Resolution of the CIPT-FOPT Discrepancy Problem for Hadronic τ Decays

André H. Hoang

University of Vienna

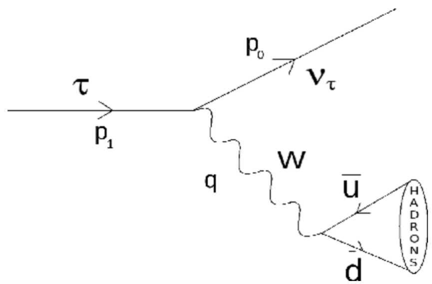
arXiv:2008.00578 , arXiv:2105.11222
(with **Christoph Regner**)
arXiv:2202.10957, arXiv:2207.xxxxx
(with **Miguel Benitez-Rathgeb, Diogo Boito** and **Matthias Jamin**)

Hadronic τ Spectral Function Moments

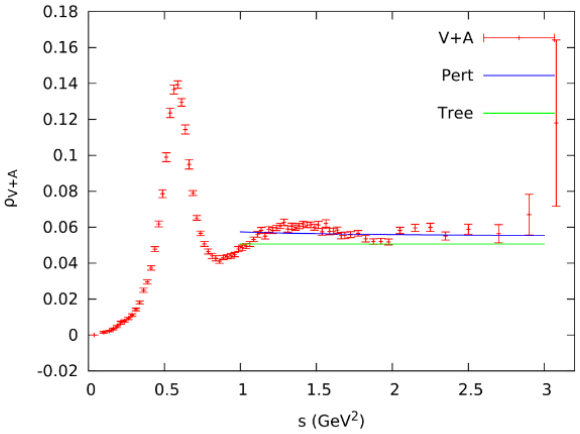
ALEPH: τ hadronic width

(HFLAV 2019)

$$R_\tau \equiv \frac{\Gamma[\tau^- \to \text{hadrons}\,\nu_\tau(\gamma)]}{\Gamma[\tau^- \to e^-\overline{\nu}_e\nu_\tau(\gamma)]} = 3.6355 \pm 0.0081$$



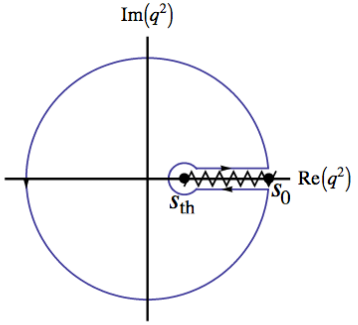
Inclusive hadronic mass spectrum



$$\left(p^\mu p^\nu - g^{\mu\nu}p^2\right)\Pi(p^2) \equiv i\int dx\, e^{ipx}\,\langle\Omega|\,T\{j^\mu_{v/av,jk}(x)\, j^\nu_{v/av,jk}(0)^\dagger\}\Omega\rangle$$

Braaten, Narison, Pich, Le Diberder, ... 90's

$$A^\omega_{V/A}(s_0) \equiv \int_{s_{\rm th}}^{s_0}\frac{ds}{s_0}\,\omega(s)\,{\rm Im}\,\Pi_{V/A}(s) = \frac{i}{2}\oint_{|s|=s_0}\frac{ds}{s_0}\,\omega(s)\,\Pi_{V/A}(s)$$



Hadronic τ Spectral Function Moments

Theory: Operator product expansion

Adler function: $$\frac{1}{4\pi^2}\Big(1+D(s)\Big) \equiv -s\frac{d\Pi(s)}{ds}$$

Braaten, Narison, Pich, Le Diberder, ... 90's

$$A_{W_i}(s_0) = \frac{N_c}{2}|V_{ud}|^2\left[\delta^{\rm tree}_{W_i} + \delta^{(0)}_{W_i}(s_0) + \sum_{d\geq 2}\delta^{(d)}_{W_i}(s_0) + \delta^{\rm DV}_{W_i}(s_0)\right]$$

$$W_i(x) = \sum_{n=0}^{m} a_n\, x^n$$

pQCD ↑ OPE ↑ Duality violation ↑

Shifman, Vainshtein, Sacharow 1978

$$\hat D(s) = \sum_{n=1}^{\infty} c_{n,1}\left(\frac{\alpha_s(-s)}{\pi}\right)^n$$ ← Perturbative

OPE non-pert. corrections

$$\hat D^{\rm OPE}(s) = \frac{C(\alpha_s(-s))}{(-s)^2}\langle\alpha_s G^2\rangle + \sum_{p=3}^{\infty}\frac{1}{(-s)^p}\Big[C_0(\alpha_s(-s))\langle\mathcal{O}_{2p,\gamma_1}\rangle + C_1(\alpha_s(-s))\langle\mathcal{O}_{2p,\gamma_2}\rangle + \ldots\Big]$$

$$\delta^{(0)}_{W_i}(s_0) = \frac{1}{2\pi i}\oint_{|s|=s_0}\frac{{\rm d}s}{s}\,W_i\!\left(\tfrac{s}{s_0}\right)\hat D(s)$$

$$\delta^{(d)}_{W_i}(s_0) \sim \frac{1}{2\pi i}\oint_{|s|=s_0}\frac{ds}{s}\,W_i\!\left(\frac{s}{s_0}\right)\frac{\Lambda^d_{\rm QCD}}{s^{d/2}}$$

FOPT-CIPT Discrepancy Problem

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi}\right)^n ,$$
$$= \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n+1} k\, c_{n,k} \ln^{k-1}\left(\frac{-s}{s_0}\right)$$

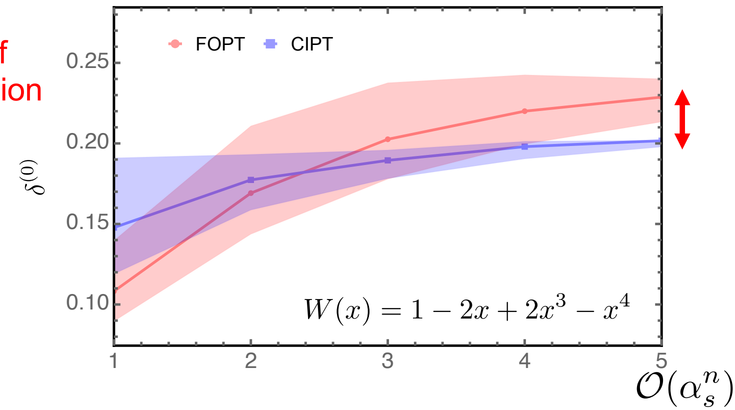
Change of renormalization scale

$$c_{0,1} = c_{1,1} = 1 , \quad c_{2,1} = 1.640$$
$$c_{3,1} = 6.371$$
$$c_{4,1} = 49.076$$
$$c_{5,1} = 280 \pm 140$$

