

# Hadronic light-by-light scattering and the muon $g-2$

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wien**

**Clues to a mysterious Universe,  
Humboldt Kolleg, Kitzbühel, 26 June - 1 July 2022**

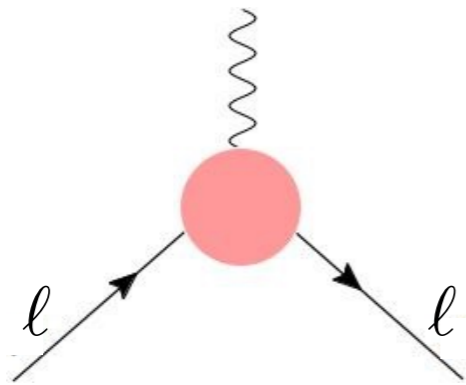
# Outline

- ✦ The muon anomalous magnetic moment: discrepancy between SM and experiment
- ✦ Hadronic contributions: **hadronic vacuum polarization** and **hadronic light-by-light**
- ✦ Approach based on **dispersion relations** for a data-driven determination of **hadronic light-by-light contribution**: basic features and numerical results
- ✦ Summary and outlook

In collaboration with G. Colangelo, M. Hoferichter, P. Stoffer and J. Lüdtkke

# Introduction

- ★ **Anomalous magnetic moments of leptons**  $a_\ell$  have played a central role in the history of particle physics by contributing to establish quantum electrodynamics



$$\vec{\mu}_\ell = g_\ell \frac{q_\ell}{2m_\ell} \vec{s} \quad a_\ell = \frac{g_\ell - 2}{2}$$

- ★ Dirac's relativistic theory of spin-1/2 particles predicts  $g_\ell = 2$   
In the Standard Model (SM), **radiative corrections** are responsible for  $g_\ell \neq 2$

- ★  $a_e^{\text{exp}} = 0.00115965218073(28)$  [0.24 ppb] [Hanneke, Fogwell, Gabrielse \(2008\)](#)

provides a stringent test of QED

# Introduction

★ **Muon** anomalous magnetic moment is particularly interesting :

- ▶ more sensitive than  $a_e$  to weak and strong interaction effects and New Physics scales ( $\Delta a_\ell \propto m_\ell^2/M^2$ )
- ▶  $4.2\sigma$  discrepancy between  $a_\mu^{\text{exp}}$  and  $a_\mu^{\text{SM}}$  : **open puzzle**

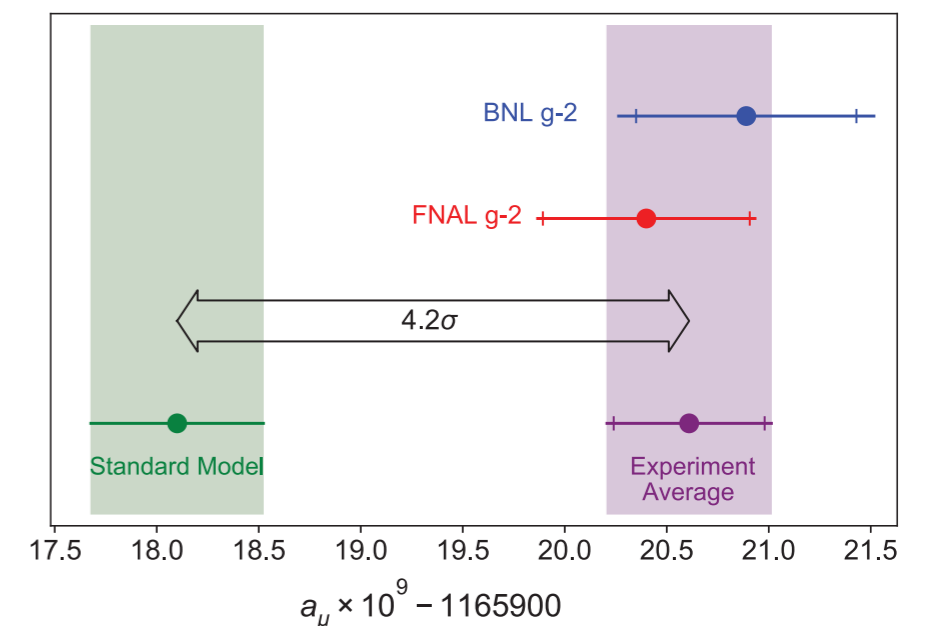
★ The experimental world average for  $a_\mu$  : **BNL (E821) and FNAL (E989) results**

Bennett et al. (2006), Abi et al. (2021)

$$a_\mu^{\text{exp}} = 116\,592\,061(41) \times 10^{-11} \quad [0.35 \text{ ppm}]$$

$$a_\mu^{\text{SM}} = 116\,591\,810(43) \times 10^{-11}$$

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 251(59) \times 10^{-11}$$



SM: Aoyama et al. (2020) [White Paper by the Muon g-2 Theory Initiative]

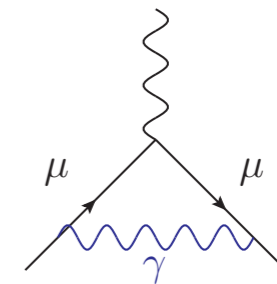


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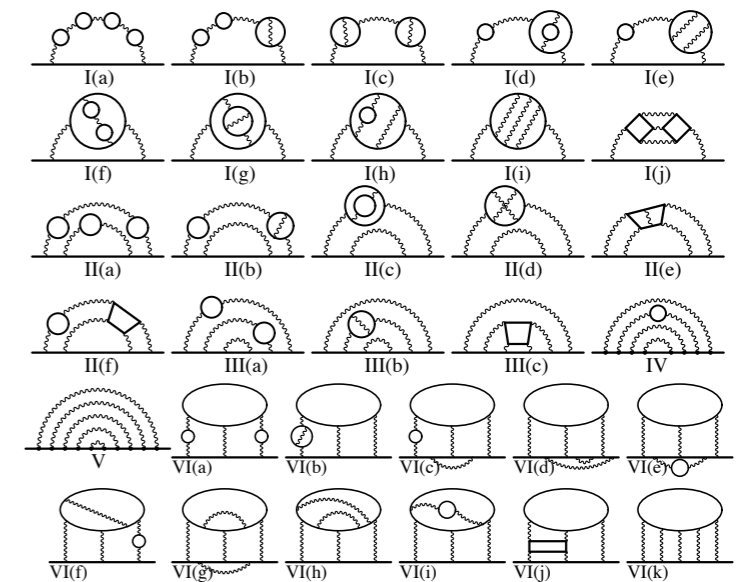
The muon anomalous magnetic moment: [experiment vs \(White Paper\) SM prediction](#)

Contribution	Value $\times 10^{11}$
Experiment (E821 + E989)	116 592 061(41)
<b>QED</b>	<b>116 584 718.931(104)</b>
Electroweak	153.6(1.0)
HVP ( $e^+e^-$ , LO + NLO + NNLO)	6845(40)
HLbL (phenomenology + lattice + NLO)	92(18)
Total SM Value	116 591 810(43)
Difference: $\Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	251(59)

