Reconstructing inflationary potential from a power spectrum using generalized slow-roll inflation

based on JCAP 06(2021) 054 with Su-beom Kang, and Rathul Raveendran

and JCAP 01(2022) 012 S.Kang, R.Raveendran, and Jinn-Ouk Gong

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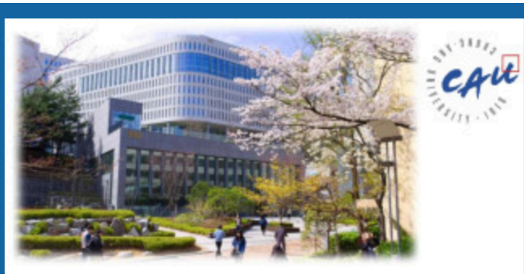

2022 Chung-Ang University Beyond the Standard Model Workshop

7-10 Feb

From Inflation to Large Scale Structure

Inflationay potential $V(\phi, \psi, \ldots)$

Primordial Power Spectrum $P_R(k)$

CMB anisotropy

Large Scale Structure

Reconstructing Inflation from primordial power spectrum by inverse

Inflationay potential $V(\phi, \psi, \ldots)$

↑

Primordial Power Spectrum $P_R(k)$

↑

CMB anisotropy | Large Scale Structure

Ki-Young Choi, Sungkyunkwan University, Korea

Contents

Introduction

Primordial Power Spectrum Reconstruction

The simplest shape of the power spectrum is the power-law form,

usually parametrized by the amplitude and spectral index

$$A_s \left(\frac{k}{k_*}\right)^{n_s - 1}$$

$$k_* = 0.05 \mathrm{Mpc}^{-1}$$

$$A_\mathrm{s} = (2.101^{+0.031}_{-0.034}) \times 10^{-9} \quad (68\,\%, \text{TT,TE,EE+lowE}).$$

[Planck 2018]

$$n_\mathrm{s} = 0.9649 \pm 0.0042 \quad (68\,\%, \textit{Planck}\ \text{TT,TE,EE+lowE +lensing}),$$

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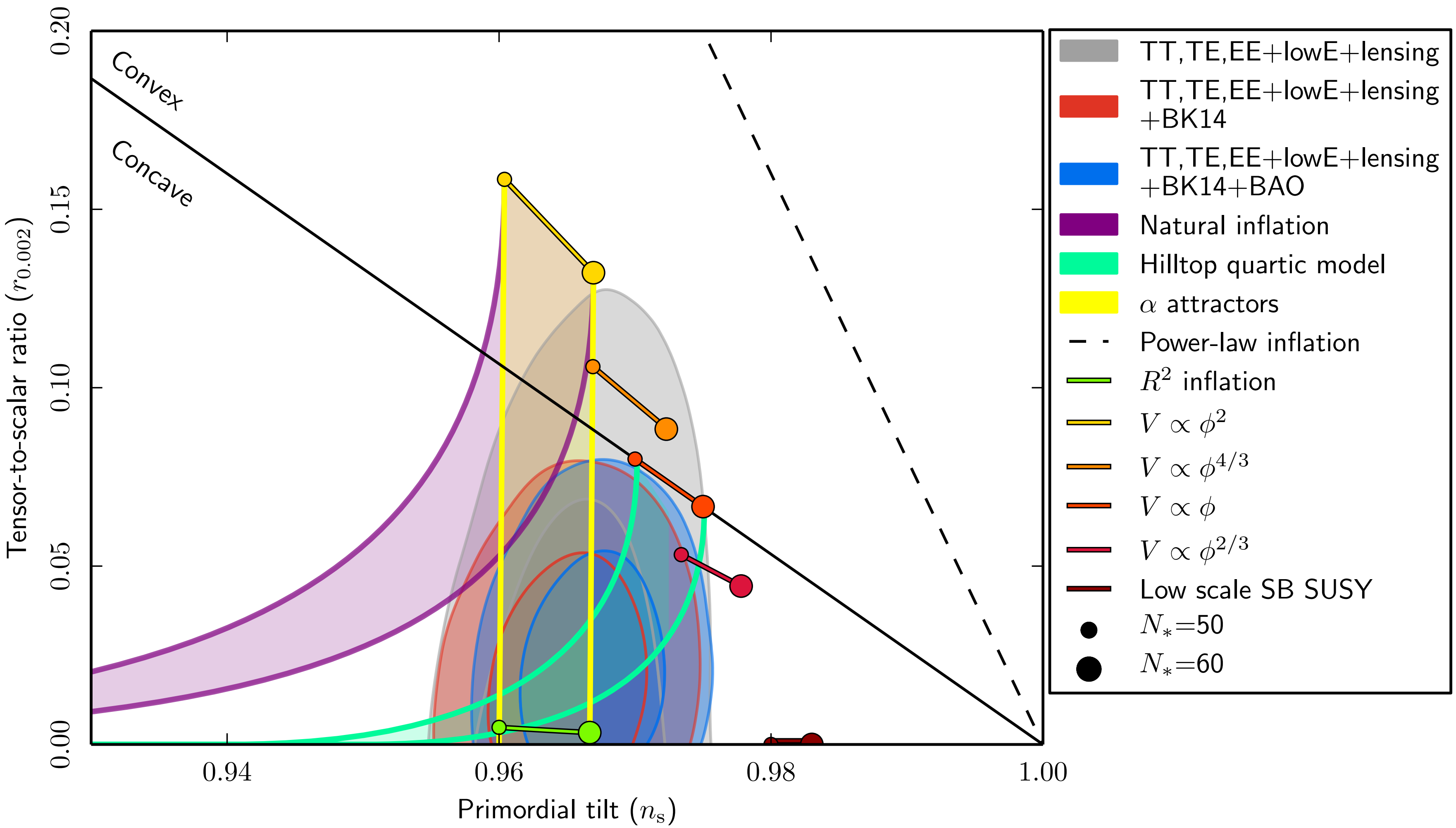
Theoretical Models for Primordial Power Spectrum