4-loop: Gorishni etal., Surguladze etal. 1991

5-loop: Baikov etal. 2008

6-loop estimate Beneke, Boito, Jamin; Caprini



Contour-improved perturbation theory (CIPT):

$$\delta_{W_i}^{(0),\mathrm{CIPT}}(s_0) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \left(\frac{\alpha_s(-x s_0)}{\pi}\right)^n$$

Fixed-order perturbation theory (FOPT):

$$x = \frac{s}{s_0}$$

$$\delta_{W_i}^{(0),\mathrm{FOPT}}(s_0) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n+1} k\, c_{n,k} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \ln^{k-1}(-x)$$

- CIPT resums powers of π with respect to FOPT
- CIPT leads in general to smaller moments than FOPT
- OPE and DV corrections assumed to be universal
- Strong coupling from CIPT larger than from FOPT
- CIPT disfavored from plausibility studies of Borel models for the Adler function

Beneke, Boito, Jamin 2008, 2012

- **Situation inconclusive !**

FOPT-CIPT Discrepancy

$$\mathcal{L}_{\rm QCD} = \bar\psi_i \left(i\gamma^\mu (D_\mu)_{ij} - m\,\delta_{ij}\right)\psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

$$G^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + g f^{abc} \mathcal{A}^b_\mu \mathcal{A}^c_\nu$$

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} - ig (T_a)_{ij} \mathcal{A}^a_\mu$$

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi}\right)^n,$$

$$= \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n+1} k\, c_{n,k} \ln^{k-1}\left(\frac{-s}{s_0}\right)$$

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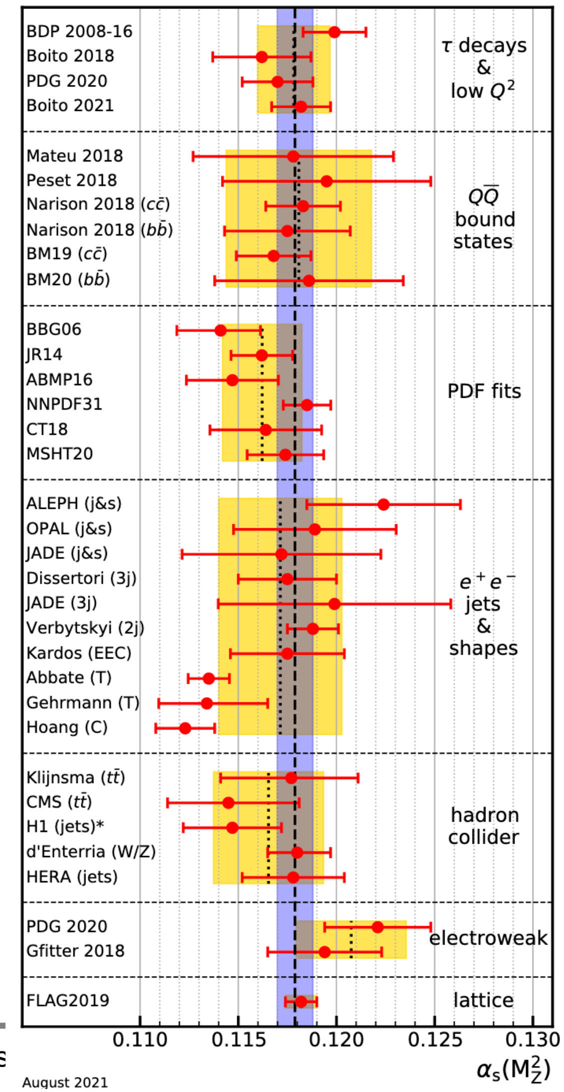


Outline

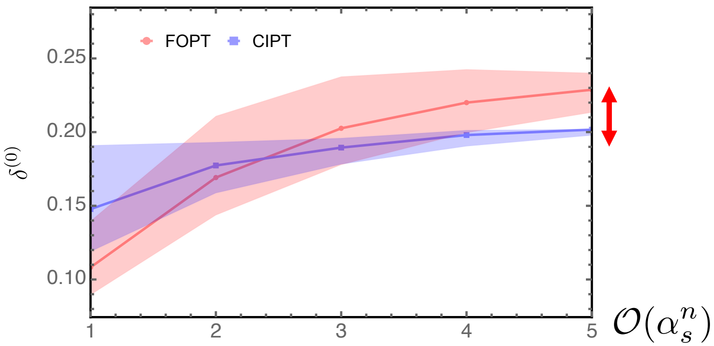
- Introduction
- FOPT and CIPT Borel representations are different:
  Asymptotic Separation Δ
  → **FOPT and CIPT expansions describe different quantities**
- Properties of the CIPT expansion:
  CIPT expansion NOT CONSISTENT with the standard form of the Operator Product Expansion and renormalon calculus
- Reconciling CIPT and FOPT:
  renormalon-free gluon condensate scheme
- Impact on determinations of $\alpha_s(m_\tau^2)$

Interesting Observations: Total Decay Rate

$$W_\tau(x) = 1 - 2x + 2x^3 - x^4 \qquad \delta^{(d)}_{W_i}(s_0) \sim \frac{1}{2\pi i}\oint_{|s|=s_0} \frac{ds}{s}\left(\frac{s}{s_0}\right)^m \frac{\Lambda^4_{\rm QCD}}{s^2} = \frac{\Lambda^4_{\rm QCD}}{s_0^2}\,\delta_{m2}$$

→ Sensitivity to leading O($\Lambda^4_{\rm QCD}$) gluon condensate strongly suppressed

Moment's perturbation series:



- Discrepancy between CIPT and FOPT scales as $\sim \frac{\Lambda^4_{\rm QCD}}{s_0^2}$

  → Accidental or indication of a quartic IR sensitivity?

  → Contradiction to standard OPE

  → How can there be O($\Lambda^4_{\rm QCD}$) sensitivity left ?

