CLUSTER OF EXCELLENCE
QUANTUM UNIVERSE

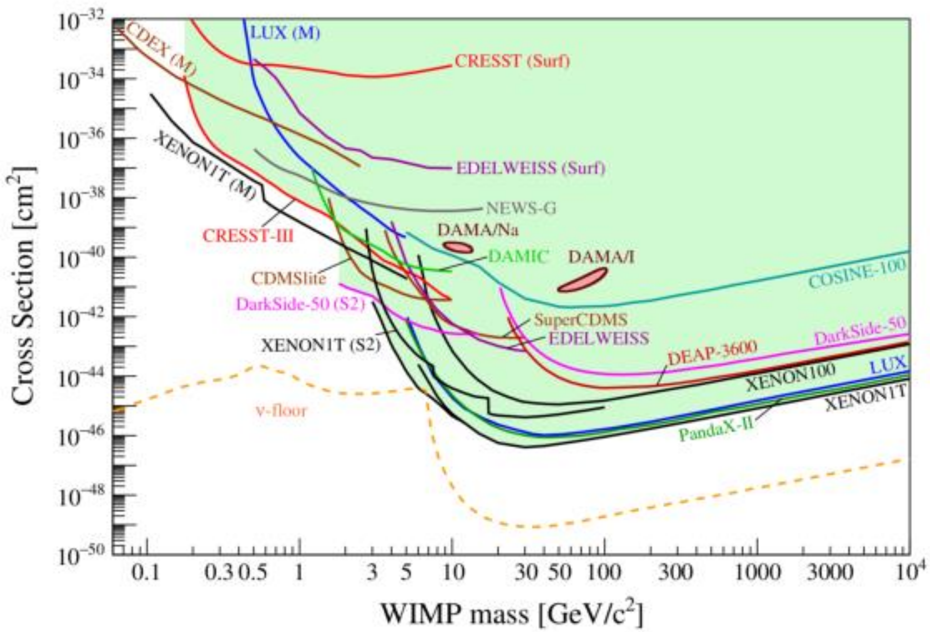
Out-of-equilibrium dark matter: production and cosmological signatures

Mathias Pierre - DESY

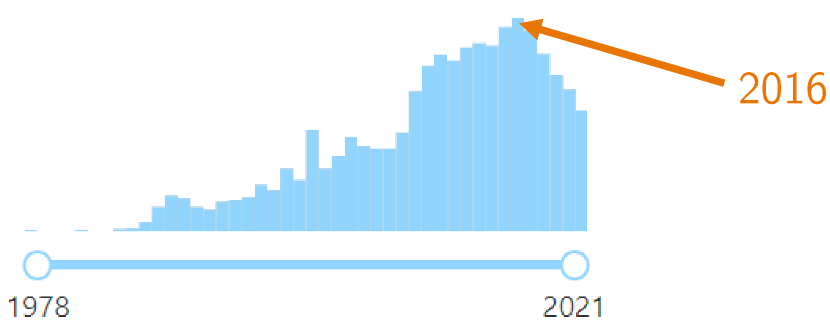
Chung-Ang University Beyond-the-Standard-Model Workshop

February 9th 2021

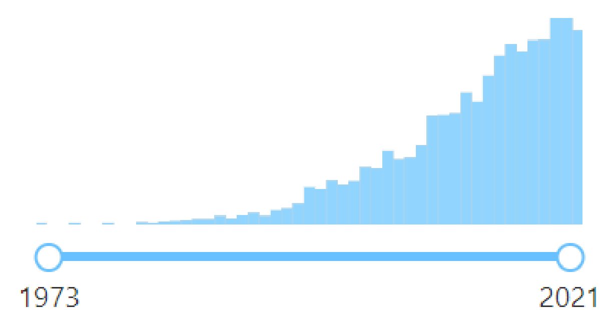
The waning of the WIMP?



No WIMP actually detected!



Inspire-HEP papers “WIMP”



Inspire-HEP papers “freeze-in”

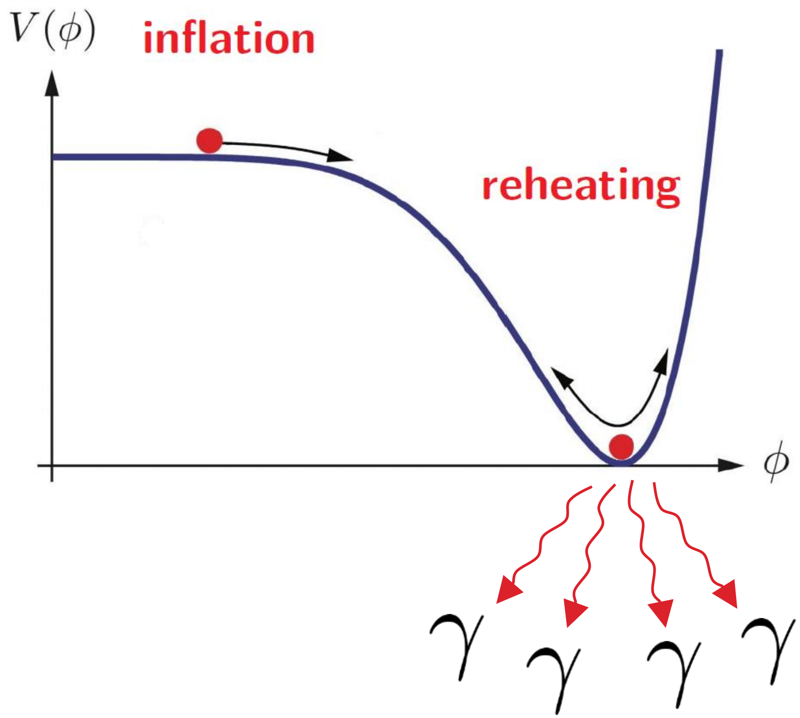
What are the cosmological signatures of out-of-equilibrium DM?

Can we probe the DM production mechanism?

Based on [**arXiv:2011.13458**] – **JCAP 21**
with **G. Ballesteros** & **M. A. G. Garcia**