[Aoyama et al. \(2020\) \[White Paper by the Muon g-2 Theory Initiative\]](#)



[Schwinger \(1948\)](#)



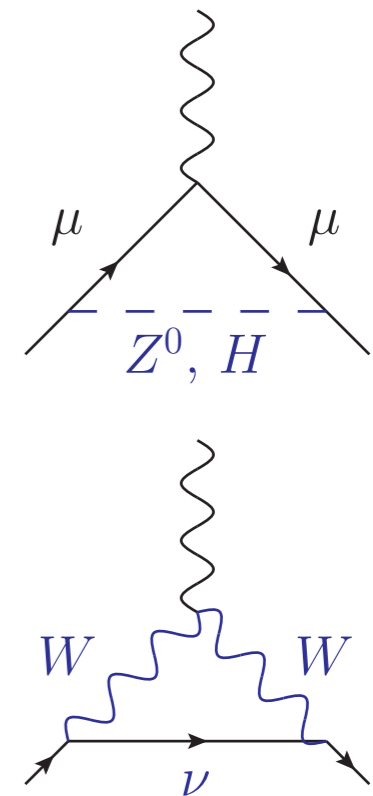
[Kinoshita et al. \(2012\)](#)

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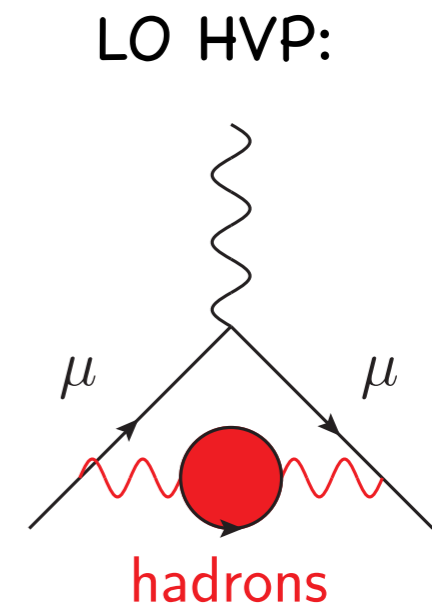


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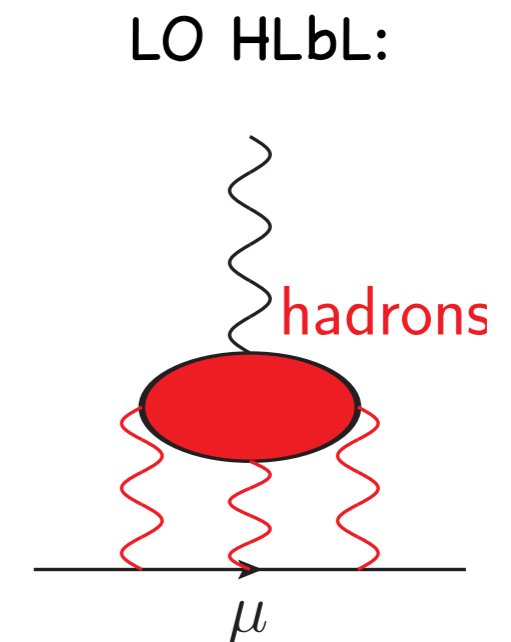
Low-energy strong interaction effects: non-perturbative!

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Low-energy strong interaction effects: non-perturbative!

► Forthcoming FNAL results also call for [further scrutiny of the SM prediction](#)

# Hadronic vacuum polarization

✦ The crucial limiting factor in the accuracy of SM predictions for  $a_\mu$  is control over **hadronic contributions**, responsible for most of the theory uncertainty

✦ The most precise determination of the LO-HVP relies on a **dispersive approach**:

▶ Gauge invariance: 
$$i \int d^4x e^{iq \cdot x} \langle 0 | T \{ j_\mu^{\text{em}}(x) j_\nu^{\text{em}}(0) \} | 0 \rangle = -(q^2 g_{\mu\nu} - q_\mu q_\nu) \Pi(q^2)$$

parameterized in terms of a **single** scalar function of **one** kinematic variable

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$$\Pi^{\text{ren}}(q^2) = \Pi(q^2) - \Pi(0) = \frac{q^2}{4\pi} \int_{s_{\text{thr}}}^{\infty} ds \frac{\text{Im } \Pi(s)}{s(s - q^2 - i\epsilon)}$$

discontinuity along a branch cut corresponding to physical processes

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▶ Unitarity (optical theorem):

$$\text{Im} \left[ \text{hadrons} \right] \Leftrightarrow \left| \text{hadrons} \right|^2 \propto \sigma_{\text{tot}}(e^+ e^- \rightarrow \text{hadrons})$$



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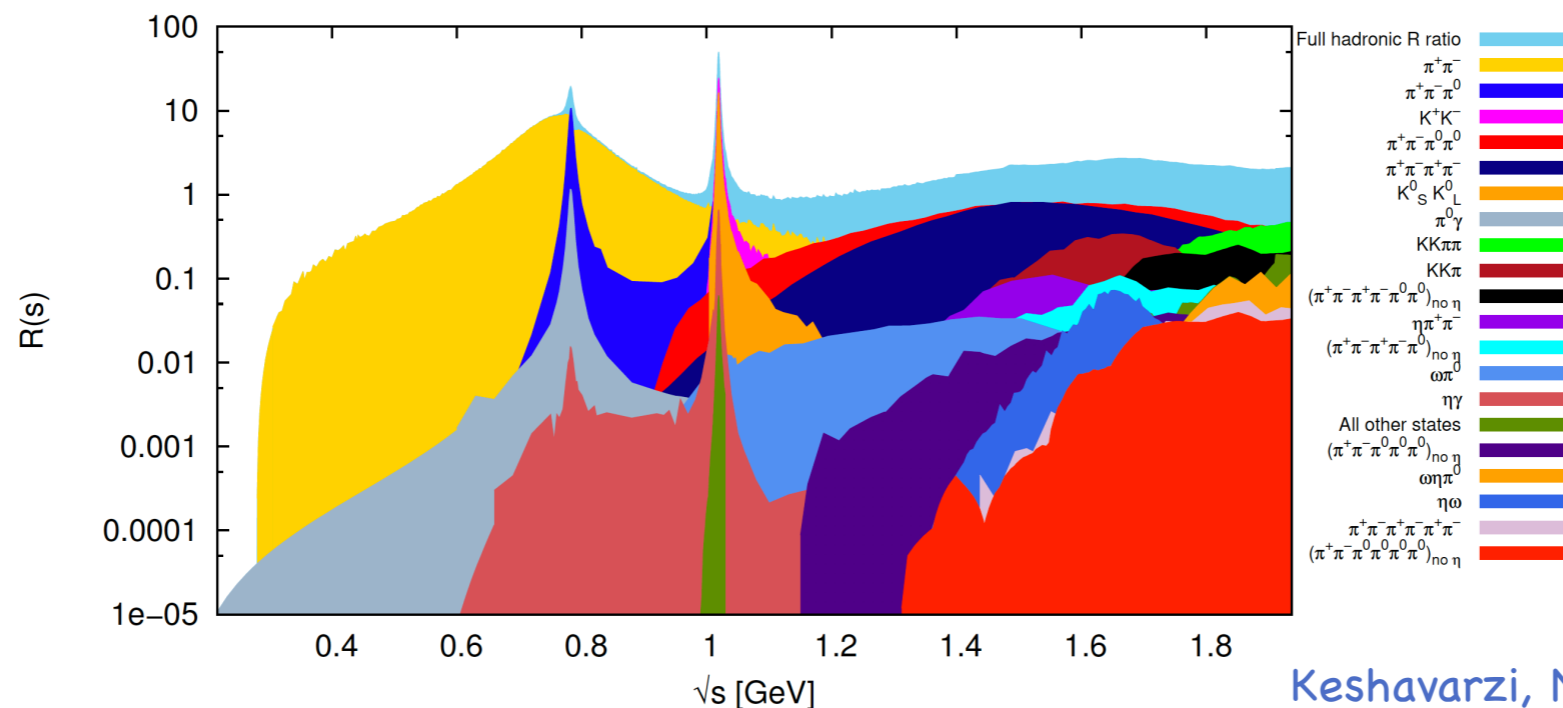
$$\text{Im } \Pi(s) = \frac{s}{4\pi\alpha(s)} \sigma_{\text{tot}}(e^+ e^- \rightarrow \text{hadrons}) = \frac{\alpha(s)}{3} R^{\text{had}}(s)$$

