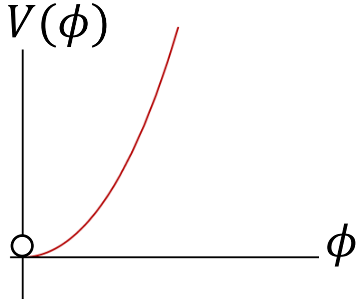


figure from
O. Gould, J. Hirvonen,
2108.04377



Cosmological first-order phase transitions

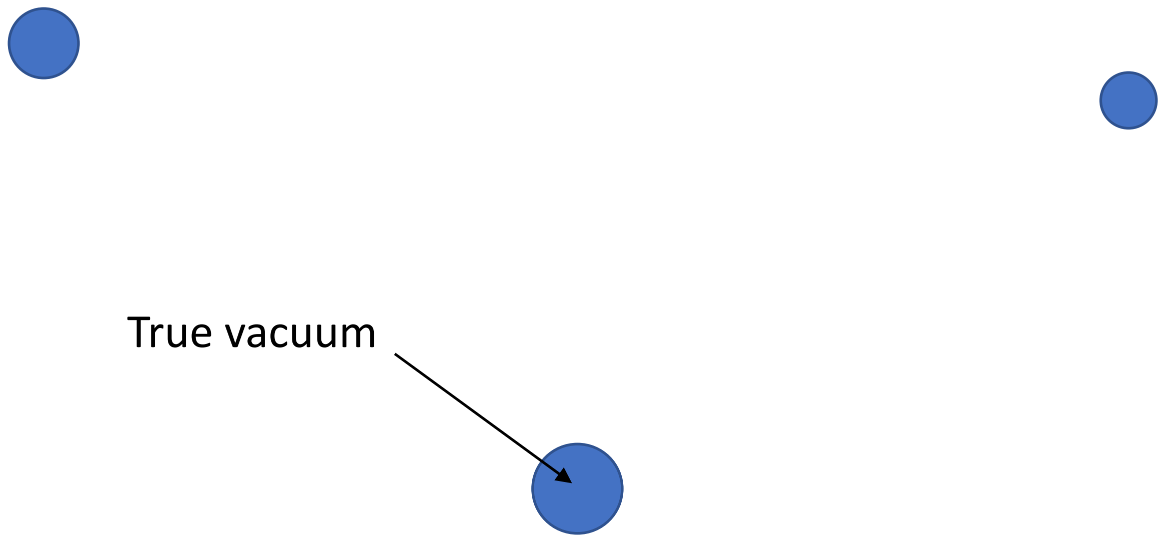
At high $T > T_c$

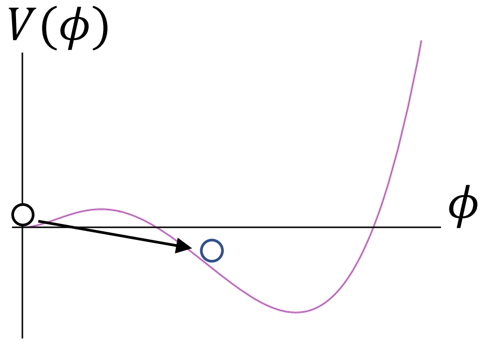


Cosmological first-order phase transitions

At $T < T_c$

bubbles nucleated

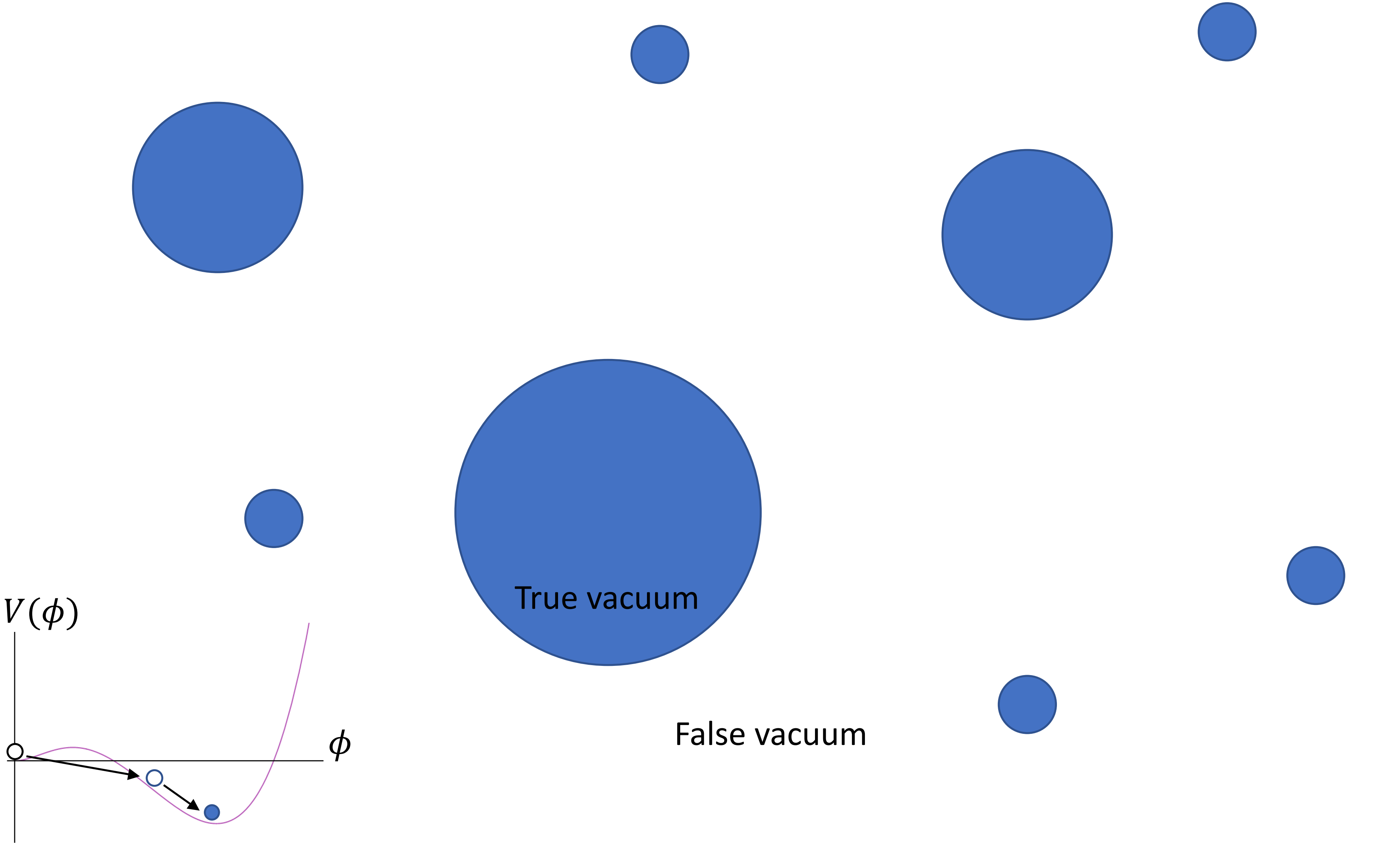




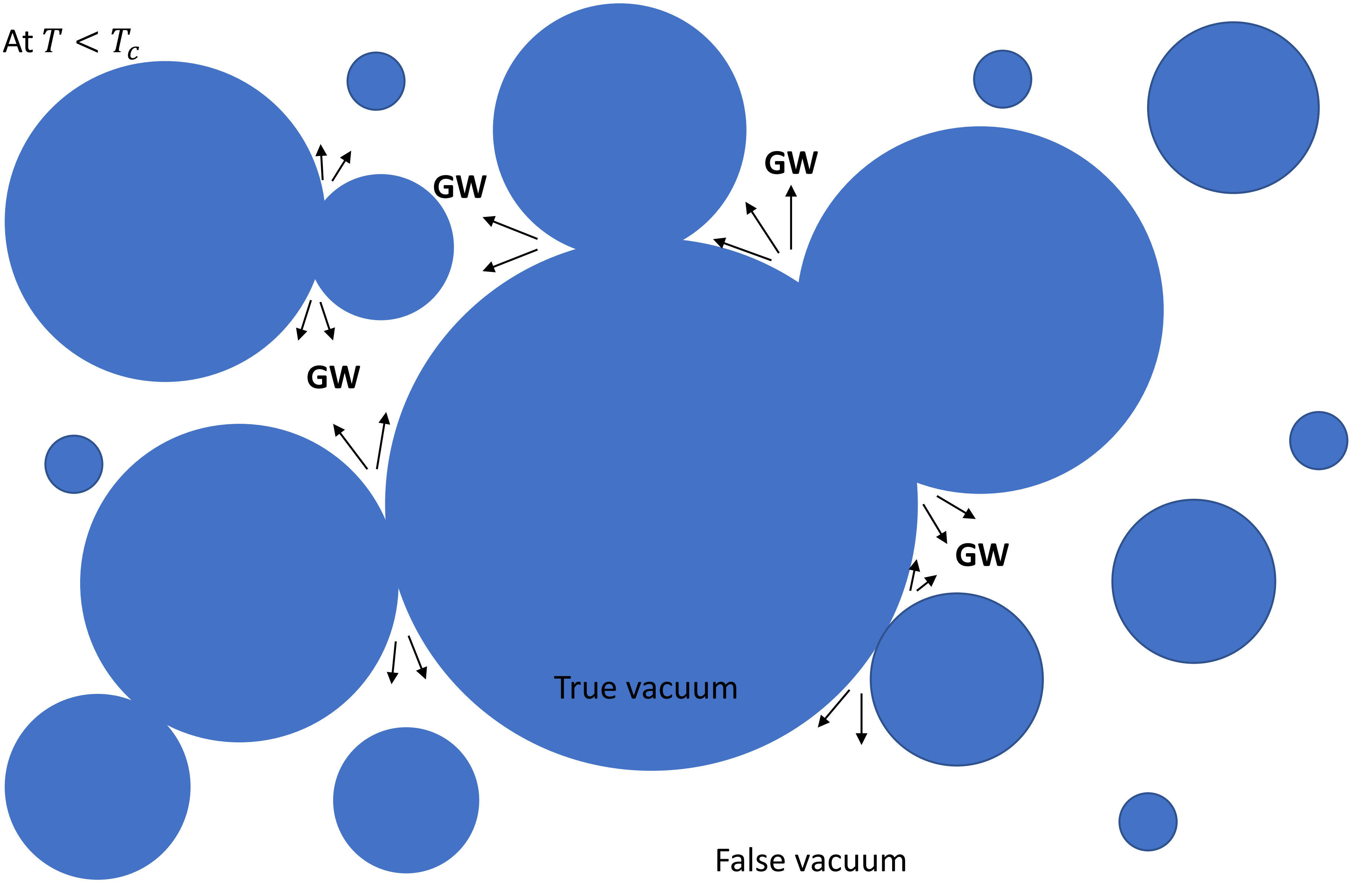
False vacuum

Cosmological first-order phase transitions

At $T < T_c$



Cosmological first-order phase transitions



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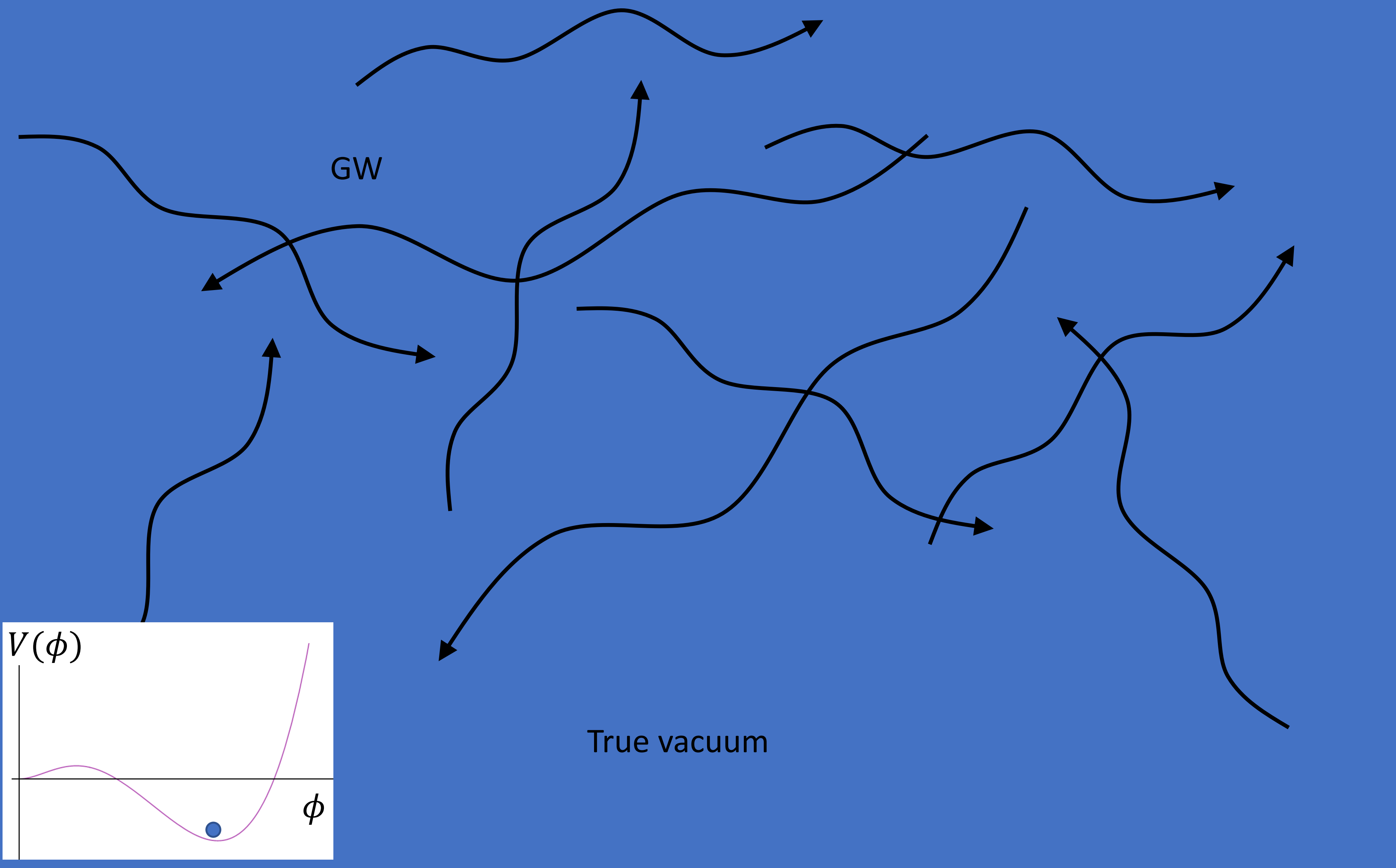