Planck Collaboration: Constraints on Inflation



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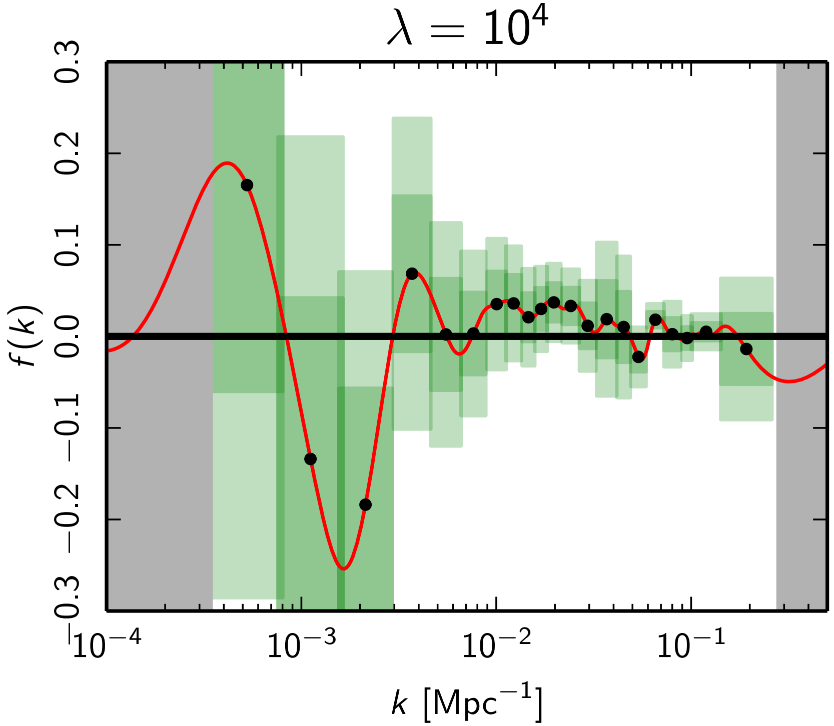
Features in the Power Spectrum?

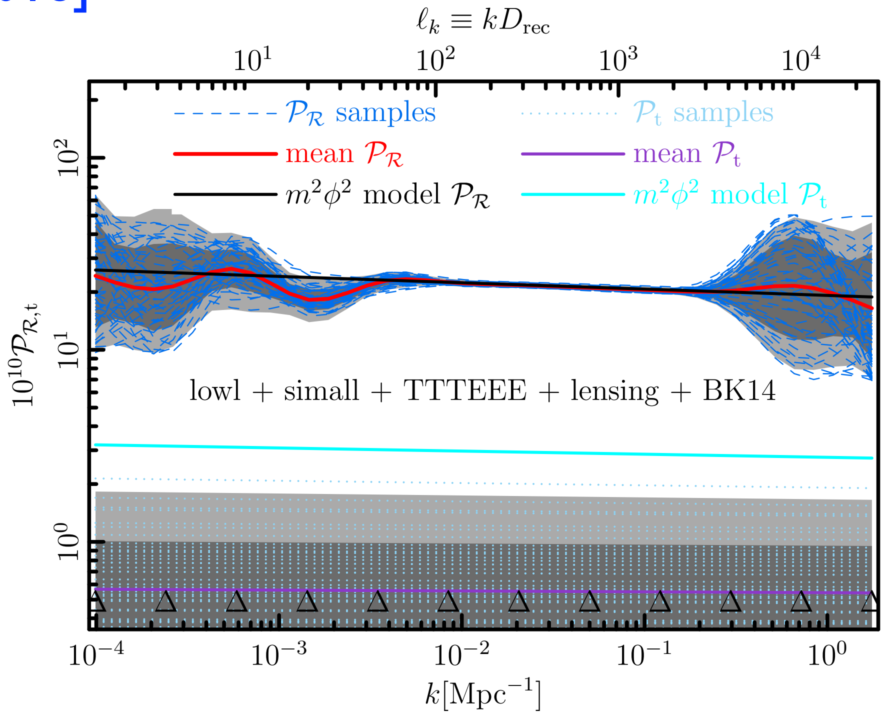
Deviation from the simple power-law form has been studied, with no significant evidence.

[Planck 2018]



Penalized likelihood method

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_0(k)\exp[f(k)],$$



Cubic spline method

Evolution of the Cosmological Perturbations

Inflation

Radiation dominated era

$\mathcal{P}_\zeta$

Power Spectrum

$$V(\phi)$$

$\delta\phi$ quantum fluctuation of a light scalar field

$$\zeta = -\psi - H\frac{\delta\rho}{\dot{\rho}}$$

$$\delta\rho \simeq \frac{dV}{d\phi}\delta\phi$$

How to reconstruct a potential?

Reconstruction

Inflation ← $\mathcal{P}_\zeta$ Power Spectrum

$$V(\phi)$$

$\delta\phi$ quantum fluctuation of a light scalar field

$$\zeta = -\psi - H\frac{\delta\rho}{\dot{\rho}}$$

$$\delta\rho \simeq \frac{dV}{d\phi}\delta\phi$$

How to reconstruct?

Reconstruction in Single Field Inflation with standard slow-roll

- Reconstructing inflationary potential in the slow-roll

[Hodges, Blumenthal, PRD (1990)]

The uncertainty in the integration constant can be eliminated if we can measure the tensor spectrum at a single scale.

