The tunneling wavefunction in Kantowski-Sachs quantum cosmology

Georgios Fanaras and Alexander Vilenkin

Institute of Cosmology, Department of Physics and Astronomy,

Tufts University, Medford, MA 02155, USA

Abstract

We use a path-integral approach to study the tunneling wave function in quantum cosmology with spatial topology $S^1 \times S^2$ and positive cosmological constant (the Kantowski-Sachs model). If the initial scale factors of both S^1 and S^2 are set equal to zero, the wave function describes (semiclassically) a universe originating at a singularity. This may be interpreted as indicating that an $S^1 \times S^2$ universe cannot nucleate out of nothing in a non-singular way. Here we explore an alternative suggestion by Halliwell and Louko that creation from nothing corresponds in this model to setting the initial volume to zero. We find that the only acceptable version of this proposal is to fix the radius of S^1 to zero, supplementing this with the condition of smooth closure (absence of a conical singularity). The resulting wave function predicts an inflating universe of high anisotropy, which however becomes locally isotropic at late times. Unlike the de Sitter model, the total nucleation probability is not exponentially suppressed, unless a Gauss-Bonnet term is added to the action.

Contents

1. INTRODUCTION

In quantum cosmology the entire universe is treated quantum mechanically and is described by a wave function, rather than by a classical spacetime. The wave function $\Psi(g, \phi)$

is defined on the space of all 3-geometries (g) and matter field configurations (ϕ), called superspace. It can be found by solving the Wheeler-DeWitt (WDW) equation [1]

$$\mathcal{H}\Psi = 0, \tag{1.1}$$

where $\mathcal{H}$ is the Hamiltonian operator. Alternatively, one can consider the transition amplitude from the initial state (g', ϕ') to the final state (g, ϕ), which can be expressed as a path integral,

$$G(g, \phi | g', \phi') = \int_{(g', \phi')}^{(g, \phi)} \mathcal{D}g \, \mathcal{D}\phi \, e^{iS}, \tag{1.2}$$

where S is the action. In general, G is a Green's function of the WDW equation [2]. But if (g', ϕ') is at the boundary of superspace, then $G(g, \phi | g', \phi')$ is a solution of the WDW equation everywhere in the bulk of superspace, and the path integral (1.2) may be used to define a wave function of the universe.

The choice of the boundary conditions for the WDW equation and of the class of paths included in the path integral representation of Ψ has been a subject of ongoing debate. The most developed proposals in this regard are the Hartle-Hawking (HH) [3] and the tunneling [4–6] wave functions.[1] The intuition behind both of these proposals is that the universe originates 'out of nothing' in a nonsingular way. But despite a large amount of work, a consensus on the precise definition of these wave functions has not yet been reached.

The two proposals have been thoroughly studied in the simple minisuperspace de Sitter model with S^3 spatial topology, where the only degree of freedom is the radius of the universe. A number of more complicated models with two or more degrees of freedom have also been considered. Among them is the Kantowski-Sachs (KS) model [13] which describes an anisotropic universe of spatial topology $S^1 \times S^2$ with different scale factors. We have recently presented a detailed analysis of the HH wave function in the KS model [14], and our goal in the present paper is to extend this analysis to the tunneling wave function.

We shall conclude this Introduction with some comments about prior work on this topic.[2] Conti and Hertog [19] considered the semiclassical wave function by studying complex Schwarzschild-de Sitter instantons of the model, imposing boundary conditions suitable for a smooth closure of the 4-geometry. An advantage of this approach is that it is straightforward

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[1] For early work closely related to the tunneling proposal, see Refs. [7–11].
[2] For earlier work on KS quantum cosmology see [15–18].

to obtain approximate expressions for the saddles by algebraic means. On the other hand, it does not allow one to rigorously define a convergent path integral and select which of these saddles contribute to the wave function. Specifically, the divergence of the tunneling wave function found by CH can be attributed to the inclusion of saddle geometries that should not contribute.

Halliwell and Louko [20] used methods similar to Picard-Lefschetz theory in order to define a steepest descent contour that renders the path integral convergent. With this approach it is usually the case that not all extremum geometries contribute to the integral, since the lapse integration contour in the complex plane does not pass through all the saddles. In this paper we will follow this procedure, but we will have to extend the analysis to a non-vanishing cosmological constant, which makes the problem significantly more complicated.

This paper is organized as follows. In Section 2 we review the classical dynamics of the KS model and its canonical quantization. The definition of the tunneling wave function is discussed in Section 3, where we outline the approach based on the outgoing wave condition in superspace, as well as the path integral definition. In this paper we adopt the path integral approach and study alternative choices of boundary conditions for the path integral in Sections 4 and 5. In Sec.4 we fix the scale factors of S^1 and S^2 on both initial and final spacetime boundaries. The path integral in this case can be calculated exactly [20]. We find however that the resulting wave function is singular and thus is not an acceptable solution of the WDW equation.

Section 5 is the main part of this work. Here we investigate the choice of boundary conditions suggested by Halliwell and Louko [20]: we fix the two scale factors on the final boundary and require a smooth closure of the 4-geometry at the initial boundary. In this case the path integral cannot be computed exactly, so we employ the methods of Picard-Lefschetz theory in order to make the integral over the lapse N absolutely convergent. This is done by finding saddle points of the action in the complex plane and deforming the initial integration contour to a steepest descent path through the contributing saddles. The wave function is then found in the WKB approximation by carrying out the Gaussian integrals in the vicinity of the saddles. Since our analysis is approximate, we singled out three regimes of interest. Denoting the scale factors of S^1 and S^2 by a and b respectively and the cosmological constant by $\Lambda = H^2/3 \ll 1$ (in Planck units), we first looked into the case $b \approx 1/H$. This is the value at which the HH wave function gives a maximum probability [14]. We found

that the tunneling wave function exhibits an opposite behavior: the probability grows away from this value. We continued by probing the region of large S^2 and found that the wave function is peaked at high anisotropy: $a/b \lesssim H \ll 1$. We note that in this regime CH found a divergence in the tunneling solution which we do not observe. Finally, we obtained the wave function in the limit of $a \ll 1/H$. This wave function does not exhibit any exponential suppression as a function of b. Thus, the familiar picture of tunneling through a potential barrier does not hold in the KS model.

Our results are summarized and discussed in Section 6.

2. KANTOWSKI-SACHS MODEL

2.1. Classical dynamics

The Kantowski-Sachs (KS) model describing a homogeneous universe with spatial sections of $S^1 \times S^2$ topology is represented by a Lorentzian metric as

$$ds^2 = -N^2 dt^2 + a^2(t)dx^2 + b^2(t)d\Omega^2. \tag{2.1}$$

The Einstein-Hilbert action is then

$$S = -\pi \int dt \left[\frac{1}{N} \left(a\dot{b}^2 + 2b\dot{a}\dot{b} \right) + Na(H^2 b^2 - 1) \right] + S_b. \tag{2.2}$$

where the integration is carried out from the initial boundary $\mathcal{B}_0$ to the final boundary $\mathcal{B}$. The inclusion of the boundary term is needed if one chooses to impose Neumann conditions on one of the scale factors at $\mathcal{B}_0$ and is given by

$$S_b = - \left[\frac{\pi}{N} \frac{d}{dt}(ab^2) \right]_{\mathcal{B}_0}. \tag{2.3}$$

Varying the action with respect to the scale factors a, b we obtain the classical equations of motion in the gauge $N = 1$:

$$\ddot{a}b + a\ddot{b} + \dot{a}\dot{b} - H^2 ab = 0, \tag{2.4}$$

$$2b\ddot{b} + \dot{b}^2 + 1 - H^2 b^2 = 0, \tag{2.5}$$

and varying with respect to the lapse N we obtain the Hamiltonian constraint:

$$\dot{b}^2 + 2\frac{\dot{a}}{a}\dot{b}b + 1 - H^2 b^2 = 0. \tag{2.6}$$

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Combining equations (2.5) and (2.6) we obtain

$$\dot{b} = \kappa a, \tag{2.7}$$

where κ is an integration constant. Plugging back into equation (2.5) and integrating we obtain

$$\kappa a = \dot{b} = \pm \sqrt{\frac{H^2 b^2}{3} - 1 + \frac{2M}{b}}, \tag{2.8}$$

where again M is an integration constant.