# Hadronic vacuum polarization

- ★ **LO-HVP** is obtained by integrating the hadronic R-ratio weighted with a known perturbative QED kernel:

$$a_{\mu}^{\text{LO-HVP}} = \frac{1}{3} \left( \frac{\alpha}{\pi} \right)^2 \int_{s_{\text{thr}}}^{\infty} \frac{ds}{s} K(s) R^{\text{had}}(s)$$

dominated by the **low-energy region** (in particular  $\pi\pi$  contribution)

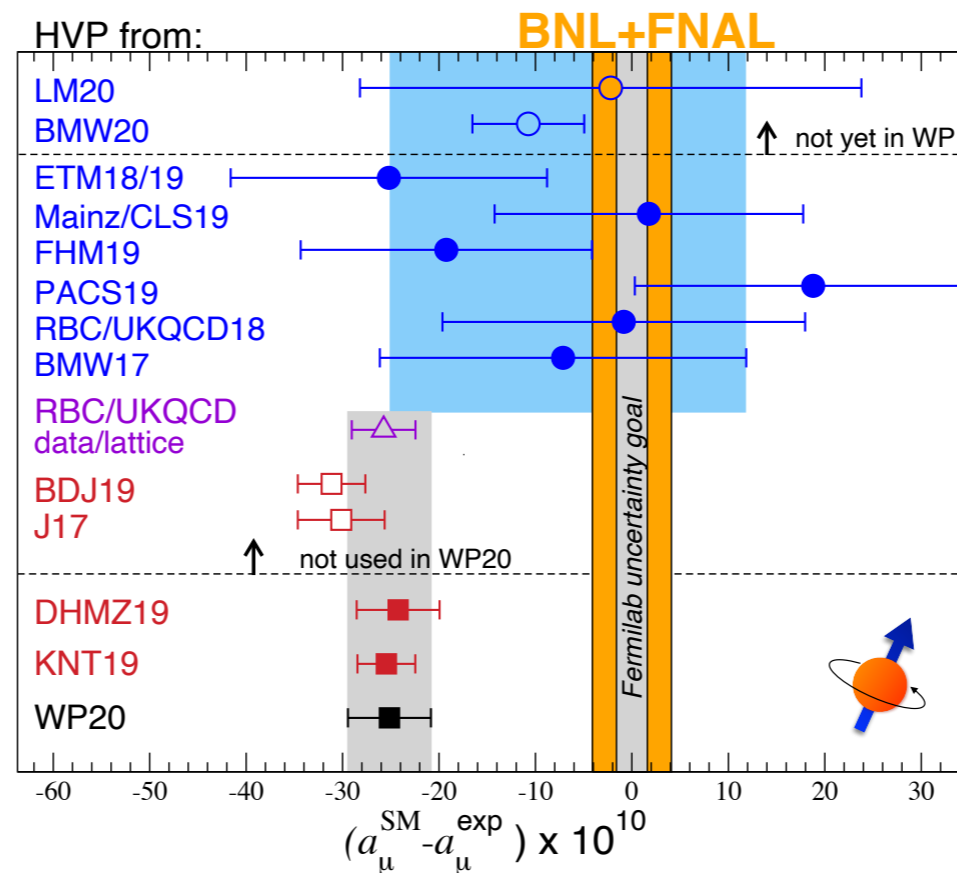


Keshavarzi, Nomura and Teubner (2018)

- ★ Dedicated  $e^+e^-$  program with the goal to improve the presently quoted sub-percent accuracy (new data expected from CMD-3, BaBar, BES-III, Belle II)

# LO-HVP using lattice QFT

- ★ Several efforts to determine LO-HVP on Euclidean discretized space-time:  
finite volume and continuum extrapolations, physical pion mass ensembles/chiral extrapolations, strong and electromagnetic isospin breaking corrections, scale setting

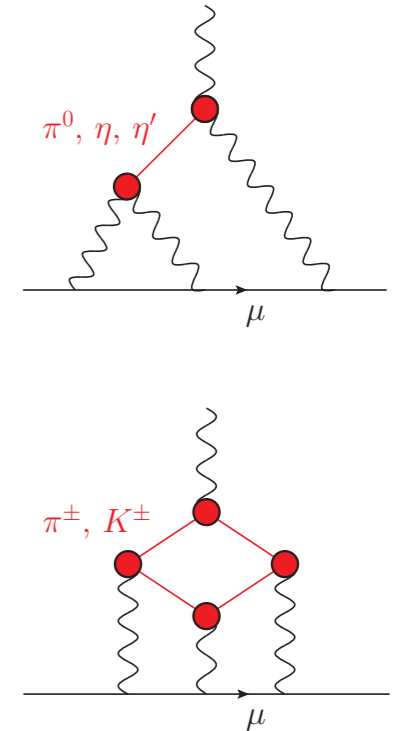


- ★ BMW20 (Borsanyi et al.) first lattice result with quoted sub-percent precision:  
ongoing cross-checks against other lattice and data-driven determinations