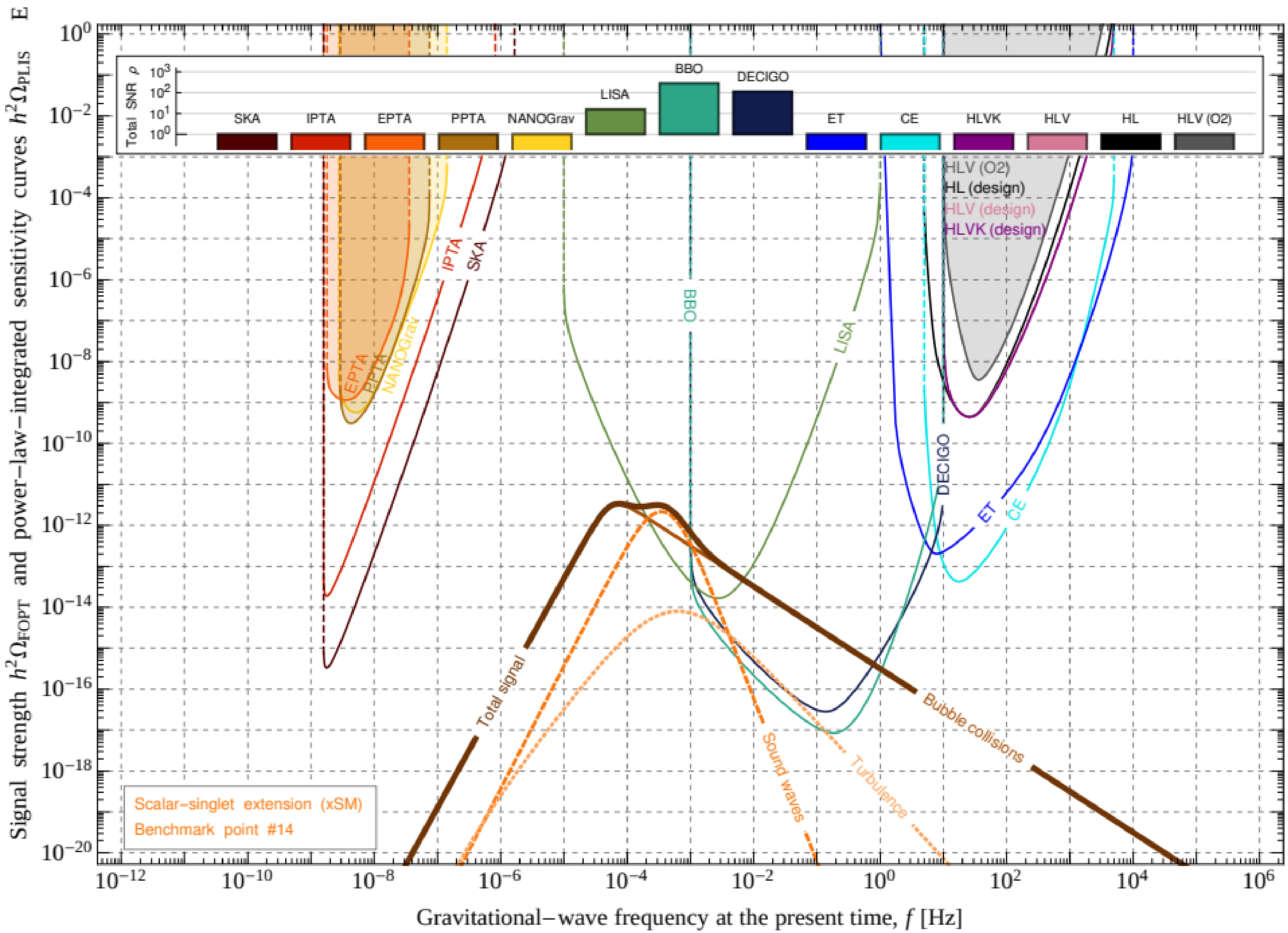
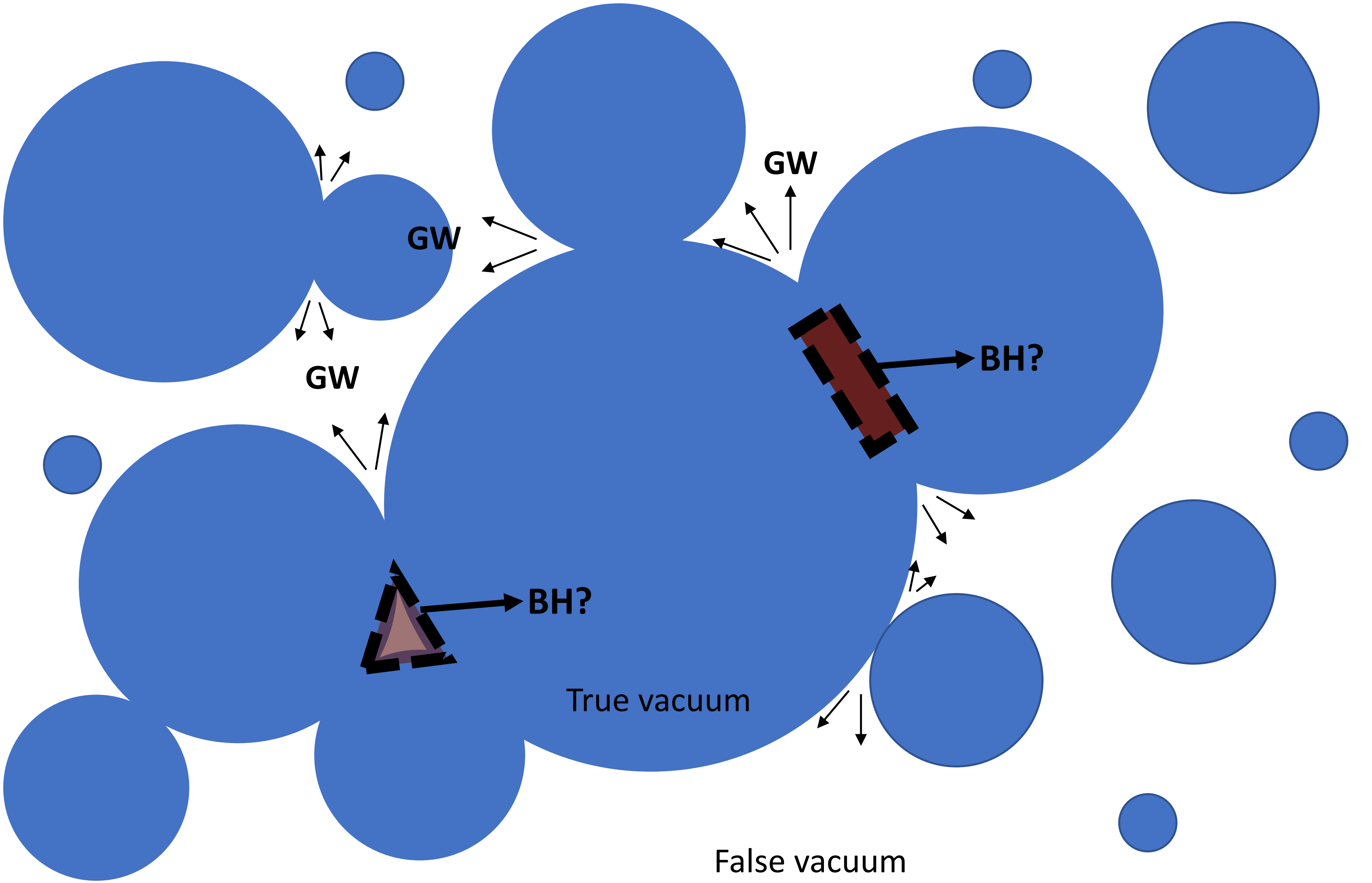


Fig. from
K. Schmitz, 2002.04615

Cosmological first-order phase transitions



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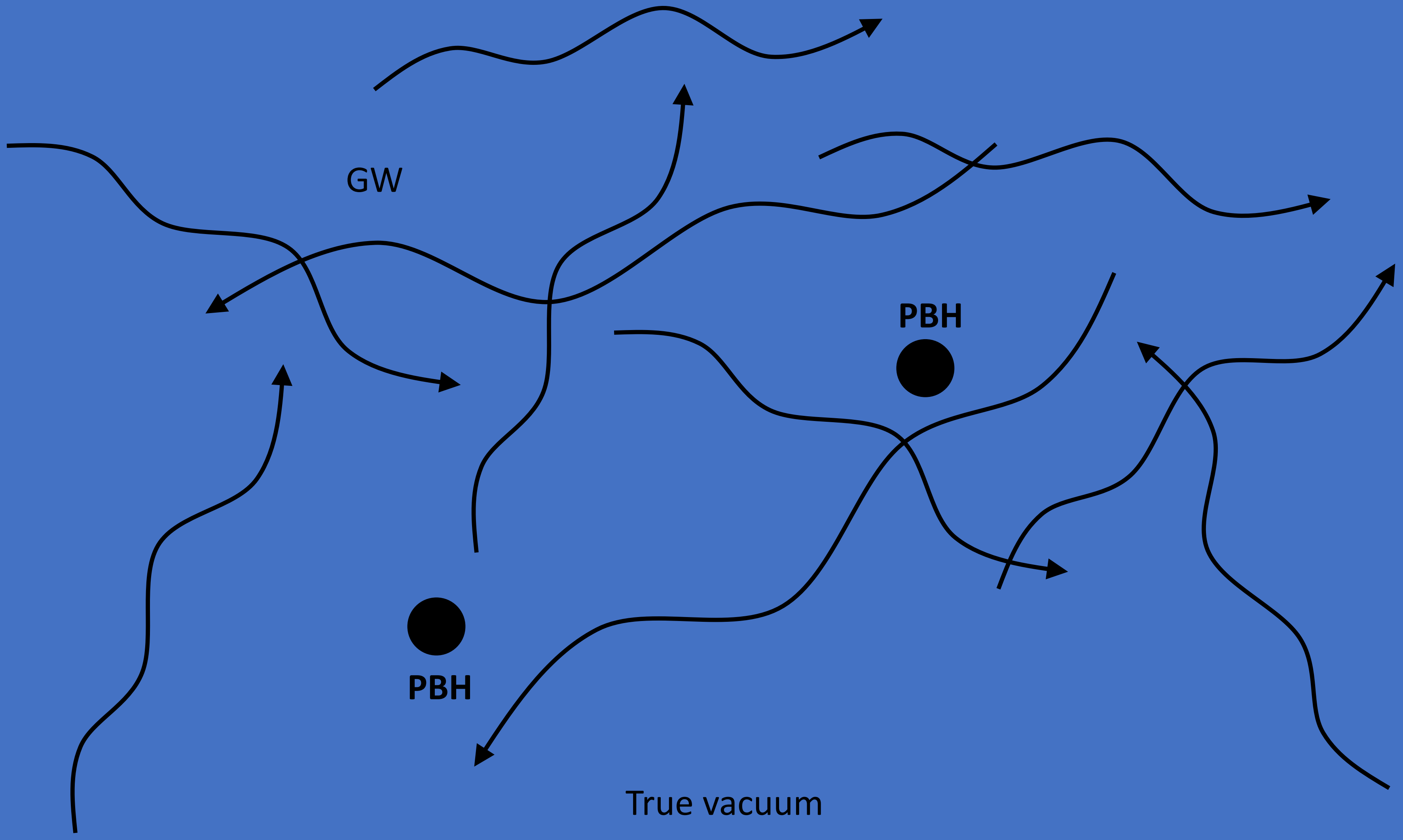