Production of out-of-equilibrium dark matter

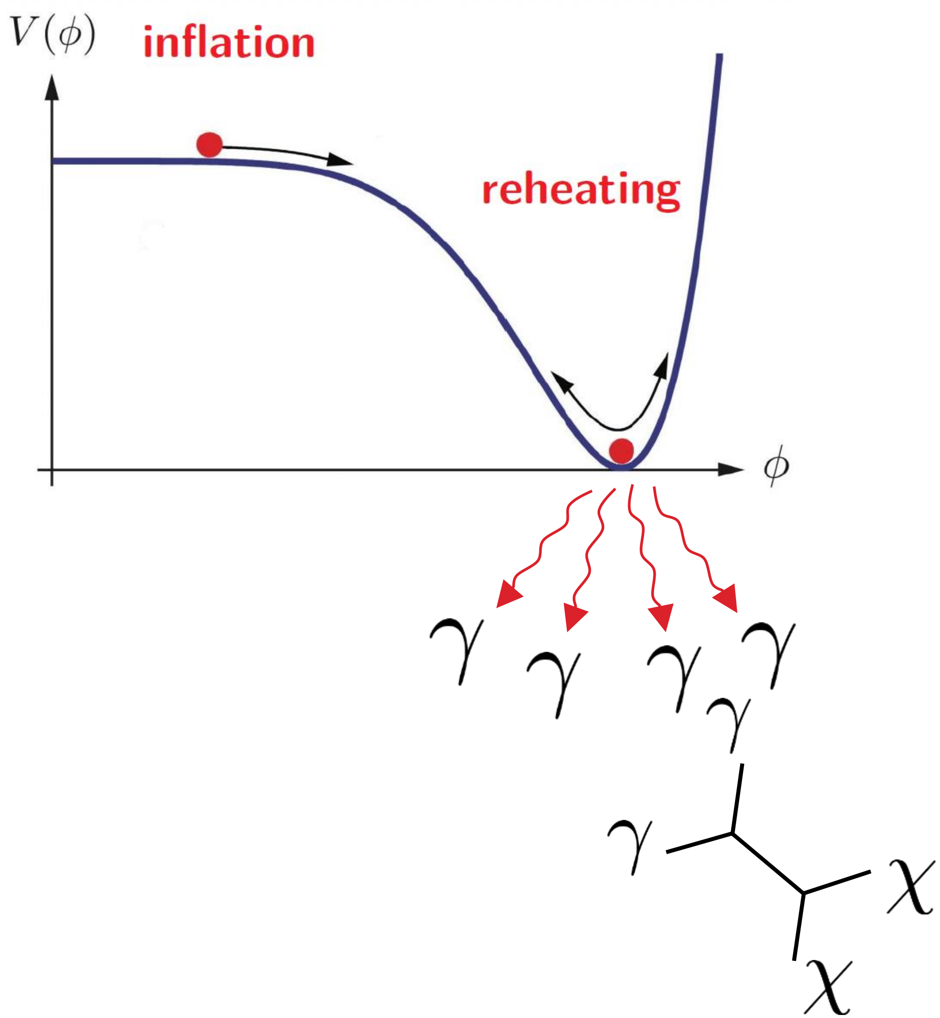
DM production during reheating



ϕ : inflaton

γ : generic SM particle

DM production during reheating



ϕ : **inflaton**

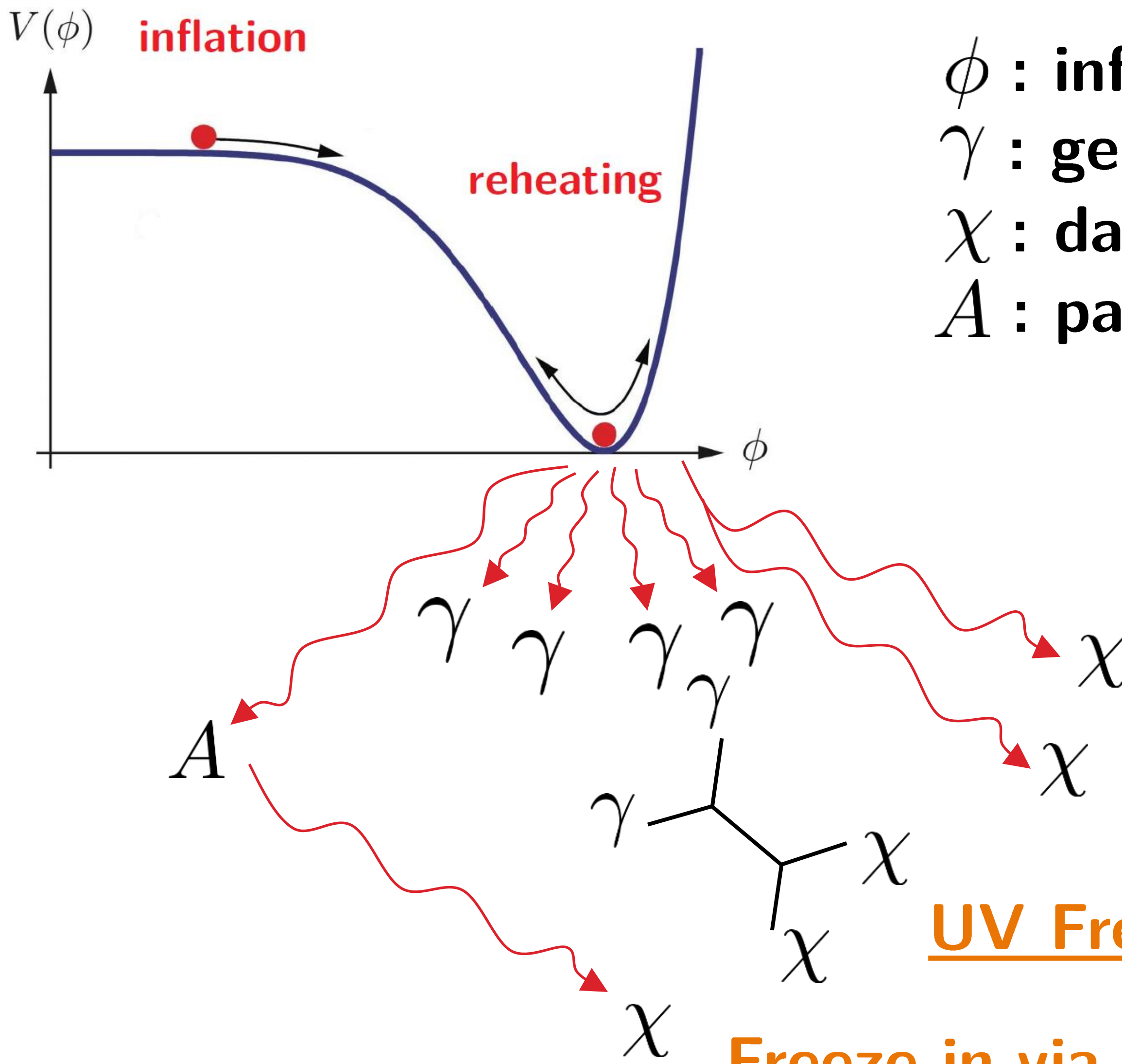
γ : **generic SM particle**

χ : **dark matter**

$$\sigma(s) = \frac{s^{\frac{n}{2}}}{\Lambda^{n+2}}$$

UV Freeze-in via scattering

DM production during reheating



ϕ : **inflaton**
γ : **generic SM particle**
χ : **dark matter**
A : **particle (SM or not)**

Decay of inflaton

UV Freeze-in via scattering

Freeze-in via decay

DM Phase space distribution

$$n_\chi(t) = \frac{g_\chi}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{p}\, f_\chi(p_0,t)$$

number-density

$$\rho_\chi(t) = \frac{g_\chi}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{p}\, p_0\, f_\chi(p_0,t)$$

energy-density

- Obtain **phase space distribution** by solving **Boltzmann** equation

$$\frac{\partial f_\chi}{\partial t} - H|\boldsymbol{p}|\frac{\partial f_\chi}{\partial |\boldsymbol{p}|} = \mathcal{C}[f_\chi(|\boldsymbol{p}|,t)]$$

- **Solution** to the Boltzmann equation:

$$f_\chi(p_0,t) = \int_{t_i}^{t} \mathcal{C}[f_\chi]\left(\frac{a(t)}{a(t')}|\boldsymbol{p}|,t'\right)\mathrm{d}t'$$

- After decoupling, distribution function only depends on the **comoving momentum**

$$q \equiv \frac{p\,a(t)}{T_\star}$$

$$n_\chi(t) = \frac{g_\chi}{(2\pi)^3}\frac{T_\star^3}{a^3}\int \mathrm{d}^3\boldsymbol{q}\,\bar{f}_\chi(q)$$

$T_\star \equiv T_{\rm NCDM}$ in **CLASS** [J. Lesgourgues & T. Tram, JCAP 09 (2011) 032]

Production from inflaton decay

- Pair of **DM** or **SM** produced from **perturbative** inflaton decay

$$\begin{matrix}\gamma\\ \chi\end{matrix} \longleftarrow \phi \longrightarrow \begin{matrix}\gamma\\ \chi\end{matrix}$$

$$\mathcal{C}[f(p,t)] = \frac{8\pi^2}{g m_\phi^2}\Gamma_\phi \,\mathrm{Br}\, n_\phi(t)\,\delta(p - m_\phi/2)$$

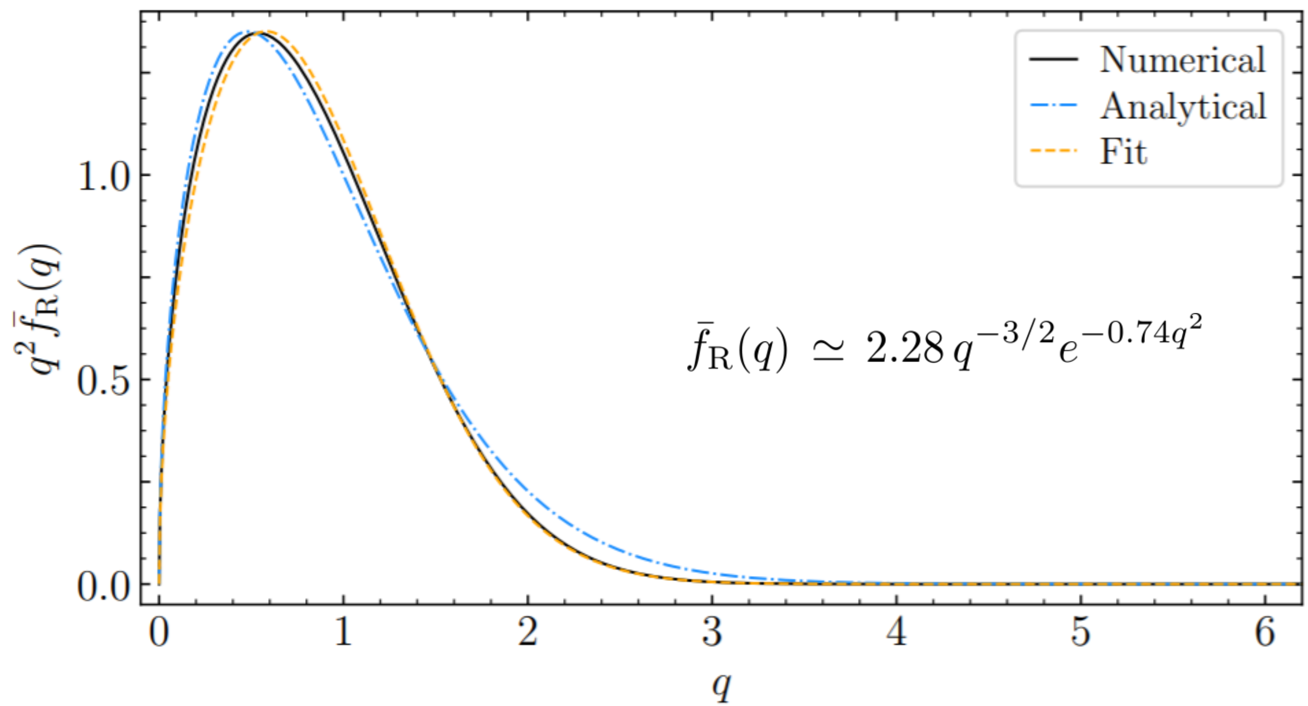
$$f(p,t) = \frac{16\pi^2 \Gamma_\phi \,\mathrm{Br}\, n_\phi(\hat{t})}{g m_\phi^3 H(\hat{t})}\theta(t-\hat{t}) \qquad \frac{a(t)}{a(\hat{t})} = \frac{m_\phi}{2p}$$

$t \ll t_{\rm reh}$

$$f(p,t) \simeq \frac{24\pi^2 n(t)}{g m_\phi^3}\left(\frac{m_\phi}{2p}\right)^{3/2}\theta(m_\phi/2 - p)$$

Inflaton decay: DM production

$$f_\chi(p,t)\,\mathrm{d}^3\boldsymbol{p} = \frac{4\pi^4 \mathrm{Br}_\chi g_{*s}^{\rm reh}}{5 g_\chi}\left(\frac{T_{\rm reh}}{m_\phi}\right)^4 \left(\frac{a_0}{a(t)}\right)^3 T_\star^3\, \bar{f}_{\rm R}(q)\,\mathrm{d}^3\boldsymbol{q} \qquad T_\star = \left(\frac{g_{*s}^0}{g_{*s}^{\rm reh}}\right)^{1/3}\frac{m_\phi}{2T_{\rm reh}}\,T_0$$



$$\Omega_\chi h^2 \simeq 0.1 \left(\frac{\mathrm{Br}_\chi}{5.5\times 10^{-4}}\right)\left(\frac{m_\chi}{1\,\mathrm{MeV}}\right)\left(\frac{T_{\rm reh}}{10^{10}\,\mathrm{GeV}}\right)\left(\frac{3\times 10^{13}\,\mathrm{GeV}}{m_\phi}\right)$$

UV freeze-in from scattering

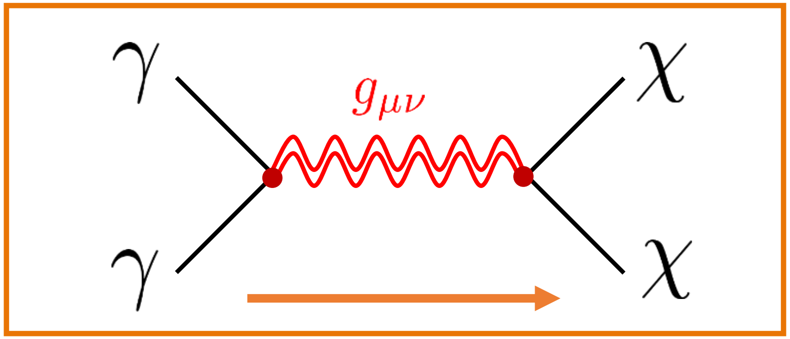
Assume **cross section** for $\gamma\gamma \to \chi\chi$: $\sigma(s) = \dfrac{s^{\frac{n}{2}}}{\Lambda^{n+2}}$

UV freeze-in from scattering

Assume **cross section** for $\gamma\gamma \to \chi\chi$: $\sigma(s) = \dfrac{s^{\frac{n}{2}}}{\Lambda^{n+2}}$

- $n = 0$ **Low-scale SUSY** for **gravitinos** $\sigma \propto 1/M_{\rm Pl}^2$ or **axinos** $\sigma \propto 1/f_a^2$
  [V. Rychkov, A. Strumia, PRD 75 (2007) 075011 - A. Strumia, JHEP 06 (2010) 036]
- $n = 2$ **Heavy Z′** from gauge unification $\sigma \propto s/m_{Z'}^4$
  [Y. Mambrini, K. A. Olive, J. Quevillon, B. Zaldívar- PRL 110, 241306]