- CIPT is not an expansion in powers of α_s at a definite renormalization scale. It is impossible to switch between the CIPT and FOPT moment series terms through a change of scheme of the strong coupling and a reexpansion of the series due to the **contour integration** !!

→ Worth to reconsider CIPT and FOPT from scratch: OPE ↔ IR renormalons

Renormalon Calculus: Euclidean Adler Function

Perturbative series in QCD are not convergent,
but asymptotic in expansion variable $\alpha_s(s_0)$. $\rightarrow$ $\hat{D}(-s_0) \sim \sum_{n=1}^{\infty} n! \left(\frac{\alpha_s(s_0)}{\pi}\right)^n$

\`t Hooft; David; Müller; ... Beneke; ...

Borel calculus:

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \quad \Longrightarrow \quad B[\hat{D}](u) = \sum_{n=1}^{\infty} \frac{c_{n,1}}{\Gamma(n)} u^{n-1}$$

Borel sum:

$$\hat{D}_{\rm Borel}(-s_0) = {\rm PV} \int_0^{\infty} {\rm d}u \; B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0 \alpha_s(s_0)}}$$

Association: IR renormalon poles/cuts $\Leftrightarrow$ (standard) OPE Corrections

$$\hat{D}^{\rm OPE}(s) = \frac{C(\alpha_s(-s))}{(-s)^2} \langle \alpha_s G^2 \rangle + \sum_{p=3}^{\infty} \frac{1}{(-s)^p} \Big[C_0(\alpha_s(-s)) \langle \mathcal{O}_{2p,\gamma_1} \rangle + C_1(\alpha_s(-s)) \langle \mathcal{O}_{2p,\gamma_2} \rangle + \ldots \Big]$$

Leading Gluon Condensate:

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \ldots \quad \Longleftrightarrow \quad \frac{\langle \bar{G}^2 \rangle}{s^2}$$

FOPT vs. CIPT Borel Representation (large-β_0)

FOPT expansion: $\rightarrow$ Expansion parameter: $\alpha_s(s_0)$

$$\hat{D}(s) = \underbrace{\sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n}_{\text{expansion variable}} \underbrace{\sum_{k=1}^{n} k\, c_{n,k}\, \ln^{k-1}(-x)}_{\text{coefficient}}$$

Borel sum:

$$\mathrm{PV}\int_0^\infty du \left[B[\hat{D}](u)\, e^{-u\ln(-x)}\right] e^{-\frac{4\pi u}{\beta_0\alpha_s(s_0)}}$$

$$e^{-u\ln(-x)}\, e^{-\frac{4\pi u}{\beta_0\alpha_s(s_0)}} = e^{-\frac{4\pi u}{\beta_0\alpha_s(-s)}}$$

➡ FOPT Borel representation = “true“ Borel representation

$$\delta^{(0),\mathrm{FOPT}}_{W_i,\mathrm{Borel}}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{d}u\, \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W_i(x)\, B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0\alpha_s(-x s_0)}}$$

FOPT vs. CIPT Borel Representation

CIPT expansion: → No obvious expansion parameter !

$$\delta^{(0),\mathrm{CIPT}}_{W_i}(s_0) = \frac{1}{2\pi i}\sum_{n=1}^{\infty} c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W_i(x) \left(\frac{\alpha_s(-xs_0)}{\pi}\right)^n = \frac{1}{2\pi i}\sum_{n=1}^{\infty} \underbrace{\left(\frac{\alpha_s(s_0)}{\pi}\right)^n}_{\text{expansion variable}} \underbrace{c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W_i(x) \left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right)^n}_{\text{coefficient}},$$

➡ CIPT Borel representation: NEW !

Regner, Hoang arXiv:2008.00578

$$\delta^{(0),\mathrm{CIPT}}_{W_i,\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\, \frac{1}{2\pi i} \oint_{\mathcal{C}_x} \frac{\mathrm{d}x}{x}\, W_i(x) \left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right) B[\hat{D}]\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\bar{u}\right) e^{-\frac{4\pi\bar{u}}{\beta_0\alpha_s(s_0)}}$$

Contour needs to be deformed from |x|=1

„Asymptotic Separation“

$$\Delta_W(s_0) \equiv \delta^{(0),\mathrm{CIPT}}_{W,\mathrm{Borel}}(s_0) - \delta^{(0),\mathrm{FOPT}}_{W,\mathrm{Borel}}(s_0)$$

$$\Delta_W(s_0) \sim \frac{\Lambda^d_{\mathrm{QCD}}}{s_0^{d/2}} \quad \text{for } \mathcal{O}(\Lambda^d_{\mathrm{QCD}}) \text{ IR renormalon contained in } \hat{D}$$

Character of the Asymptotic Separation

FOPT Borel representation

$$\delta^{(0),\mathrm{FOPT}}_{W_i,\mathrm{Borel}}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{d}u\,\frac{1}{2\pi i}\oint_{|x|=1}\frac{\mathrm{d}x}{x}\,W_i(x)\,B[\hat{D}](u)\,e^{-\frac{4\pi u}{\beta_0\alpha_s(-xs_0)}}$$

CIPT Borel representation

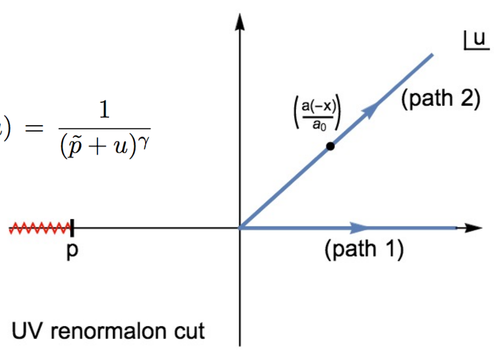
$$\delta^{(0),\mathrm{CIPT}}_{W_i,\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\,\frac{1}{2\pi i}\oint_{\mathcal{C}_x}\frac{\mathrm{d}x}{x}\,W_i(x)\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right)B[\hat{D}]\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\bar{u}\right)e^{-\frac{4\pi\bar{u}}{\beta_0\alpha_s(s_0)}}$$