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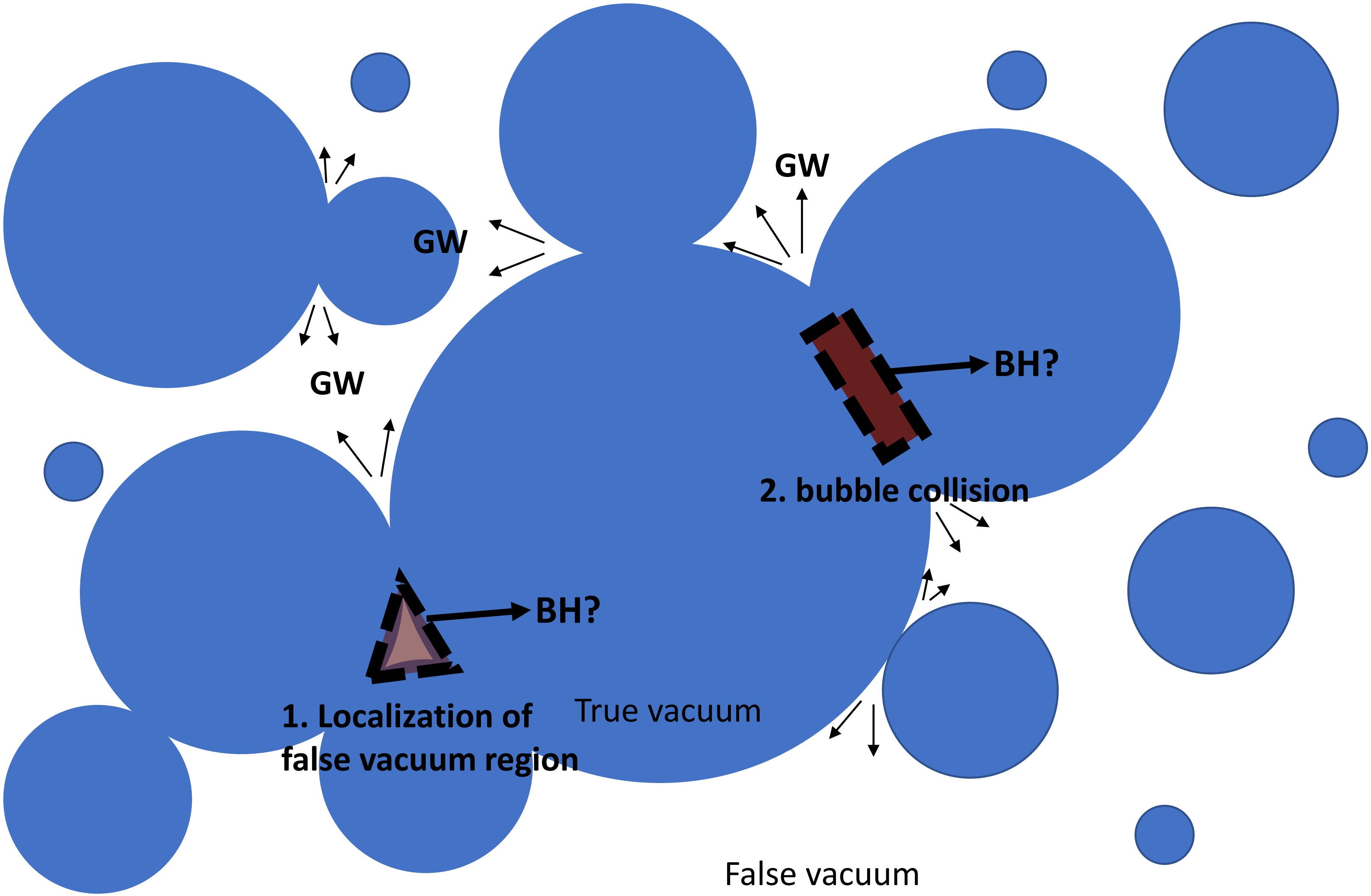

1. From localization of false vacuum region

From a localization of false vacuum

- K. Sato, M. Sasaki, H. Kodama, K. Maeda, 1981,
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With plasma in false vacuum region

- L. J. Hall and S. D. H. Hsu, 1990
  M. J. Baker, M. Breitbach, J. Kopp, L. Mittnacht, 2021
  2105.07481

During the inflation

- A. Kusenko, M. Sasaki, S. Sugiyama, M. Takada, V. Takhistov, E. Vitagliano, 2020
  2001.09160

Semi-bubble collision (bubble collision → false vacuum created → BH)

- M. Khlopov, R. Konoplich, S. Rubin, A. Sakharov, 1998, 1999
  hep-ph/9912422

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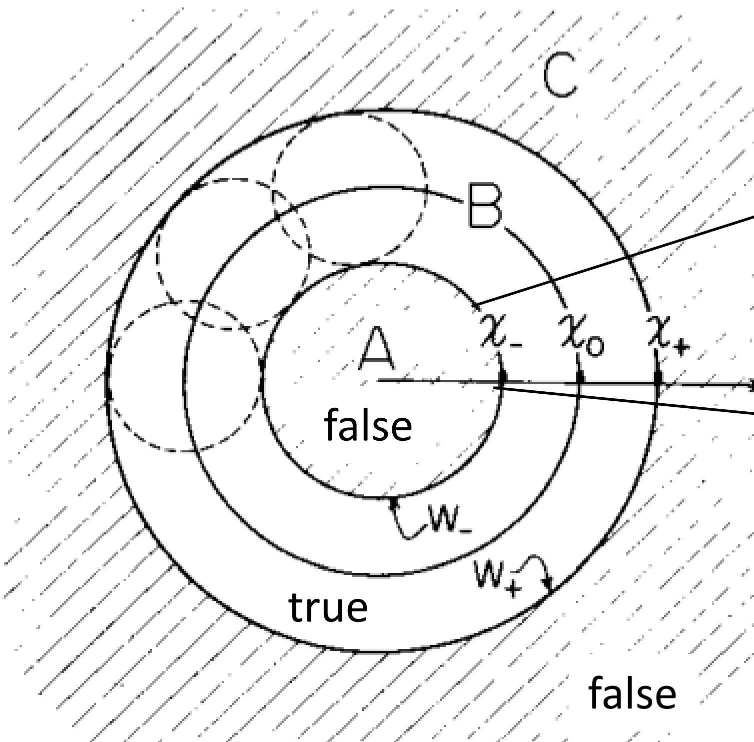
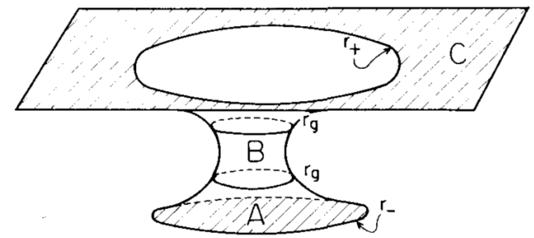


A is large
Volume $> H^{-3}$



→ heavy ($\sim H^{-3}\Delta V$) BH forms

A is small
Volume $< H^{-3}$

A shrinks
→ kinetic energy of domain wall ↑
→ (light) BH forms

figures from Sato et. al., 1981

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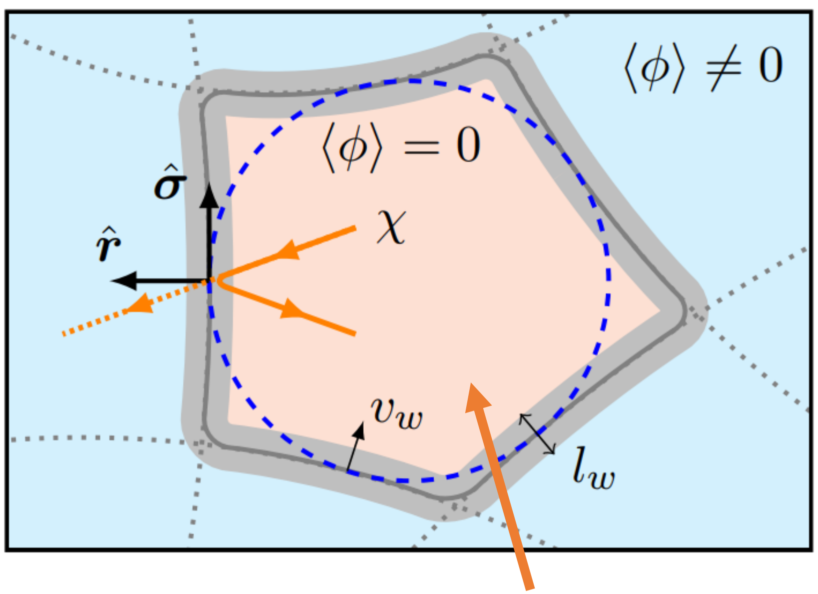

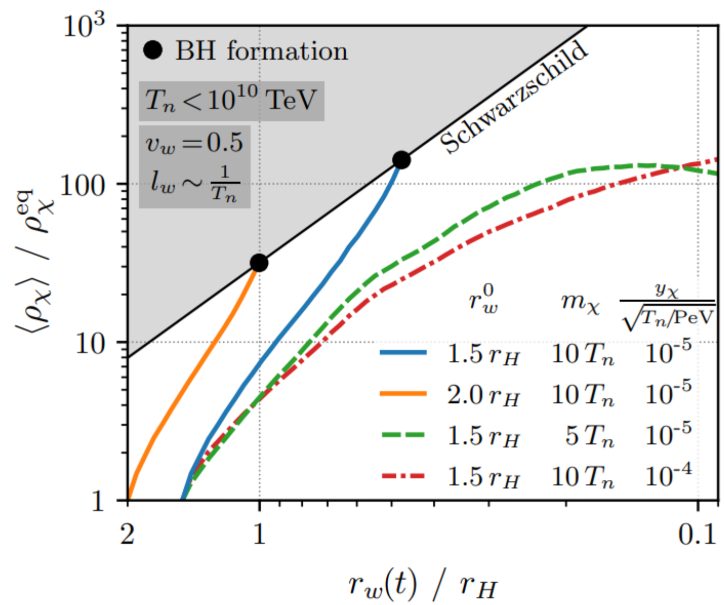

plasma heated

(As discussed in Kopp's talk)

figures from Baker et. al., 2105.07481

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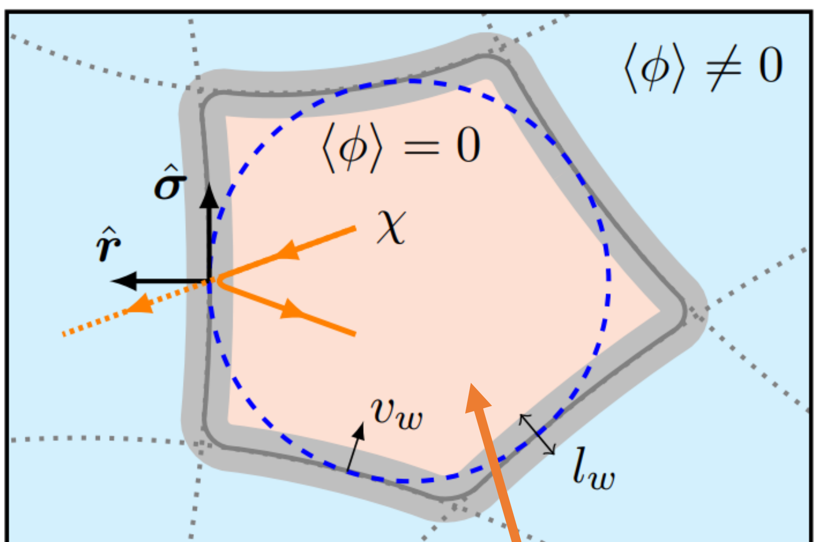
surrounded region
from random nucleations



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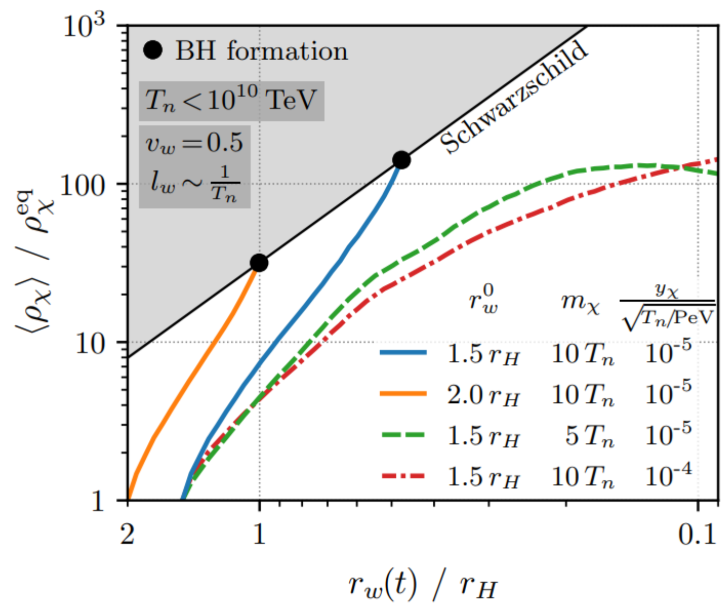

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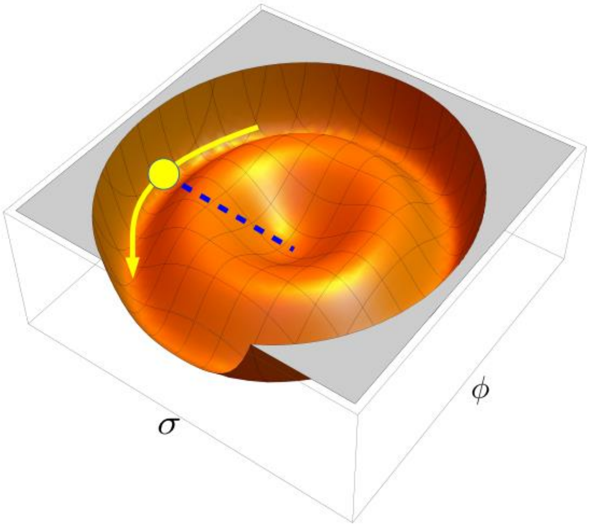

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Bubble collision → false vacuum created → BH

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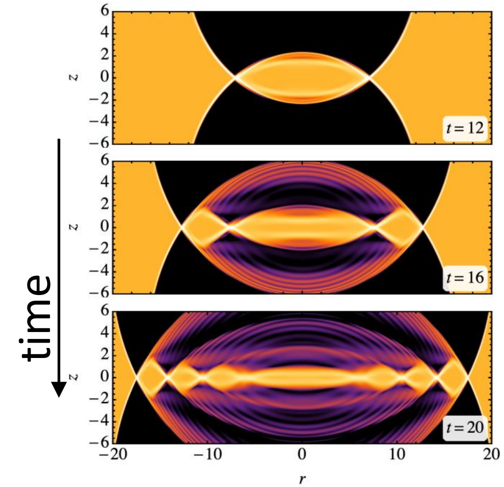

M. Lewicki, V. Vaskonen, 1912.00997

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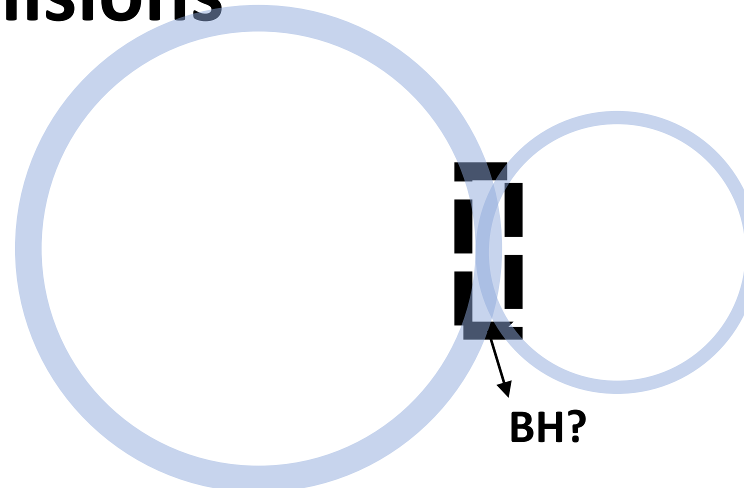
(And many other scenarios. I apologize if I missed your work.)