- Reconstructing Hubble parameter from tensor spectrum

[Grishchuk, Solokhin, PRD (1991) 43, 2566]

- Reconstructing inflation potential, in principle and in practice

[Copeland, Kolb, Liddle, Lidsey, PRD (1993) 48, 2529]
[Copeland, Kolb, Liddle, Lidsey, PRL (1993) 71, 219]
[Lidsey, Liddle, Kolb, Copeland, RMP (1997) 69, 373]

Reconstruction in Single Field Inflation with standard slow-roll

Hodges, Blumenthal, PRD (1990)

1. Using the slow-roll relation

$$d\ln k = d(\ln(aH)) \simeq -dN_e = -Hdt,$$

$$\mathcal{P}_\zeta(k) \simeq \frac{H^2}{8\pi^2\epsilon_H}$$

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2} \simeq -\frac{1}{2H^2}\frac{dH^2}{d\ln k}$$

with a proper identification of the horizon exit (w/ Slow-Roll)

2. Solve the differential equation $\frac{1}{H^4}\frac{dH^2}{d\ln k} = \frac{1}{4\pi^2\mathcal{P}_\zeta}$

$$\frac{1}{H^2(k)} - \frac{1}{H_*^2} = -\frac{1}{4\pi^2}\int_{k_*}^{k}\frac{1}{\mathcal{P}_\zeta(k')}d\ln k'$$

H_* undetermined integration constant

3. Find a potential in terms of k $U(k) = 3H^2 - \frac{H^4}{8\pi^2\mathcal{P}_\zeta(k)}.$

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + U(\phi)\right)$$

$$\dot{H} = -\frac{\dot{\phi}^2}{2}$$

Reconstruction in Single Field Inflation

4. From equations,

$$d\ln k = d(\ln(aH)) \simeq -dN_e = -Hdt,$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dU}{d\phi} = 0,$$

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + U(\phi)\right),$$

$\Rightarrow$

$$\dot{H} = -\frac{\dot{\phi}^2}{2} \quad \text{or} \quad \frac{d\phi}{dt} = \sqrt{\frac{dH^2}{d\ln k}}$$

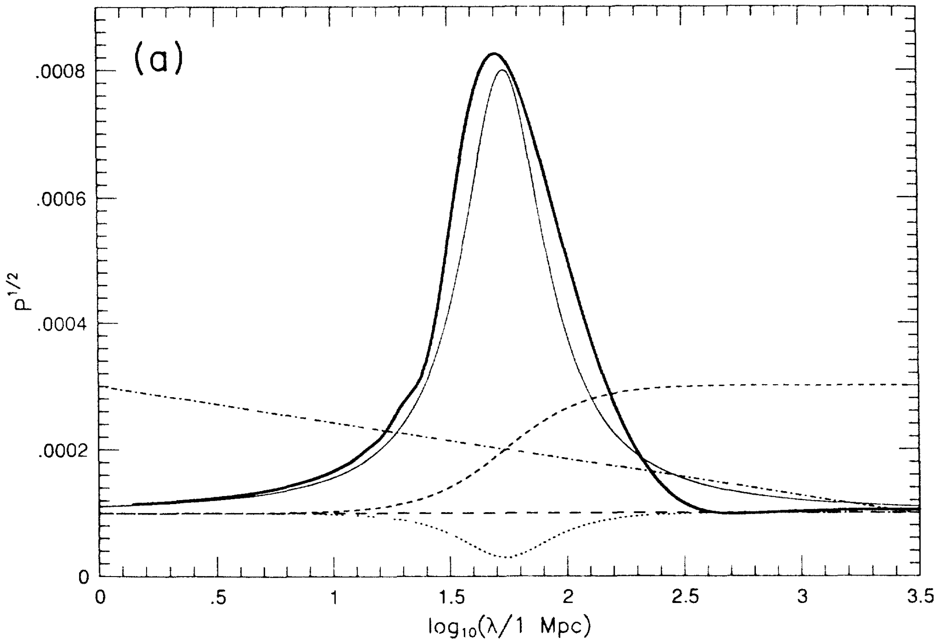
solve it $\Rightarrow$

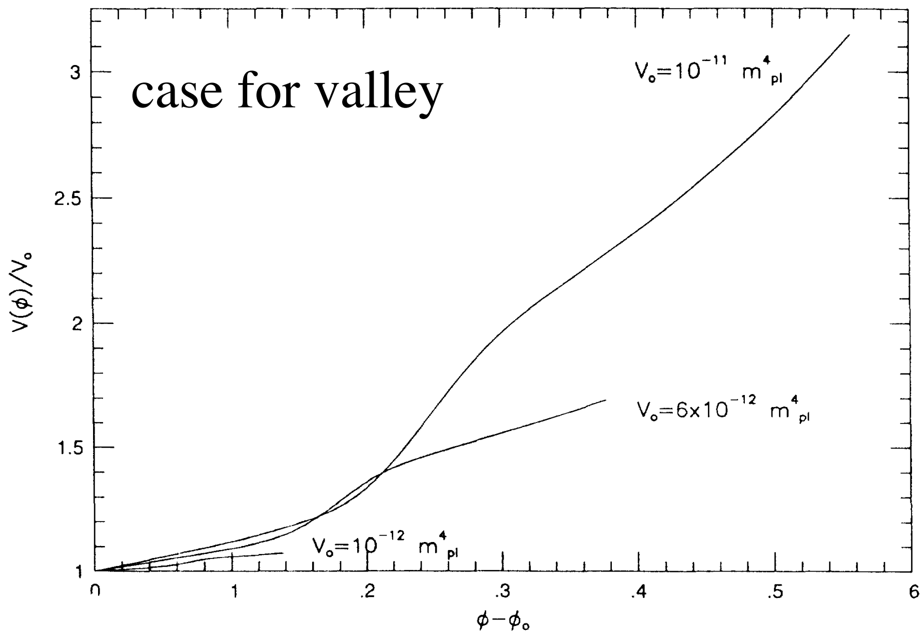
$$\phi(k) - \phi_* = -\int_{k_*}^{k} \sqrt{\frac{H^2}{4\pi^2 \mathcal{P}_{\mathcal{R}}}} d\ln k'$$

5. Put the inverse function into U(k), to find $U(\phi)$

Power spectrum and reconstructed potential

Hodges, Blumenthal, PRD (1990)





different potential shapes for different integration constant H_*
- the reconstructed potential is not just scaled one from each other

Limitation in the standard slow-roll

$$\mathcal{P}_\zeta(k) \simeq \frac{H^2}{8\pi^2\epsilon_H},$$

It assumes hierarchical standard slow-roll approximation

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{d\log H}{d\log a},$$
$$\delta_1 \equiv \frac{\ddot{\phi}}{H\dot{\phi}} = \frac{d\log\dot{\phi}}{d\log a},$$

are small and nearly constant

It cannot be applied to the case

- Higher order of slow-roll parameters are not neglected
  ex) oscillating power spectrum