- Related through complex-valued change of variables

$$u = \frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\bar{u}$$

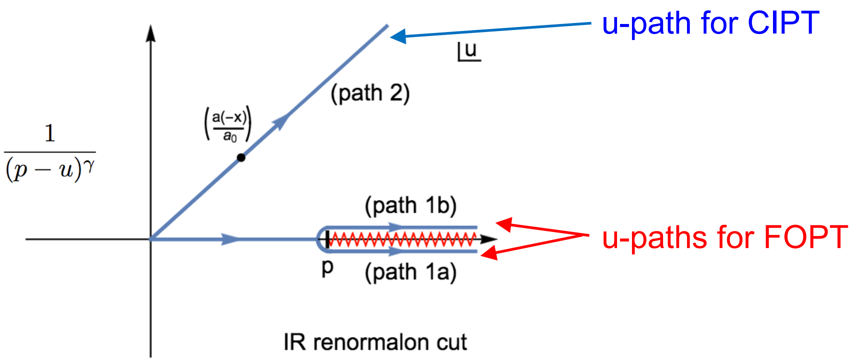
- Equivalent in perturbation theory (u-Taylor series)
- Agree at Euclidean point x = -1
- Difference in presence of IR renormalon cuts

$$B^{\mathrm{UV}}_{\hat{D},-\tilde{p},\gamma}(u) = \frac{1}{(\tilde{p}+u)^\gamma}$$



UV renormalon cut

$$B^{\mathrm{IR}}_{\hat{D},p,\gamma}(u) = \frac{1}{(p-u)^\gamma}$$



IR renormalon cut

UV renormalons:

FOPT and CIPT Borel representations equivalent because closing up paths 1 and 2 does not contain cuts

IR renormalons: finite difference !

FOPT and CIPT Borel representations inequivalent

- FOPT: PV prescription needs to be imposed
- CIPT: automatically well-defined by complex-valued α_s
- Difference because closing paths 1a/1b and 2 always contains cuts

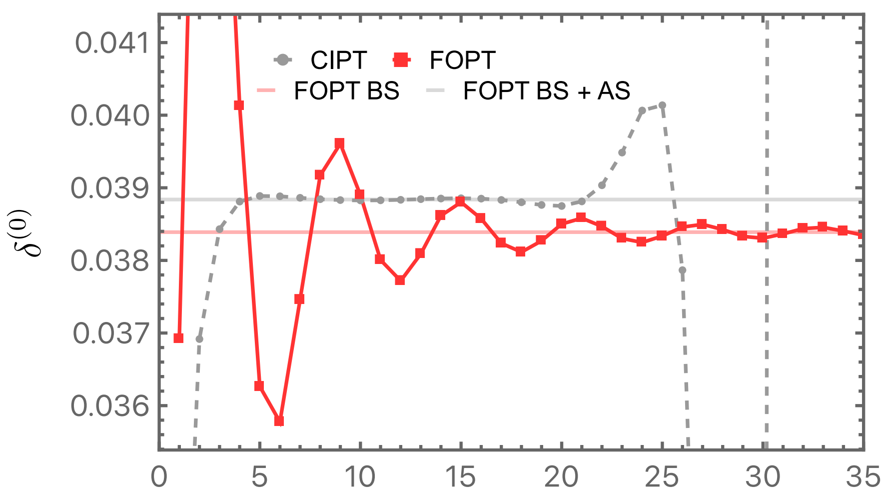
Brief Numerical Analysis

Single renormalon model:

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \dots \iff \frac{\langle \bar{G}^2 \rangle}{s^2}$$

$$W(x) = 1$$

$\bar{c}^{(1)}_{4,0} = 0,\ R = 0.8 m_\tau,\ W(x) = 1$



$s_0 = m_\tau^2$ $\qquad \mathcal{O}(\alpha_s^n)$

Pure $O(\Lambda^4_{QCD})$ renormalon in Adler function

→ Gluon condensate corrections vanishes

→ Per. series should be convergent

- CIPT series is **divergent** !
  FOPT series convergent.

  This fact was overlooked in the past

- CIPT not compatible with standard OPE !

  CIPT Borel representation should not be considered as "true", but it correctly characterizes the CIPT expansion

- Excellent description of CIPT-FOPT discrepancy by asymptotic separation $\Delta_W(s_0)$

- Moments with small asymptotic separation can be identified.

Renormalon-Free GC Scheme

Conclusion from the asymptotic separation:

AHH, Regner 2008.00578

- Asymptotic separation vanishes if IR renormalons are absent
- CIPT and FOPT should become consistent for IR-subtracted perturbation theory

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Idea: "short-distance" scheme for the gluon condensate

Benitez-Rathgeb, Boito, Jamin, AHH 2202.10957

$$\langle \bar{G}^2 \rangle^{\overline{\rm MS},(n)}$$

Original $\overline{\rm MS}$ GC contains pure O(Λ^4_{QCD}) renormalon (scale invariant)

Expand perturbatively with Adler function

$$\langle \bar{G}^2 \rangle^{\overline{\rm MS},(n)} \equiv \underbrace{\langle G^2 \rangle (R^2)}_{\text{renormalon-free, R-dependent}} - R^4 \overbrace{\sum_{\ell=1}^{n} \underbrace{N_g}_{\text{renormalon norm (approximately known)}} r_\ell^{(4,0)} \, \bar{a}^\ell (R^2)}\,,$$

renormalon-free

R-dependent

renormalon norm

(approximately known)

IR factorization scale R

$$r_\ell^{(4,0)} = \left(\frac{1}{2}\right)^{\ell + 4\hat{b}_1} \frac{\Gamma(\ell + 4\hat{b}_1)}{\Gamma(1 + 4\hat{b}_1)} \qquad \bar{a}(R^2) = \frac{\beta_0 \, \bar{\alpha}_s(R^2)}{4\pi}$$

C-scheme (C=0)

Boito, Jamin, Miravitllas 2016

Renormalon-Free GC Scheme

Benitez-Rathgeb, Boito, Jamin, AHH 2202.10957

$$\langle G^2\rangle(R^2) - \langle G^2\rangle(R'^2)$$

Renormalon-free (convergent series)

$$\frac{\mathrm{d}}{\mathrm{d}\ln R^2}\langle G^2\rangle(R^2) = \frac{N_g}{2^{4\hat{b}_1}}\,\frac{R^4\,\bar{a}(R^2)}{1-2\hat{b}_1\bar{a}(R^2)}$$

(R-evolution equation)

Convergent series!