2. From bubble collisions

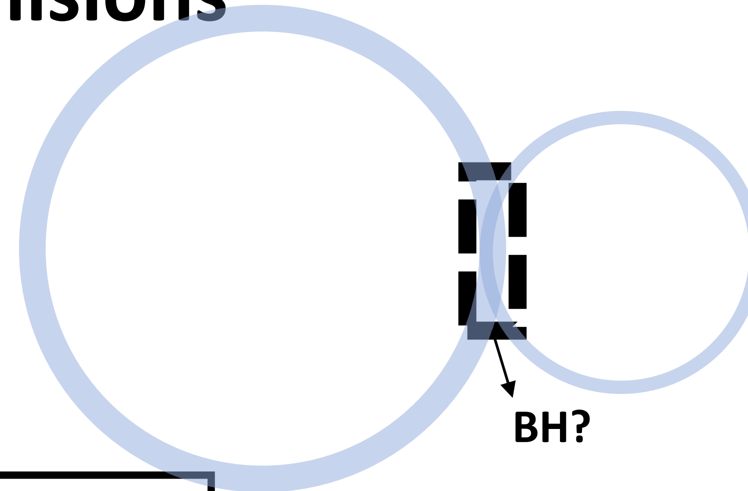


- S. W. Hawking, I. G. Moss, J. M. Stewart, 1982
  Naïve estimation, **flat spacetime** ($R \ll H^{-1}$),
  ⇒ collision of a large number of bubbles is required

- I. G. Moss, 1994
  $R > H^{-1}$ (GR), **infinitesimally thin wall** approximation
  ⇒ a collision of two bubbles **cannot** produce a black hole

- THJ, T. Okui, 2021
  $R > H^{-1}$ (GR), **non-negligible thickness of energy density around the wall**,
  ⇒ a collision of two bubbles **can** produce a black hole
  ⟹ Main subject of this talk

2. From bubble collisions



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How to realize it?

- THJ, T. Okui, 2021
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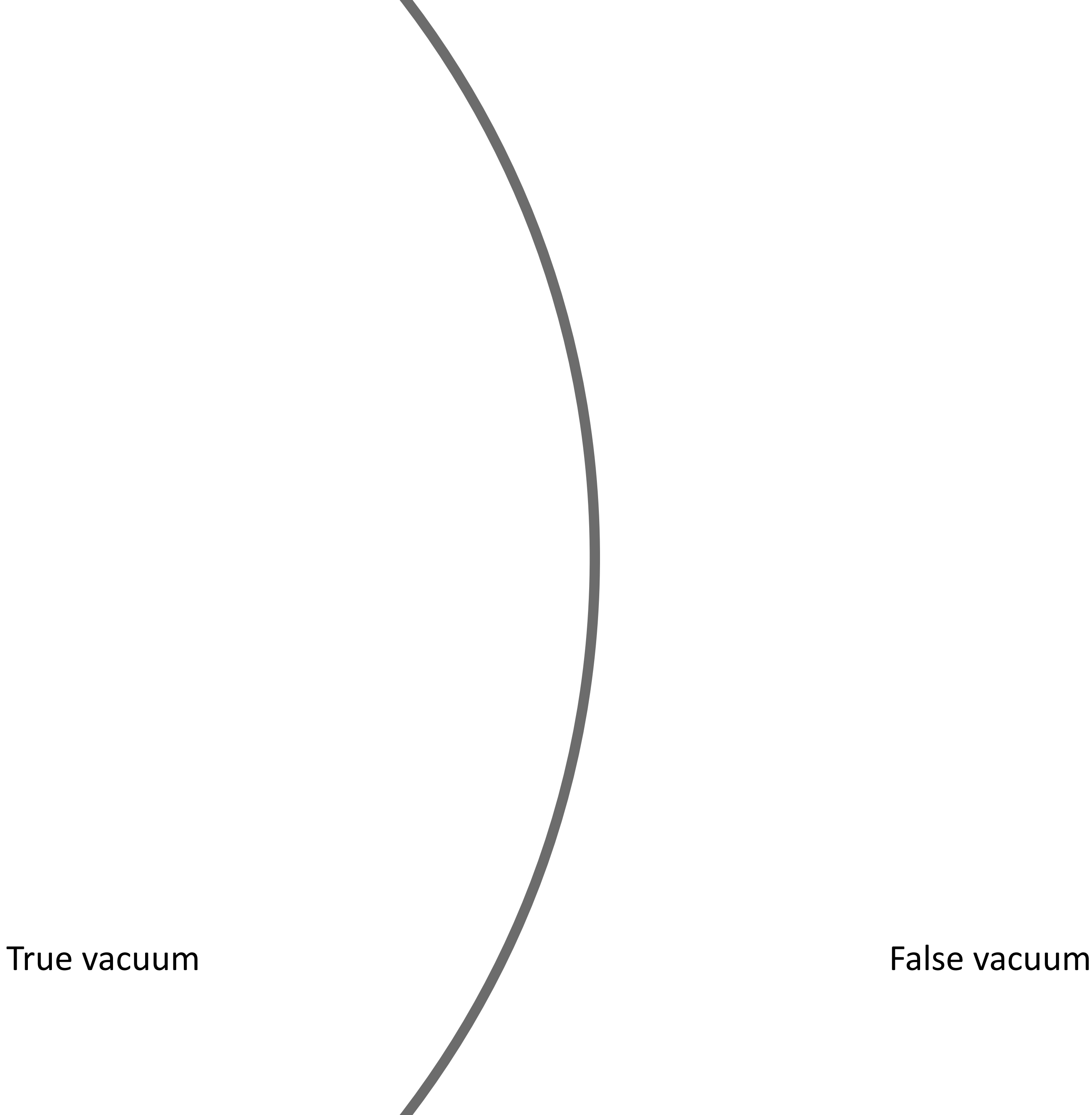
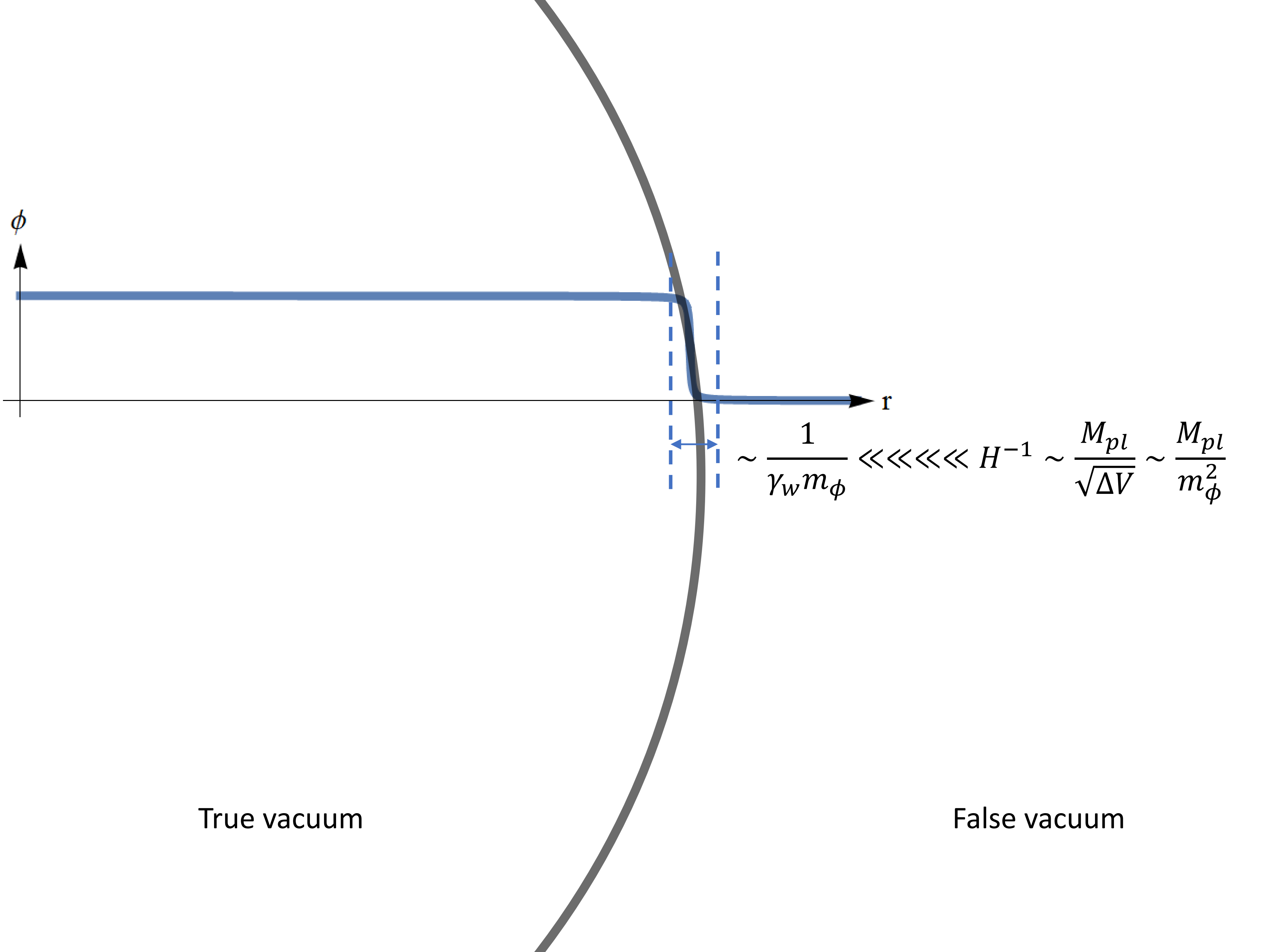
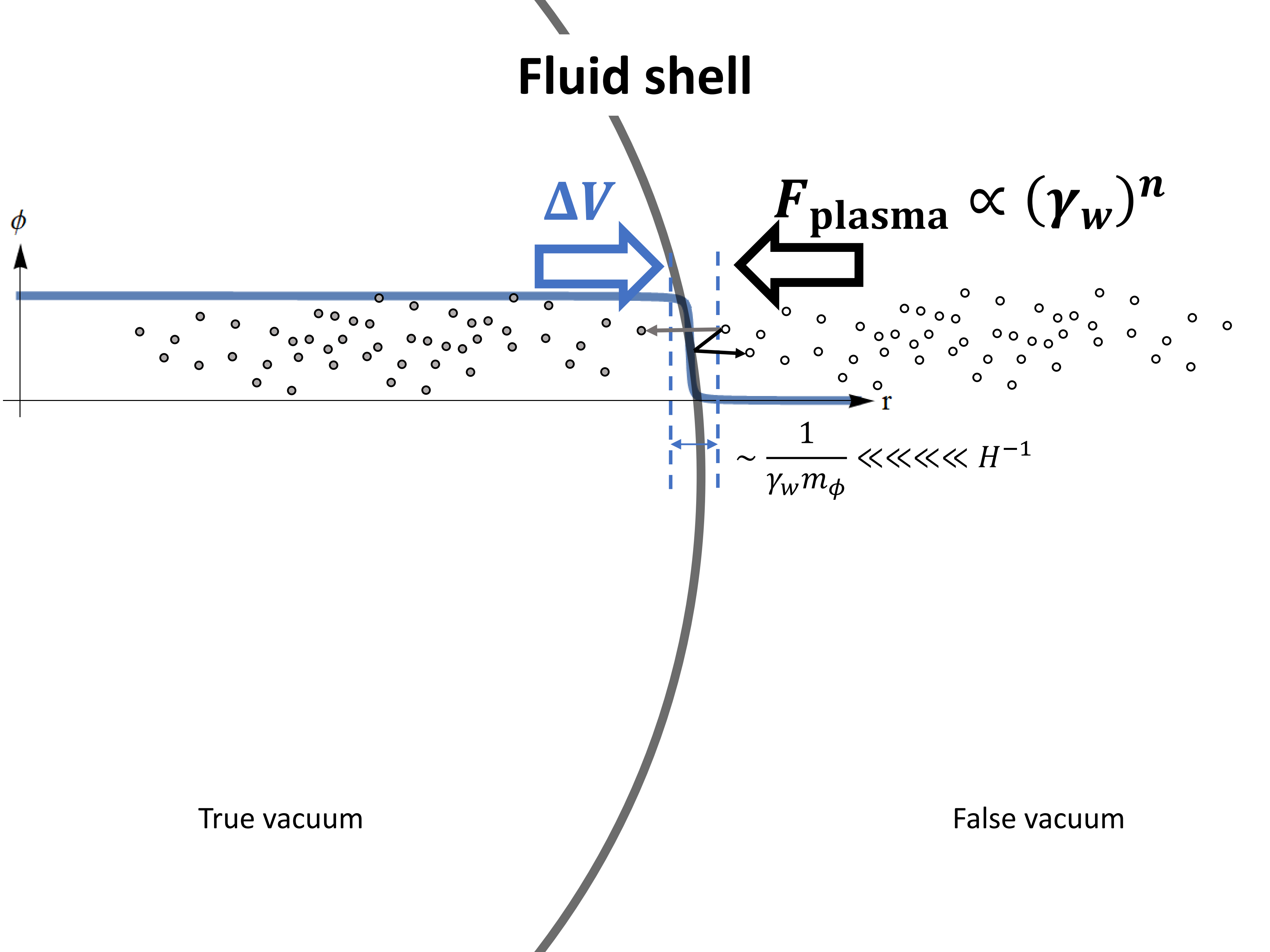
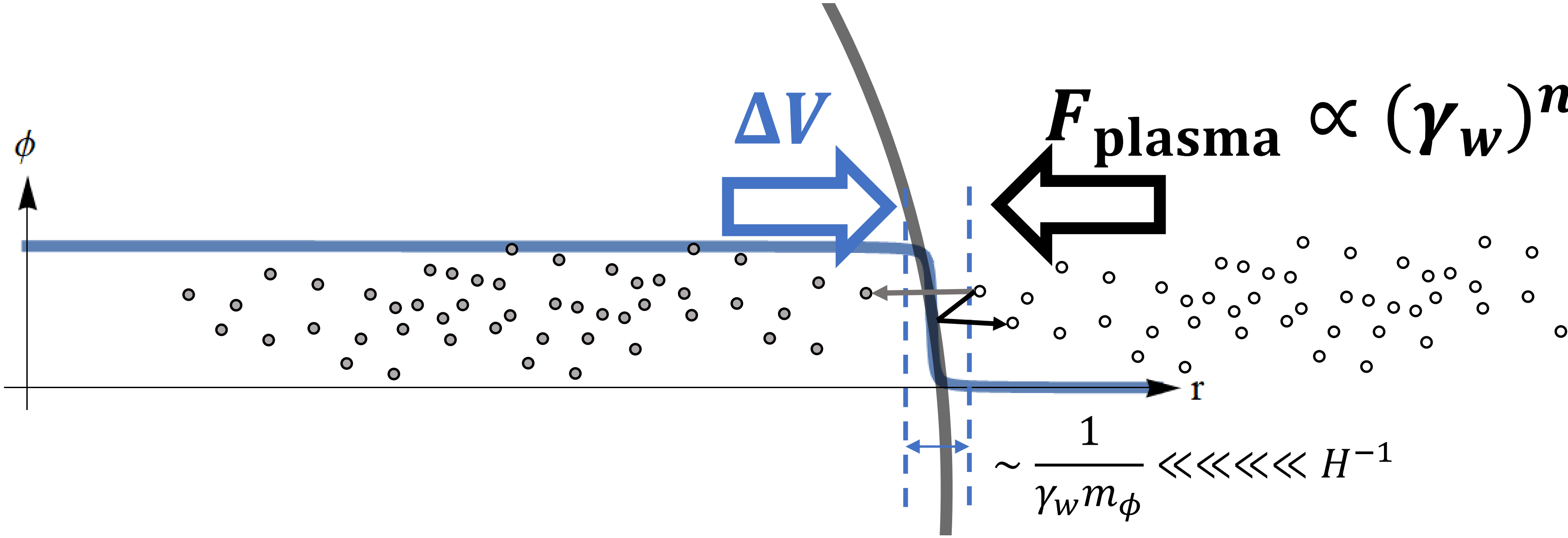