The above solution corresponds to a Euclidean Schwarzschild-deSitter black hole of mass M. In Euclidean time $t_E = it$ and in the gauge $N = 1$ the metric (2.1) becomes

$$ds_E^2 = \left(1 - \frac{2M}{b} - \frac{H^2 b^2}{3}\right) d\lambda^2 + \frac{db^2}{\left(1 - \frac{2M}{b} - \frac{H^2 b^2}{3}\right)} + b^2 d\Omega_2^2, \tag{2.9}$$

where $\lambda = \kappa x$. The black hole mass can be expressed in terms of the boundary data. Setting $a(t_E = 0) = a'$ and $b(t_E = 0) = b'$ we have

$$M = \frac{b'}{2} \left(1 - \frac{H^2 b'^2}{3} + \kappa^2 a'^2\right). \tag{2.10}$$

The limiting case in which $\kappa = 0$ corresponds to the Nariai solution [23] in which:

$$b = \frac{1}{H} \quad , \quad \ddot{a} = H^2 a \quad , \quad M = \frac{1}{3H} . \tag{2.11}$$

It describes a 4-geometry $dS_2 \times S^2$, where the characteristic radius of both dS_2 and S^2 is H^{-1}.

2.2. The WdW equation

The WdW equation of the KS model can be more conveniently realized by switching to time variable $d\tau = a(t)dt$. In this representation the metric is

$$ds^2 = -\frac{N^2}{a^2} d\tau^2 + a^2 dx^2 + b^2 d\Omega^2, \tag{2.12}$$

where N, a and b are functions of time τ, which we can choose to vary in the range $0 < \tau < 1$. After substituting this in the Lorentzian Einstein-Hilbert action and integrating over x and over the angular variables, the action reduces to

$$S = -\pi \int_0^1 d\tau \left[\frac{b\dot{c}}{N} + N(H^2 b^2 - 1)\right], \tag{2.13}$$

where we have introduced a new variable $c = a^2 b$.

The constraint equation is obtained by varying the action with respect to N:

$$\frac{b\dot{c}}{N} - N(H^2 b^2 - 1) = 0, \tag{2.14}$$

where overdots stand for derivatives with respect to τ.

The momenta conjugate to the variables b and c are

$$p_b = -\pi \dot{c}/N, \quad p_c = -\pi \dot{b}/N. \tag{2.15}$$

Using this in the constraint equation (2.14) and replacing $p_b \to -i\partial/\partial b$, $p_c \to -i\partial/\partial c$, we obtain the WDW equation

$$\pi \mathcal{H} \Psi = \left[\partial_b \partial_c + \pi^2 (H^2 b^2 - 1) \right] \Psi = 0. \tag{2.16}$$

This can be rewritten in the form of a Klein-Gordon (KG) equation,

$$\frac{1}{\sqrt{-f}} \partial_\alpha \left(\sqrt{-f} f^{\alpha\beta} \partial_\beta \Psi \right) + V \Psi = 0, \tag{2.17}$$

where the potential is $V = \pi^2 (H^2 b^2 - 1)$, $f_{\alpha\beta}$ is the minisuperspace metric, and $f = \det(f_{\alpha\beta})$. With b and $c = a^2 b$ used as coordinates, the minisuperspace metric is given by $f_{bb} = f_{cc} = 0$, $f_{bc} = f_{cb} = 2$, and $\sqrt{-f} = 2$. The contravariant components of the metric are $f^{bb} = f^{cc} = 0$ and $f^{bc} = f^{cb} = 1/2$. The conserved current density corresponding to this KG equation is

$$j^\alpha = i\sqrt{-f} f^{\alpha\beta} (\Psi^* \partial_\beta \Psi - \Psi \partial_\beta \Psi^*). \tag{2.18}$$

It satisfies

$$\partial_\alpha j^\alpha = 0, \tag{2.19}$$

or

$$\frac{\partial j^b}{\partial b} + \frac{\partial j^c}{\partial c} = 0. \tag{2.20}$$

This suggests that

$$dP_b = j^c db \tag{2.21}$$

can be interpreted as the probability distribution for b at a fixed value of c; then c plays the role of a "clock" variable. Similarly,

$$dP_c = j^b dc \tag{2.22}$$

can be interpreted as the probability distribution for c at a fixed value of b, with b being the clock variable [24].

3. TUNNELING WAVE FUNCTION

The tunneling wave function Ψ_T was originally defined by imposing a boundary condition in superspace [4]. The tunneling boundary condition states that Ψ_T includes only outgoing modes, with the probability flux directed toward the boundary, at singular boundaries of superspace.[3] The division of the superspace boundary into regular and singular parts has not been specified in the general case; here is a somewhat heuristic prescription. The boundary of superspace can be thought of as consisting of singular configurations which have some regions of infinite 3-curvature or infinite matter density, as well as configurations of infinite 3-volume and infinite values of matter fields. The regular part of the boundary includes singular 3-geometries which can be obtained by slicing regular Euclidean 4-geometries. For example, if a 4-sphere embedded in a $5D$ Euclidean space is sliced by parallel planes, one gets 3-spheres of vanishing radius and infinite curvature at the two poles, even though the 4-geometry is perfectly regular there. The outgoing wave boundary condition is sometimes supplemented by requiring that the wave function is normalizable or that its modulus is bounded from above.

In the simplest minisuperspace model, describing a de Sitter universe, the wave function depends only on the scale factor a and the superspace is a half-line, $0 \leq a < \infty$. The regular boundary in this case is at $a = 0$ and the singular boundary is at $a = \infty$. The general picture is that the probability flux is injected into superspace through the regular boundary and flows out through the singular boundary.

The origin of the universe in the de Sitter model can be pictured semiclassically as illustrated in Fig.1. The purple hyperboloid at the top is the classical de Sitter space and the blue hemisphere at the bottom is its Euclidean continuation. Such a continuation is necessary because Lorentzian geometries cannot close off at the bottom without a singularity. For this reason the regular boundary of superspace is specified in terms of Euclidean geometries.

The classical de Sitter model describes a universe contracting from infinite size, bouncing at the turnaround radius $a = 1/H_{dS}$, and re-expanding. (Here H_{dS} is the de Sitter expansion rate.) The tunneling wave function represents a universe that transits from $a = 0$ through the classically forbidden range $0 < a < 1/H_{dS}$ and expands from there. It is formally similar

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[3] The definition of outgoing modes in quantum cosmology is discussed in Ref.[6].
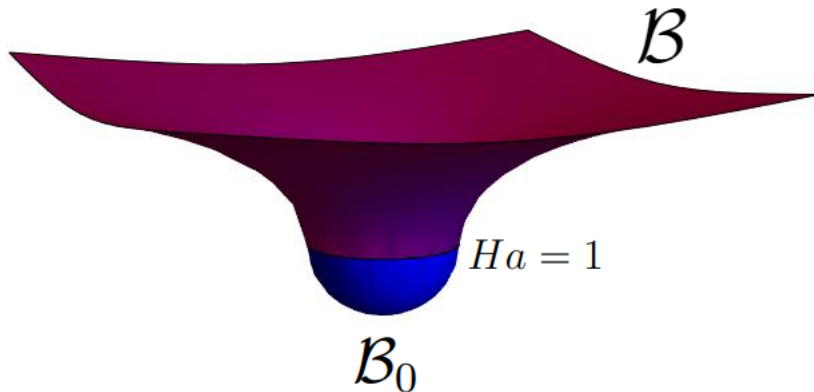
FIG. 1: Nucleation of a de Sitter universe. Half of de Sitter space is matched to Euclidean hemisphere at the bounce radius $a = 1/H_{dS}$.

to a wave function describing quantum tunneling through a potential barrier.

It should be noted that the analogy between quantum cosmology and quantum tunneling often breaks down in models with more than one degree of freedom. Consider for example a model with coordinates q^α and conjugate momenta p_α, satisfying the Hamiltonian constraint $\mathcal{H} = K(q,p) + V(q) = 0$, where the kinetic energy K is a non-negative-definite quadratic form in momenta. Then classically we have $K \geq 0$, so the range $V(q) > 0$ is classically forbidden. On the other hand, in minisuperspace cosmology the quadratic form K is not generally positive-definite and not even bounded from below – not even at the classical level. Hence, in multi-dimensional models superspace cannot generally be divided into classically allowed and classically forbidden regions.