  Gravity mediated freeze-in $\sigma \propto s/M_{\rm Pl}^4$
  [M. Garny, M. Sandora, M. S. Sloth - PRL 116 (2016) 10, 101302
  N. Bernal, M. Dutra, Y. Mambrini, K. Olive, M. Peloso, **MP** - PRD 97 (2018) 11, 115020]
  Y. Mambrini & K. Olive - PRD 97 (2021) 11, 115009]
- $n = 4$ Non-SUSY **Spin-3/2 DM** + sterile neutrino $\sigma \propto s^2/(m_{3/2} m_R M_{\rm Pl})^2$
  [M A. G. Garcia, Y. Mambrini, K. A. Olive, S. Verner - PRD 102 (2020) 8, 083533]



UV freeze-in from scattering

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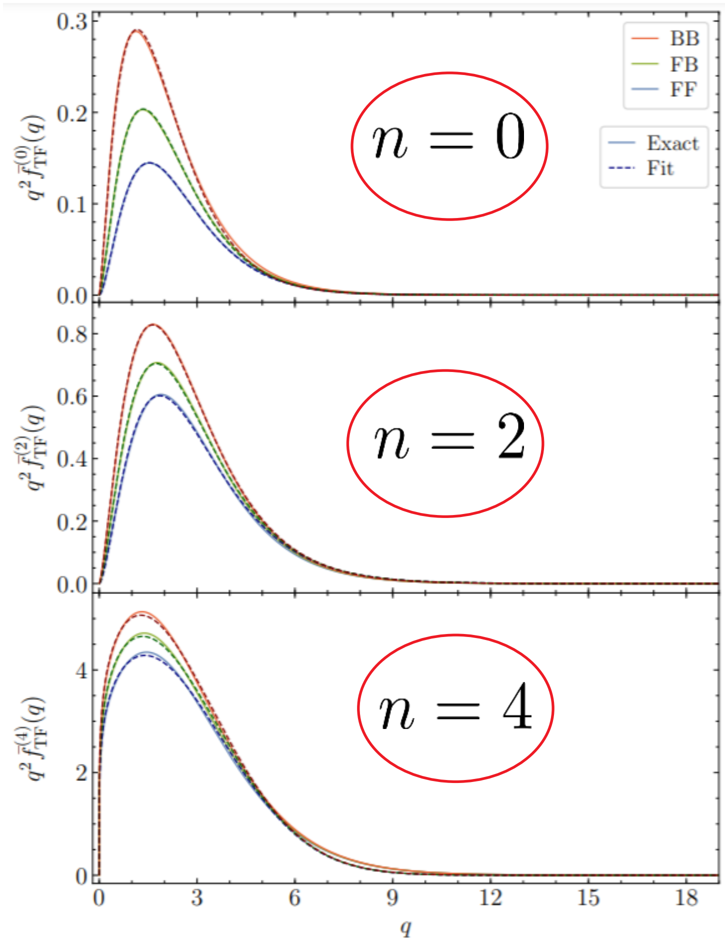
- $n \geq 6$ Constructions inspired by **modified gravity**
  [N. Bernal, M. Dutra, Y. Mambrini, K. Olive, M. Peloso, **MP** - PRD 97 (2018) 11, 115020]
  [P. Anastasopoulos, K. Kaneta, Y. Mambrini, **MP** - PRD 102 (2020) 5, 055019]
  [P. Brax, K. Kaneta, Y. Mambrini, **MP** - PRD 103 (2021) 5, 015028]
  [P. Brax, K. Kaneta, Y. Mambrini, **MP** - PRD 103 (2021) 5, 115016]

UV freeze-in from scattering $(n < 6)$

$$f_\chi(p,t)\,\mathrm{d}^3\boldsymbol{p} \simeq \left(\frac{6b}{g_{*s}^{\rm reh}}\right)^{1/2} \frac{3\cdot 2^{n+6}\Gamma(\frac{n+4}{2}) g_A g_B g_\psi M_P T_{\rm reh}^{n+1}}{5(2\pi)^3\Lambda^{n+2}} \left(\frac{a_0}{a(t)}\right)^3 T_\star^3\, \bar{f}_{\rm TF}^{(n)}(q)\,\mathrm{d}^3\boldsymbol{q}$$

$$T_\star = \left(\frac{g_{*s}^0}{g_{*s}^{\rm reh}}\right)^{1/3} T_0$$



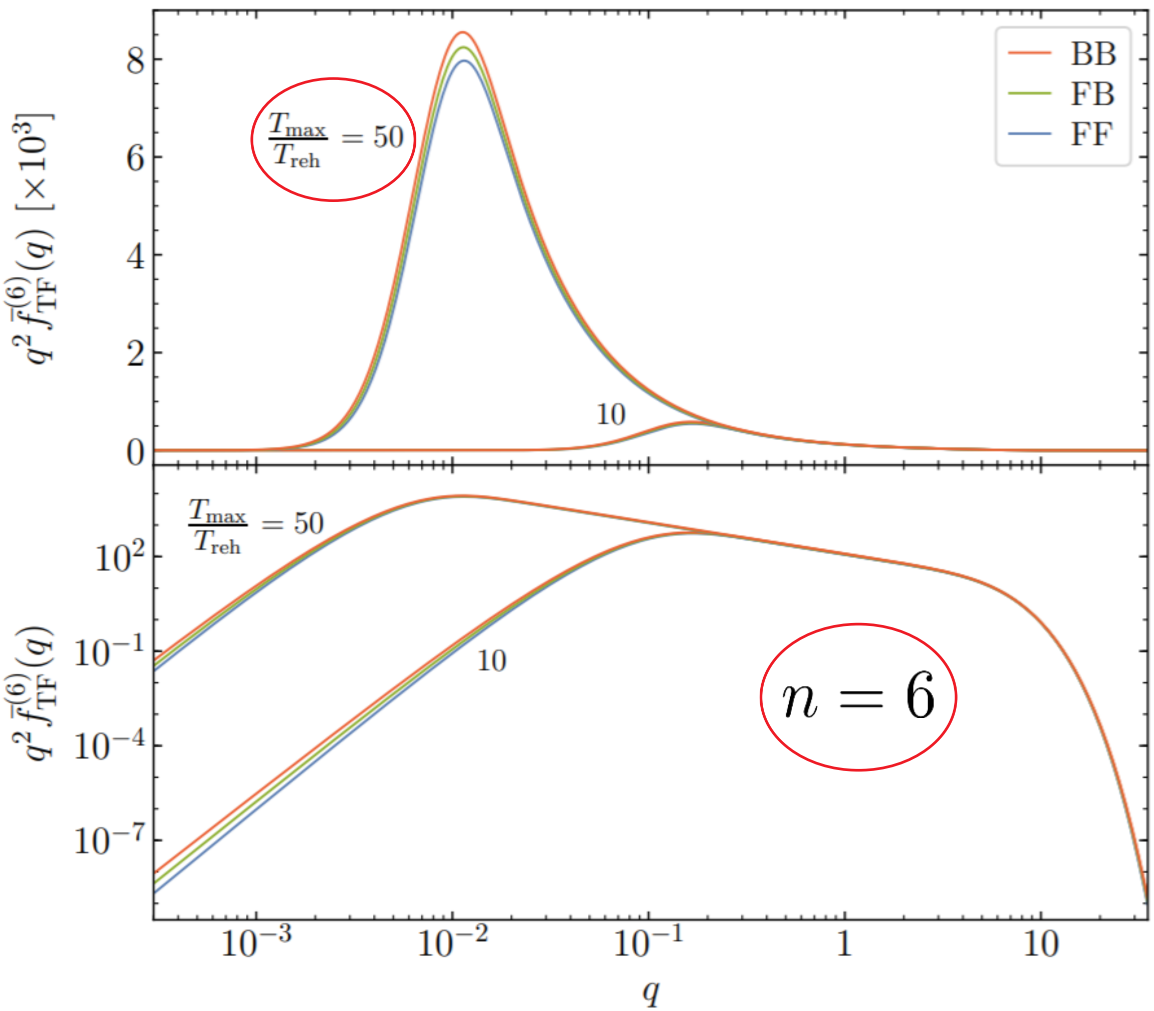
- **Mild** dependence on progenitor **spin**
  B : Bose-Einstein
  F : Fermi-Dirac