# Hadronic light-by-light

$a_{\mu}^{\text{HLbL}}$  in  $10^{-11}$  units

Contribution	PdRV(09)	N/JN(09)	J(17)	Dispersive
$\pi^0, \eta, \eta'$ -poles	114(13)	99(16)	95.45(12.40)	93.8(4.0)
$\pi, K$ -loops/boxes	-19(19)	-19(13)	-20(5)	-16.4(2)
$S$ -wave $\pi\pi$ rescattering	-7(7)	-7(2)	-5.98(1.20)	-8(1)
subtotal	88(24)	73(21)	69.5(13.4)	69.4(4.1)
scalars	-	-	-	} -1(3)
tensors	-	-	1.1(1)	
axial vectors	15(10)	22(5)	7.55(2.71)	6(6)
$u, d, s$ -loops / short-distance	-	21(3)	20(4)	15(10)
$c$ -loop	2.3	-	2.3(2)	3(1)
total	105(26)	116(39)	100.4(28.2)	92(19)



Based on model calculations:  
uncertainties are guesstimates

Data-driven determination  
with reliable uncertainties

► Dispersive approach to HLbL

Colangelo, Hoferichter, MP, Stoffer

► Lattice:

$$a_{\mu}^{\text{HLbL}} = 78.7 (30.6)_{\text{stat}} (17.7)_{\text{sys}} \times 10^{-11}$$

RBC/UKQCD, Blum et al. (2020)

$$a_{\mu}^{\text{HLbL}} = 107 (15) \times 10^{-11}$$

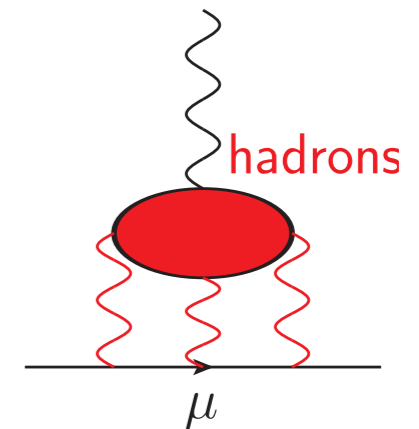
Mainz, Chao et al. (2021)

# Dispersive approach to HLbL

✦ Exploits fundamental principles:

- ▶ gauge invariance and crossing symmetry
- ▶ unitarity and analyticity

to relate HLbL to experimentally accessible quantities



✦ Much more challenging task than LO-HVP due to the complexity of the **HLbL tensor**

✦ Defines and relates **single** contributions to HLbL to form factors and cross sections

Colangelo, Hoferichter, Procura, Stoffer, JHEP 1505 (2015), JHEP 1704 + PRL 118 (2017)

Colangelo, Hoferichter, Procura, Stoffer, JHEP 1409 (2014)

Colangelo, Hoferichter, Kubis, Procura, Stoffer, PLB 738 (2014)

# HLbL tensor and master formula

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int d^4x d^4y d^4z e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \langle 0 | T \{ j_{\text{em}}^\mu(x) j_{\text{em}}^\nu(y) j_{\text{em}}^\lambda(z) j_{\text{em}}^\sigma(0) \} | 0 \rangle$$

- ★ Lorentz covariance: 138 structures, which are redundant due to Ward identities
- ★ Derived 54 generating Lorentz structures that are manifestly gauge invariant and crossing symmetric. The scalar functions  $\Pi_i$  are **free of kinematic singularities and zeros**: their analytic structure is dictated by dynamics only

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- ★ Obtained a **general** master formula:

$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$



# Contributions to $a_\mu^{\text{HLbL}}$

- ★ Unitarity in direct and crossed channel (poles and branch cuts)

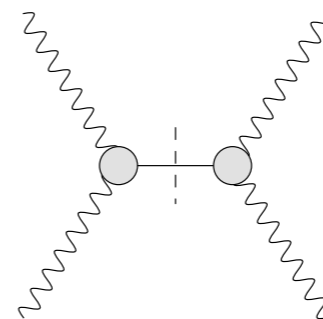
$$\Pi_i^t(s, t, u) = c_i^t + \frac{\rho_{i;s}^t}{s - M_\pi^2} + \frac{\rho_{i;u}^t}{u - M_\pi^2} + \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\text{Im}_s \Pi_i^t(s', t, u')}{s' - s} + \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} du' \frac{\text{Im}_u \Pi_i^t(s', t, u')}{u' - u}$$

- ★ The **lightest intermediate states** dominate (in agreement with models)

- ★ HLbL tensor can be split up into contributions with different topologies:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$

one-pion intermediate state :





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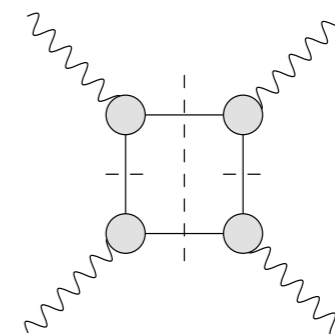
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two-pion intermediate state in both channels :



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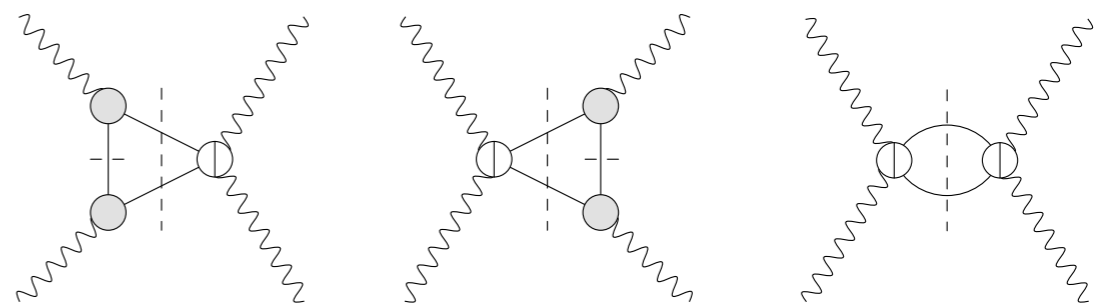
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two-pion state only in the direct channel:



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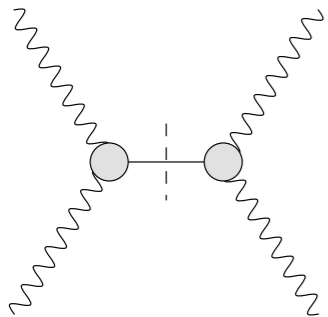
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higher intermediate states: ongoing work

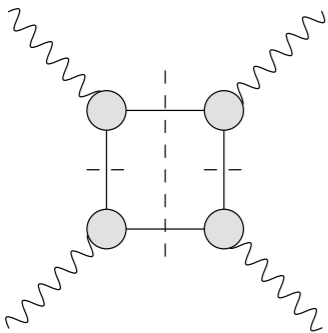
# Numerical results for dispersive $a_\mu^{\text{HLbL}}$



Info on pion **transition form factor**:  $a_\mu^{\pi^0\text{-pole}} = 63.0_{-2.1}^{+2.7} \times 10^{-11}$