We can define an R-independent „short-distance" GC:

treated like a tree-level term
(Do not expand !)

$$\langle G^2\rangle(R^2) \equiv \langle G^2\rangle^{\mathrm{RF}} + N_g\,\bar{c}_0(R^2)\,.$$

→ R-invariance of scheme at infinite truncation order

$$\bar{c}_0(R^2) \equiv R^4\ \mathrm{PV}\int_0^\infty \frac{\mathrm{d}u\ e^{-\frac{u}{\bar{a}_R}}}{(2-u)^{1+4\hat{b}_1}} = -\frac{R^4\,e^{-\frac{2}{\bar{a}(R^2)}}}{(\bar{a}(R^2))^{4\hat{b}_1}}\,\mathrm{Re}\left[e^{4\pi\hat{b}_1 i}\,\Gamma\Big(-4\hat{b}_1, -\frac{2}{\bar{a}(R^2)}\Big)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln R^2}\langle G^2\rangle^{\mathrm{RF}} = 0$$

Scale-invariant "short-distance" scheme for the gluon condensate

→ „true" Borel sum value unchanged (i.e. N_g-independent) ! („minimal scheme")

CIPT and FOPT: RF GC Scheme

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957

Single renormalon model:

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \ldots \quad \Longleftrightarrow \quad \frac{\langle \bar{G}^2 \rangle}{s^2}$$

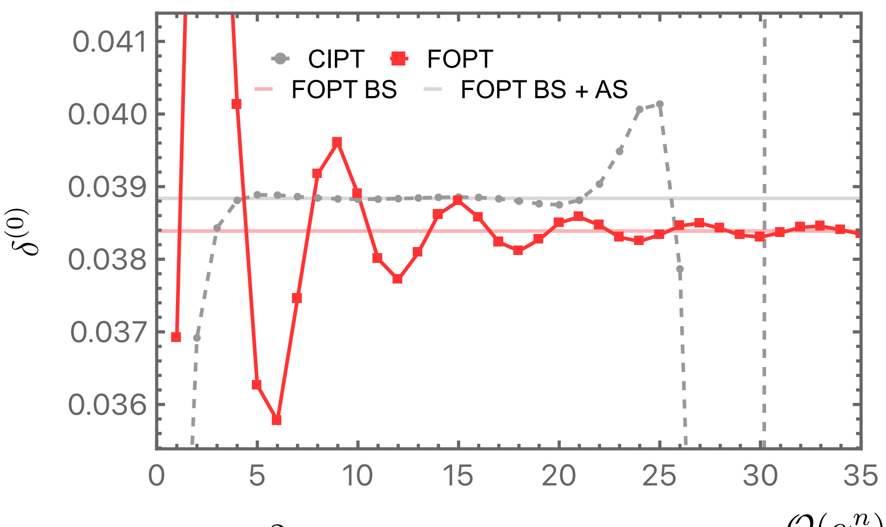
$$W(x) = 1 \qquad N_g = \frac{3}{2\pi^2}$$

Pure O(Λ^4_{QCD}) renormalon in Adler function

→ Gluon condensate corrections vanishes !

→ Nevertheless dramatic impact of changing to the RF GC scheme

$\bar{c}^{(1)}_{4,0} = 0, \; R = 0.8 m_\tau, \; W(x) = 1$



$s_0 = m_\tau^2$

- FOPT same as in the original GC scheme
- CIPT^{RS} series is convergent
- CIPT^{RS} consistent with FOPT !
- CIPT^{RS} compatible with standard OPE !
- CIPT^{RS} Borel sum = FOPT Borel sum
- CIPT^{RS} converges much faster than FOPT (oscillating behavior absent)

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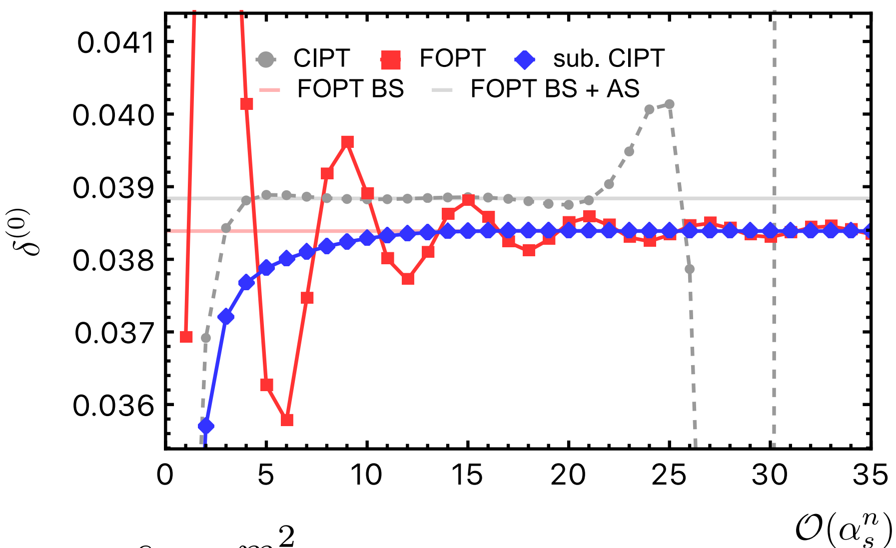

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CIPT and FOPT: RF GC Scheme

Realistic Multi renormalon model: GC, $O(\Lambda^4_{QCD}, \Lambda^6_{QCD})$ + UV renormalons in Adler function