- **Maxwell-Boltzmann $\simeq$ FB**

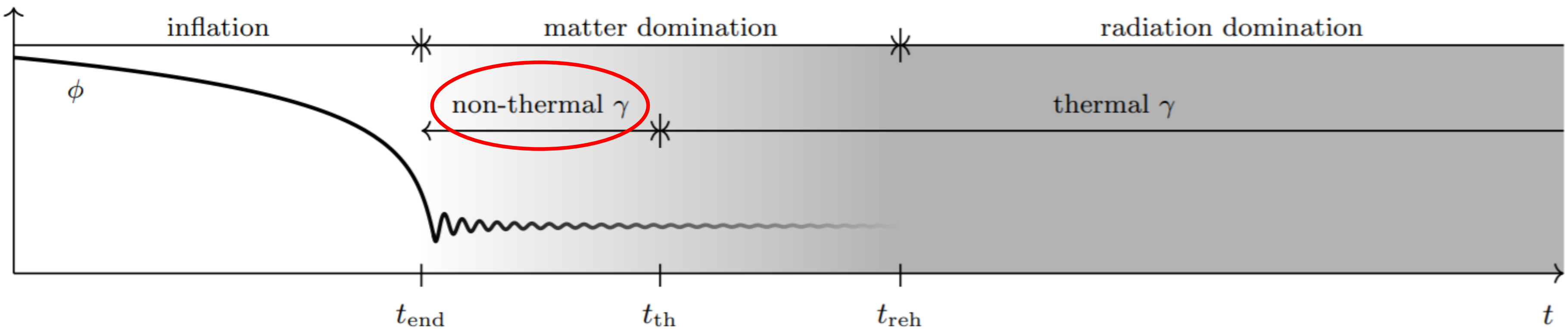
- Not thermal but well **fitted** by

$$f(q) \propto q^\alpha \exp\left(-\beta\, q^\gamma\right)$$

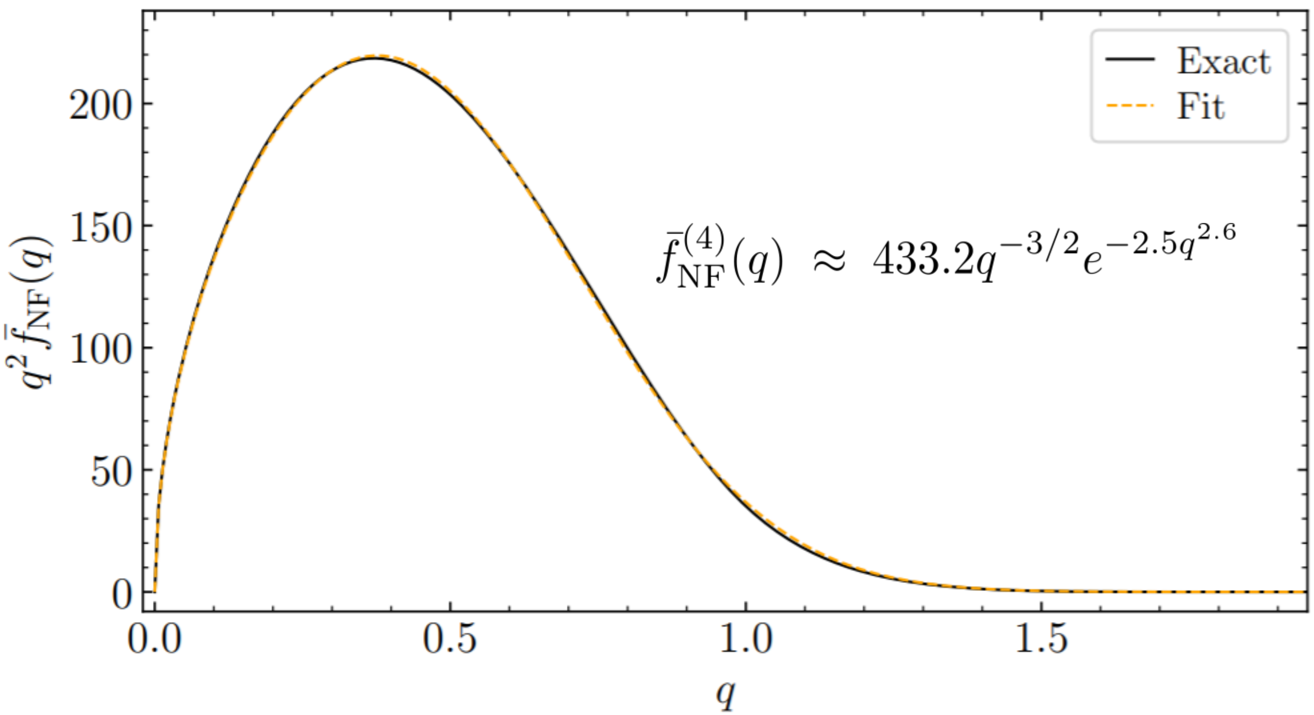
UV freeze-in from scattering $(n = 6)$



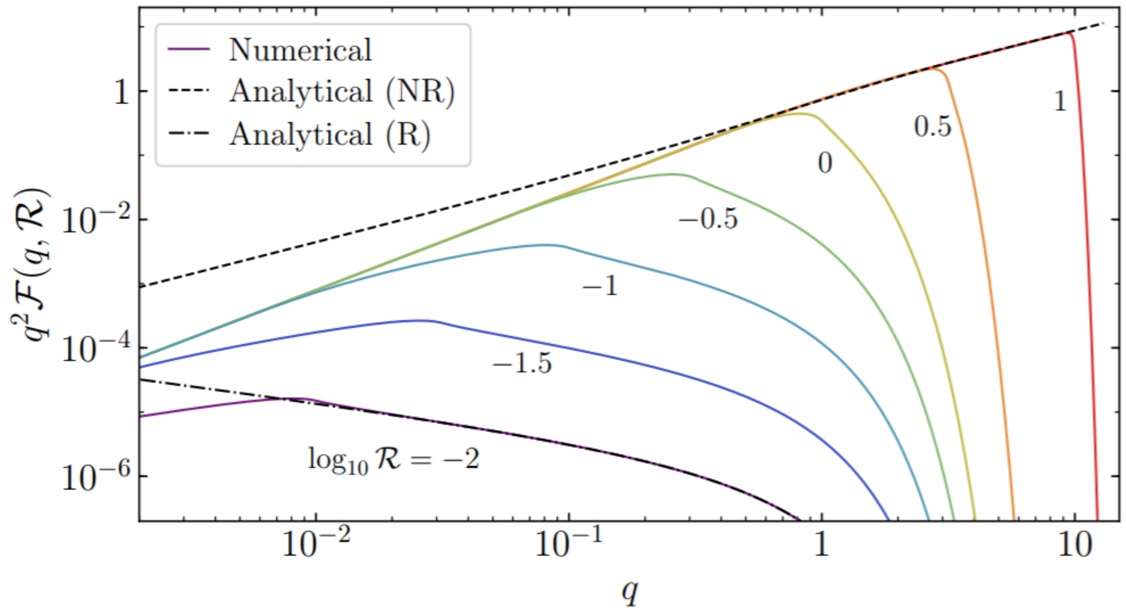
Non-termal UV freeze-in $(n = 4)$



[M. A. G. Garcia & M. A. Amin Phys. Rev. D 98, 103504 (2018)]

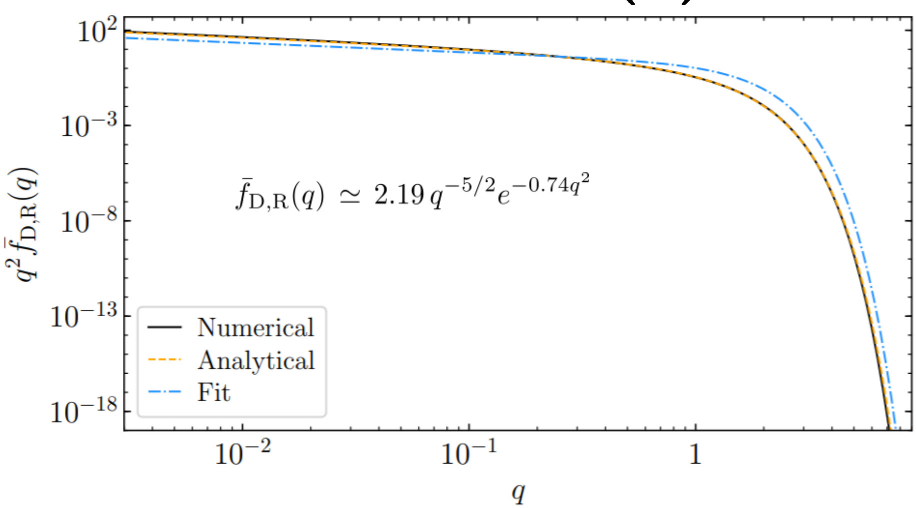


Freeze-in via decay $\phi \to A \to \chi$

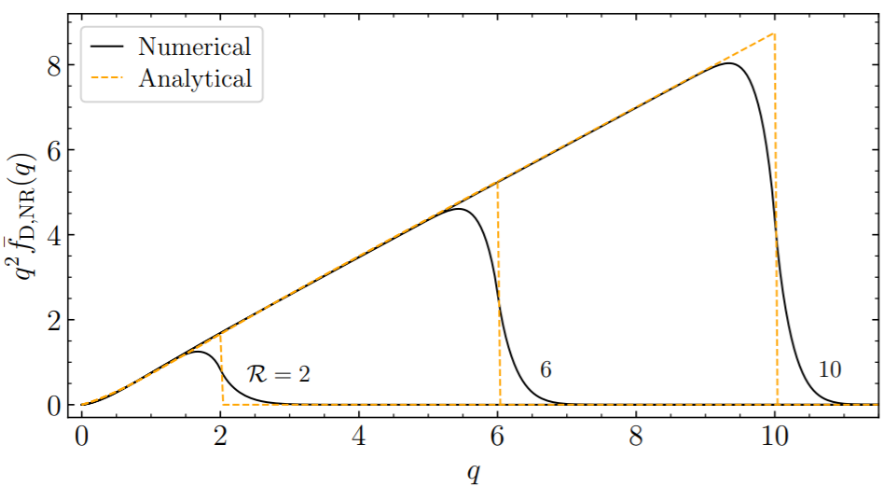


$$\mathcal{R} \equiv \left(\frac{g_{*s}^{\rm reh}}{g_{*s}^{\rm dec}}\right)^{1/3} \frac{m_A T_{\rm reh}}{m_\phi T_{\rm dec}} \propto \frac{m_A}{\langle p\rangle}$$

Relativistic limit (R) $\mathcal{R} \ll 1$

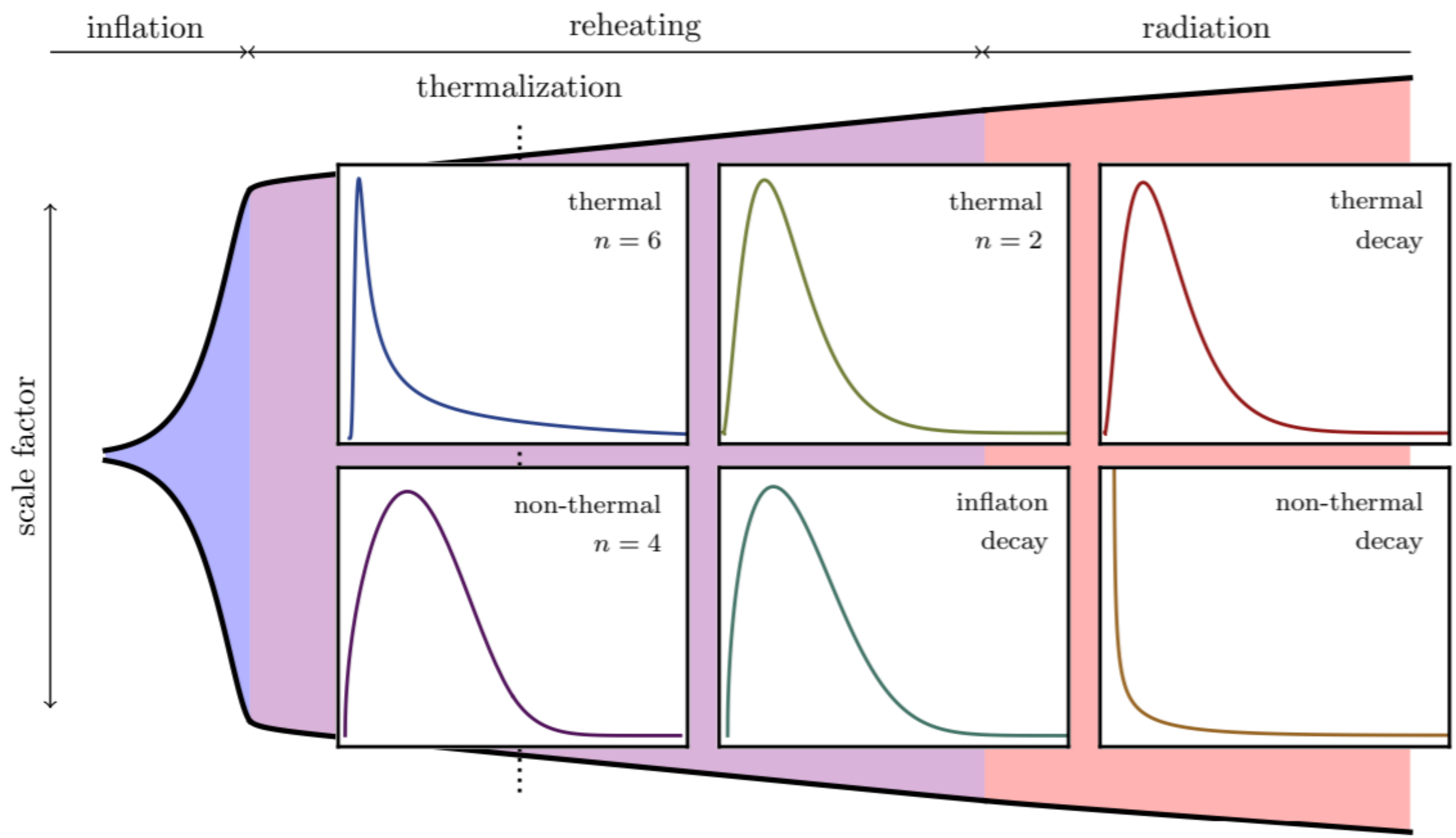


Non-relativistic limit (NR) $\mathcal{R} \gg 1$



Summary

- **Phase space distribution of out-of-equilibrium DM**



What is the cosmological imprint of out-of-equilibrium dark matter?