An alternative definition of the tunneling wave function has been given in terms of a path integral [6]. It states that $\Psi_T(g)$ is given by a path integral over Lorentzian histories interpolating between a 3-geometry of vanishing size ("nothing") and a given configuration g, with the lapse integration taken over the positive range $0 < N < \infty$. As discussed in [2], a lapse integral over a half-infinite range gives a Green's function of the WDW equation. But in this case the source term has support only at geometries of vanishing size, which are at the boundary of superspace. Hence one can expect that Ψ_T is a solution of the WDW equation everywhere in the bulk of superspace. The positive lapse condition can be thought of as a causality requirement [2]: the histories included in the path integral are to the future of the origin event of "nothing". In the present paper we shall adopt the path integral approach

to the tunneling wave function.

In the case of de Sitter model the path integral is taken over the scale factor $a(t)$ with the boundary condition $a(0) = 0$. We note that this boundary condition, combined with the classical constraint equation $\dot{a}^2 + 1 = H_{dS}^2 a^2$ implies $\dot{a}(0) = \pm i$. After Euclidean continuation $t = i\tau$, this gives $da/d\tau = \pm 1$, which is the condition of a smooth closure (absence of a conical singularity) in the Euclidean geometry at $\tau = 0$. Thus the boundary condition $a = 0$ enforces the regularity condition at a semiclassical level. Alternatively, one could impose the regularity requirement $da/d\tau(0) = \pm 1$ as a boundary condition. Then the constraint equation would imply semiclassical closure, $a(0) = 0$. The Lorentzian path integral for the de Sitter model with both of these boundary conditions has been calculated in Refs.[12, 21]. In both cases the result coincides with the wave function obtained from the outgoing-wave boundary condition.

Turning now to the KS model, we first need to specify the class of histories included in the path integral. By analogy with the de Sitter model, one might consider histories originating from a configuration of vanishing 3-geometry, $a = b = 0$. However, it has been shown in Ref.[20] that Euclidean 4-geometries admitting $S^1 \times S^2$ slicing with radii a and b necessarily have a divergent 4-curvature in the limit $a, b \to 0$ and are therefore singular even at the semiclassical level.

The conclusion could be that a universe of topology $S^1 \times S^2$ cannot be created from nothing. In this paper we shall explore an alternative idea, suggested by Halliwell and Louko in Ref.[20]. We shall relax the condition $a = b = 0$ and require that only one of the two scale factors, a or b, is equal to zero. One possibility is then to fix the other scale factor at a nonzero value that is consistent with a non-singular geometry. Alternatively, we can leave the other scale factor unspecified and impose a regularity condition excluding conical singularities, as we mentioned for the de Sitter model. We shall discuss both of these approaches here.

4. FIXING INITIAL SCALE FACTORS

The transition amplitude from the initial state $\{a', b'\}$ to the final state $\{a, b\}$ in the KS model has been calculated by Halliwell and Louko in Ref.[20]. It is given by

$$G(a, b|a', b') = \int_0^\infty \frac{dN}{N} \exp\left[i\pi\left(\alpha N - \frac{\beta}{N}\right)\right], \tag{4.1}$$

where

$$\alpha = 1 - \frac{H^2}{3}(b^2 + bb' + b'^2), \quad \beta = (a^2 b - a'^2 b')(b - b'). \tag{4.2}$$

The contour of N-integration is generally complex; here we choose it to lie along the positive real axis, as required for the tunneling wave function.

The integral over N in (4.1) can be expressed in terms of Bessel functions [20]. The resulting wave function is

$$\Psi_T = -i\pi H_0^{(2)}(2\pi(-X)^{1/2}) \tag{4.3}$$

for $X < 0$ and

$$\Psi_T = 2K_0(2\pi(X)^{1/2}) \tag{4.4}$$

for $X > 0$, where $X = \alpha\beta$. If we set $a' = 0$, then

$$-X = a^2 b^2 \left(\frac{H^2 b^2}{3} - 1 + \frac{2M}{b}\right) \tag{4.5}$$

with M from Eq.(2.10), and if we set $b' = 0$, then $M = 0$ and X is independent of a',

$$-X = a^2 b^2 \left(\frac{H^2 b^2}{3} - 1\right). \tag{4.6}$$

We note that the same value of $M = 0$ is obtained for $a' = 0$, $b' = H^{-1}\sqrt{3}$; hence this choice of parameters gives the same wave function as $b' = 0$ with arbitrary a'.

Let us now consider the wave function with $M = 0$. For $Hb > \sqrt{3}$ it is given by

$$\Psi_T(Hb > \sqrt{3}) = -i\pi H_0^{(2)}\left(2\pi ab\sqrt{\frac{H^2 b^2}{3} - 1}\right). \tag{4.7}$$

To verify that this wave function satisfies the outgoing flux criterion, we first note its asymptotic form for large argument:

$$\Psi_T \propto \exp\left[-2i\pi ab\sqrt{\frac{H^2 b^2}{3} - 1}\right], \tag{4.8}$$

where we have ignored the pre-exponential factor. This gives a good approximation for the exponent, as long as b is not very close to $H^{-1}\sqrt{3}$. Acting with the momentum operators and using the gauge $N = 1$ we obtain:

$$\Pi_a \Psi_T = -i\frac{\partial \Psi_T}{\partial a} \longrightarrow \dot{b} = +\sqrt{\frac{H^2 b^2}{3} - 1} > 0 \tag{4.9}$$

$$\Pi_b \Psi_T = -i\frac{\partial \Psi_T}{\partial b} \longrightarrow \frac{\dot{a}}{a} = \frac{bH^2}{3\sqrt{\frac{H^2 b^2}{3} - 1}} > 0. \tag{4.10}$$

The solution of these equations for b and a is

$$b(t) = \frac{\sqrt{3}}{H}\cosh\left(\frac{Ht}{\sqrt{3}}\right), \quad a(t) = D\sinh\left(\frac{Ht}{\sqrt{3}}\right), \tag{4.11}$$

where $D > 0$ is a constant parameter.

In the semiclassical approximation, the wave function (4.7) describes a congruence of expanding classical trajectories (4.11) with different values of D. The trajectories start at $a' = 0$, $b' = H^{-1}\sqrt{3}$ and extend to $a, b \to \infty$. The trajectory with $D = H^{-1}\sqrt{3}$ describes a de Sitter space with expansion rate $H/\sqrt{3}$, while for other values of D the geometries (4.11) have conical singularities at $t = 0$. For large values of Ha and Hb all these geometries approach an expanding de Sitter space.

The wave function for $X > 0$ can be similarly analyzed. The resulting congruence of trajectories is a Euclidean continuation of (4.11):

$$b(\tau) = \frac{\sqrt{3}}{H}\cos\left(\frac{H\tau}{\sqrt{3}}\right), \quad a(\tau) = iD\sin\left(\frac{H\tau}{\sqrt{3}}\right), \tag{4.12}$$

where τ is the Euclidean time. It describes trajectories staring at $a' = 0$, $b' = H^{-1}\sqrt{3}$ and ending at $a = D$, $b = 0$. Once again, the trajectories with $D \neq H^{-1}\sqrt{3}$ have conical singularities at $\tau = 0$. We note that even though the wave function (4.7) is obtained for several choices of the initial values a', b', the congruence of trajectories that it describes corresponds to only one of these choices: $a' = 0$, $b' = H^{-1}\sqrt{3}$.

We can use the conserved current (2.18) to find the probability distribution for $c = a^2 b$ at a fixed value of b, with b playing the role of a clock. Using the Wronskian products of the Hankel functions, we find that the current is given by

$$j^\alpha = 4\pi\sqrt{-f}f^{\alpha\beta}\frac{\partial \ln(X)}{\partial y^\beta}, \tag{4.13}$$

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where y^β are the superspace coordinates b, c and $X = 2\pi\sqrt{cb\left(\frac{H^2b^2}{3} - 1\right)}$. From Eq.(2.22) we obtain the probability distribution

$$dP_c \propto j^b dc \propto \frac{dc}{c}. \tag{4.14}$$

Since b is fixed, the distribution for a is $dP_a \propto da/a$. This distribution is not normalizable, but it admits a simple interpretation: values of a in each logarithmic interval are equally probable (at any given value of b).