Beneke, Jamin 2008

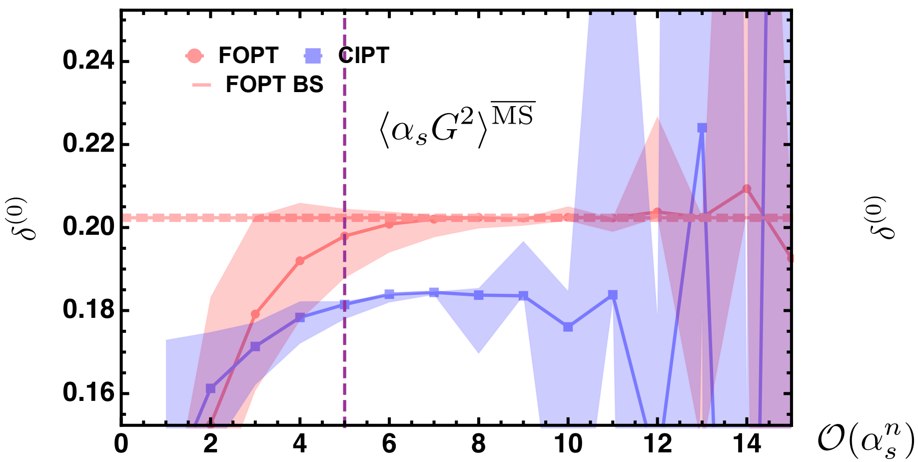
$$W(x) = 1 - 2x + 2x^3 - x^4$$

→ GC suppressed

$$s_0 = m_\tau^2\,, \quad \frac{1}{2} \le \xi \le 2\,, \quad N_g = 0.64$$

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957

$\bar{c}^{(1)}_{4,0} = -22/81,\ W(x) = (1-x)^3(1+x)$



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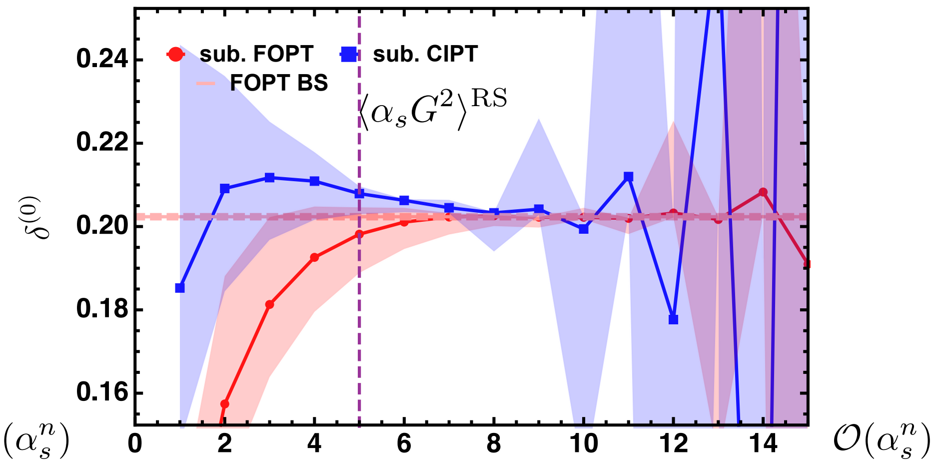


New RF GC Scheme !

- Discrepancy between CIPT and FOPT removed
- CIPT becomes consistent with FOPT (which is only slightly modified)
- Higher precision for α_s determinations from hadronic tau decays achievable
- Additional uncertainty from uncertainties in N_g

CIPT and FOPT: RF GC Scheme

Realistic Multi renormalon model: GC, $O(\Lambda^4_{QCD}, \Lambda^6_{QCD})$ + UV renormalons in Adler function

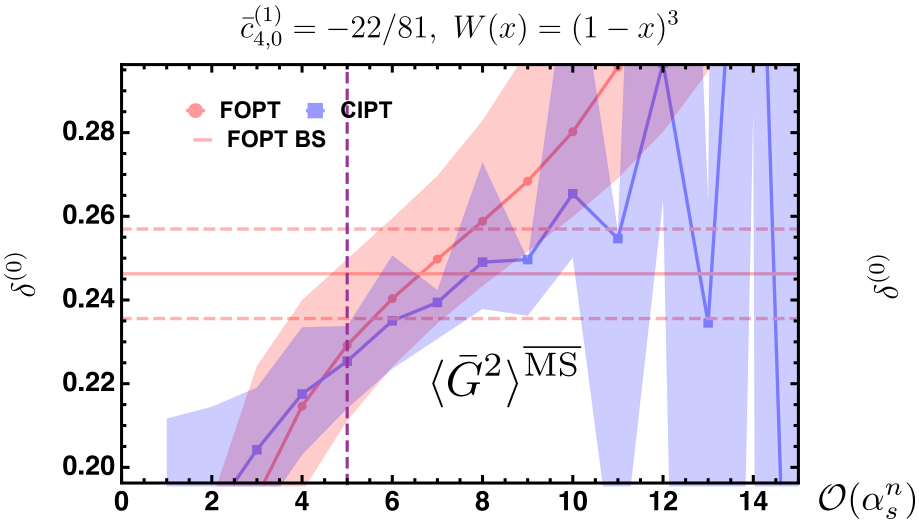
$\rightarrow$ GC enhanced

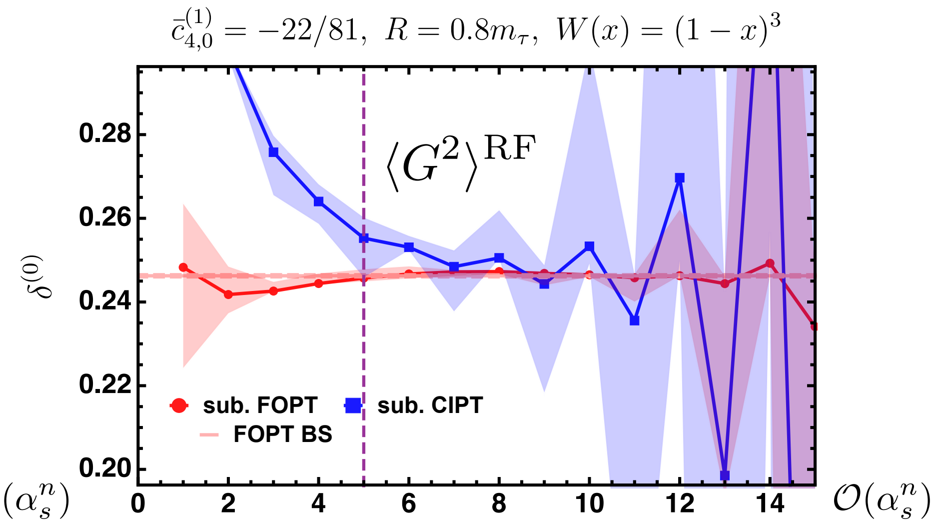
Beneke, Jamin 2008

$$W(x) = (1-x)^3$$

$$s_0 = m_\tau^2\,, \quad \frac{1}{2} \le \xi \le 2\,, \quad N_g = 0.64$$

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957





New RF GC Scheme !