Cosmological imprint

Cosmological imprint

- **Cosmological role** of out-of-equilibrium dark matter via

$$\bar{\rho} = 4\pi \left(\frac{T_\star}{a}\right)^4 \int q^2 \epsilon \bar{f}(q)\,\mathrm{d}q$$

energy-density

$$\bar{P} = \frac{4\pi}{3} \left(\frac{T_\star}{a}\right)^4 \int q^2 \frac{q^2}{\epsilon} \bar{f}(q)\,\mathrm{d}q$$

pressure

$$q \equiv \frac{p\,a(t)}{T_\star}$$ **: comoving momentum**

$$\epsilon = \sqrt{q^2 + \left(\frac{m_{\mathrm{DM}}\,a}{T_\star}\right)^2}$$

- Define $w \equiv \bar{P}/\bar{\rho}$ **: equation-of-state parameter**

- In pure $\Lambda\mathrm{CDM}$: $w = 0$ precisely (**Cold = pressureless**)

- But $w \neq 0$! ➡ **Non-Cold Dark Matter** cosmology

Non-Cold Dark Matter $w \neq 0$

- Expanding quantities around **homogenous background**

$$f(\boldsymbol{x}, \boldsymbol{p}, \tau) = \bar{f}(|\boldsymbol{p}|, \tau)[1 + \Psi(\boldsymbol{x}, \boldsymbol{p}, \tau)]$$

- In matter domination, matter **overdensities** δ follow

$$\ddot{\delta} + \mathcal{H}\dot{\delta} - \frac{3}{2}\mathcal{H}^2 \left(1 - \frac{k^2}{k_{\rm FS}^2}\right)\delta = 0 \qquad w \ll 1$$

where $k_{\rm FS}^2(a) = \frac{9}{10}\frac{\mathcal{H}^2}{w(a)}$ is the **Free-Streaming wavenumber**

$\mathrm{d}\tau \equiv a\,\mathrm{d}t$: **Conformal time** τ

$\mathcal{H} \equiv a\,H$: **Conformal Hubble rate**

$w \equiv \bar{P}/\bar{\rho}$: **Equation-of-state parameter**

[C. Ma & E. Bertschinger. ApJ 455 (1995) 7-25]
[J. Lesgourgues & T. Tram, JCAP 09 (2011) 032]
[M. Kunz, S. Nesseris, I. Sawicki, PRD 94, 023510 (2016)]
[G. Ballesteros, M. A. G. Garcia & **MP**, 2011.13458]

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where $k_{\rm FS}^2(a) = \frac{9}{10}\frac{\mathcal{H}^2}{w(a)}$ is the **Free-Streaming wavenumber**

- If $w = 0$ all modes grow "**democratically**" : **CDM** limit
  $w \neq 0$ **cutoff in power spectrum** at $k_{\rm H}(a) \equiv \left[\int_0^a \frac{1}{k_{\rm FS}(\tilde{a})} \frac{{\rm d}\tilde{a}}{\tilde{a}}\right]^{-1}$

- **Only w controls the cutoff scale!**

${\rm d}\tau \equiv a\,{\rm d}t$: **Conformal time τ**

$\mathcal{H} \equiv a\,H$: **Conformal Hubble rate**

$w \equiv \bar{P}/\bar{\rho}$: **Equation-of-state parameter**

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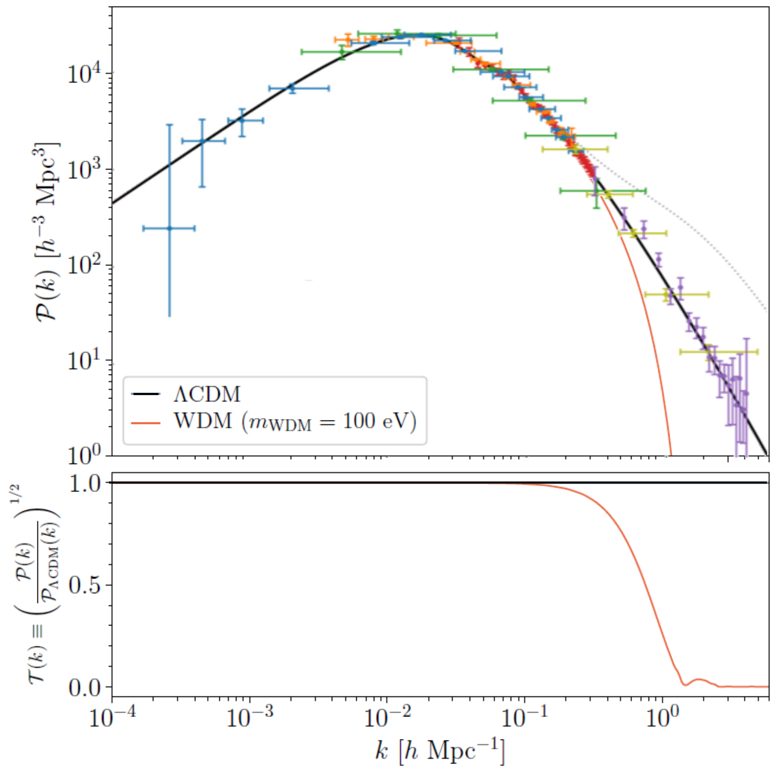
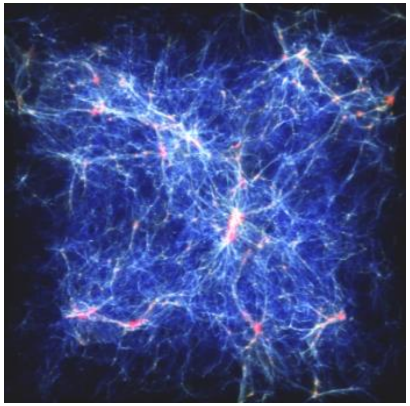
Non-Cold Dark Matter $w \neq 0$

- Lyman-alpha forest constraints Warm Dark Matter (**WDM**)

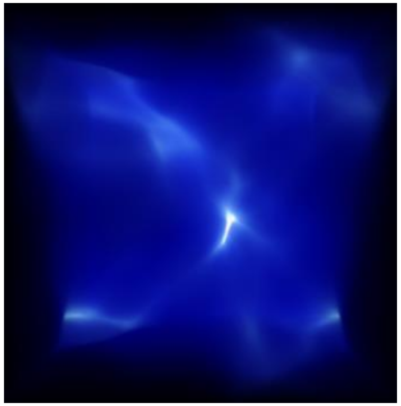
$$\bar{f}_{\rm WDM}(q) = \frac{1}{1 + e^{q/T_{\rm WDM}}} \quad \Longrightarrow \quad \Omega_{\rm WDM} h^2 \simeq \left(\frac{m_{\rm WDM}}{94\ \rm eV}\right)\left(\frac{T_{\rm WDM}}{T_\nu}\right)^3 \simeq 0.12$$



ΛCDM



WDM



$m_{\rm WDM} = 100\,\rm eV$

[J. Baur, N. Palanque-Delabrouille, C. Yèche, C. Magneville, M. Viel, JCAP 08 (2016) 012]

How warm is Non-Cold Dark Matter?

- From **Lyman-alpha** forest $m_{\rm WDM}^{\rm Ly-\alpha} = (1.9 - 5.3)$ keV at 95% C.L.

[Braur et al. JCAP 08 (2016) 012 – Iršič et al. PRD 96 (2017) 2, 023522
Palanque Delabrouille et al. JCAP 04 (2020) 038 – Viel et al. PRD 88 (2013) 043502
Viel et al. PRD 71 (2005) 063534 – Narayanan et al. ApJ 543 (2000) L103-L106]

$$w_{\rm WDM}(a) \simeq 6 \times 10^{-15}\, a^{-2} \left(\frac{\rm keV}{m_{\rm WDM}}\right)^{8/3} \Rightarrow w_{\rm WDM}(a=1) < 10^{-15}$$

- Constraints much **stronger** than **CMB**! $w_{\rm WDM}(a=1) < 10^{-10}$

[M. Kunz, S. Nesseris, I. Sawicki, PRD 94, 023510 (2016)]

- How **cold** are **WIMPs** ?

$$w(a) \simeq 10^{-29} \left(\frac{1}{a^2}\right) \left(\frac{20\, T_F}{m_{\rm DM}}\right) \left(\frac{100\ \rm GeV}{m_{\rm DM}}\right)^2 \left(\frac{100}{g_*^F}\right)^{2/3} \Rightarrow$$

- **How to translate Lyman-alpha WDM bounds on any DM?**

$$w(m_{\rm DM}) = w_{\rm WDM}(m_{\rm WDM}^{\rm Ly-\alpha})$$

[S. Colombi, S. Dodelson, L. M. Widrow ApJ. 458 (1996) 1 - Kamada, N. Yoshida, K. Kohri, T. Takahashi JCAP 03 (2013) 008
K. J. Bae, R. Jinno, A. Kamada, K. Yanagi JCAP 03 (2020) 042 - A. Kamada & K. Yanagi JCAP 1911 (2019) 029]

How warm is Non-Cold Dark Matter?

w – matching

$$w \simeq \frac{\delta P}{\delta \rho} = \frac{T_\star^2}{3 m_{\rm DM}^2} \frac{\langle q^2 \rangle}{a^2}$$

$$m_{\rm DM} = m_{\rm WDM}^{\rm Ly-\alpha} \left(\frac{T_\star}{T_{\rm WDM}} \right) \sqrt{\frac{\langle q^2 \rangle}{\langle q^2 \rangle_{\rm WDM}}}$$