True vacuum
False vacuum



ϕ
r
$\sim \frac{1}{\gamma_w m_\phi} \lll\lll\lll\lll H^{-1} \sim \frac{M_{pl}}{\sqrt{\Delta V}} \sim \frac{M_{pl}}{m_\phi^2}$
True vacuum
False vacuum

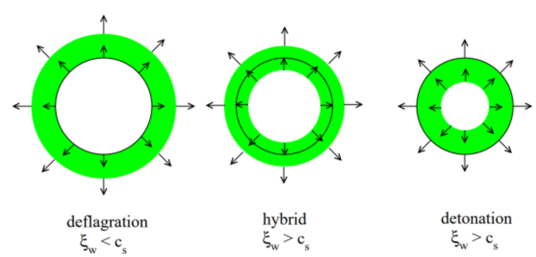

Fluid shell
ΔV
$F_{\mathrm{plasma}} \propto (\gamma_w)^n$
ϕ
r
$\sim \frac{1}{\gamma_w m_\phi} \lll\lll\lll\lll H^{-1}$
True vacuum
False vacuum

Fluid shell



1. If $n > 0$, bubble wall velocity is frozen by $\Delta V = F_{\text{plamsa}}$.
ΔV is transferred to fluid energy ⇒ bulk motion is generated
(bubble wall works on plasma)

$$\partial_\mu \left(T^{\mu\nu}_{\text{fluid}} + T^{\mu\nu}_{\phi} \right) = 0 \quad \Rightarrow$$



(thick energy density profile)

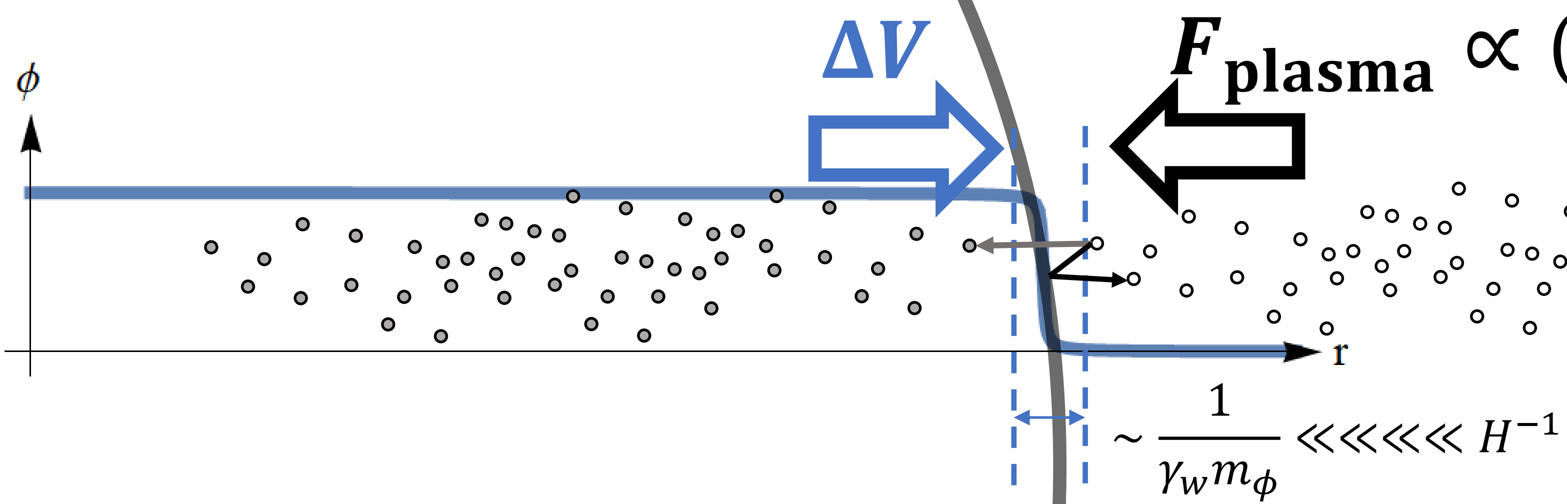
$$L \sim \frac{1}{\gamma_w^2} R$$

e.g. see Espinosa et. al., 1004.4187

2. If $n \to 0$ & $\Delta V \gg F_{fluid}$, bubble wall runs away.
ΔV is transferred to scalar profile ⇒ kinetic energy of scalar profile ↑ ⇒ wall width ↓ ⇒ No BH

$$L \propto 1/R$$

Fluid shell



1. If $n > 0$, bubble wall velocity is frozen by $\Delta V = F_{\text{plamsa}}$.
ΔV is transferred to fluid energy $\Rightarrow$ bulk motion is generated
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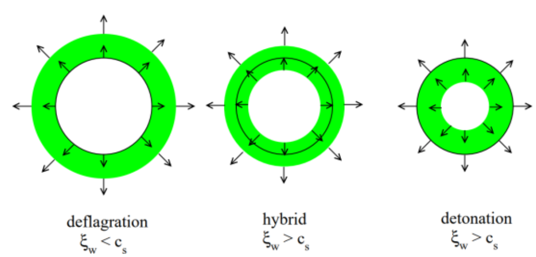

(thick energy density profile)

$$L \sim \frac{1}{\gamma_w^2} R$$

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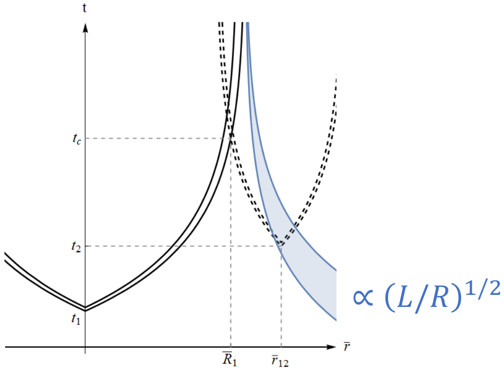
THJ, T. Okui, 2110.04271

Preview of our findings

THJ, T. Okui, 2110.04271



Condition: $R \gtrsim \dfrac{H^{-1}}{\sqrt{L/R}}$ $\quad L/R \ll 1$ in our framework

$$N(t_1) = \int dV\, \Gamma_n(t_2) P_f(t_c)$$

Number density:

$$n_{PBH} = \frac{1}{R_f^3} \int dt_1\ e^{3Ht_1} N(t_1) \Gamma_n(t_1)$$

Mass:

$$M_{PBH} > \frac{4\pi}{3} H^{-3} \Delta V$$

(mostly on the threshold since $P(t_c)$ decreases quickly as $r_{12} \uparrow$)

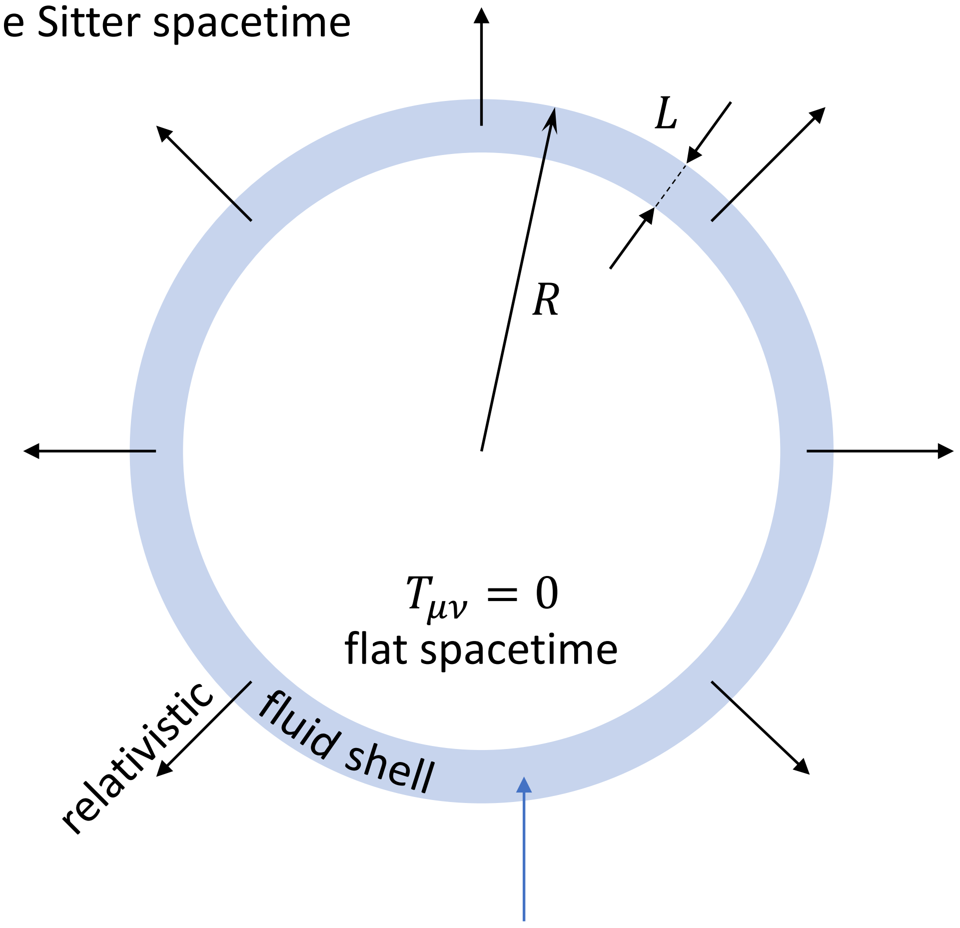
Condition for PBH formation

PBH production from bubble collisions

THJ, T. Okui, 2110.04271

$T_{\mu\nu} = \Delta V\ g_{\mu\nu}$

de Sitter spacetime



Large $T_{\mu\nu}$,
Energy density $\simeq$ energy flux $\sim$ large
Mass density $\sim$ microscopic scale