- FOPT and CIPT expansions both get improved substantially
- Spectral function moments with high sensitivity to the GC can now be used for high-precision determinations of the strong coupling and the GC

GC Renormalon Normalization

Benitez-Rathgeb, Boito, Jamin, AHH: 2206.xxxxx

GC Norm in the Adler function's Borel function:

$$B[\hat{D}(s)]_{\rm GC}(u) = \frac{2\pi^2}{3}\frac{N_g\left[1-\frac{22}{81}\bar{a}(-s)\right]}{(2-u)^{1+4\hat{b}_1}} \qquad \bar{a}(\mu^2) \equiv \frac{\beta_0\,\bar{\alpha}_s(\mu^2)}{4\pi} \qquad \hat{b}_1 \equiv \frac{\beta_1}{2\beta_0^2}$$

Multi-renormalon model approach

Beneke, Jamin 2008

$$B[\hat{D}(s)]_{\rm mr}(u) = b^{(0)} + b^{(1)}u + \frac{2\pi^2}{3}\frac{N_g\left[1-\frac{22}{81}\bar{a}(-s)\right]}{(2-u)^{1+4\hat{b}_1}} + \frac{N_6}{(3-u)^{1+6\hat{b}_1}} + \frac{N_{-2}}{(1+u)^{2-2\hat{b}_1}}$$

$$\begin{aligned} c_{0,1} &= c_{1,1} = 1\,, \quad c_{2,1} = 1.640 \\ c_{3.1} &= 6.371 \\ c_{4,1} &= 49.076 \\ c_{5,1} &= 280 \pm 140 \end{aligned}$$

→ $N_g = 0.64 \pm 0.27$

Conformal mapping approach

Lee 2012

$$\tilde{B}(u) \equiv \frac{3\,(2-u)^{1+4\hat{b}_1}}{2\pi^2}\,B[\hat{D}(s)](u)$$

GC renormalon-free

$$w(u,p) = \frac{\sqrt{1+u}-\sqrt{1-\frac{u}{p}}}{\sqrt{1+u}+\sqrt{1-\frac{u}{p}}}$$

u=2 closest to the origin in the w plane

$N_g = \tilde{B}(w(2,p))$

Use $c_{1,1}$ to $c_{5,1}$ and w-expansion → $N_g = 0.71 \pm 0.26$

GC Renormalon Normalization

Benitez-Rathgeb, Boito, Jamin, AHH: 2206.xxxxx

GC Norm in the Adler function's Borel function:

$$B[\hat{D}(s)]_{\rm GC}(u) = \frac{2\pi^2}{3}\,\frac{N_g\left[1-\frac{22}{81}\bar{a}(-s)\right]}{(2-u)^{1+4\hat{b}_1}} \qquad \bar{a}(\mu^2) \equiv \frac{\beta_0\,\bar{\alpha}_s(\mu^2)}{4\pi} \qquad \hat{b}_1 \equiv \frac{\beta_1}{2\beta_0^2}$$

Optimal subtraction approach

New !

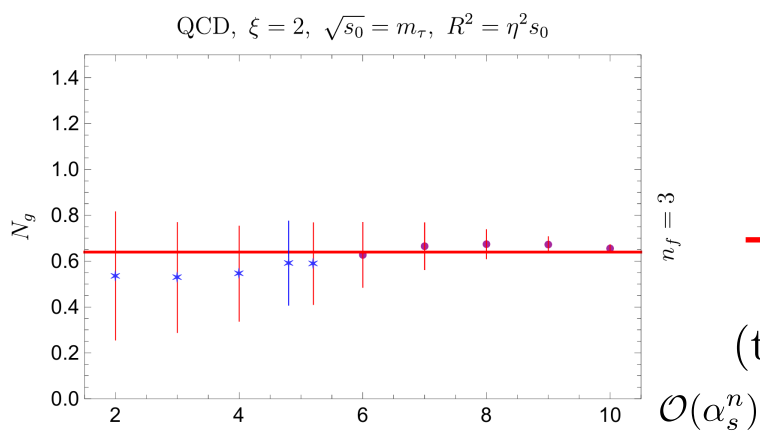
Use quantiative measure for improvements for GC suppresed and GC enhanced moments in the RF GC scheme

$$\chi^2_m(N_g) = \chi^2_{m,\rm GCS}(N_g) + \chi^2_{m,\rm GCE}(N_g)$$

Good convergence of 5 GC enhanced moments

Small discrepancy for 5 GC supressed moments

Can precisely determine N_g for the Beneke-Jamin model



$$N_g = 0.57 \pm 0.23$$

(take $\mathcal{O}(\alpha_s^4)$ result)

universität wien

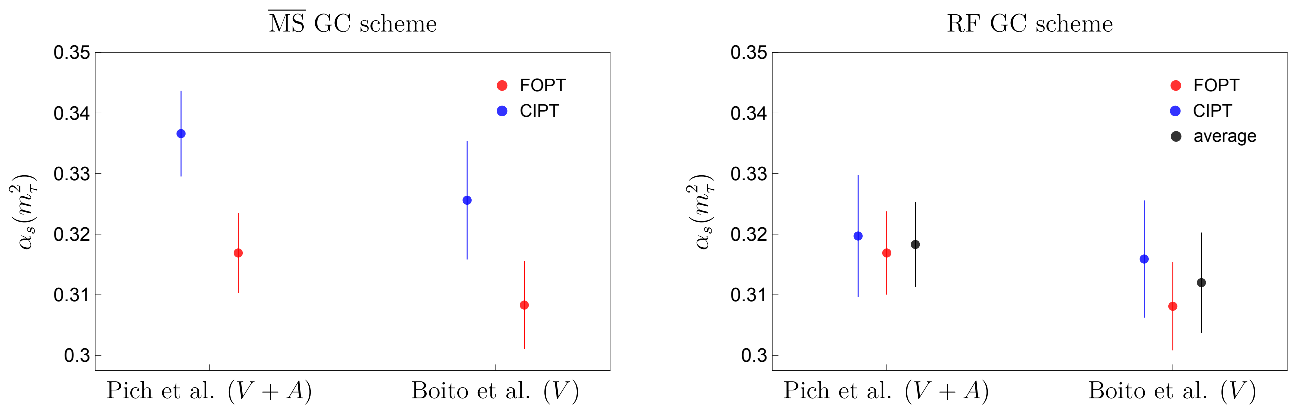
Strong Coupling Determinations

We repeat (in detail!) two state-of-the-art determination methods in the RF GC scheme:

Truncated OPE approach: Pich, Rodriguez-Sanchez 2016

Duality Violation model approach: Boito, Golterman, Maltman, Peris, Rodriguesz, Scharf 2021

→ Include uncertainties: $N_g = 0.57 \pm 0.23 \qquad 0.7 m_\tau \leq R \leq m_\tau$



- FOPT-CIPT for GC suppressed moments remedied
- Taking average of FOPT and CIPT results now meaningful
- Spectral function moments with high sensitivity to the GC can now be used for high-precision determinations of the strong coupling and the GC
- Uncertainties due GC renormalon norm N_g and R variations very small !