The problem with the wave function (4.7) is that it exhibits a logarithmic singularity at $Hb = \sqrt{3}$. Hence it is not a solution of the WDW equation in the entire superspace, $0 \le a, b < \infty$. Furthermore, the semiclassical geodesic congruence described by this wave function has conical singularities and thus does not correspond to a non-singular origin of the universe. The reason is that the boundary conditions that we used for initial scale factors do not enforce regular geometry even at the semiclassical level. Similar features are obtained for wave functions specified by a vanishing initial scale factor $a' = 0$ with an arbitrary value of b'. We therefore conclude that this class of wave functions is not a suitable choice for the tunneling wave function of the universe.

5. SMOOTH CLOSURE

We now consider the boundary condition of a smooth closure of Euclidean geometry with one of the scale factors vanishing. In the rest of the paper it will be more convenient to switch to Euclidean signature, by replacing $N \to -iN$.

It has been shown in [20] that in a classical Euclidean KS geometry a smooth closure can be achieved by one of the following two sets of boundary conditions:

$$a' = 0, \quad \frac{1}{N}\dot{a}' = \pm 1, \quad \frac{1}{N}\dot{b}' = 0, \tag{5.1}$$

$$b' = 0, \quad \frac{1}{N}\dot{b}' = \pm 1, \quad \frac{1}{N}\dot{a}' = 0, \tag{5.2}$$

where a prime indicates evaluation at the initial boundary and the Euclidean lapse parameter N is now real. The former set corresponds to a smooth closing of S^1, while the latter to a smooth closing of S^2.

Neither set of boundary conditions can be implemented in quantum theory. This becomes apparent if we introduce new variables

$$A = b^2, \quad B = ab \tag{5.3}$$

with conjugate momenta

$$P_A = -\frac{\dot{a}}{2N}, \quad P_B = -\frac{\dot{b}}{N}. \tag{5.4}$$

In this representation, the regularity conditions (5.1), (5.2) take the form

$$B' = 0, \quad P_A{}' = \mp\frac{1}{2}, \quad P_B' = 0 \tag{5.5}$$

and

$$A' = 0, \quad P_A{}' = 0, \quad P_B' = \mp 1. \tag{5.6}$$

Quantum mechanically, however, one is not allowed to impose these regularity conditions in their entirety, since that would violate the uncertainty principle: we cannot fix both a superspace variable and its conjugate momentum at the boundary. The best we can do is to enforce two of the three conditions. The third can then be inferred from the classical equations of motion, indicating that the semiclassical wave function would approximately describe a regular geometry. As we shall see in the next subsection, setting $b' = 0$ does not allow one to specify the momenta, since they appear in the action factored to b'. In fact, with $b' = 0$ one necessarily gets the same wave function (4.7) as we discussed in Sec 4, which we have concluded should be disqualified. We therefore focus on the boundary conditions

$$B' = 0, \quad P_A{}' = \mp\frac{1}{2}. \tag{5.7}$$

5.1. General formalism

The formalism for calculating the propagator with specified values of a and b at the future boundary $\mathcal{B}$ and boundary conditions (5.7) at the initial boundary $\mathcal{B}_0$ has been developed in [14, 20]. Here we will outline the approach for constructing the propagator; the details can be found in these references.

The fist step is to compute the action by integrating Eq.(2.2) while also evaluating the boundary term (2.3). After substituting $a' = 0$, the result is

$$S_E = \pi \left[\frac{H^2}{3} N(b^2 + bb' + b'^2) - N - \frac{a^2 b}{N}(b - b') + 2b'^2 P_A{}' \right]. \tag{5.8}$$

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This is the action for $a(t)$ and $b(t)$ satisfying the classical equations of motion and boundary conditions, but not the constraint. The quantity b' in Eq.(5.8) has to be expressed in terms of the boundary data $\{a, b\}$, $\{P_A'\}$ and N using the equations of motion. This gives

$$b' = -\frac{b}{2N} \frac{a^2 + H^2 N^2/3}{2P_A' + H^2 N/3}.$$

(5.9)

Substituting this expression in the action (5.8) and simplifying, we have

$$S_E = \frac{\pi N}{3}(H^2 b^2 - 3) - \frac{\pi}{N}a^2 b^2 - \frac{\pi b^2}{4N^2} \frac{\left(a^2 + \frac{H^2 N^2}{3}\right)^2}{2P_A' + \frac{H^2 N}{3}}$$

(5.10)

The transition amplitude from the initial state $Z^\beta = \{B', P_A'\}$ to the final state $Q^\alpha = \{A, B\}$ can now be expressed as

$$\Psi_T(Q^\alpha) = \int_C \mu(Q^\alpha, Z^\beta, N) \exp\left[-S_E\left(Q^\alpha; N | Z^\beta\right)\right] dN$$

(5.11)

where μ is the semiclassical prefactor and C is a Lorentzian integration contour along the positive imaginary axis.

At this point the transition amplitude is not yet fully defined since we have not specified which of the values of $P_A' = \pm 1/2$ should be used. We make this choice by requiring that the integral (5.11) is convergent. Representing $N = iy$ with $0 < y < \infty$, let us examine the behavior of the integrand at $y \to 0$. In this limit the action (5.10) becomes

$$S_E \approx \left(\frac{\pi b^2 a^4}{8P_A'}\right)\frac{1}{y^2}$$

(5.12)

In order for the integral of $\exp(-S_E)$ to converge near the origin, the appropriate choice is $P_A' = +1/2$. This is opposite to the choice $P_A' = -1/2$ made in Refs.[14] for the Hartle-Hawking wave function.

In the limit $y \to +\infty$ the action becomes

$$S_E \approx i\pi(H^2 b^2 - 4)\frac{y}{4}.$$

(5.13)

Thus the integral of $\exp(-S_E)$ will be convergent in the following cases:

$$\arg(y) \in \left(-\frac{\pi}{2}, 0\right) \quad , \quad Hb > 2$$

(5.14)

or

$$\arg(y) \in \left(0, \frac{\pi}{2}\right) \quad , \quad Hb < 2.$$

(5.15)

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So, depending on the sign of $(Hb - 2)$, the integration contour must be given an appropriate tilt in order to converge.

Unlike in the case of transition between fixed scale factors, the amplitude (5.11) cannot be evaluated exactly. We shall therefore compute the integral following the methods of Picard-Lefschetz, namely distorting the integration contour C to a steepest descent/ascent path going through (at least) one of the saddle points of the action.

In order to simplify the analysis we will rescale our variables:

$$u = H^2 a^2 \ , \quad v = H^2 b^2 \ , \quad \tilde{N} = H^2 N \ , \quad \tilde{S}_E = \frac{H^2 S_E}{\pi}. \tag{5.16}$$

In this representation the rescaled action becomes

$$\tilde{S}_E = \frac{\tilde{N}(v - 3)}{3} - \frac{uv}{\tilde{N}} - \frac{v}{4\tilde{N}^2} \frac{\left(u + \frac{\tilde{N}^2}{3}\right)^2}{2P'_A + \frac{\tilde{N}}{3}}. \tag{5.17}$$

The saddle points of the action are found by solving the algebraic equation

$$\frac{\partial \tilde{S}_E}{\partial \tilde{N}} = 0. \tag{5.18}$$

This is a quintic equation for $\tilde{N}$ that cannot be solved exactly for all values of u, v, except in some limits.

Once the steepest descent contour has been specified, along with the contributing saddle points, the integral can be approximated by expanding the action about the extrema and carrying out a Gaussian approximation. Specifically, for each contributing saddle N_i the action can be expanded in the vicinity of N_i as

$$S_E(N - N_i) \approx S_E(N_i) + \frac{S_{NN}(N_i)}{2}(N - N_i)^2, \tag{5.19}$$

where $S_{NN} = \partial^2 S_E / \partial N^2$. Inserting in the integral (5.11) and integrating over $d(N - N_i)$ we can approximate the transition amplitude as

$$\Psi \propto \sum_i \frac{\mu(N_i)}{\sqrt{S_{NN}(N_i)}} \exp[-S_E(N_i)]. \tag{5.20}$$

Thus, the overall WKB pre-exponential factor for each contributing saddle will be given by

$$\tilde{\mu} = \frac{\mu}{\sqrt{S_{NN}}}. \tag{5.21}$$

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The semiclassical prefactor μ for the propagator (5.11) has been derived in []. For the choice of $P_A' = +1/2$ it is given by

$$\mu(a, b, N) \propto \frac{b'}{N\sqrt{H^2 N + 3}}, \qquad (5.22)$$

where b' is the initial radius of S^2 defined in (5.9). The denominator of μ introduces a branch cut. For convenience, we can choose its orientation to be along the real axis at $H^2 N < -3$, but as we will see this will not affect the calculation of the propagator, since any integration along the branch cut is exponentially suppressed.