PBH production from bubble collisions

THJ, T. Okui, 2110.04271

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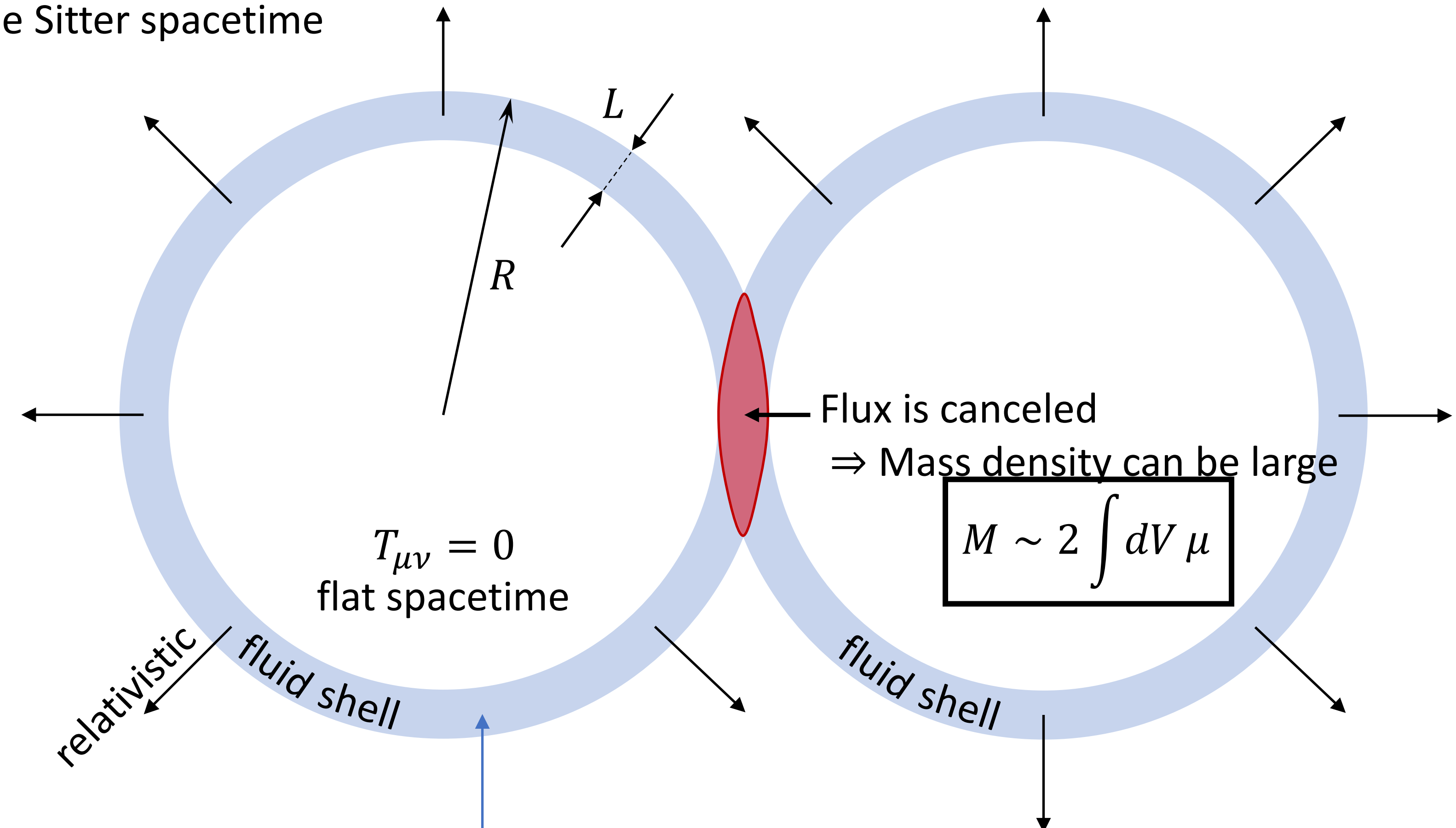

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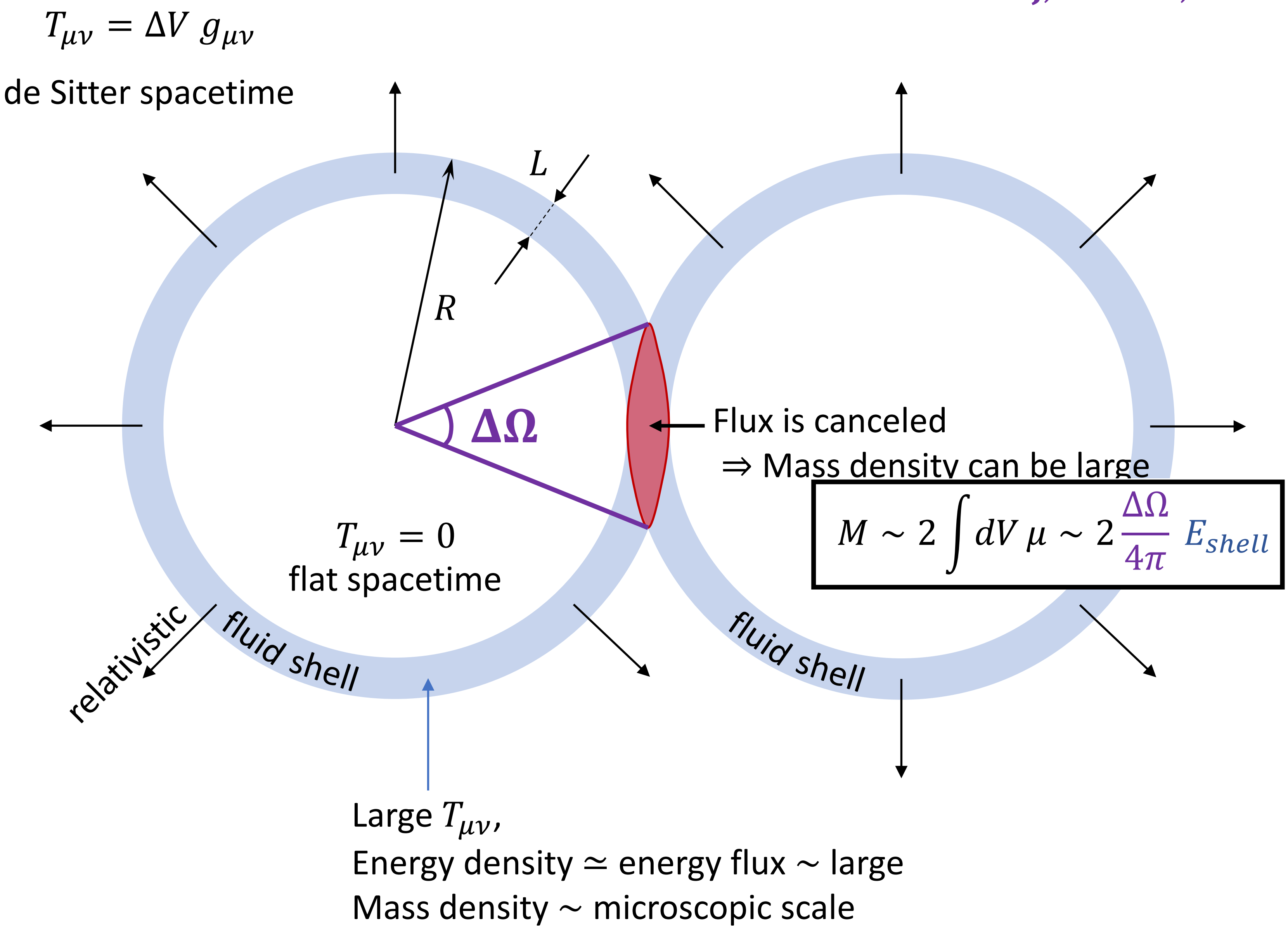

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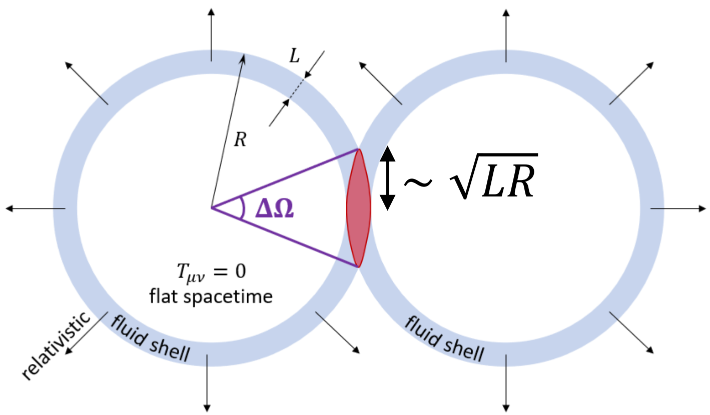


$$\Rightarrow \boldsymbol{M} \sim 2\,\frac{\Delta\Omega}{4\pi}\,E_{\text{Shell}}$$

$$\Delta\Omega \simeq \pi\frac{L}{R}$$

When $\boldsymbol{R \ll H^{-1}}$,

$\boldsymbol{E_{shell} \simeq \frac{4\pi}{3} R^3 \Delta V}$ since energy is conserved $\left(\partial_\mu T^{\mu\nu} = 0\right)$.

A BH forms if

$$1 < \frac{2GM}{\sqrt{RL}} \sim H^2 R^2 \sqrt{L/R} \qquad : \boldsymbol{R > \frac{H^{-1}}{(L/R)^{1/4}}}$$

$\boldsymbol{E_{\text{shell}}(R)}$ for $R \gtrsim H^{-1}$ (R: outer surface area $= 4\pi R^2$)

With $8\pi G = 1$, Vaidya-de Sitter metric:

$$ds^2 = -\left(1 - \frac{\bar{\rho}(u)}{3}\bar{r}^2\right)dt^2 - 2dtdr + r^2 d\Omega^2$$

$\bar{\rho}(u) = 0$ for $u > u_1 + \epsilon$
ΔV for $u < u_1$

$u = t - \frac{1}{H}\log(1 + e^{Ht}Hr)$, $\bar{r} = re^{Ht}$

$$ds^2 = -dt^2 + e^{2Ht}(dr^2 + r^2 d\Omega^2) - \frac{\Delta H^2(u) r^2 e^{2Ht}}{(1 + Hre^{Ht})^2}(dt - [illegible])^2$$

$T_{\alpha\beta} = G_{\alpha\beta} \quad \rightarrow \quad \mu = T_{\alpha\beta}n^\alpha n^\beta$ where $n \perp H$

For H: $t = t_c$,

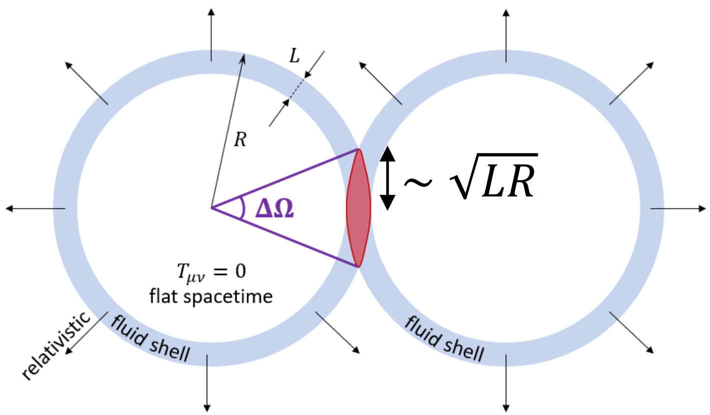
$$E_{shell} = \int_{shell} [illegible] \sqrt{\det g_{ij}}\ \mu$$

$$\rightarrow \begin{cases} \frac{4\pi}{3}R^3\Delta V & \text{for } R \ll H^{-1} \\ \frac{4\pi}{3}R^3\Delta V \times \frac{1}{2HR} & \text{for } R \gg H^{-1} \end{cases}$$

some messy calculations

$$E_{\text{shell}} \simeq \frac{4\pi}{3}R^3\Delta V \times \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{HR}} \qquad \text{for } R \gg H^{-1}$$

PBH production from bubble collisions



$$\Rightarrow M \sim 2\frac{\Delta\Omega}{4\pi} E_{\text{Shell}}$$

$$\Delta\Omega \simeq \pi\frac{L}{R}$$

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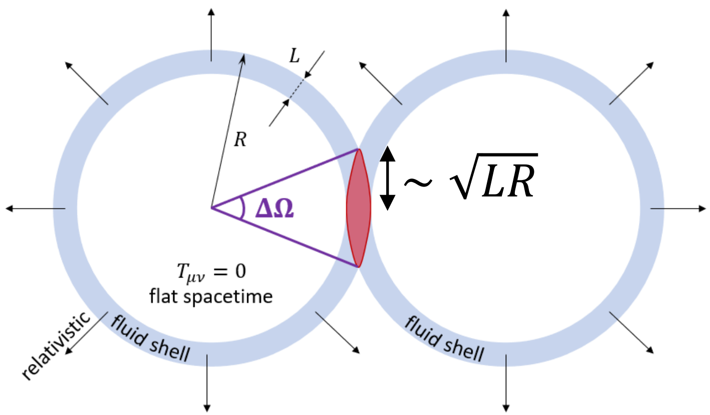


$$\Rightarrow M \sim 2\,\frac{\Delta\Omega}{4\pi}\,E_{\text{Shell}} \qquad \Delta\Omega \simeq \pi\,\frac{L}{R}$$

When $R \ll H^{-1}$, $\gg$

$$E_{shell} \simeq \frac{4\pi}{3} R^3 \Delta V \times 1/2HR$$

~~since energy is conserved ($\partial_\mu T^{\mu\nu} = 0$)~~.