Summary and Conclusions

- CIPT Borel representation different from FOPT Borel representation in the presence of IR renormalons. $\longrightarrow$ **Asymptotic Separation**

  $\longrightarrow$ **CIPT expansion NOT consistent with standard OPE approach**

- Problems of CIPT resolved largely in renormalon-free RF gluon condensate (GC) scheme.

  $\longrightarrow$ CIPT^{RF} “cured” and still useful

- We have devised such a GC scheme in the most minimalistic and transparent way. (Additional uncertainty from N_g (GC renormalon norm), and factorization scale R.)

- RF GC scheme: Disparity between CIPT and FOPT reconciled

- RF GC scheme: Moments with high sensitivity to the GC can be used for high precision analyses

- Excellent prospects for new high-precision determinations of the strong coupling

(1) CIPT Borel Sum Contour Integration

The contour integration for the CIPT Borel representation must be deformed away from |x| = 1.

(Leaves FOPT Borel sum unchanged!)

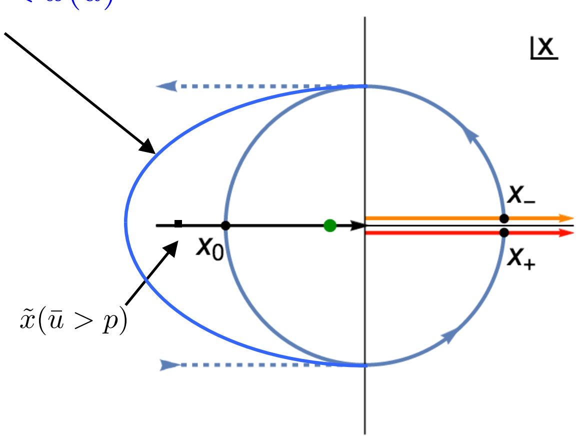
Do the contour-integral first:

$$\delta^{(0),\mathrm{CIPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\,\frac{1}{2\pi i}\oint_{\mathcal{C}_x}\frac{\mathrm{d}x}{x}\,(-x)^m\left(\frac{a(-x)}{a_0}\right)\frac{e^{-\frac{\bar{u}}{a_0}}}{\left(p-\frac{a(-x)}{a_0}\bar{u}\right)^\gamma}$$

$$= \int_0^\infty \mathrm{d}\bar{u}\,e^{-\frac{\bar{u}}{a_0}}\,\tilde{C}(p,\gamma,m,s_0;\bar{u})\,.$$

pole in x-plane at (arge-β_0)

Contour must always cross real axis for $x < \tilde{x}(\bar{u})$



$$\tilde{x}(\bar{u}) = -e^{(\bar{u}-p)/pa_0} = -\left(\frac{\Lambda^2_{\mathrm{QCD}}}{s_0}\right)^{\frac{p-\bar{u}}{p}}$$

$$< -1 \quad \text{for} \quad \bar{u} > p$$

$$\tilde{x}(0) = -\left(\frac{\Lambda^2_{\mathrm{QCD}}}{s_0}\right) \quad \text{(Landau pole)}$$

$$\tilde{x}(\bar{u}\to\infty) \to -\infty$$

(2) CIPT Borel Sum Contour Integration

The contour integration for the CIPT Borel representation must be deformed away from |x| = 1.

(Leaves FOPT Borel sum unchanged!)

Do the Borel-u-integral first:

$$\Delta(m,p,\gamma,s_0) \equiv \delta^{(0),\mathrm{CIPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0) - \delta^{(0),\mathrm{FOPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0)$$

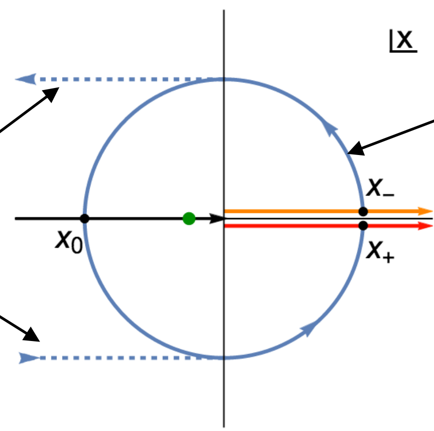
„Asymptotic Separation“

$$= \frac{1}{2\Gamma(\gamma)} \oint_{\mathcal{C}_x} \frac{\mathrm{d}x}{x} (-x)^m \,\mathrm{sig}[\mathrm{Im}[a(-x)]]\, (a(-x))^{1-\gamma}\, e^{-\frac{p}{a(-x)}} \,.$$

Cut along the negative real s-axis!

Power-suppressed $\sim \left(\frac{\Lambda^2_{\mathrm{QCD}}}{s}\right)^p$

Remaining contour integration must be deformed (to negative real infinity in the x-plane)



Contour along the unit circle

Deformed contour for the asymptotic separation

(3) Computation of the CIPT Borel Sum

An analytic continuation is mandatory to compute the CIPT Borel sum for m>p

$$W(x) \sim x^m$$

$$B(u) \sim \frac{1}{(p-u)^\gamma}$$

$$\Delta(m,p,\gamma,s_0) \equiv \delta^{(0),\mathrm{CIPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0) - \delta^{(0),\mathrm{FOPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0)$$

$$= \frac{1}{2\Gamma(\gamma)} \oint_{\mathcal{C}_x} \frac{\mathrm{d}x}{x} (-x)^m \,\mathrm{sig}[\mathrm{Im}[a(-x)]]\, (a(-x))^{1-\gamma}\, e^{-\frac{p}{a(-x)}} \,.$$

$\sim x^{-p}$

Properties of the asymptotic separation:

- Renormalization scheme invariant
- **Much larger than canonical FOPT Borel sum ambiguity estimate if the Borel function has a sizeable gluon condensate cut**
- Fully analytic results
- **Properties of CIPT Borel representation imply that OPE corrections for CIPT do not have the common standard form C x <condensate> / s^p**