5.2. S^2 of radius $Hb \approx 1$

The classical KS model has a Nariai solution (2.11), which describes a 4-geometry $dS_2 \times S^2$ with the radius of S^2 being $b = 1/H$. The dS_2 part of the geometry is a circle S^1 undergoing inflation with an expansion rate H. Our analysis of the Hartle-Hawking wave function Ψ_{HH} for this model [14] showed that it gives a probability distribution for b at fixed values of a which is peaked at $b = H^{-1}$. So this wave function predicts Nariai-type evolution as the most probable scenario. To compare this prediction with that of the tunneling wave function Ψ_T, we shall now study the behavior of Ψ_T in the regime $Hb \approx 1$.

5.2.1. Saddles and contours

Following the method of Ref.[14], we shall first find the saddle points for $Hb = 1$ and then consider small perturbations of those points. Using the rescaled variables (5.16) and setting $v = H^2 b^2 = 1$, the action (5.17) becomes

$$\tilde{S}_{E0} = -\frac{2\tilde{N}}{3} - \frac{u}{\tilde{N}} - \frac{\left(3u + \tilde{N}^2\right)^2}{12\tilde{N}^2\left(\tilde{N} + 3\right)}, \qquad (5.23)$$

where the subscript 0 indicates zeroth order with respect to $(1 - v)$. The saddle equation (5.18) for $v = 1$ is

$$\left(\tilde{N}^2 + 2\tilde{N} + u\right)\left(\tilde{N}^3 + 4\tilde{N}^2 - 3\tilde{N}u - 6u\right) = 0. \qquad (5.24)$$

It has two solutions,

$$\tilde{N}_{1,2} = -1 \pm \sqrt{1 - u}, \qquad (5.25)$$

which are real for $u \leq 1$ and form a complex conjugate pair for $u > 1$, along with three solutions $\tilde{N}_{3,4,5}$ which are real for all $u \geq 0$ and whose explicit form will not be needed.

The saddle points and steepest descent/ascent lines for $Ha > 1$ are shown in Fig.2 Our nearly Lorentzian contour can be distorted into a contour running from $N = 0$ along the arc through the saddle N_1, all the way to N_5. Then it takes a turn and runs along the negative N axis to $N \to -\infty$. This contour is dominated by the saddle point at N_1.
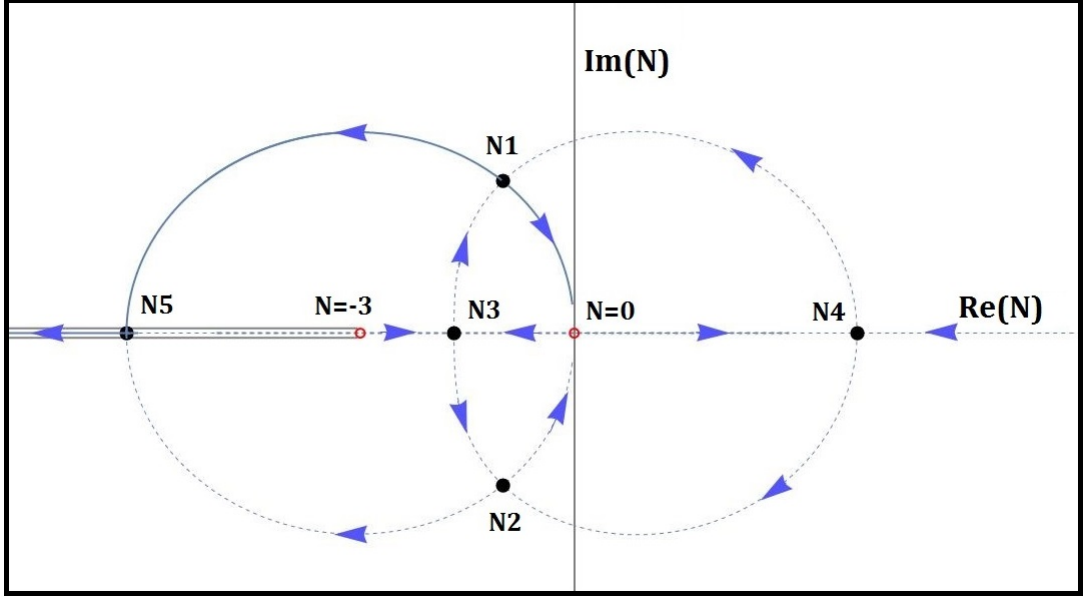

FIG. 2: The steepest descent contours for $Ha > 1$ and $Hb = 1$. The arrowheads point to the direction where $Re(-\tilde{S}_E)$ decreases. The saddles $\tilde{N}_i$ are marked with solid dots and the singularities with circles. Note the branch cut at $\tilde{N} \in (-\infty, -3)$. Our nearly-Lorentzian contour corresponds to the solid curve starting from the singularity $N = 0$; it is dominated by the saddle N_1.

We now consider small deviations from $Hb = 1$. In the region $Ha > 1$ we have identified the saddle $\tilde{N}_1$ as the dominant one, so we will introduce a shift x defined by

$$\tilde{N} = \tilde{N}_1 + x, \tag{5.26}$$

where $\tilde{N}_1$ is given by (5.25) and $|x| << 1$. We insert this into the action (5.17) and expand to second order in x:

$$\tilde{S}_E \approx 1 - i(1-v)\sqrt{u-1} - (1-v)x + \frac{v}{f(u)}x^2 + O(x^3), \tag{5.27}$$

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where

$$3f(u) = \left(2i + 2\sqrt{u-1} - iu - iu^2\right)(u-1)^{-3/2}. \tag{5.28}$$

The action is extremized with

$$x = \frac{1-v}{2v}f(u). \tag{5.29}$$

To lowest order in $(1 - Hb)$, we have $x \propto (1 - Hb)$, so the contribution of the x-dependent terms to the action is $\mathcal{O}[(1 - Hb)^2]$ and we can write

$$S_E \approx \frac{\pi}{H^2} - \frac{2i\pi}{H^2}(1 - Hb)\sqrt{H^2a^2 - 1} + \mathcal{O}[(1 - Hb)^2]. \tag{5.30}$$

The higher order correction term $\mathcal{O}[(1 - Hb)^2]$ is proportional to $f(u)$ and will play an important role in the probability distribution that the tunneling wave function predicts.

5.2.2. WKB tunneling wave function

We are now in a position to compute the WKB tunneling wave function in the region $Hb \approx 1$ and $Ha > 1$ following the prescription (5.20). The WKB prefactor $\tilde{\mu}$ defined in (5.21) can be computed for the saddle N_1 as:

$$\tilde{\mu} \propto \frac{1}{\sqrt{H^2a^2 - 1}} \tag{5.31}$$

Keeping only the lowest non-trivial orders of $(1 - Hb)$ in the action (5.30), we arrive at an expression for the tunneling wave function:

$$\Psi_T(Ha > 1) \propto \frac{1}{\sqrt{H^2a^2 - 1}} \exp\left(-\frac{\pi}{H^2}\right) \exp\left(\frac{2\pi i}{H^2}(1 - Hb)\sqrt{H^2a^2 - 1}\right). \tag{5.32}$$

There are a few things to note about this solution. It exhibits a WKB-type divergence at the turning point $Ha = 1$, as expected. Additionally, the tunneling exponential suppression factor $\exp(-\pi/H^2)$ is present. This is a consequence of the choice $P'_A = +1/2$ for our boundary condition. Finally, it is easily verified that the wave function $\Psi_T(Ha > 1, Hb = 1)$ describes an outgoing wave at $Ha \gg 1$, and thus Ψ_T satisfies the outgoing wave boundary condition in the region $Hb \approx 1$.