A BH forms if

$$1 < \frac{2GM}{\sqrt{RL}} \sim H^{1} R^{1} \sqrt{L/R} \qquad : R > \frac{H^{-1}}{(L/R)^{1/2}}$$

$$M_{PBH} > \frac{4\pi}{3} H^{-3} \Delta V$$

Summary

Condition:

$$R \gtrsim \frac{H^{-1}}{\sqrt{L/R}}$$

$L/R \ll 1$
in our framework

$$N(t_1) = \int dV\, \Gamma_n(t_2) P_f(t_c)$$

$$P_f(t) = \exp\left(-\frac{4\pi}{3} H^{-3} \int_{-\infty}^{t} dt' \left(1 - e^{-H(t-t')}\right)^3 \Gamma_n(t')\right)$$

Number density:

$$n_{PBH} = \frac{1}{R_f^3} \int dt_1\ e^{3Ht_1} N(t_1) \Gamma_n(t_1)$$

Mass:

$$M_{PBH} > \frac{4\pi}{3} H^{-3} \Delta V$$

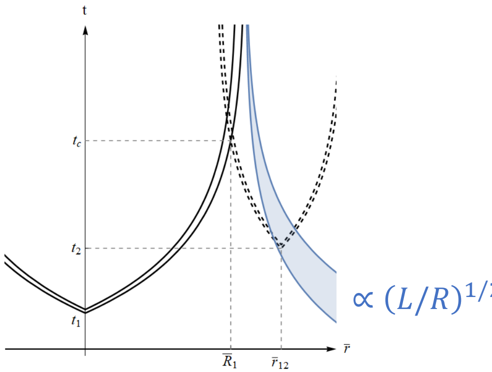
(mostly on the threshold
since $P(t_c)$ decreases quickly as $r_{12} \uparrow$)

Summary

Condition:

$$R \gtrsim \frac{H^{-1}}{\sqrt{L/R}}$$

$L/R \ll 1$
in our framework



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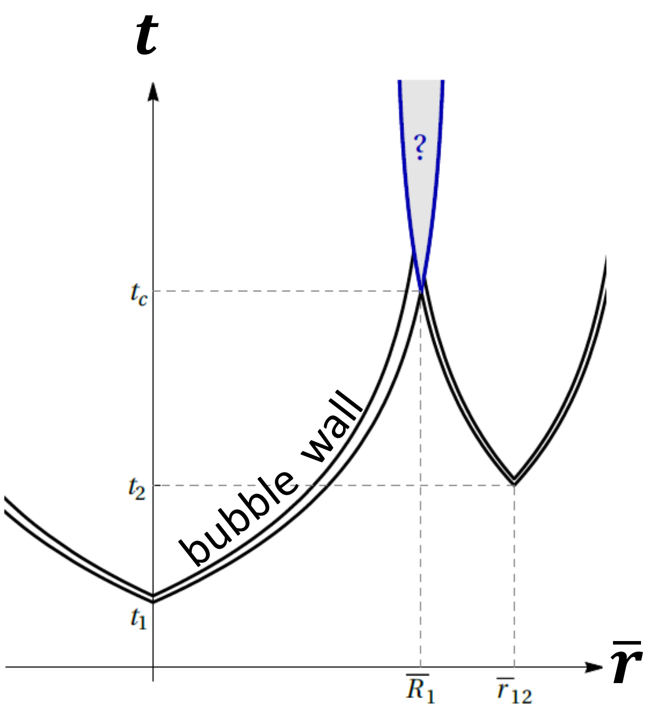
⇩

In FRW(-like) coordinates ?

PBH abundance



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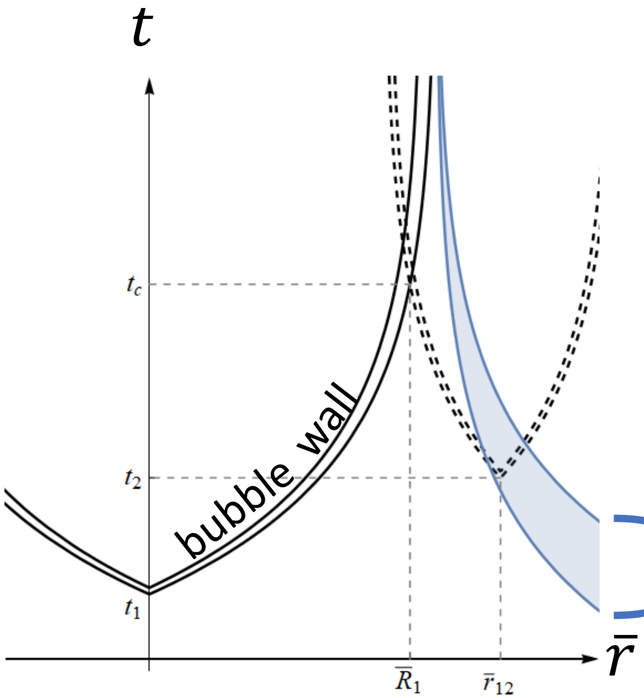

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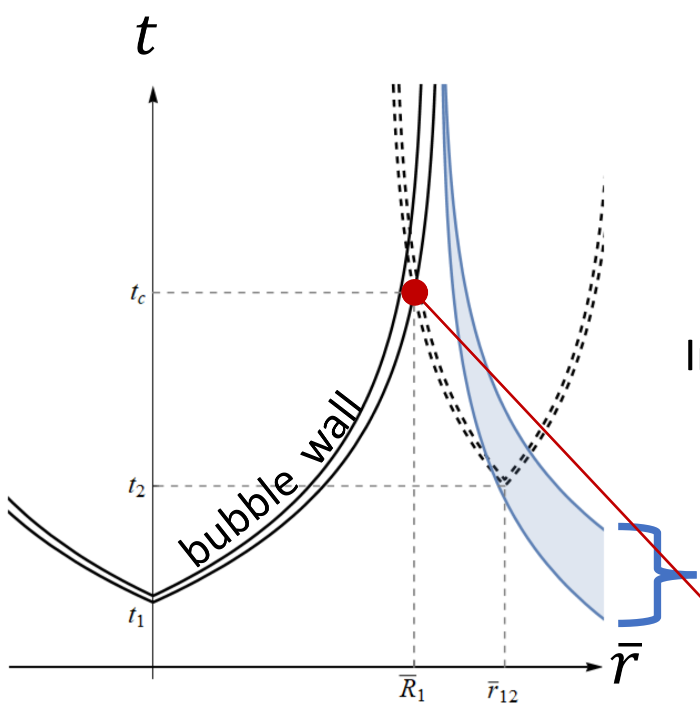

For a bubble nucleated at t_1,

1. Another bubble nucleated in the band
2. Collision point should be in the FV

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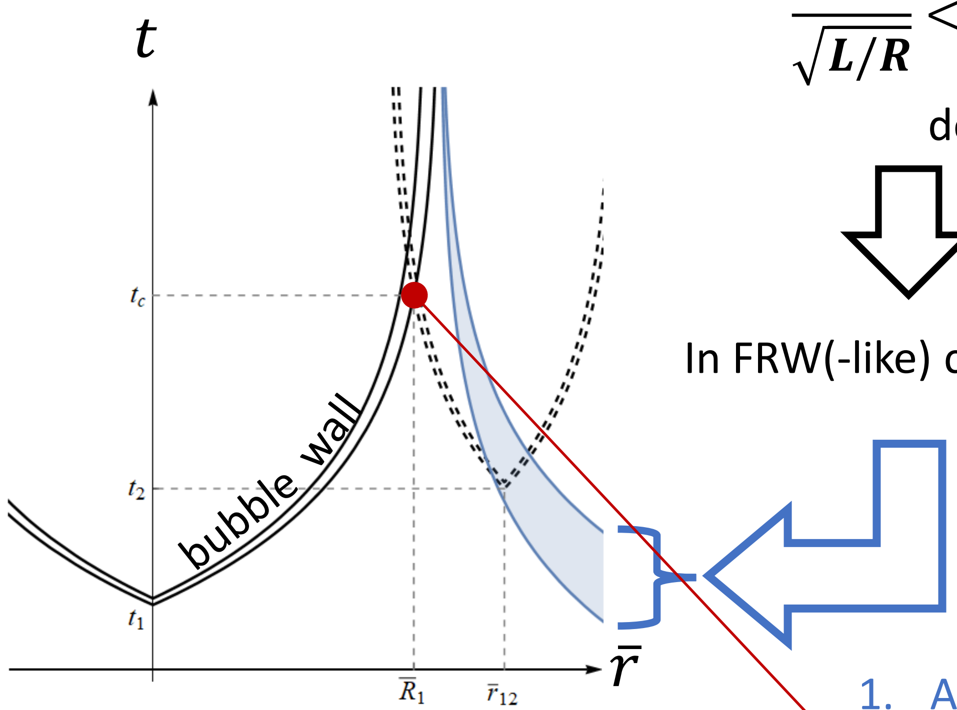

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$$N(t_1) = \int dV_2\, \Gamma_n(t_2) P_f(t_c)$$

$P_f(t_c)$ = prob. that a point remains in FV at t_c

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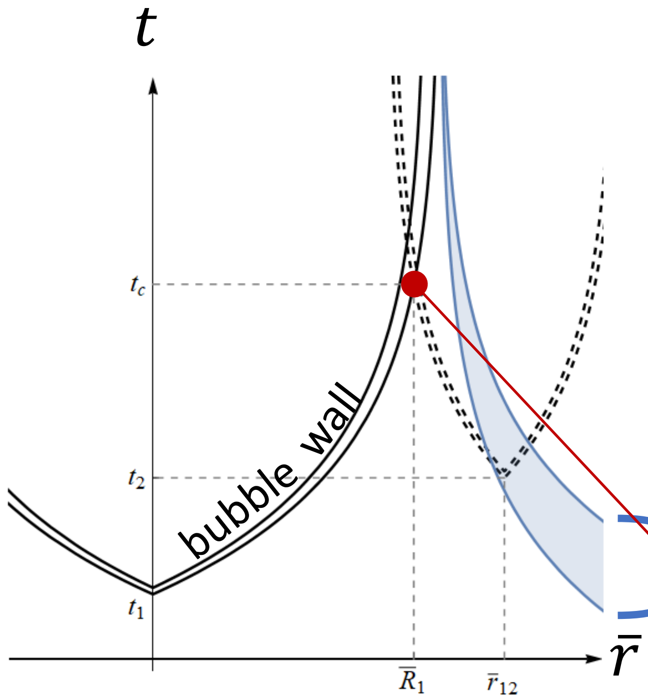

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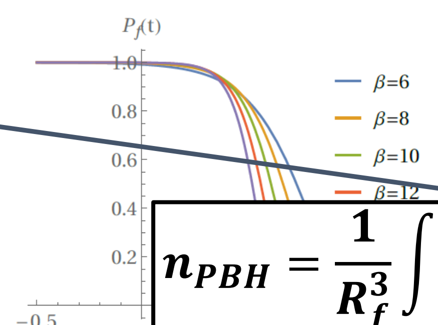

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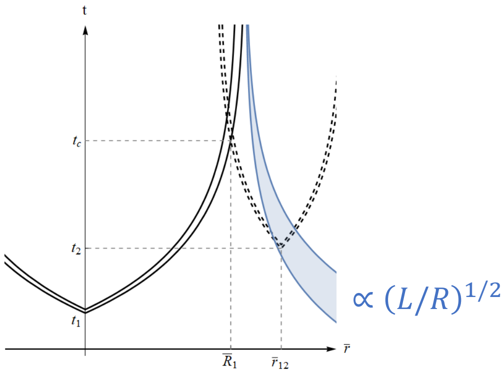

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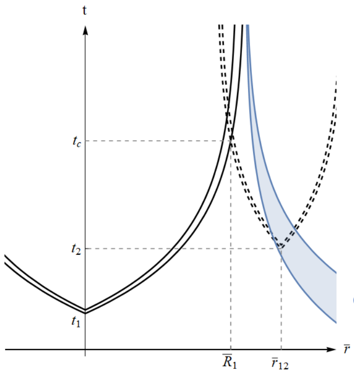

$\propto (L/R)^{1/2}$

$$N(t_1) = \int dV\, \Gamma_n(t_2) P_f(t_c)$$

Number density: $$n_{PBH} = \frac{1}{R_f^3}\int dt_1\ e^{3Ht_1} N(t_1)\boldsymbol{\Gamma_n(t_1)}$$