Generalized Slow-Roll Approximation

Generalized slow-roll approximation

[Dodelson, Stewart, PRD (2002)] [Stewart, PRD (2002)]

For each Fourier mode, the equation of the curvature perturbation

$$\frac{d^2\varphi}{d\xi^2} + \left(k^2 - \frac{1}{z}\frac{d^2 z}{d\xi^2}\right)\varphi = 0 \qquad \text{with} \qquad \varphi \equiv z\mathcal{R}\,, \quad z \equiv \frac{a\dot{\phi}}{H}\,,$$

$$\xi \equiv -\int \frac{dt}{a} = \frac{1}{aH}[1 + \mathcal{O}(\epsilon)]$$

Or in other way, scale-invariant eq in the LHS and the deviation in the RHS

$$\frac{d^2 y}{dx^2} + \left(1 - \frac{2}{x^2}\right)y = \frac{1}{x^2}g(\log\xi)y\,, \qquad g \equiv \frac{f'' - 3f'}{f},$$

$$f(\log\xi) \equiv \frac{2\pi x}{k}z = 2\pi\xi\frac{a\dot{\phi}}{H}\,,$$

$$f' \equiv df/d\log\xi.$$

When g <<1, we can sole the eq. with perturbation.

Generalized slow-roll approximation

[Dodelson, Stewart, PRD (2002)] [Stewart, PRD (2002)]

$$\ln \mathcal{P}(k) = \int_0^\infty \frac{d\xi}{\xi} \left[-k\xi\, W'(k\xi)\right] \left[\ln\left(\frac{1}{f^2}\right) + \frac{2}{3}\frac{f'}{f}\right] + \mathcal{O}\left(g^2\right)$$

where the window function $W(x) = \frac{3\sin(2x)}{2x^3} - \frac{3\cos(2x)}{x^2} - \frac{3\sin(2x)}{2x} - 1$

This relation is valid when $g \ll 1$ (generalized slow-roll, GSR)

GSR can be used for the potential with sharp step, bump, or dip.

with the relations to the slow-roll parameters

$$g \equiv \frac{f'' - 3f'}{f}, \qquad \epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2}\left(\frac{\dot{\phi}}{H}\right)^2 \quad \text{and} \quad \delta_n = \frac{1}{H^n \dot{\phi}} \frac{d^n \dot{\phi}}{dt^n},$$

$$\begin{aligned}
\frac{1}{f^2} &= \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot{\phi}}\right)^2 \left[1 - 2\epsilon - 3\epsilon^2 - 4\epsilon\delta_1 - 4\epsilon\delta_2 - 16\epsilon^3 - 28\epsilon^2\delta_1 - 4\epsilon\delta_1^2 + \mathcal{O}(\xi^4)\right], \\
\frac{f'}{f} &= -2\epsilon - \delta_1 - 4\epsilon^2 - 3\epsilon\delta_1 - 2\epsilon\delta_2 - 18\epsilon^3 - 25\epsilon^2\delta_1 - 4\epsilon\delta_1^2 + \mathcal{O}(\xi^4), \\
\frac{f''}{f} &= \delta_2 + 8\epsilon^2 + 9\epsilon\delta_1 + 4\epsilon\delta_2 + 36\epsilon^3 + 50\epsilon^2\delta_1 + 8\epsilon\delta_1^2 + \mathcal{O}(\xi^4),
\end{aligned}$$

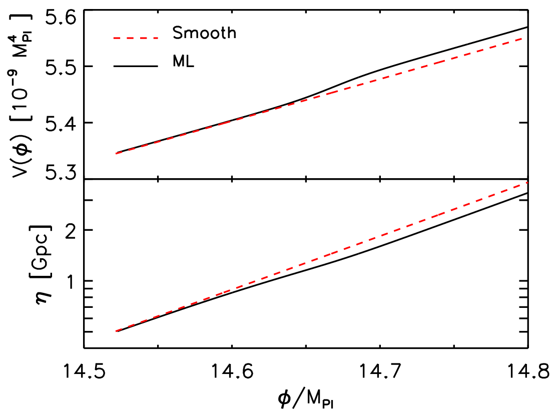
[Choe, Gong, Stewart, JCAP (2004)]