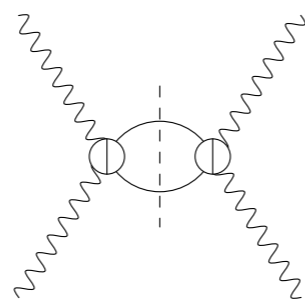
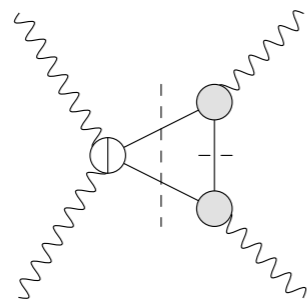
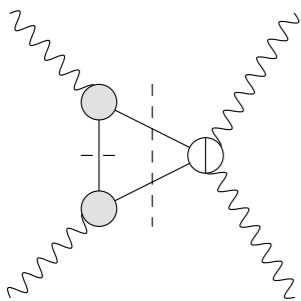
Hoferichter, Hoid, Kubis, Leupold, Schneider (2018)

In agreement with lattice: Gérardin, Meyer, Nyffeler (2019)



Info on pion **vector form factor**:  $a_\mu^{\pi\text{-box}} = -15.9(2) \times 10^{-11}$

Colangelo, Hoferichter, MP, Stoffer (2017)



Info on **helicity partial waves** for  $\gamma^*\gamma^* \rightarrow \pi\pi$   
with S-wave  $\pi\pi$  rescattering effects:

$$a_{\mu, J=0}^{\pi\pi} = -8(1) \times 10^{-11}$$

Colangelo, Hoferichter, MP, Stoffer (2017)

# Ongoing work on dispersive $a_{\mu}^{\text{HLbL}}$

Reduce model dependence in the remaining contributions:

- ★ Include rescattering contributions for **higher partial waves** to account for prominent features in photon-photon to two mesons cross sections.  
Extension of the solution of partial-wave dispersion relations for  $\gamma^* \gamma^* \rightarrow \pi\pi$  to **D-waves** to capture effects of  $f_2(1270)$  beyond narrow width approximation  
Hoferichter and Stoffer (2019)
- ★ Contributions from **higher intermediate states** (axial and tensor mesons)
- ★ Asymptotic regime: **short-distance constraints** on HLbL (OPE and pQCD)

Hadronic models: Melnikov and Vainshtein (2004), Colangelo et al. (2020),  
Leutgeb and Rebhan (2020), Capiello et al. (2020)

Interpolants: Lüdtkke and MP (2020)

Will lead to a **more precise SM evaluation** of the muon  $g-2$  !

# Summary

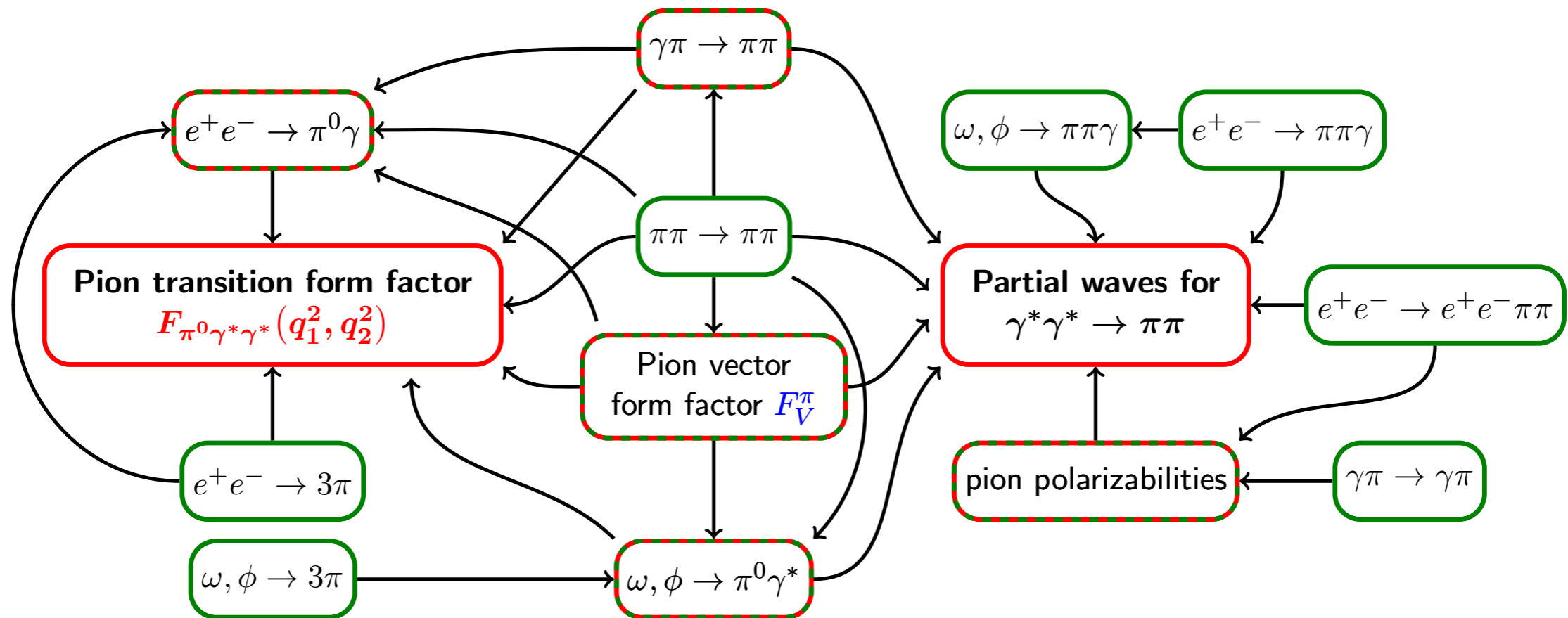
- ✦ The discrepancy between SM prediction and experimental determination of the muon anomalous magnetic moment is (still) an **open puzzle: new physics ?**
- ✦ Theoretical uncertainties are dominated by **hadronic contributions**
- ✦ **Hadronic vacuum polarization:** quoted sub-percent precision from a data-driven approach based on dispersion relations.  
**Ongoing work:** improved experimental input, further accurate lattice evaluations, MUonE experiment.
- ✦ **Hadronic light-by-light:** data-driven dispersive approach delivered robust evaluations of dominant and sub-dominant contributions, in agreement with lattice.  
**Ongoing work:** refined analysis of two-meson intermediate states, higher intermediate states and asymptotic constraints from OPE and perturbative QCD.