Mass: $$M_{PBH} > \frac{4\pi}{3} H^{-3}\boldsymbol{\Delta V}$$

(mostly on the threshold
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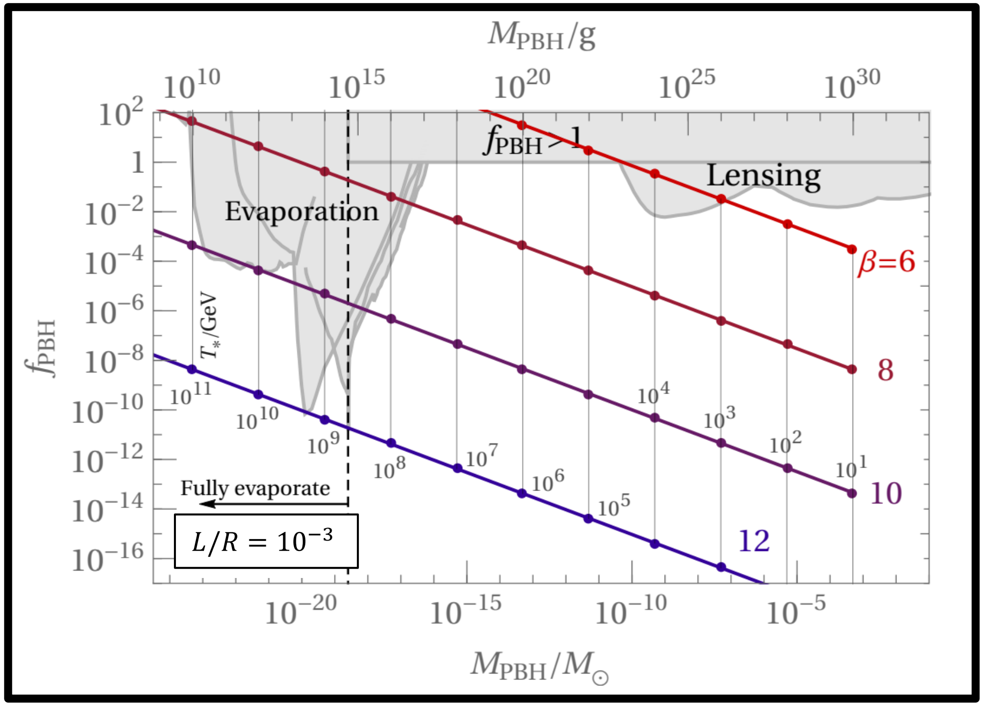
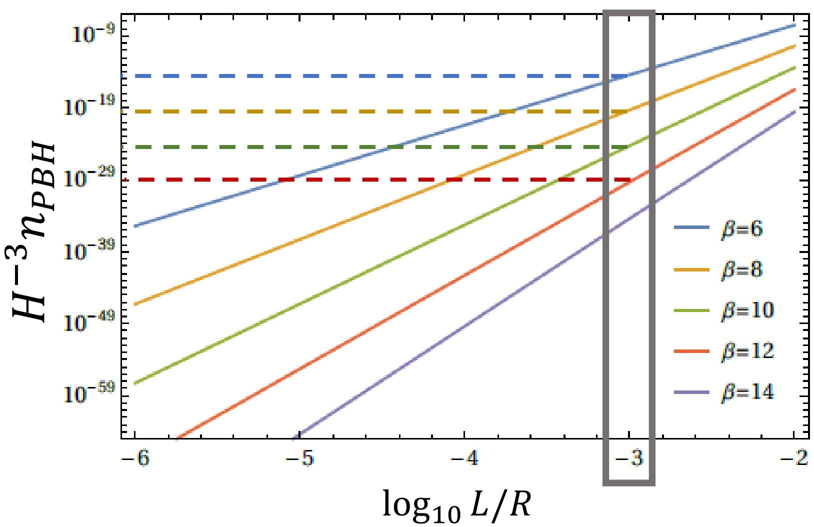
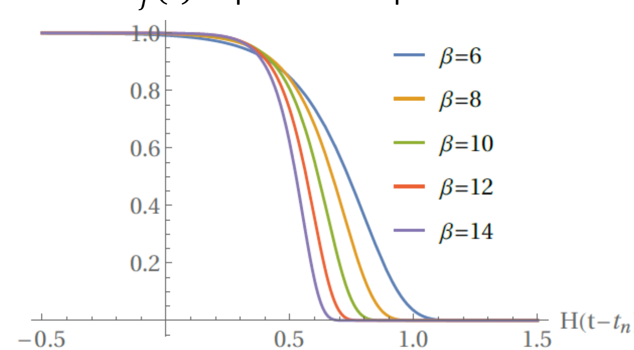
We can calculate $Y_{PBH} = M_{PBH} n_{PBH}/s$ if we know $\begin{cases} \boldsymbol{L/R} \\ \boldsymbol{\Gamma_n(t)} \\ \boldsymbol{\Delta V} \end{cases}$

Example: $\Gamma_n(t) = H^4 e^{\beta H(t-t_n)}$ & $L/R = 10^{-3}$



- $R > 30\, H^{-1}$ for PBH formation
- t_n doesn't change the result
- $(\beta H)^{-1}$: duration of PT
- $T_* \sim (\Delta V)^{1/4}$: reheating temp.
  or
  critical temp.
  or
  some characteristic scale
- Exponential $\Gamma_n(t)$: no eternal inflation





Thank you for your attention!!

Next steps

- Small β (=long duration of PT)? Estimation from a concrete particle physics model?

- $L/R = \text{const}$ in $R \gg H^{-1}$?

- With help of non-relativistic outcome from the collision?

- Numerical relativity?

Backup slides for those who don’t like hand-waving arguments

Hoop conjecture

original

A BH forms if and only if a mass M gets compacted in a region whose largest circumference is less than $2\pi \cdot (2GM)$.

refined

A BH forms if and only if there exist a hypersurface H and its subset V that satisfies

$$1 < \frac{2\pi \cdot \left(2GM(V)\right)}{C(V)},$$

where $M(V)$ is the mass inside V and $C(V)$ is the largest circumference of V.

Definition of mass $M(V)$

For a given spacelike hypersurface H and $V \subset H$

$$M(V) = \int_V d^3x \sqrt{\det(g_{ij})} \left(\mu - \sqrt{J_\alpha J^\alpha}\right),$$

where $\mu = T_{\alpha\beta} n^\alpha n^\beta$, $J^\alpha = T^{\alpha\beta} n_\beta + \mu\, n^\alpha$ and $n^\alpha \perp H$.

: well-motivated from Schoen-Yau theorem, 1983

Theorem (Schoen and Yau, 1983)

Suppose H is any spacelike hypersurface in spacetime, and V is a bounded region in H on which $\mu - \sqrt{J_\alpha J^\alpha} \geq \Lambda > 0$, for some $\Lambda > 0$. If $R(V) > \pi \sqrt{\frac{3}{2\Lambda}}$, where $R(V)$ is a suitably defined measure of the radius of V, then V contains an apparent horizon.

$R(V) \sim$ the radius of the largest torus that can be embedded in V

$\boldsymbol{E_{\mathrm{shell}}(R)}$ (R: outer surface area $= 4\pi R^2$)

With $8\pi G = 1$, Vaidya-de Sitter metric:

$$ds^2 = -\left(1 - \frac{\bar{\rho}(u)}{3}\bar{r}^2\right)dt^2 - 2dtdr + r^2 d\Omega^2$$

$\bar{\rho}(u) = 0$ for $u > u_1 + \epsilon$

ΔV for $u < u_1$

$u = t - \frac{1}{H}\log(1 + e^{Ht}Hr)\,, \bar{r} = re^{Ht}$

$$ds^2 = -dt^2 + e^{2Ht}(dr^2 + r^2 d\Omega^2) - \frac{\Delta H^2(u) r^2 e^{2Ht}}{(1 + Hre^{Ht})^2}(dt - e^{Ht}dr)^2$$

$T_{\alpha\beta} = G_{\alpha\beta} \quad \rightarrow \quad \mu = T_{\alpha\beta}n^\alpha n^\beta$ where $n \perp H$

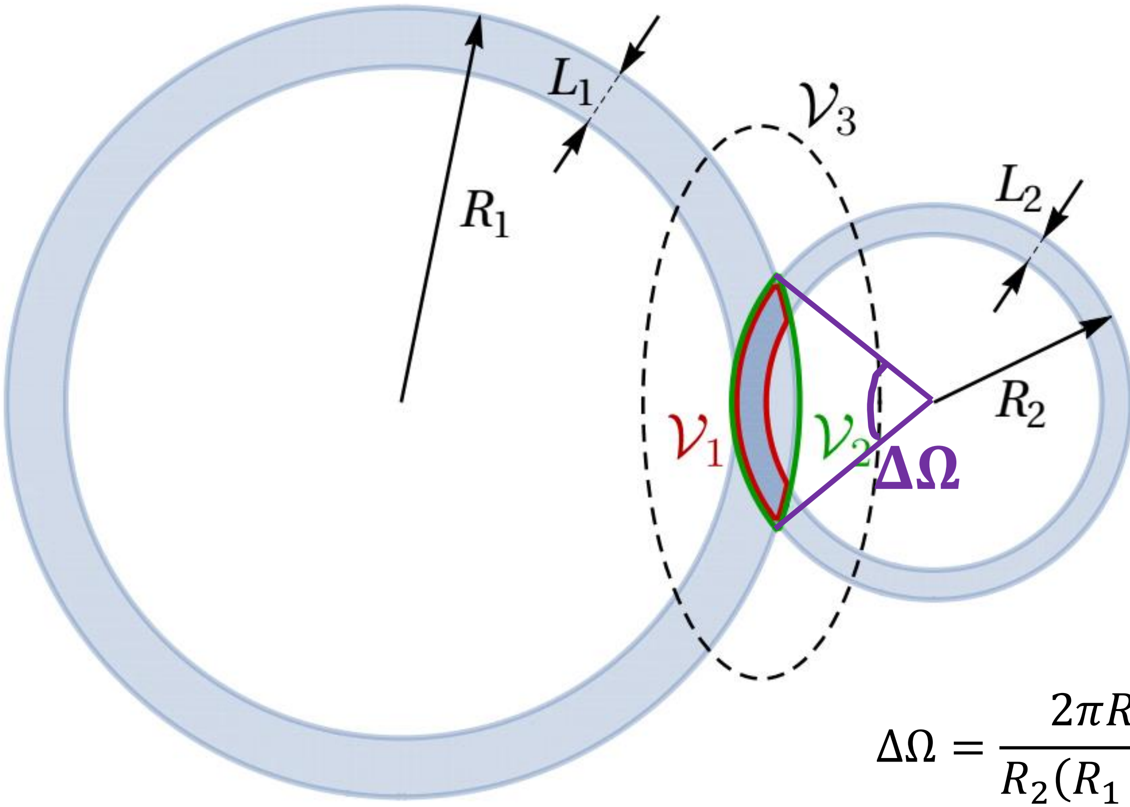
For H: $t = t_c$,

$$E_{shell} = \int_{shell} d^3x \sqrt{\det g_{ij}}\ \mu$$

$$\rightarrow \begin{cases} \frac{4\pi}{3}R^3\Delta V & \text{for } R \ll H^{-1} \\ \frac{4\pi}{3}R^3\Delta V \times \boldsymbol{\frac{1}{2HR}} & \text{for } R \gg H^{-1} \end{cases}$$

$$E_{\mathrm{shell}} \simeq \frac{4\pi}{3}R^3\Delta V \times \boldsymbol{\frac{1}{2HR}} \qquad \text{for } R \gg H^{-1}$$

When $R_1 \neq R_2$



$$M(V_1) \sim 2\,\frac{\Delta\Omega}{4\pi}\,E_{shell}(R_2)$$

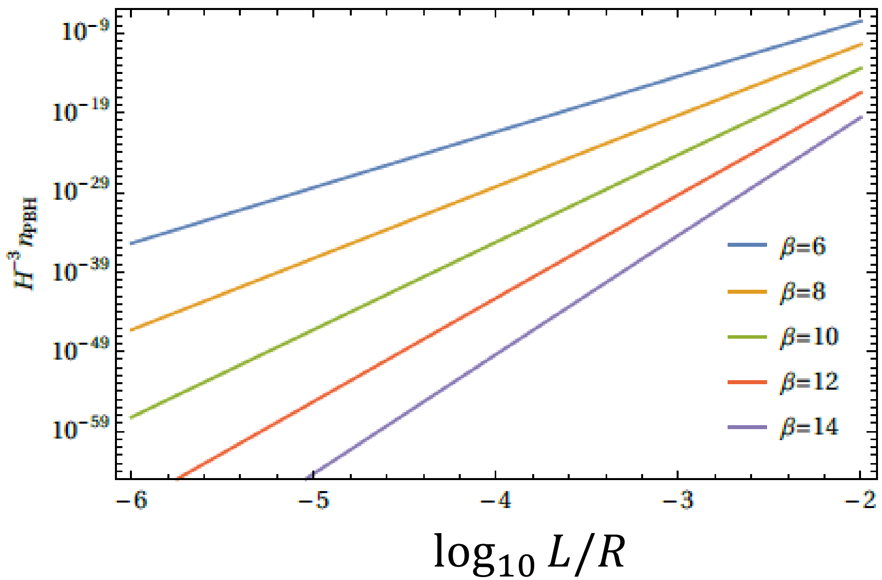
$$\begin{aligned}
&\mu - \sqrt{J_\alpha J^\alpha} \\
&\simeq (\mu_1 + \mu_2) - \sqrt{(J_1 + J_2)_\alpha (J_1 + J_2)^\alpha} \\
&\simeq (\mu_1 + \mu_2) - \mu_1\left(1 + \frac{J_1 \cdot J_2}{\mu_1^2}\right) \\
&\simeq 2\mu_2
\end{aligned}$$

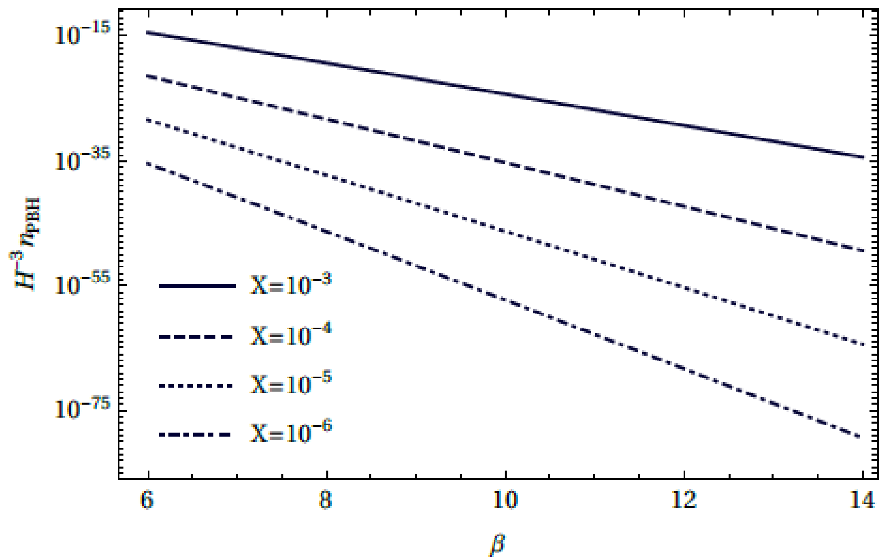
$$\Delta\Omega = \frac{2\pi R_1 L_1}{R_2(R_1 + R_2)}, \qquad C(V_1) = 2\pi\sqrt{\frac{2R_1 R_2 L_1}{R_1 + R_2}}$$

$$1 < \frac{2\pi \cdot 2GM}{C(V_1)} \qquad \Rightarrow \qquad R_2 > f(1{,}2)\,\frac{H^{-1}}{\sqrt{L/R}}$$

Backup slides for detailed analysis

Example: $\Gamma_n(t) = H^4 e^{\beta H(t-t_n)}$ & $L/R = 10^{-3}$





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