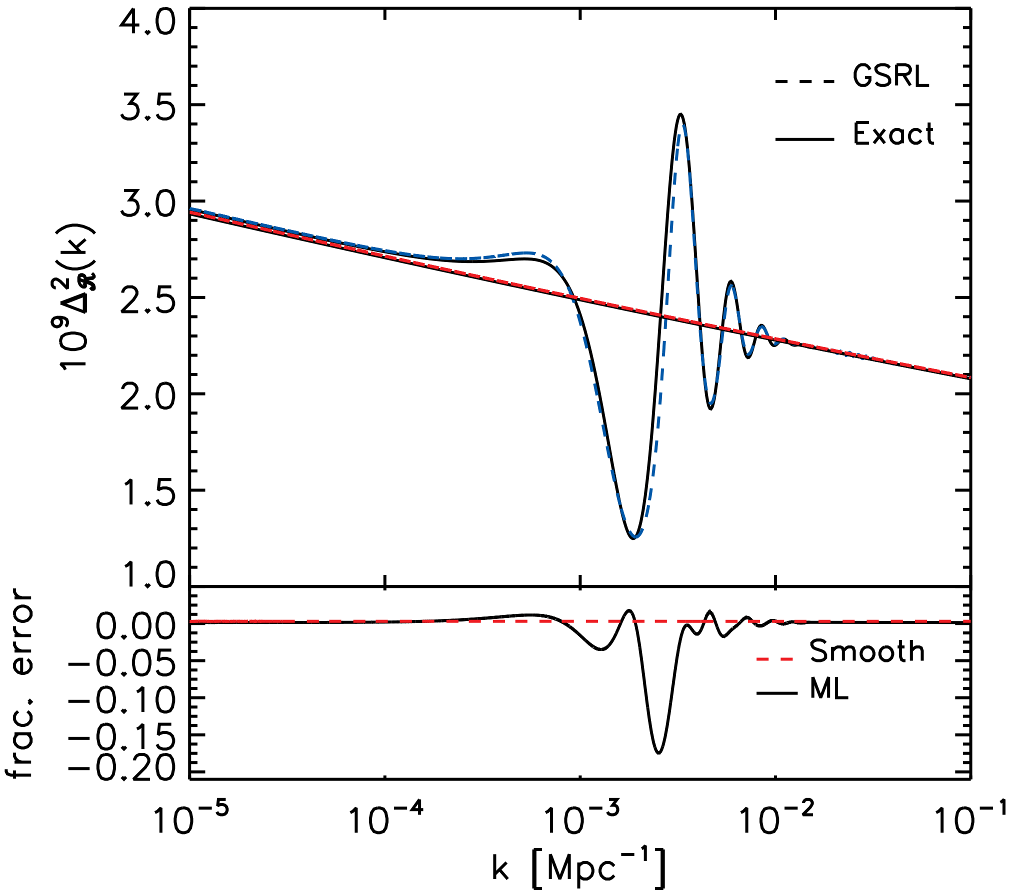
Example

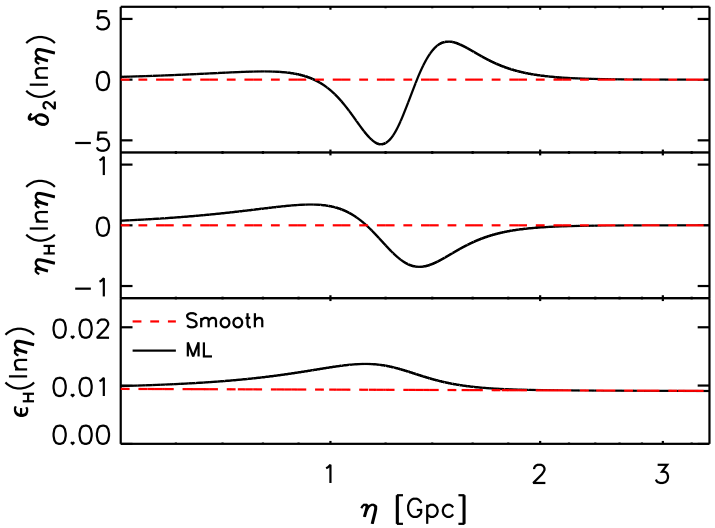
[Dvorkin, Hu (2009)]

$$V(\phi) \;=\; m^2_{\rm eff}(\phi)\phi^2/2$$

$$m^2_{\rm eff}(\phi) = m^2\left[1 + c\tanh\left(\frac{\phi - b}{d}\right)\right]$$







$$\epsilon_H \equiv \frac{1}{2}\left(\frac{\dot\phi}{H}\right)^2, \qquad \delta_2 = \frac{\dddot\phi}{H^2\dot\phi},$$

$$\eta_H \equiv -\left(\frac{\ddot\phi}{H\dot\phi}\right),$$

Reconstructing potential using general slow-roll (GSR)

Reconstructing potential using general slow-roll (GSR)

Inverse formula [Joy, Stewart, Gong, Lee, JCAP (2005)]

$$\ln\left(\frac{1}{f^2}\right) = \int_0^\infty \frac{dk}{k} m(k\xi) \ln \mathcal{P}\,,$$

$f(\ln \xi)$ from $\mathcal{P}(k)$

with $m(x) = \frac{2}{\pi}\left[\frac{1}{x} - \frac{\cos(2x)}{x} - \sin(2x)\right]$

From

$$f = \frac{2\pi a \xi \dot{\phi}}{H}$$

$$\xi = -\int \frac{dt}{a} \simeq \frac{1}{aH},$$ for $\epsilon \ll 1$

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_\mathrm{P}^2}$$

$$\rightarrow \quad H^{-3}\frac{dH}{d\xi} = \frac{1}{2(2\pi)^2 M_\mathrm{P}^2}\frac{f^2(\xi)}{\xi}$$

Hubble parameter

$$\rightarrow \quad \frac{1}{H^2(\xi)} = \frac{1}{H_i^2} - \frac{1}{(2\pi)^2 M_\mathrm{P}^2}\int_{\xi_i}^{\xi} f^2(\xi')d\ln\xi'.$$

[JCAP (2022) Choi, Gong, Kang, Raveendran]

with an integration constant H_i

Reconstructing potential using GSR

[JCAP (2022) Choi, Gong, Kang, Raveendran]

$$f = \frac{2\pi a \xi \dot{\phi}}{H} \quad \Rightarrow \quad \frac{d\phi}{d\ln\xi} = -\frac{fH}{2\pi}$$

Field evolution

$$\Rightarrow \quad \phi(\xi) = \phi_i - \int \frac{fH(\xi)}{2\pi} d\ln\xi .$$

Potential

$$\begin{aligned} V(\xi) =& 3M_{\rm P}^2 H^2 - \frac{1}{2}\dot{\phi}^2 \\ =& 3M_{\rm P}^2 H^2(\xi) - \frac{f^2 H^4(\xi)}{2(2\pi)^2} \\ =& 3M_{\rm P}^2 H^2(\xi)\left(1 - \frac{f^2 H^2(\xi)}{6(2\pi)^2 M_{\rm P}^2}\right) . \end{aligned}$$

by eliminating ξ

$$\Rightarrow \quad V(\phi)$$

(Analytic) Example $\mathcal{P}(k) = \mathcal{P}_0\left(\frac{k}{k_*}\right)^{n_s-1}$

$$\ln\left(\frac{1}{f^2}\right) = \int_0^\infty \frac{dk}{k}\, m(k\xi)\ln\mathcal{P}\,,$$

$$\log\left(\frac{1}{f^2}\right) = \log A_s + (n_s - 1)\left[\alpha - \log(k_*\xi)\right]\,,$$

where $\alpha \equiv 2 - \log 2 - \gamma \approx 0.729637$, with $\gamma \approx 0.577216$