In order to obtain a probability distribution for b in the region $Hb \approx 1$ we must include higher order corrections to the wave function (5.32). Making use of the perturbed action (5.27) and ignoring corrections to the prefactor, we arrive at

$$\Psi_T(Ha > 1) \propto \frac{1}{\sqrt{H^2a^2 - 1}} \exp\left(-\frac{\pi}{H^2}\right) \exp\left[\frac{2\pi i}{H^2}(1 - Hb)\sqrt{H^2a^2 - 1} + f(a)(Hb - 1)^2\right], \tag{5.33}$$

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where the function $f(a)$ is given by (5.28). Utilizing the methods of section 2.2, we calculate the current density:

$$dP_b \propto j^a db \propto \frac{db}{\sqrt{H^2 a^2 - 1}} \exp\left[2(Hb - 1)^2 \mathrm{Re} f(a)\right].\qquad(5.34)$$

This can be interpreted as the distribution for b on surfaces of constant a. The real part of $f(a)$ is given by:

$$\mathrm{Re} f(a) = \frac{2\pi}{3H^2(H^2 a^2 - 1)}.\qquad(5.35)$$

Thus the probability distribution can be written explicitly as:

$$dP \propto \frac{db}{\sqrt{H^2 a^2 - 1}} \exp\left[\frac{4\pi(Hb - 1)^2}{3H^2(H^2 a^2 - 1)}\right].\qquad(5.36)$$

This distribution grows as we move away from $Hb = 1$, favoring large values of Hb. This is in contrast with the HH state for which the probability distribution is peaked at $Hb = 1$ [14].

5.3. Large S^2 region $Hb \gg 1$

5.3.1. Saddles and contours

Since the distribution (5.36) appears to favor large values of Hb, we shall now explore the behavior of the wave function at $Hb \gg 1$. In order to find the saddle points in this regime, we neglect the -3 term in the first parenthesis of (5.17). Then the saddle equation (5.18) can be factored:

$$\left(\tilde{N}^2 + 3u\right)\left(\tilde{N}^3 + 6\tilde{N}^2 + 12\tilde{N} + 3u\tilde{N} + 6u\right) = 0.\qquad(5.37)$$

The corresponding solutions include two complex conjugate pairs and one real saddle. Two obvious solutions are

$$\tilde{N}_{1,2} = \pm i\sqrt{3u}\qquad(5.38)$$

and we label the other pair as $\tilde{N}_{4,5}$ and the real saddle as $\tilde{N}_3$. We note that the numbering of saddle points here is not related to the one used in Section 5.2.

The steepest descent contours for this set of saddles are shown in Fig.3. A nearly Lorentzian contour can be distorted into a contour that follows the steepest ascent path from the origin $N = 0$ to the saddle N_1 and continues on the steepest descent path all
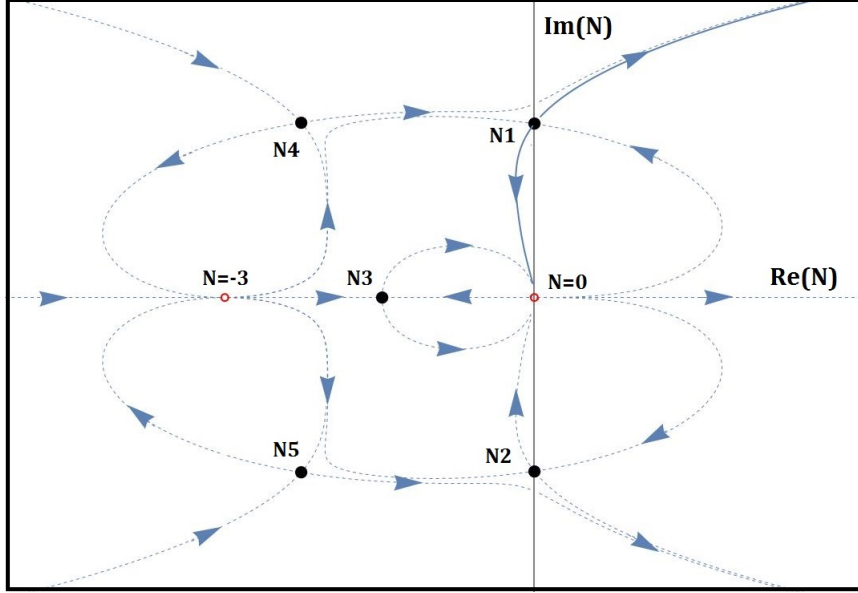
FIG. 3: The steepest descent contours for $Hb \gg 1$. The tunneling contour corresponds to the solid curve starting from the singularity $N = 0$ and dominated by saddle N_1.

the way to infinity in the first quadrant. The contour can be closed with an infinite arc at $N \to +i\infty$. The dominant contribution to the wave function comes from the saddle N_1.

To obtain a more accurate expression for the saddle point, we set $\tilde{N} = \tilde{N}_1 + x$ with $\tilde{N}_1$ from (5.38) and $x \ll \tilde{N}_1$. We then substitute it in the action (5.17), without neglecting the -3 term and expand the action to second order in x:

$$\tilde{S}_E \approx \frac{i\sqrt{u}(2v-3)}{\sqrt{3}} - x - \frac{vx^2}{3(u - i\sqrt{3u})} + \mathcal{O}(x^3). \tag{5.39}$$

Extremizing with respect to x we find

$$x = \frac{3(i\sqrt{3u} - u)}{2v}. \tag{5.40}$$

This approximation is valid in the region of superspace where $v \gg 1$ and $u \ll v^2$ in order for the condition $\tilde{N}_1 \gg x$ to be satisfied. The x-dependent terms in the action introduce corrections $\mathcal{O}(v^{-1})$. Switching back to variables a and b, the perturbed action can be expressed as

$$S_E \approx \frac{2i\pi ab^2 H}{\sqrt{3}} - \frac{\sqrt{3}i\pi a}{H} - \frac{3\sqrt{3}i\pi a}{4H^3b^2} + \frac{3\pi a^2}{4H^2b^2} + \mathcal{O}\left(\frac{1}{H^3b^3}\right). \tag{5.41}$$

As with the case of $Hb \approx 1$, the higher order correction terms will be important for obtaining the probability distribution that the tunneling wave function predicts.

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To lowest order in $(Hb)^{-1}$ the prefactor of the WKB wave function is:

$$\tilde{\mu} \propto \frac{\sqrt{a}}{b^2} e^{i\frac{\pi}{4}}. \tag{5.42}$$

Thus to the leading order of WKB approximation the tunneling wave function is given by

$$\Psi_T(Hb \gg 1) \propto \frac{\sqrt{a}}{b^2} \exp\left[-\frac{2i\pi ab^2 H}{\sqrt{3}}\right], \tag{5.43}$$

where we have kept only the dominant contribution in $Hb \gg 1$. It is straightforward to show that the above solution describes expanding asymptotically de Sitter universes satisfying

$$\dot{b}/b \approx \dot{a}/a \approx H/\sqrt{3}, \tag{5.44}$$

just as in the case of fixed initial scale factors. Thus the outgoing wave criterion is satisfied.

To find the probability distribution for the scale factors, we have to include higher order corrections to the action, as given by Eq.(5.41). Ignoring corrections to the prefactor, we have

$$\Psi_T(Hb \gg 1) \propto \frac{\sqrt{a}}{b^2} \exp\left[-\frac{2i\pi ab^2 H}{\sqrt{3}} + \frac{\sqrt{3}i\pi a}{H} + \frac{3\sqrt{3}i\pi a}{4H^3 b^2} - \frac{3\pi a^2}{4H^2 b^2}.\right] \tag{5.45}$$

Noticing that the first three terms are the expansion of a square root, we can tidy this result to

$$\Psi_T(Hb \gg 1) \propto \frac{\sqrt{a}}{b^2} \exp\left[-2i\pi ab\sqrt{\frac{H^2 b^2}{3} - 1} - \frac{3\pi a^2}{4H^2 b^2}\right], \tag{5.46}$$

which is valid up to order $(Hb)^{-2}$. This wave function has the same asymptotic form as the transition amplitude (4.8) that we obtained for fixed initial scale factors, the only difference being its amplitude, which is controlled by the last term in the bracket. We will show that this term is responsible for the predictions of the tunneling wave function.