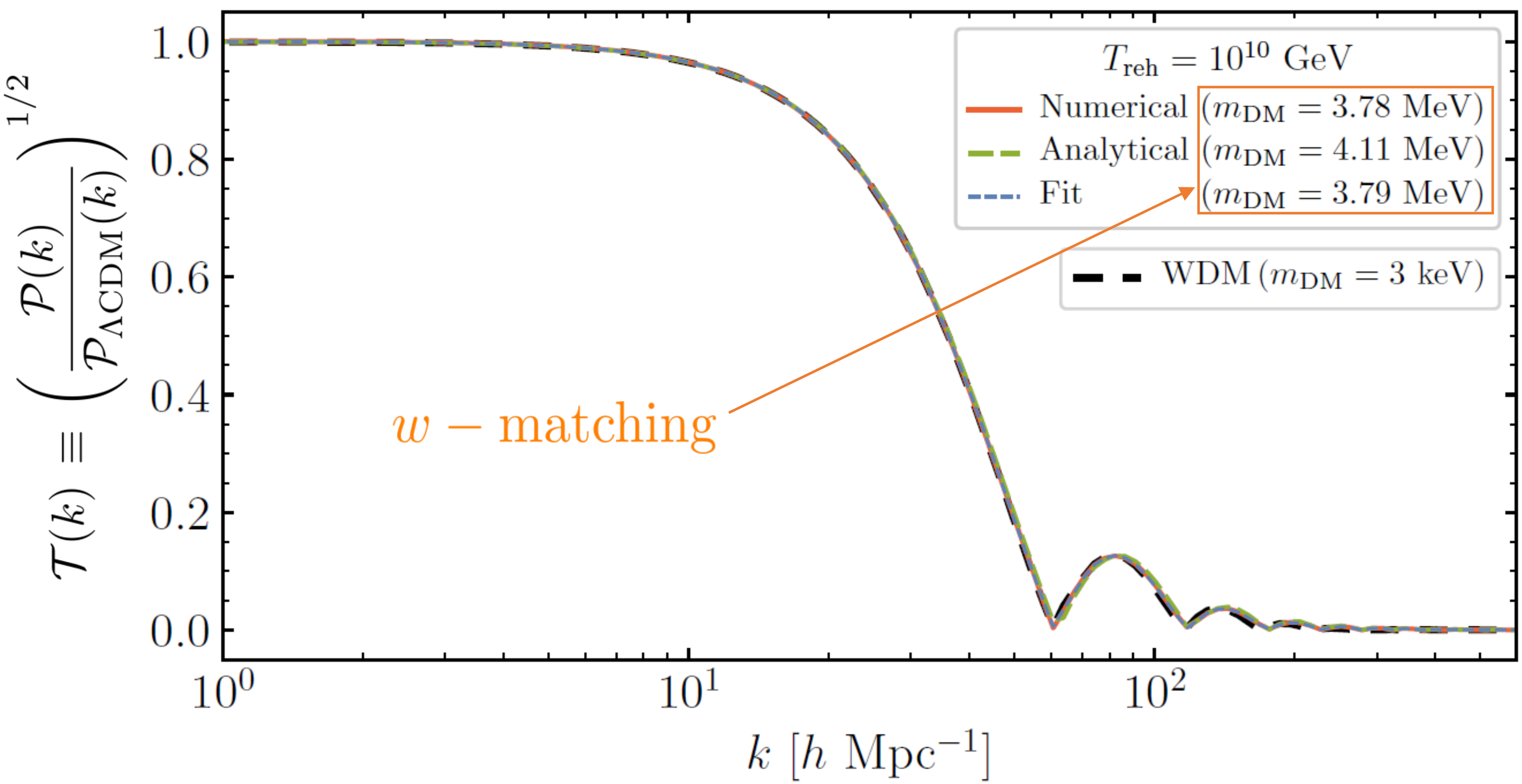
- Compute **2nd moment of distribution** + **determine** $T_\star$
- If **distribution** can be fitted by $f(q) \propto q^\alpha \exp\left(-\beta\, q^\gamma\right)$

w – matching ⟹ $$m_{\rm DM} \simeq 7.56\ {\rm keV} \left(\frac{m_{\rm WDM}^{\rm Ly-\alpha}}{3\ {\rm keV}} \right)^{4/3} \left(\frac{\langle p \rangle_0}{T_0} \right) \sqrt{\frac{\Gamma\left(\frac{3+\alpha}{\gamma}\right) \Gamma\left(\frac{5+\alpha}{\gamma}\right)}{\Gamma^2\left(\frac{4+\alpha}{\gamma}\right)}}$$

How warm is Non-Cold Dark Matter?

- Example: **inflaton decay** case computed using **CLASS**

  [D. Blas, J. Lesgourgues & T. Tram JCAP 07 (2011) 034 - J. Lesgourgues & T. Tram, JCAP 09 (2011) 032]



- **Excellent agreement** with $w -$ matching for **all distributions**!

Inflaton decay

- **Lyman-alpha** bounds translate into

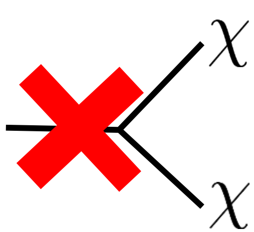
$$m_{\rm DM} \gtrsim \left(\frac{m_{\rm WDM}^{\rm Ly-\alpha}}{3~{\rm keV}}\right)^{4/3} \left(\frac{106.75}{g_{*s}^{\rm reh}}\right)^{1/3} \left(\frac{m_\phi}{3\times 10^{13}~{\rm GeV}}\right) \left(\frac{10^{10}~{\rm GeV}}{T_{\rm reh}}\right) \begin{cases} 3.78~{\rm MeV}\,, & {\rm Numerical}\,, \\ 4.11~{\rm MeV}\,, & {\rm Analytical}\,, \\ 3.79~{\rm MeV}\,, & {\rm Fit}\,. \end{cases}$$

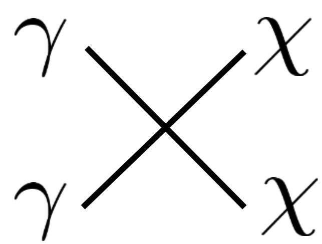
- For **low reheating temperature** $T_{\rm reh} \ll m_\phi$

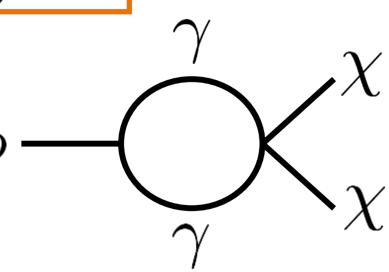
$$\boxed{m_{\rm DM} \gtrsim {\rm EeV}}$$

- **Combining** with **relic density** condition

$$\boxed{{\rm Br}_\chi < 1.5\times 10^{-4} \left(\frac{g_{*s}^{\rm reh}}{106.5}\right)^{1/3} \left(\frac{3\,{\rm keV}}{m_{\rm WDM}^{\rm Ly-\alpha}}\right)^{4/3}}$$

- Even if ϕ 

, since 

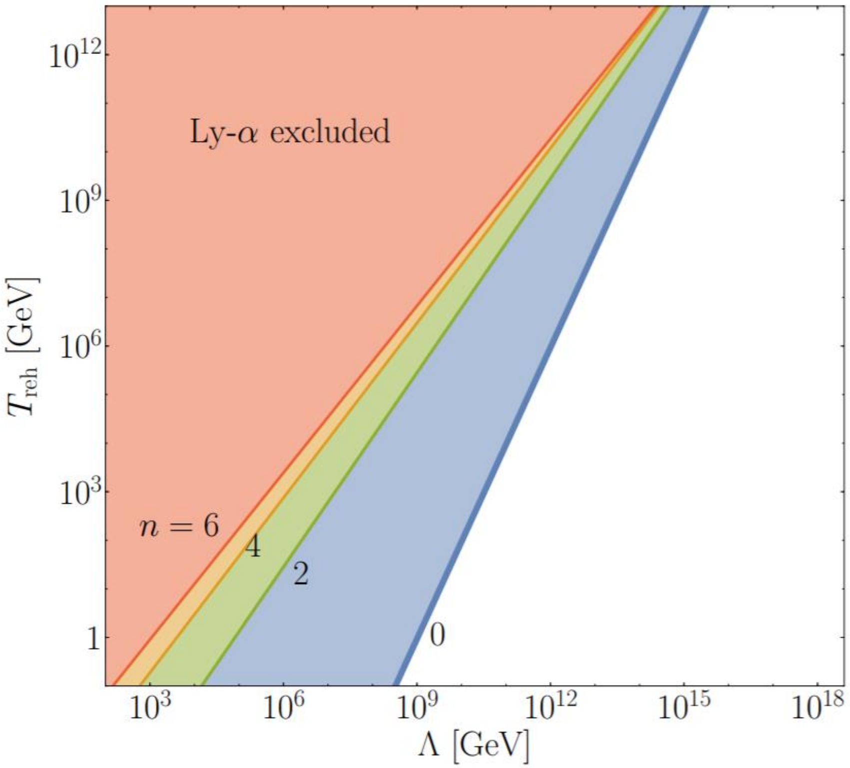
then ϕ 

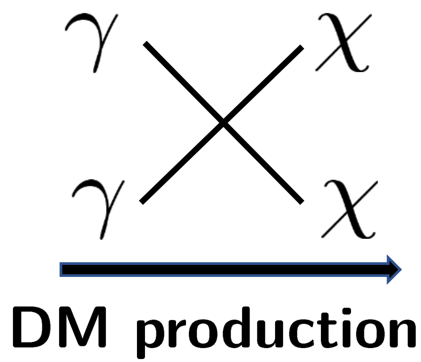
[K. Kaneta, Y. Mambrini & Keith A. Olive Phys.Rev.D 99 (2019) 6, 063508]

UV freeze-in via scattering

$$m_{\rm DM} \gtrsim \left(\frac{m_{\rm WDM}^{\rm Ly-\alpha}}{3\ {\rm keV}}\right)^{4/3}\left(\frac{106.75}{g_{*s}^{\rm reh}}\right)^{1/3}\begin{cases}7.27\ (7.17)\ {\rm keV}\,, & {\rm FF} \quad {\rm Numerical\ (Fit)}\,, \quad n=0\\ 8.48\ (8.73)\ {\rm keV}\,, & {\rm FF} \quad {\rm Numerical\ (Fit)}\,, \quad n=2\\ 8.52\ (8.05)\ {\rm keV}\,, & {\rm FF} \quad {\rm Numerical\ (Fit)}\,, \quad n=4\end{cases}$$