Euler-Mascheroni constant

$$\rightarrow \quad f^2 = \frac{(k_*\xi)^{n_s-1}}{A_s e^{\alpha(n_s-1)}}$$

$$\frac{1}{H^2(\xi)} = \frac{1}{H_i^2} - \frac{1}{(2\pi)^2 M_{\rm P}^2}\int_{\xi_i}^{\xi} f^2(\xi')\,d\ln\xi'.$$

$$\rightarrow \quad \frac{1}{H^2} = \frac{1}{H_i^2}\left[1 - \frac{(k_*\xi_i)^{n_s-1}}{\beta} + \frac{(k_*\xi)^{n_s-1}}{\beta}\right],$$

with $\beta \equiv 4\pi^2(1-n_s)A_s\frac{m_{\rm Pl}^2}{H_i^2}e^{\alpha(n_s-1)}$

$$\phi(\xi) = \phi_i - \int \frac{f H(\xi)}{2\pi} d\ln\xi.$$

$$\Delta\phi \equiv \phi - \phi_i = -\frac{2m_{\rm Pl}}{\sqrt{1-n_s}} \sinh^{-1}\left[\sqrt{\frac{(k_*\xi')^{n_s-1}}{\beta - (k_*\xi_i)^{n_s-1}}}\right]\Bigg|_{\xi'=\xi_i}^{\xi'=\xi},$$

$$\begin{aligned} V(\xi) &= 3M_{\rm P}^2 H^2 - \frac{1}{2}\dot\phi^2 \\ &= 3M_{\rm P}^2 H^2(\xi) - \frac{f^2 H^4(\xi)}{2(2\pi)^2} \\ &= 3M_{\rm P}^2 H^2(\xi)\left(1 - \frac{f^2 H^2(\xi)}{6(2\pi)^2 M_{\rm P}^2}\right). \end{aligned}$$

with a redefined constant $\Delta\phi + \sinh^{-1}\left[\sqrt{\frac{(k_*\xi_i)^{n_s-1}}{\beta - (k_*\xi_i)^{n_s-1}}}\right] \equiv \phi - \phi_0$

$$V(\phi) = \frac{3m_{\rm Pl}^2 H_i^2 \beta}{\beta - (k_*\xi_i)^{n_s-1}} \frac{1 - \frac{1}{6}(1-n_s)\tanh^2\left[\sqrt{1-n_s}\frac{\phi-\phi_0}{2m_{\rm Pl}}\right]}{1 + \sinh^2\left[\sqrt{1-n_s}\frac{\phi-\phi_0}{2m_{\rm Pl}}\right]}.$$

Reconstructed inflationary potential from a power-law power spectrum

Reconstructed inflationary potential from a power-law power spectrum

$$V(\phi) = \frac{3m_{\rm Pl}^2 H_i^2 \beta}{\beta - (k_* \xi_i)^{n_s-1}} \frac{1 - \frac{1}{6}(1-n_s)\tanh^2\left[\sqrt{1-n_s}\frac{\phi-\phi_0}{2m_{\rm Pl}}\right]}{1+\sinh^2\left[\sqrt{1-n_s}\frac{\phi-\phi_0}{2m_{\rm Pl}}\right]} .$$

Two limits:

$\beta \gg (k_* \xi_i)^{n_s-1}$ it becomes small field and vacuum-dominated potential

$$V(\phi) \approx 3m_{\rm Pl}^2 H_i^2 \left[1 - \frac{1-n_s}{4}(\Delta\phi)^2\right]$$

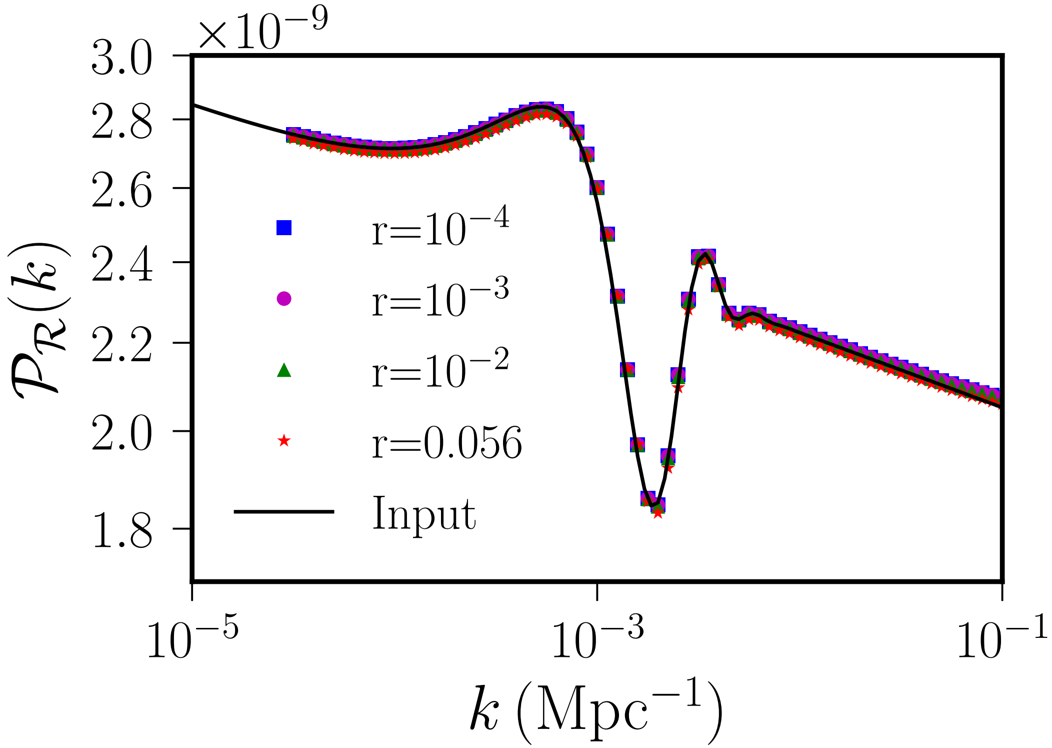
$\beta \ll (k_* \xi_i)^{n_s-1}$ it becomes an exponential function, inconsistent with CMB

$$V(\phi) = 3m_{\rm Pl}^2 H_i^2 \beta \exp\left(\sqrt{1-n_s}\frac{\phi-\phi_0}{m_{\rm Pl}}\right)$$