**Additional slides**



# A roadmap for HLbL

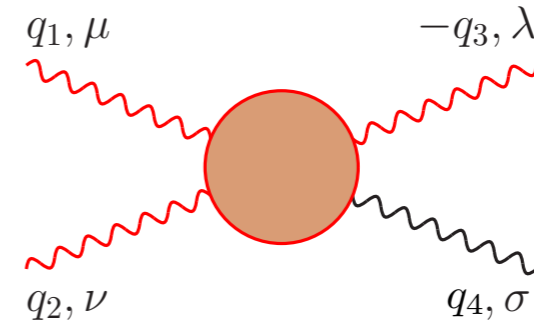
Colangelo, Hoferichter, Kubis, MP, Stoffer (2014)



Artwork by M. Hoferichter

# The HLbL tensor

★ The fully off-shell HLbL tensor :



$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int d^4x d^4y d^4z e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \langle 0 | T \{ j_{\text{em}}^\mu(x) j_{\text{em}}^\nu(y) j_{\text{em}}^\lambda(z) j_{\text{em}}^\sigma(0) \} | 0 \rangle$$

★ Mandelstam variables:

$$s = (q_1 + q_2)^2, t = (q_1 + q_3)^2, u = (q_2 + q_3)^2$$

★ In order to extract  $a_\mu^{\text{HLbL}}$ ,  $q_4 \rightarrow 0$  afterwards

# Lorentz structure of HLbL tensor

- Based on Lorentz covariance the HLbL tensor can be decomposed in 138 structures

$$\begin{aligned}
 \Pi^{\mu\nu\lambda\sigma} = & g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 \\
 & + \sum_{\substack{k=1,2,4 \\ l=1,2,3}} g^{\mu\nu} q_k^\lambda q_l^\sigma \Pi_{kl}^4 + \sum_{\substack{j=1,3,4 \\ l=1,2,3}} g^{\mu\lambda} q_j^\nu q_l^\sigma \Pi_{jl}^5 + \sum_{\substack{j=1,3,4 \\ k=1,2,4}} g^{\mu\sigma} q_j^\nu q_k^\lambda \Pi_{jk}^6 \\
 & + \sum_{\substack{i=2,3,4 \\ l=1,2,3}} g^{\nu\lambda} q_i^\mu q_l^\sigma \Pi_{il}^7 + \sum_{\substack{i=2,3,4 \\ k=1,2,4}} g^{\nu\sigma} q_i^\mu q_k^\lambda \Pi_{ik}^8 + \sum_{\substack{i=2,3,4 \\ j=1,3,4}} g^{\lambda\sigma} q_i^\mu q_j^\nu \Pi_{ij}^9 \\
 & + \sum_{\substack{i=2,3,4 \\ j=1,3,4}} \sum_{\substack{k=1,2,4 \\ l=1,2,3}} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi_{ijkl}^{10}
 \end{aligned}$$

- In 4 space-time dimensions there are 2 linear relations among these 138 structures

Eichmann, Fischer, Heupel, Williams (2014)

- Scalar functions encode the hadronic dynamics and depend on 6 kinematic variables

- This set of functions is hugely redundant: Ward identities imply 95 linear relations among these scalar functions (kinematic zeros)

# Lorentz structure of HLbL tensor

- ★ Following Bardeen and Tung (1968) - "BT"- we contracted the HLbL tensor with

$$I_{12}^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}, \quad I_{34}^{\lambda\sigma} = g^{\lambda\sigma} - \frac{q_4^\lambda q_3^\sigma}{q_3 \cdot q_4}$$

▶ 95 structures project to zero

- ★  $1/q_1 \cdot q_2$  and  $1/q_3 \cdot q_4$  poles eliminated by taking linear combinations of structures

- ★ This procedure introduces **kinematic singularities in the scalar functions**: degeneracies in these BT Lorentz structures, e.g. as  $q_1 \cdot q_2 \rightarrow 0$ ,  $q_3 \cdot q_4 \rightarrow 0$

$$\sum_k c_k^i T_k^{\mu\nu\lambda\sigma} = q_1 \cdot q_2 X_i^{\mu\nu\lambda\sigma} + q_3 \cdot q_4 Y_i^{\mu\nu\lambda\sigma}$$

# Lorentz structure of HLbL tensor

- ★ Following Tarrach (1975) we extended BT set to incorporate  $X_i^{\mu\nu\lambda\sigma}$ ,  $Y_i^{\mu\nu\lambda\sigma}$  to obtain a ("BTT") generating set of structures even for  $q_1 \cdot q_2 \rightarrow 0$ ,  $q_3 \cdot q_4 \rightarrow 0$

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- ▶ Lorentz structures are manifestly **gauge invariant**
- ▶ **crossing symmetry** is manifest (only 7 genuinely different structures, the remaining ones being obtained by crossing)
- ▶ the BTT scalar functions are **free of kinematic singularities and zeros**: their analytic structure is dictated by dynamics only. This makes them **suitable for a dispersive treatment**

# Master formula for $a_\mu^{\text{HLbL}}$

★ From  $\Pi_{\mu\nu\lambda\sigma}$  to  $a_\mu^{\text{HLbL}}$  :

By expanding the photon-muon vertex function around  $q_4 = 0$ ,

$$a_\mu^{\text{HLbL}} = -\frac{1}{48m_\mu} \text{Tr} \left( (\not{p} + m_\mu) [\gamma^\rho, \gamma^\sigma] (\not{p} + m_\mu) \Gamma_{\rho\sigma}^{\text{HLbL}}(p) \right)$$

Aldin, Brodsky, Dufner, Kinoshita (1970)

where  $p^2 = m_\mu^2$  and

$$\Gamma_{\rho\sigma}^{\text{HLbL}}(p) = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \gamma^\mu \frac{(\not{p} + \not{q}_1 + m_\mu)}{(p + q_1)^2 - m_\mu^2} \gamma^\lambda \frac{(\not{p} - \not{q}_2 + m_\mu)}{(p - q_2)^2 - m_\mu^2} \gamma^\nu$$
$$\times \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \frac{\partial}{\partial q_4^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_4 - q_1 - q_2) \Big|_{q_4=0}$$

# Master formula for $a_\mu^{\text{HLbL}}$

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★ Since there are **no kinematic singularities** in the BTT scalar functions, the limit  $q_4 \rightarrow 0$  can be taken explicitly

$$a_\mu^{\text{HLbL}} = -\frac{e^6}{48m_\mu} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \frac{1}{(p + q_1)^2 - m_\mu^2} \frac{1}{(p - q_2)^2 - m_\mu^2} \\ \times \text{Tr} \left( (\not{p} + m_\mu) [\gamma^\rho, \gamma^\sigma] (\not{p} + m_\mu) \gamma^\mu (\not{p} + \not{q}_1 + m_\mu) \gamma^\lambda (\not{p} - \not{q}_2 + m_\mu) \gamma^\nu \right) \\ \times \sum_{i=1}^{54} \left( \frac{\partial}{\partial q_4^\rho} T_{\mu\nu\lambda\sigma}^i(q_1, q_2, q_4 - q_1 - q_2) \right) \Big|_{q_4=0} \Pi_i(q_1, q_2, -q_1 - q_2)$$