- **Combine** with **relic density** condition

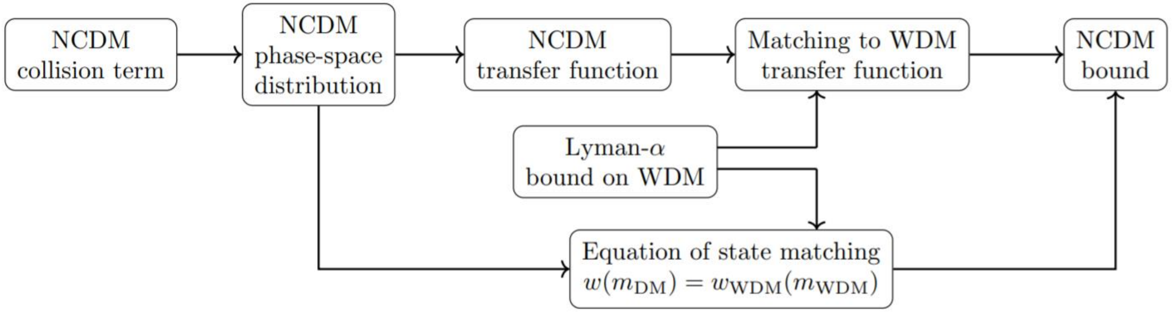


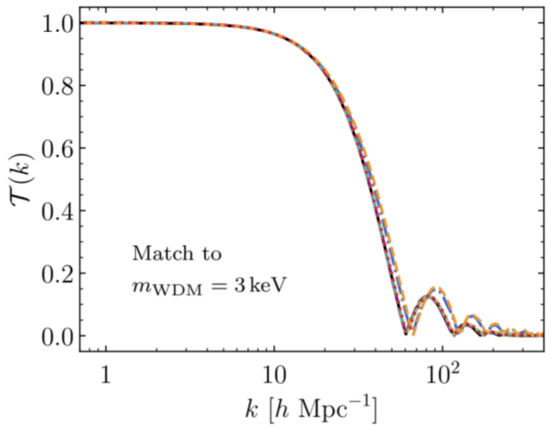


$$\sigma(s) = \frac{s^{\frac{n}{2}}}{\Lambda^{n+2}}$$

- **Apply to any UV freeze-in model!**

Summary





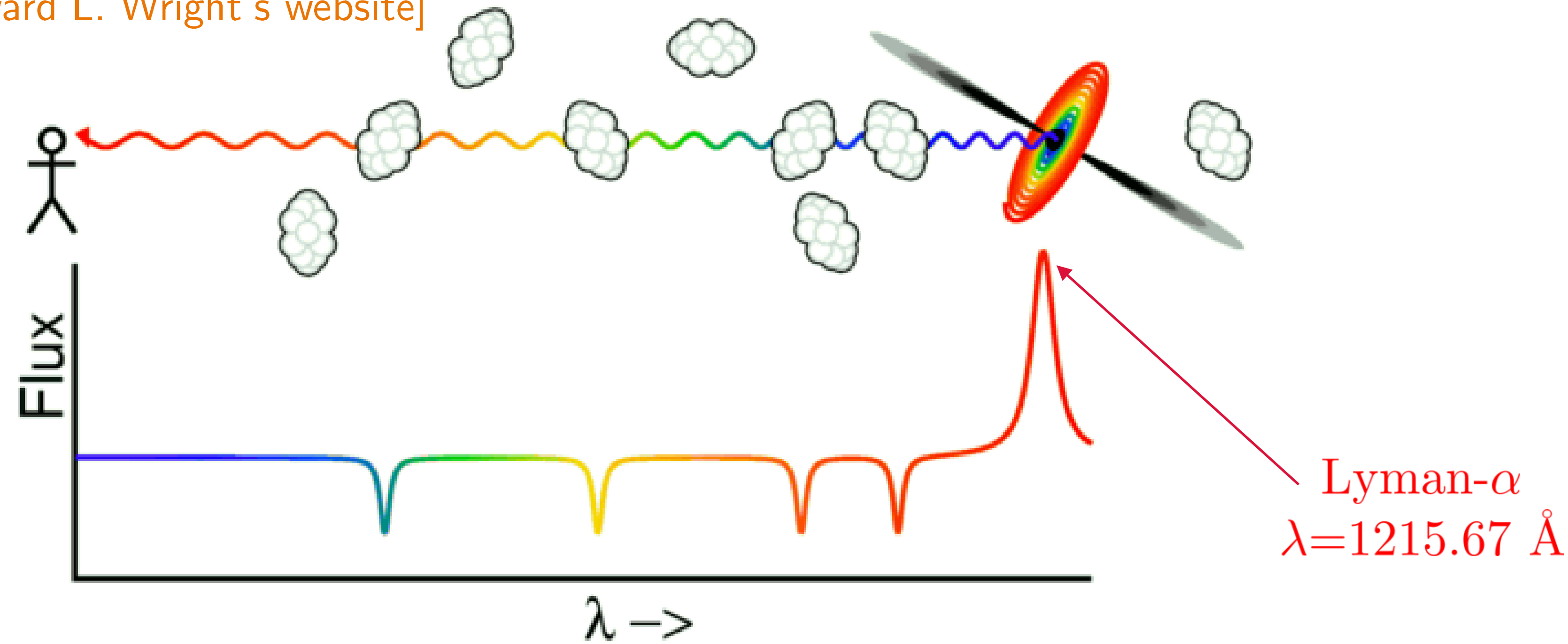
- Lyman-alpha is a **powerful tool** to **probe out-of-equilibrium dark matter** and **early universe dynamics**
- Many more cases **in the paper!** [arXiv:2011.13458]
- **Dark matter is cold.**

Thank you for your attention

Back-up Slides

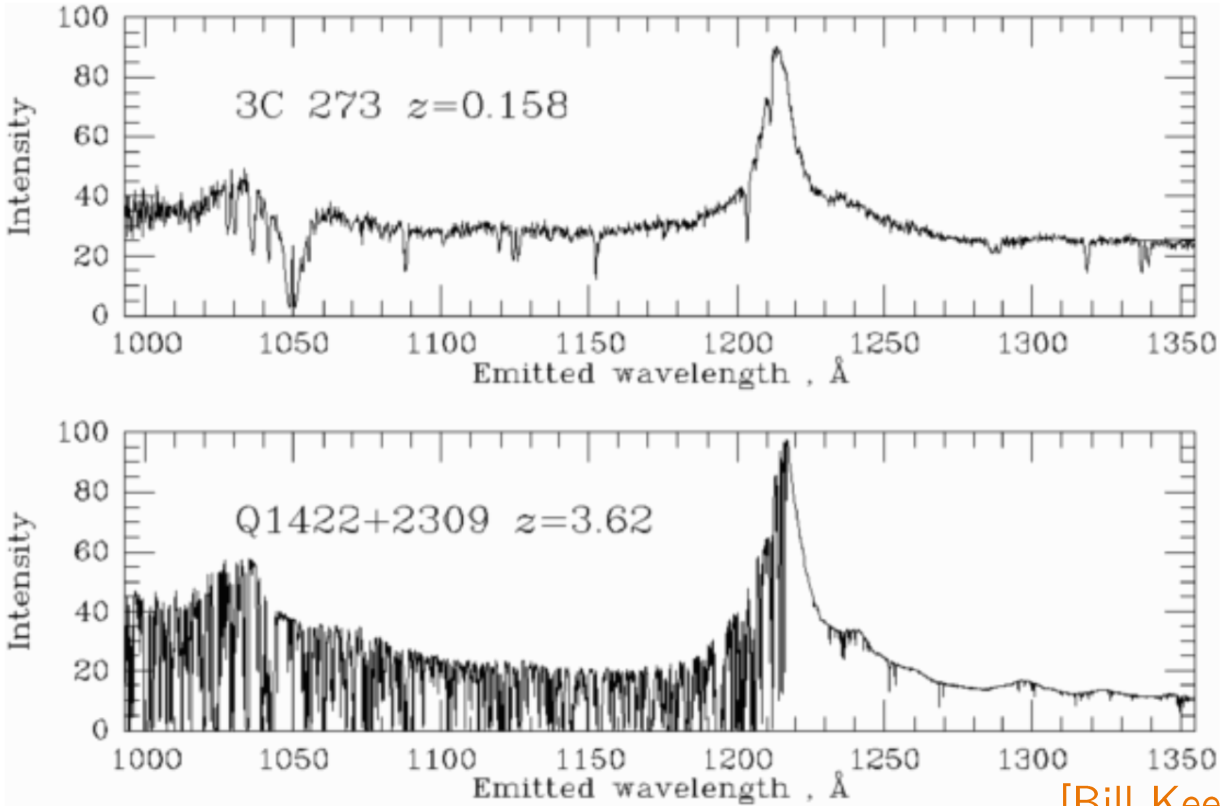
Lyman-alpha forest

[Edward L. Wright's website]



- Quasi-Stellar Objects (QSO) are luminous astrophysical objects powered by gas spiraling at high velocity into an massive black hole
- Light emitted by distant QSO is absorbed in foreground structures
- Allows for a 1D measure of overdensities along line of sight

Lyman-alpha forest



[Bill Keel's website]

- Comparison of QSO spectra at low and high redshift in QSO rest frame

NCDM Cosmology

- Expand around (homogenous) **background quantities**

$$f(\boldsymbol{x},\boldsymbol{p},\tau)=\bar{f}(|\boldsymbol{p}|,\tau)[1+\Psi(\boldsymbol{x},\boldsymbol{p},\tau)]$$

- Expand **fluctuations** in term of **Legendre polynomials**

$$\Psi(\boldsymbol{k},\hat{\boldsymbol{n}},q,\tau)=\sum_{\ell=0}^{\infty}(-i)^{\ell}(2\ell+1)\Psi_{\ell}(\boldsymbol{k},q,\tau)P_{\ell}(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}})$$

- Express **fluctuations** in terms of **Legendre coefficients**

$$\delta\bar{\rho} = 4\pi\left(\frac{T_\star}{a}\right)^4\int q^2\epsilon\bar{f}(q)\Psi_0\,\mathrm{d}q, \qquad \text{energy density fluctuation}$$

$$\delta\bar{P} = \frac{4\pi}{3}\left(\frac{T_\star}{a}\right)^4\int q^2\frac{q^2}{\epsilon}\bar{f}(q)\Psi_0\,\mathrm{d}q, \qquad \text{pressure (density) fluctuation}$$

$$(\bar{\rho}+\bar{P})\theta = 4\pi k\left(\frac{T_\star}{a}\right)^4\int q^3\bar{f}(q)\Psi_1\,\mathrm{d}q, \qquad \text{velocity divergence}$$

$$(\bar{\rho}+\bar{P})\sigma = \frac{8\pi k}{3}\left(\frac{T_\star}{a}\right)^4\int q^2\frac{q^2}{\epsilon}\bar{f}(q)\Psi_2\,\mathrm{d}q, \qquad \text{anisotropic stress.}$$