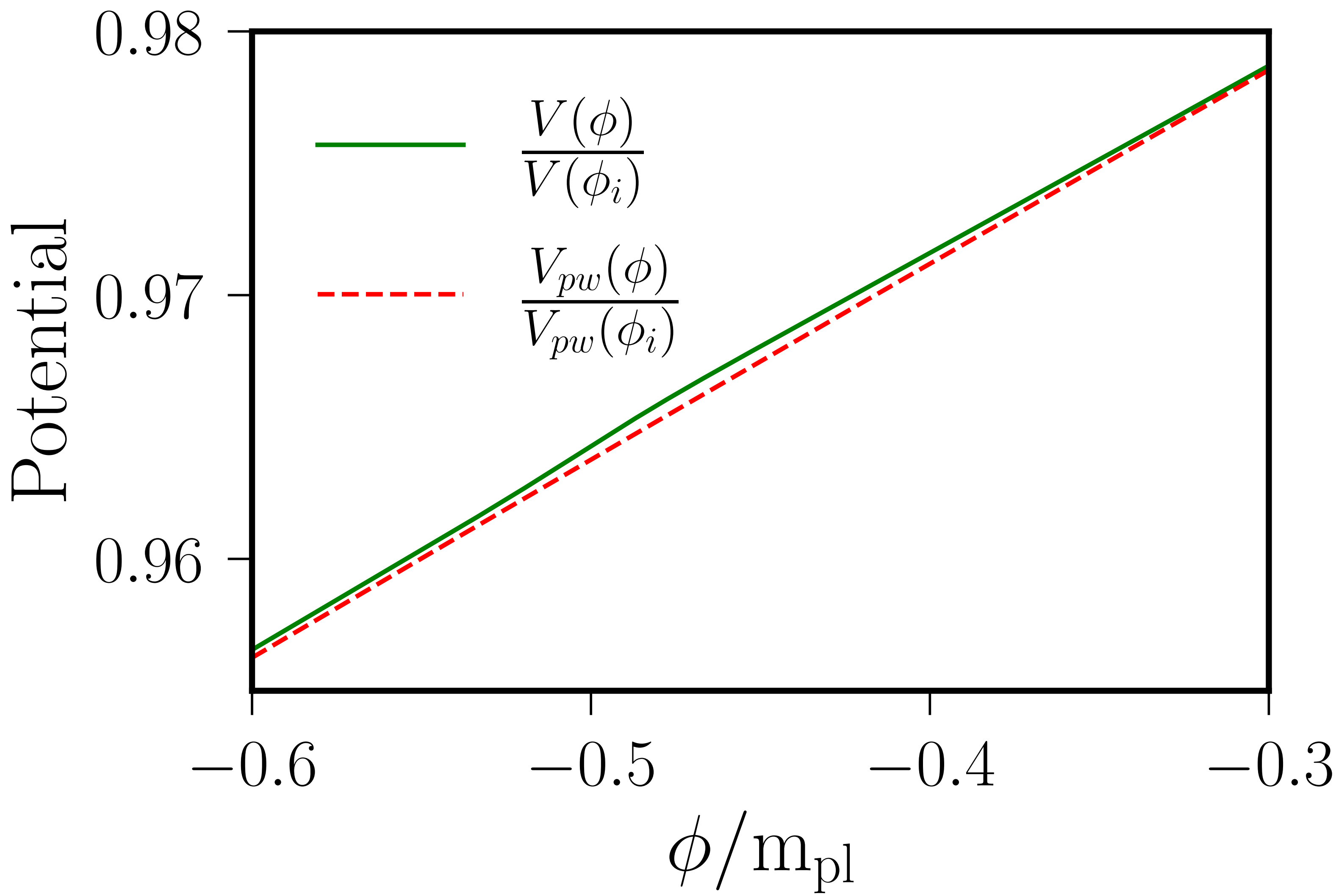
Featured Power Spectrum

[JCAP (2022) Choi, Gong, Kang, Raveendran]