# Master formula for $a_\mu^{\text{HLbL}}$

- ★ We obtained a **general** master formula

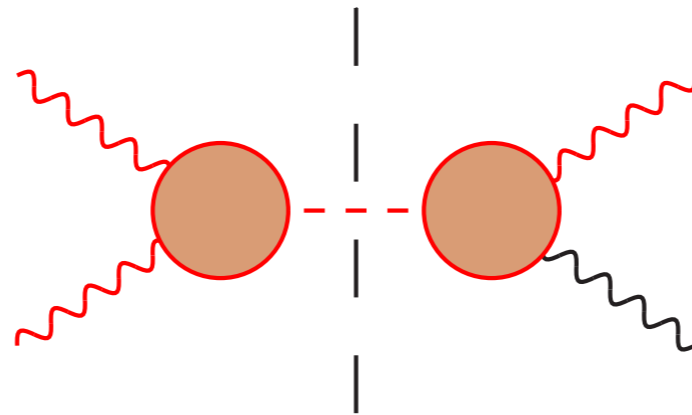
$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

- ★  $Q_i^2 = -q_i^2$  are Euclidean momenta and  $Q_1 \cdot Q_2 = Q_1 Q_2 \tau$ : **space-like kinematics**
- ★ Generalization of the formula for the pion pole by **Knecht and Nyffeler (2002)**
- ★ We determined the integration kernels  $T_i$ .  
The scalar functions  $\bar{\Pi}_i$  are linear combinations of the BTT  $\Pi_i$
- ★ **Our goal**: dispersive representation of HLbL scalar functions at fixed photon virtualities to be evaluated at the reduced kinematics in the master formula,

$$s = -Q_3^2 = -Q_1^2 - 2Q_1 Q_2 \tau - Q_2^2, \quad t = -Q_2^2, \quad u = -Q_1^2,$$
$$q_1^2 = -Q_1^2, \quad q_2^2 = -Q_2^2, \quad q_3^2 = -Q_3^2 = -Q_1^2 - 2Q_1 Q_2 \tau - Q_2^2, \quad q_4^2 = 0$$

# The pion-pole contribution

- From the unitarity relation with only  $\pi^0$  intermediate state, the pole residues in each channel are given by products of **doubly-virtual and singly-virtual pion transition form factors** ( $\mathcal{F}_{\gamma^*\gamma^*\pi^0}$  and  $\mathcal{F}_{\gamma^*\gamma\pi^0}$ , input for our analysis)



$$a_{\mu}^{\pi^0\text{-pole}} = \frac{2\alpha^3}{3\pi^2} \int_0^{\infty} dQ_1 \int_0^{\infty} dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \left( T_1(Q_1, Q_2, \tau) \bar{\Pi}_1^{\pi^0\text{-pole}}(Q_1, Q_2, \tau) + T_2(Q_1, Q_2, \tau) \bar{\Pi}_2^{\pi^0\text{-pole}}(Q_1, Q_2, \tau) \right)$$

with

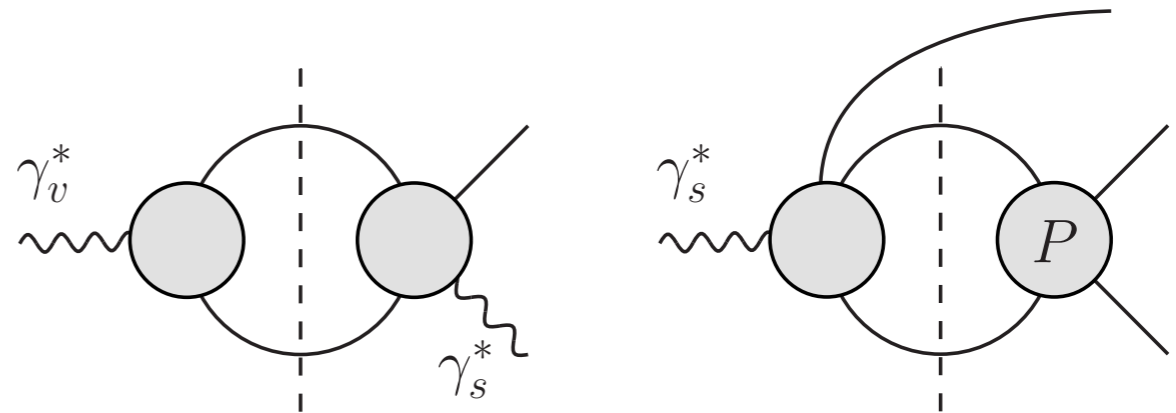
$$\bar{\Pi}_1^{\pi^0\text{-pole}} = -\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_3^2, 0)}{Q_3^2 + M_{\pi}^2} \quad \bar{\Pi}_2^{\pi^0\text{-pole}} = -\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_3^2) \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0)}{Q_2^2 + M_{\pi}^2}$$

# The pion-pole contribution

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★ These form factors can be reconstructed dispersively using

- ▶ pion vector form factor
- ▶  $\gamma^* \rightarrow 3\pi$  amplitude
- ▶ elastic  $\pi\pi$  scattering amplitude



Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)

→  $a_{\mu}^{\pi^0\text{-pole}} = 62.6_{-2.5}^{+3.0} \times 10^{-11}$

Hoferichter, Hoid, Kubis, Leupold, Schneider (2018)

★ Pseudoscalar poles with higher masses can be treated analogously

# Pion-box contribution

- ★ Defined by simultaneous **two-pion cuts in two channels**
- ★ Contribution to scalar functions as dispersive integral of double spectral functions

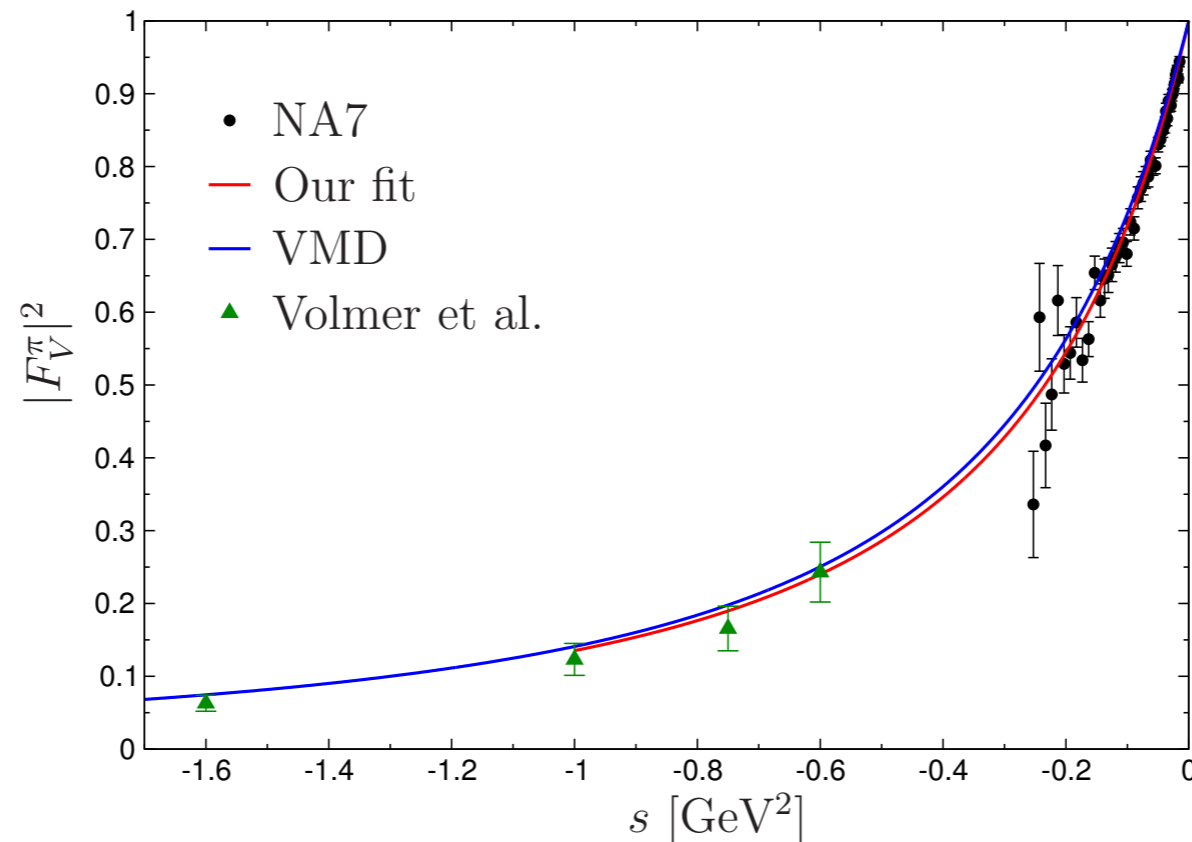
$$\Pi_i = \frac{1}{\pi^2} \int ds' dt' \frac{\rho_i^{st}(s', t')}{(s' - s)(t' - t)} + (t \leftrightarrow u) + (s \leftrightarrow u)$$