NCDM Cosmology

- The phase space distribution satisfies **collisionless Boltzmann equation**

$$\frac{\partial f}{\partial \tau} + \frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\partial f}{\partial x^i} + \frac{\mathrm{d}q}{\mathrm{d}\tau}\frac{\partial f}{\partial q} + \frac{\mathrm{d}n_i}{\mathrm{d}\tau}\frac{\partial f}{\partial n_i} = 0\,,$$

- **Plugging** distribution expansion in **Legendre polynomials** give

$$\mathrm{d}s^2 = a(\tau)\big(-\mathrm{d}\tau^2 + (\delta_{ij} + h_{ij})\,\mathrm{d}x^i\,\mathrm{d}x^j\big)$$

in **synchronous** gauge $h \equiv h_{ii}$

$$\begin{aligned}
\dot{\Psi}_0 &= -\frac{qk}{\epsilon}\Psi_1 + \frac{1}{6}\dot{h}\frac{\mathrm{d}\ln\bar{f}}{\mathrm{d}\ln q}\,,\\
\dot{\Psi}_1 &= \frac{qk}{3\epsilon}\Big(\Psi_0 - 2\Psi_2\Big)\,,\\
\dot{\Psi}_2 &= \frac{qk}{5\epsilon}\Big(2\Psi_1 - 3\Psi_3\Big) - \Big(\frac{1}{15}\dot{h} + \frac{2}{5}\dot{\eta}\Big)\frac{\mathrm{d}\ln\bar{f}}{\mathrm{d}\ln q}\,,\\
\dot{\Psi}_\ell &= \frac{qk}{(2\ell+1)\epsilon}\Big(\ell\Psi_{\ell-1} - (\ell+1)\Psi_{\ell+1}\Big)\,, \quad [\ell \geq 3]
\end{aligned}$$

For a non-relativistic species, higher multipoles are typically suppressed by (positive) powers of $q/\epsilon \sim p/m_{\mathrm{DM}}$, making any Ψ_ℓ with $\ell \geq 2$ much smaller than Ψ_0 and Ψ_1. In this case, the Boltzmann hierarchy can be truncated imposing $\Psi_\ell = 0$ for $\ell > 1$. In this (non-relativistic) case Ψ_0 depends only mildly on the variable q, and the integrals are dominated by the low $q \ll \epsilon$ regime so that we can identify $\delta P/\delta\rho \simeq \bar{P}/\bar{\rho} = w$.

NCDM Cosmology

- **Neglecting higher multipoles**, for very **non-relativistic** DM, integrating over momenta gives

$$\dot{\delta} = -(1+w)\left(\theta + \frac{\dot{h}}{2}\right) - 3\mathcal{H}\left(\hat{c}_s^2 - w\right)\delta + 9\mathcal{H}^2(1+w)\left(\hat{c}_s^2 - c_a^2\right)\frac{\theta}{k^2},$$

$$\dot{\theta} = -\mathcal{H}\left(1 - 3\hat{c}_s^2\right)\theta + \frac{\hat{c}_s^2}{1+w}k^2\delta,$$

- In **matter domination**, from **Einstein equations**, metric perturbation follow

$$\ddot{h} + \mathcal{H}\dot{h} + 3(1+3w)\mathcal{H}^2\delta = 0,$$

- Which can be translated to evolution of **matter density fluctuations**

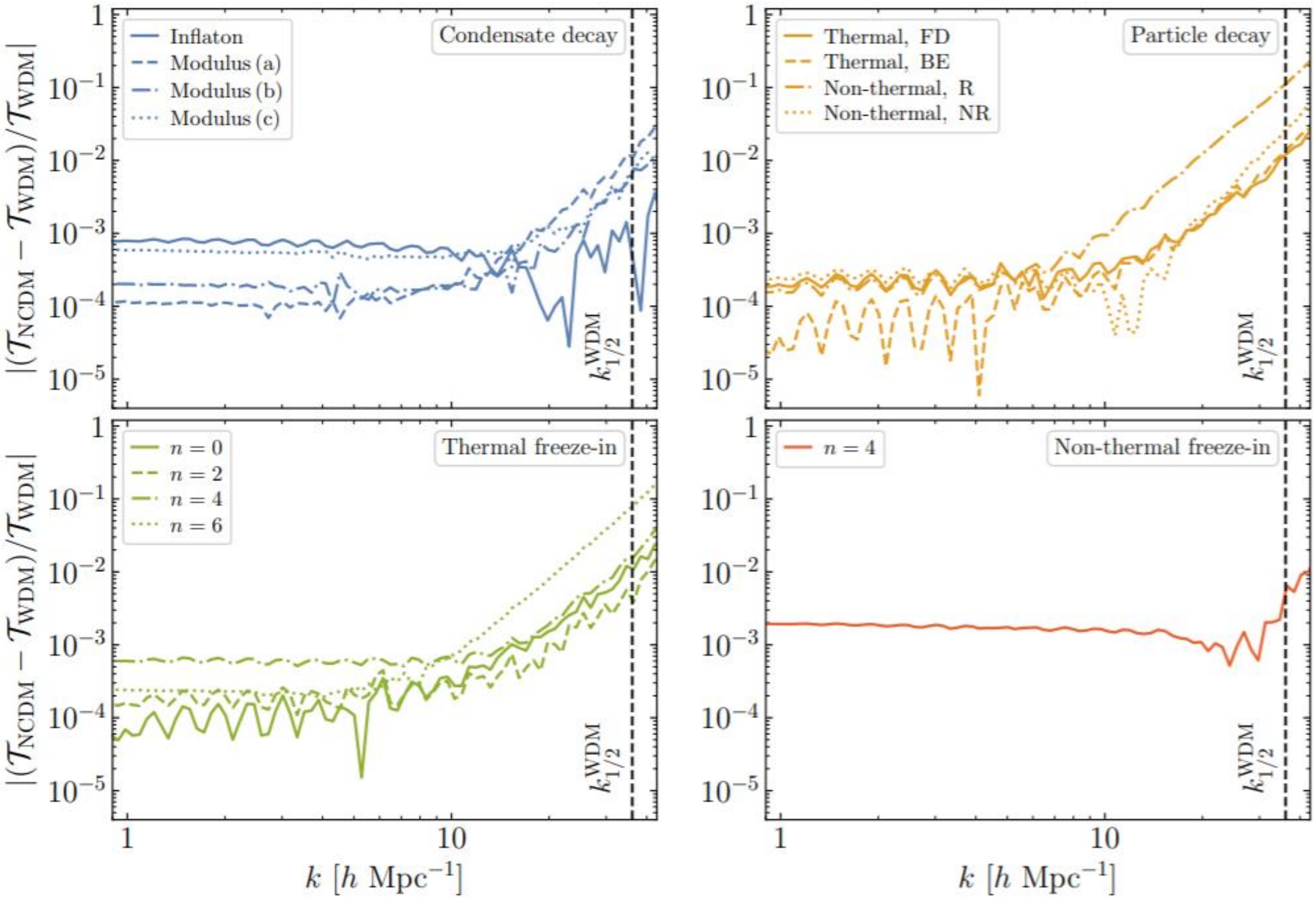
$$\ddot{\delta} + \mathcal{H}\dot{\delta} - \frac{3}{2}\mathcal{H}^2\left(1 - w\frac{10}{9}\frac{k^2}{\mathcal{H}^2}\right)\delta = 0.$$

General phase space distribution

$$f(q) \propto q^{\alpha} \exp\left(-\beta q^{\gamma}\right)$$

Scenario		α	β	γ
Inflaton decay		-3/2	0.74	1.00
Moduli decay	during reheating	-3/2	1.00	3/2
	after reheating	-1.00	1.00	2.00
Thermal decay		-1/2	1.00	1.00
Non-thermal decay	non-relativistic	-	-	-
	relativistic	-5/2	0.74	2.00
UV Freeze-in ($n = 0$)	BB	0.70	1.13	1.00
	FB	0.51	1.10	1.00
	FF	0.29	1.11	1.00
UV Freeze-in ($n = 2$)	BB	0.51	0.91	1.00
	FB	0.42	0.90	1.00
	FF	0.33	0.90	1.00
UV Freeze-in ($n = 4$)	BB	0.21	0.06	1.98
	FB	0.21	0.06	2.04
	FF	0.21	0.05	2.10
UV Freeze-in ($n = 6$)	BB	-	-	-
	FB	-	-	-
	FF	-	-	-
Non-thermal UV Freeze-in		-3/2	2.5	2.6

Precision on transfer functions



Contribution to $N_{\rm eff}$?

$$\begin{aligned}
\Delta N_{\rm eff} &= \frac{8}{7}\left(\frac{T}{T_\nu}\right)^4 \frac{\rho_\chi - m_{\rm DM} n_\chi}{\rho_\gamma} \\
&= \frac{8\pi\Omega_\chi}{7\Omega_\gamma}\left(\frac{g_{*s}(T)}{g^0_{*s}}\right)^{4/3}\left(\frac{T}{T_\nu}\right)^4\left(\frac{T_\star}{m_{\rm DM}}\right) \\
&\quad\times\left[\left\langle\sqrt{q^2+\left(\frac{g^0_{*s}}{g_{*s}(T)}\right)^{2/3}\left(\frac{m_{\rm DM}}{T_\star}\right)^2\left(\frac{T_0}{T}\right)^2}\right\rangle - \left(\frac{g^0_{*s}}{g_{*s}(T)}\right)^{1/3}\left(\frac{m_{\rm DM}}{T_\star}\right)\left(\frac{T_0}{T}\right)\right].
\end{aligned}$$

- **Saturating** the Lyman-alpha bound gives

$$\begin{aligned}
\Delta N_{\rm eff,max} &\simeq \frac{1.4\times 10^{-4}}{\sqrt{\langle q^2\rangle}}\left(\frac{g_{*s}(T)}{g^0_{*s}}\right)^{4/3}\left(\frac{\Omega_\chi h^2}{0.1}\right)\left(\frac{3\,\mathrm{keV}}{m_{\rm WDM}}\right)^{4/3}\left(\frac{T}{T_\nu}\right)^4 \\
&\quad\times\left[\left\langle\sqrt{q^2+\mu_*(T)^2}\right\rangle - \mu_*(T)\right],
\end{aligned}$$

$$\mu_*(T) \equiv \sqrt{\langle q^2\rangle}\left(\frac{g^0_{*s}}{g_{*s}(T)}\right)^{1/3}\left(\frac{3\,\mathrm{keV}}{m_{\rm WDM}}\right)^{4/3}\left(\frac{7.56\,\mathrm{keV}}{T}\right).$$

$$\Delta N_{\rm eff}(T_{\rm BBN}) \lesssim 5.4\times 10^{-4}\left(\frac{\langle q\rangle}{\sqrt{\langle q^2\rangle}}\right)\left(\frac{\Omega_\chi h^2}{0.1}\right)\left(\frac{3\,\mathrm{keV}}{m_{\rm WDM}}\right)^{4/3},$$