Following the formalism of Section 2.2 we calculate the probability current on surfaces of constant a. The calculation is simplified if we use the compact form (5.46), but take the limit $Hb \gg 1$ in the final step. The resulting expression is

$$dP \propto \frac{a^2}{b^3} \exp\left[-\frac{3\pi a^2}{2H^2 b^2}\right] db, \tag{5.47}$$

which is valid to lowest order in $(Hb)^{-1}$. Introducing a new variable l which characterizes the relative size of S^1 and S^2,

$$l = \frac{a}{b}, \tag{5.48}$$

we can express (5.47) as a distribution for l^2:

$$dP \propto \exp\left[-\frac{3\pi l^2}{2H^2}\right] dl^2.$$ (5.49)

This is a normalizable distribution with the average value

$$\bar{l}^2 = \frac{2H^2}{3\pi}.$$ (5.50)

Thus, in the regime of large a and b the tunneling wave function predicts an ensemble of classical de Sitter-like universes (5.44) with $a/b \lesssim H \ll 1$. This is in contrast with the HH state which gives a distribution peaked at the Nariai solution (2.11).

5.4. Small S^1 region $Ha \ll 1$

In the KS model there is no clear distinction between classically allowed and forbidden regions, except when $Hb \approx 1$. In this part of superspace the point $Ha = 1$ divides the classical and quantum regimes. On the contrary, the classical universes (4.11) do not have a well defined bounce point since $\dot{a}$ and $\dot{b}$ do not vanish simultaneously. Thus, we do not expect the wave function to bear much resemblance to the usual tunneling picture established in quantum cosmology. A related reason, mentioned in Sec.3, is that the WDW eq. is a hyperbolic equation, so the kinetic terms in the WDW operator have opposite signs. In this subsection we will try to obtain some insight into the behavior of the tunneling wave function close to the superspace boundary at $a = 0$, without assuming b to be small.

5.4.1. Saddles and contours

We are interested in finding the saddles in the region $u \ll 1$. Setting $u = 0$ in the action (5.17) and solving for the saddles through (5.18) we obtain

$$N_{1,2} \approx -3 \pm \sqrt{\frac{3v}{4-v}}$$ (5.51)

where we have dropped the tildes. This approximation is valid as long as $N^2 \gg 3u$ and v is not too close to 3. The first constraint justifies setting $u = 0$ in the parenthesis of the third term in the action. The second constraint is imposed so that the first term in the action is larger than the second, which justifies dropping it for $u \ll 1$.

The rest of the saddles can be approximately found by neglecting the third term in the action (5.17). Solving for the saddles yields

$$N_{3,4} \approx \pm\sqrt{\frac{3uv}{3-v}}. \qquad (5.52)$$

This approximation is valid everywhere in the region $u \ll v$, apart from values of v that approach $v \approx 3$, for which N becomes large.

Finally, in order to find the 5th saddle we will use the insight obtained from numerical results. The 5 saddles in the region $u \ll 1$ and $v \sim 1$ have the following characteristics. Two of them are $\mathcal{O}(1)$, the other two are $\mathcal{O}(\sqrt{u})$, while the 5th is $\mathcal{O}(u)$. In fact, we observe that when v is not too small, the 5th saddle is given approximately by

$$N_5 \approx -\frac{u}{2}, \qquad (5.53)$$

which is in agreement with the lowest order expansion for $\tilde{N}_1$ given by Eq.(5.25). It is reassuring that all our saddle points can be matched in the appropriate limits in the different regions of KS superspace.

Overall, we expect the above saddles to be valid when $u \ll 1$, $u \ll v$, $v \neq \{3,4\}$. We must also note that a consequence of these restrictions is that b is not allowed to approach zero.

The analytic expressions for the saddles in the region $u \ll 1$ suggest that there exist three qualitatively different steepest descent contour configurations depending on the value of v. In this subsection we are mostly interested in investigating the behavior of the wave function at small overall volume, so we will focus on the region $u \ll 1$ and $v < 3$.

In this case all five saddles are placed on the real-N axis. The steepest descent contours are similar to the ones in Fig.4. The only difference is that when $u \ll 1$, the loop surrounding the singularity $N = 0$ and the loop passing through $N = 0$ are shrunk to a very small size. These loops are defined by the saddles $\mathcal{O}(\sqrt{u})$ and $\mathcal{O}(u)$ respectively, $N_{3,4}$ and N_5. The loop encircling the singularity $N = -3$ does not shrink, since it is defined by the saddles of zero order in u, $N_{1,2}$.

The steepest descent/ascent lines for $u \ll 1$ and $v < 3$ are illustrated in Fig.4 . In this case the nearly Lorentzian contour can be distorted into a contour running from $N = 0$ along the upper arc to the saddle N_5, then along the real axis through the saddle N_4 until the saddle N_1. At that point it takes a turn and follows the upper arc to N_2, and runs from there to $N \to -\infty$. This contour is dominated by the saddle point N_4.
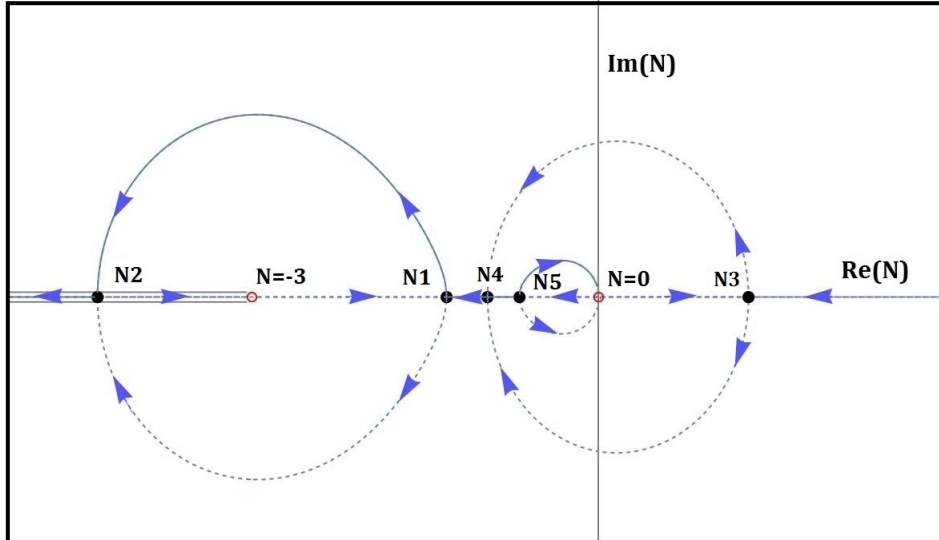
FIG. 4: The steepest descent contours for $Ha \ll Hb < \sqrt{3}$. In this case, all the saddles are real. The tunneling contour corresponds to the solid curve starting from the singularity $N = 0$ and dominated by saddle N_4.

5.4.2. WKB tunneling wave function

The WKB wave function corresponding to this saddle can be found along the same lines as previously demonstrated:

$$\Psi_T(Ha \ll 1, Hb < \sqrt{3}) \propto \exp\left(-2\pi ab\sqrt{1 - \frac{H^2b^2}{3}}\right), \qquad (5.54)$$

where we have omitted the pre-exponential factor. This expression is valid in the region of superspace where $Ha \ll 1$ and $Ha \ll Hb \neq \sqrt{3}$. We notice that for $a =$const, Ψ_T is a decreasing function of Hb until it reaches a minimum at $Hb = \sqrt{3/2}$. For larger values of Hb the wave function increases until it approaches $Hb \approx \sqrt{3}$ where our approximation breaks down.

In the case of $4 > v > 3$, the saddles $N_{1,2}$ become a complex conjugate pair, while the rest of the saddles remain real. Steepest descent analysis shows that the path integral receives dominant contributions from the saddles N_1 and N_5. Thus this region can be thought of as a transition from a fully quantum regime $v < 3$ to a hybrid one, in which classical trajectories do penetrate, however they are accompanied by Euclidean components.

Finally, the case for which $u \ll 1$, $v > 4$ is qualitatively similar to the region $v \gg 1$

discussed in section 5.3. So as long as $u \ll v$ the character of the steepest descent contours will not change and we are able to define a nearly-Lorentzian contour picking a contribution from the saddle N_1 as defined in Sec. 5.3 The WKB wavefunction will then be approximatelly given by the analytic continuation of (5.54) to $Hb > \sqrt{3}$ and will be valid for as long as $a \ll b$.