- ★ Dependence on  $q_i^2$  carried by the **pion vector FFs for each off-shell photon**
- ★ one-loop sQED projected onto the BTT structures fulfills the same Mandelstam representation of the pion box, the only difference being the **pion vector FFs** :

$$\equiv F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2) \times \left[ \text{Box} + \text{Crossed Box} + \text{Self-Energy} \right]$$

# Numerics for the pion-box contribution

- ★ The only input: pion vector form factor in the **space-like** region

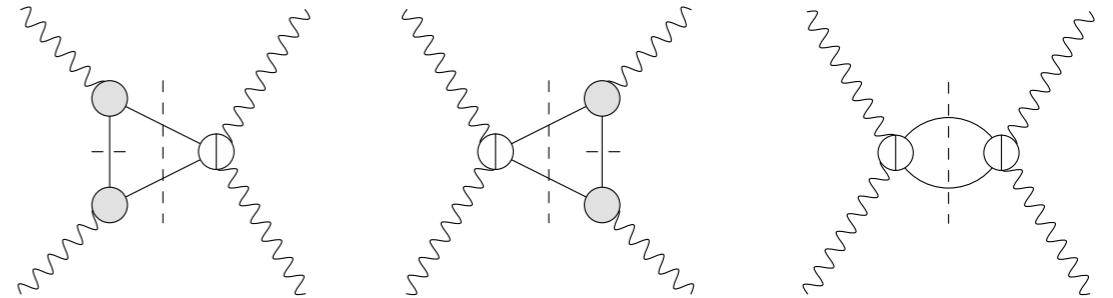


- ★ Numerical results:  $a_\mu^{\pi\text{-box}} = -15.9(2) \times 10^{-11}$  vs  $a_\mu^{K\text{-box}, \text{VMD}} \simeq -0.5 \times 10^{-11}$

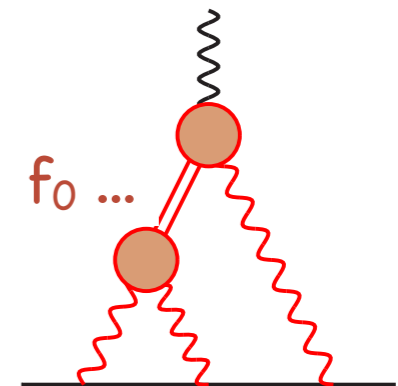
- ★ Rapid convergence:  $Q_{\text{max}} = \{1, 1.5\}$  GeV  $\Rightarrow a_\mu^{\pi\text{-box}} = \{95, 99\}$ % of full result

# The remaining $\pi\pi$ contribution

- Two-pion cut only in the direct channel:  
LH cut due to multi-particle intermediate states in the crossed channel neglected



- Unitarity relates this contribution to the subprocess  $\gamma^*\gamma^{(*)} \rightarrow \pi\pi$
- By generalizing previous analyses of  $\gamma\gamma \rightarrow \pi\pi$  and  $\gamma\gamma^* \rightarrow \pi\pi$  [Moussallam et al. \(2010, 2013\)](#) our goal is a **dispersive reconstruction** (based on analyticity, unitarity and crossing) of **helicity partial waves** for  $\gamma^*\gamma^* \rightarrow \pi\pi$  [Colangelo, Hoferichter, MP, Stoffer \(2014\)](#)
- The solution of the resulting coupled set of dispersion relations involves elastic  $\pi\pi$  phase shifts, which allows for the summation of  $\pi\pi$  rescattering effects in the direct channel (**effects of resonances coupling to  $\pi\pi$** )

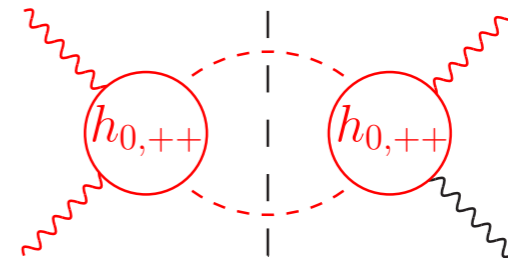




# The remaining $\pi\pi$ contribution

- ★ Contribution to  $a_\mu^{\text{HLbL}}$  from  $\gamma^*\gamma^* \rightarrow \pi\pi$  helicity partial waves :

$$\text{Im } h_{++,++}^J(s; q_1^2, q_2^2; q_3^2, 0) = \frac{\sigma(s)}{16\pi} h_{J,++}^*(s; q_1^2, q_2^2) h_{J,++}(s; q_3^2, 0)$$



projecting onto BTT basis determines  $\text{Im } \Pi_i$ , from which  $\Pi_i$  for master formula.  
Our framework holds for arbitrary partial waves.

- ★ We solved dispersion relations for  $\gamma^*\gamma^* \rightarrow \pi\pi$  S-waves taking:

- ▶ pion pole as only LH singularity and phenomenological  $\pi\pi$  phase shifts

$a_\mu^{\text{HLbL}}$  in  $10^{-11}$  units

	$\Lambda$	1 GeV	1.5 GeV	2 GeV	$\infty$
$f_0(500) \longrightarrow$	$l=0$	-9.2	-9.5	-9.3	-8.8
	$l=2$	2.0	1.3	1.1	0.9

$$a_{\mu, J=0}^{\pi\pi, \pi\text{-pole LHC}} = -8(1) \times 10^{-11}$$