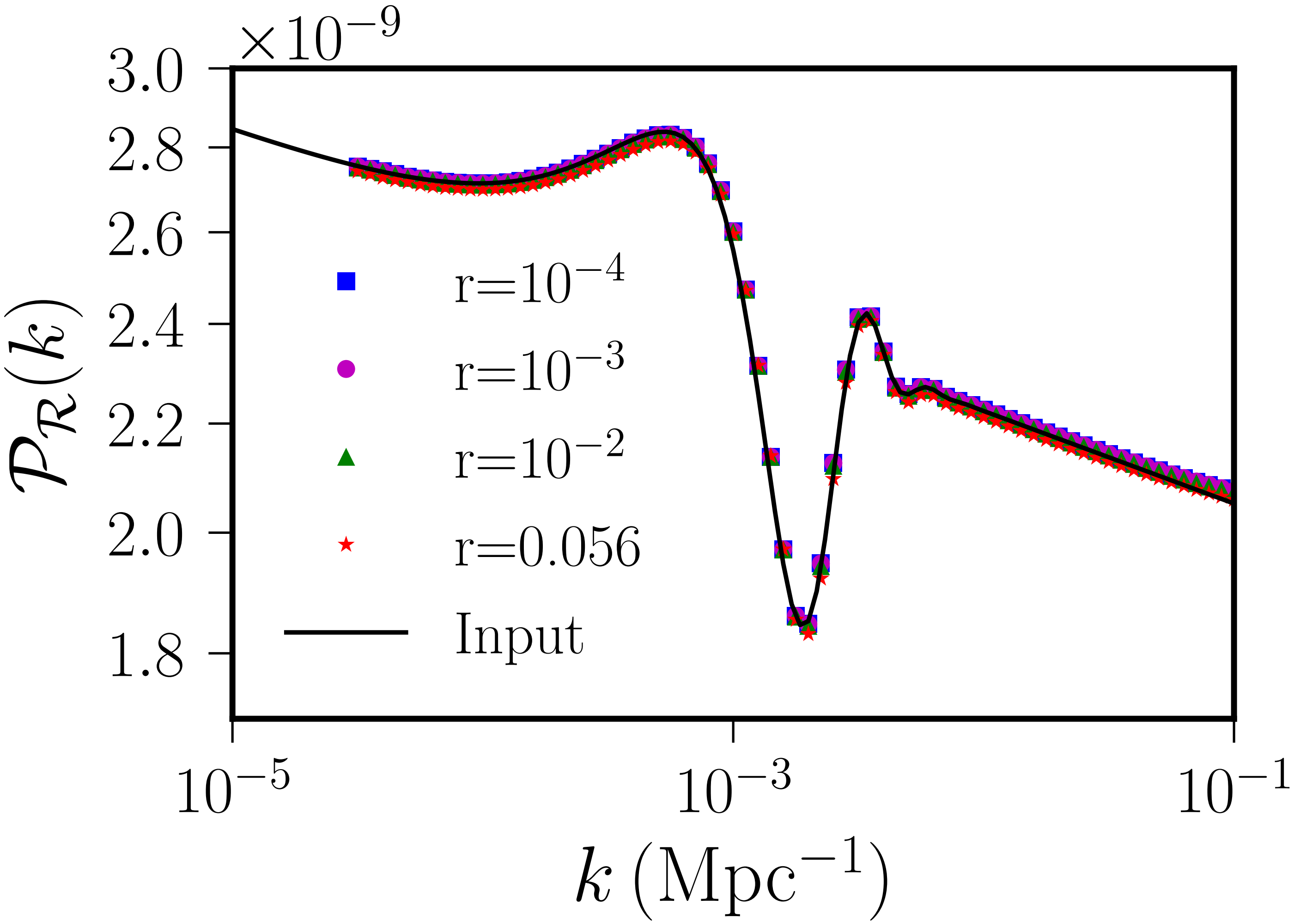
For a featured power spectrum, as an example, we choose an
Power Spectrum with a localized oscillatory feature from Planck 2018 paper



The reconstructed potential is obtained numerically

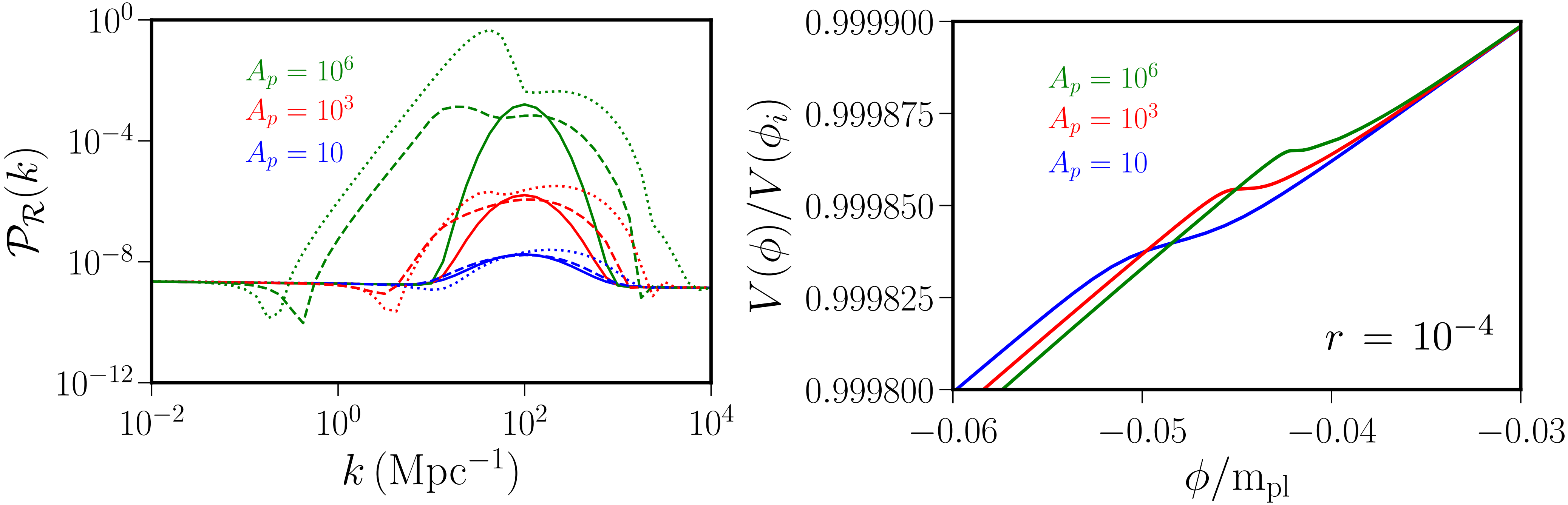


Again calculate the power spectrum numerically from the reconstructed potential, and compare with the input power spectrum



Power Spectrum with a Peak on small scale

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}^0_{\mathcal{R}}(k)\left\{1 + A_p \exp\left[-\left(\frac{\log_{10}(k/k_c)}{\Delta}\right)^2\right]\right\},$$



solid: input power spectrum

dashed: from reconstructed potential with GSR

dotted: from reconstructed potential with SR

g becomes larger for higher peak

Discussion

We proposed a new method to reconstruct an inflationary potential for a single scalar field.

- it uses the GSR (generalized slow-roll) approximation

- it is valid for small value of a function $g \equiv \frac{f'' - 3f'}{f},$

- it can be applied to a power spectrum with a deviation from a scaler-invariance and oscillation

- We showed a few examples of the reconstructed potential and its validity by comparing with the input power spectrum

Thank You!