Overall, the behavior of Ψ_T that we have found is not in line with the familiar quantum tunneling through a barrier. We note especially that the nucleation of a universe with $b \gg a$ is not exponentially suppressed. This can be seen from

$$\left| \frac{\Psi_T(Hb \gg Ha \gg 1)}{\Psi_T(Ha \ll Hb < \sqrt{3})} \right| \sim 1. \tag{5.55}$$

On the other hand, exponential suppression is present in the narrow region $Hb \approx 1$, where there is a clear bounce point at $Ha = 1$ and

$$\left| \frac{\Psi_T(Ha > 1)}{\Psi_T(a \approx 0)} \right| \approx \exp(-\frac{\pi}{H^2}) \ll 1. \tag{5.56}$$

5.5. Total nucleation probability

The total nucleation probability of an $S^1 \times S^2$ universe can be found by integrating the distribution (5.49) over l^2. We obtain

$$P \propto H^2. \tag{5.57}$$

As in the de Sitter model, this probability is maximized at large values of H. But the dependence on H in (5.57) is a power law, while in de Sitter model it is exponential [5],

$$P_{dS} \sim \exp\left(-\frac{3\pi}{H^2}\right). \tag{5.58}$$

If one adopts a minisuperspace framework that allows both $S^1 \times S^2$ and S^3 topologies, then Eqs.(5.57),(5.58) suggest that for $H \ll 1$ nucleation of $S^1 \times S^2$ universes is exponentially favored. However, if different topologies are allowed, one can also consider adding a topological Gauss-Bonnet term to the action,

$$S_{GB} = \frac{\alpha}{16\pi} \int d^4x \sqrt{-g} \left(R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) + S_{boundary}, \tag{5.59}$$

where α is a constant of dimension (length)^2 and $S_{boundary}$ is a boundary term which generalizes the Gibbons-Hawking term [25]. The addition of this term is necessary to make the boundary value problem well defined.

In $(3+1)$ dimensions the variation of the integrand in Eq.(5.59) is a total derivative, so the Gauss-Bonnet term has no effect on dynamics. This term is a topological invariant,

$$S_{GB} = -2\pi\alpha\chi, \tag{5.60}$$

where χ is the Euler character. It can nevertheless have physical implications (see, e.g., [26] and references therein). In the context of quantum cosmology, the extra term (5.59) adds only a constant factor to the wave function, $f_{GB} = \exp(-2\pi\alpha\chi)$, if the topology of the universe is fixed. But if the topology is allowed to vary, different topologies will be weighted with different factors. The Euler character is $\chi = 2$ for a de Sitter universe and $\chi = 4$ for the $S^1 \times S^2$ universe[4]. Hence the nucleation probabilities of such universes are

$$P_{dS} \sim \exp\left(-\frac{\pi}{H^2} - 4\pi\alpha\right), \tag{5.61}$$

$$P_{S^1 \times S^2} \sim H^2 \exp\left(-8\pi\alpha\right). \tag{5.62}$$

This shows that de Sitter universes may dominate if α is sufficiently large,[5]

$$\alpha > 1/4H^2 \gg 1. \tag{5.63}$$

6. CONCLUSIONS

In this project we applied the tunneling proposal for the wave function of the universe to the Kantowski-Sachs (KS) minisuperspace model of spatial topology $S^1 \times S^2$. The path integral version of this proposal defines the wave function $\Psi_T(g)$ as a path integral over histories interpolating between a vanishing 3-geometry ("nothing") and a given configuration g, with the lapse integration taken over semi-infinite Lorentzian contour. It turns out, however, that all histories with vanishing initial radii of S^1 and S^2 in KS model necessarily have an initial curvature singularity, even after Euclidean continuation [20]. It follows that the wave function defined in this way does not describe a non-singular origin of the universe, even at the semiclassical level.

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[4] These are the Euler characters for S^4 and $S^2 \times S^2$ instantons, respectively.

[5] A Gauss-Bonnet term (5.59) appears in the low-energy effective action of heterotic string theory [27], together with an infinite series of higher-order curvature corrections $\sim (\alpha R)^n$. However, the higher-order terms can be neglected only if $\alpha R \ll 1$ [26], which is in conflict with (5.63). Thus the Gauss-Bonnet term should have a different origin if it is to play a role in suppressing $S^1 \times S^2$ universes.

The conclusion could be that the tunneling wave function cannot be defined in the KS model. This could mean that a universe of topology $S^1 \times S^2$ cannot originate out of nothing – assuming that the tunneling approach to the wave function of the universe is on the right track.

Here we have explored an alternative possibility, introduced in Ref.[20] by Halliwell and Louko. They suggested that the boundary condition of vanishing 3-geometry should be replaced by the condition of a vanishing 3-volume. That is, only one of the initial scale factors should be set equal to zero. The interpretation of such a degenerate but non-vanishing 3-geometry as "nothing" may be found objectionable. On the other hand, such geometries can be obtained as limiting slices of regular Euclidean 4-geometries of topology $S^2 \times S^2$, which may be regarded as instantons describing a non-singular origin of the universe. We took an agnostic attitude to this issue and pursued the HL proposal in this paper, to see where it leads.

With one of the scale factors set to zero, the other one can be set at a nonzero value that is consistent with a non-singular 4-geometry. Alternatively, leaving the other scale factor unspecified one can impose a regularity condition excluding conical singularities. We have studied both of these approaches.

In the first approach, when the path integral is taken between fixed values of the scale factors with one of them set to zero, we found that the resulting wave function diverges at a finite value of the radius of S^2, $b = H^{-1}\sqrt{3}$. Hence it is not a solution of the WDW equation in the entire superspace and is not a suitable choice for the tunneling wave function of the universe.

The main body of the paper is devoted to the second approach. Here we found that the choice of $b = 0$ cannot be supplemented by a regularity condition that would give an acceptable wave function. One always gets the same singular wave function that was obtained for fixed initial scale factors. The only option is then to set $a = 0$ and impose a regularity condition of smooth closure on S^1. We found that the resulting wave function is normalizable, with a probability distribution peaked at $b \gtrsim a/H \gg a$. It predicts a highly anisotropic initial universe. However, the universe expands exponentially in all directions, and after a large amount of inflation observers will see an isotropic local universe. In contrast, the Hartle-Hawking wave function gives a probability distribution peaked at Nariai-type universes with $b \approx 1/H$, which remain locally anisotropic until late times.

We have emphasized that the wave function we obtained here is rather different from wave functions describing tunneling in quantum mechanics. The reason is that the WDW equation of the KS model is a hyperbolic equation, with the two kinetic terms having opposite signs. There is therefore no clear division of superspace into classically allowed and forbidden regions. In particular, there is no exponential suppression of the probability distribution in the region of $b \gg a$.

We find that the total nucleation probability is $P \propto H^2$. As in de Sitter model, it is maximized at large values of H, but unlike de Sitter the dependence on H is a power law, not exponential. It follows that for $H \ll 1$ the nucleation probability is much higher for $S^1 \times S^2$ than for de Sitter universes. We noted that this situation can be reversed if the gravitational action is supplemented with a Gauss-Bonnet term with a sufficiently large coefficient.

It would be interesting to extend our analysis by including a homogeneous scalar field ϕ, replacing the cosmological constant with a slowly varying potential $V(\phi)$. Another interesting extension would be to study a "midi-superspace" model, including perturbatively an infinite number of inhomogeneous modes of a quantum field. In the de Sitter model one finds that the universe nucleates with the field in a de Sitter invariant Bunch-Davies state, and in a more realistic model this initial state may lead to a nearly scale-invariant spectrum of density fluctuations, in agreement with observations. On the other hand, in the KS model the initial quantum state will not have de Sitter, and not even rotational symmetry. It is possible however that the field will approach the Bunch-Davies state at late times in the course of inflation, due to the no-hair "theorem" (e.g., [28]).

We finally mention that KS model is closely related to the $2D$ Jackiw-Teitelboim gravity, which has recently attracted much attention. This relation has been discussed in Ref.[14] in the case of the HH wave function, and it would be interesting to explore it for the tunneling wave function.

Acknowledgements

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