On Possible Minimal Length Deformation of Metric Tensor and Affine Connection

Fady T. Farouk[1], Abdel Nasser Tawfik[2], F. Salah Tarabia[1], Muhammad Maher[1]

[1]*Helwan University, Faculty of Science,*

Physics Department, 11792 Ain Helwan, Cairo, Egypt

[2]*Future University in Egypt (FUE), 11835 New Cairo, Egypt*

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When the minimal length approach emerging from noncommutative Heisenberg algebra, generalized uncertainty principle (GUP), and thereby integrating gravitational fields to this fundamental theory of quantum mechanics (QM) is thoughtfully extended to Einstein field equations, the possible deformation of the metric tensor could be suggested. This is a complementary term combining the effects of QM and general relativity (GR) and comprising noncommutative algebra together with maximal spacelike four–acceleration. This deformation compiles with GR as curvature in relativistic eight–dimensional spacetime tangent bundle, generalization of Riemannian spacetime, is the recipe applied to derive the deformed metric tensor. This dictates how the affine connection on Riemannian manifold is straightforwardly deformed. We have discussed the symmetric property of deformed metric tensor and affine connection. Also, we have evaluated the dependence of a parallel transported tangent vector on the spacelike four–acceleration given in units of L, where $L=\sqrt{\frac{\hbar G}{c^3}}$ is a universal constant, c is speed of light, and $\hbar$ is Planck constant, and G is Newton's gravitational constant.

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I. Introduction

The general theory of relativity (GR) assumes that the gravitational field has a geometrical nature. This is a four–dimensional Riemannian manifold having a symmetric metric tensor and an affine (linear) connection. The latter is torsion–free and metric compatible and thus can be determined in terms of the metric tensor, itself. An affine connection is defined as a geometric object on a *smooth* manifold connecting nearby tangent spaces. The tangent vector fields are covariant derivatives on that manifold. An affine connection dictates how to perform parallel transport of tangent vectors on manifold. In general relativity, the connection plays the role of the gravitational force field, where the metric tensor is the corresponding gravitational potential.

The present script introduces a minimal length approach, in which the inherent uncertainties emerging in detecting a quantum state are constrained in noncommutative operators as governed by Heisenberg uncertainty principle (HUP), which limits these to simultaneous measurements but obviously doesn't incorporate the impacts of the gravitational fields. The extended version of HUP known as generalized uncertainty principle (GUP) is also predicted in string theory, loop quantum gravity, doubly special relativity, and various gedanken experiments [1, 2]. GUP could be seen as an approach emerging from the gravitational impacts on the quantum measurements. The latter are essential components of the underlying quantum theory. In other words, GUP helps explaining the origin of the gravitational field and how a particle behaves in it [3, 4]. Recently, the effects of the minimal length approach on the line element, the metric tensor, and the geodesics have been evaluated [3]. While in line element and metric tensor an additional term of the GUP parameter and squared spacelike four–acceleration appears in each quantity, multiple terms with higher–order derivatives appear in the geodesics [3].

Extending the four–dimensional manifold M to an eight-dimentional spacetime tangent bundle TM, the possible deformation of the metric tensor is derived in section II. In section III, a short review of the minimal measurable length is outlined. The possible deformation of the affine connection on Riemannian manifold is discussed in section IV. The parallel transport of a vector on Riemannian manifold is elaborated in section V. The symmetry properties of the deformed metric tensor and affine connection are discussed in section VI. Section VII is devoted to the conclusions and outlook.

II. The deformation of metric tensor

Caianiello suggested that the deformed GR can be described by the four dimensional spacetime embedded as a hypersurface in the eight dimensions manifold M_8 [5–7]. The eight dimensions x^A is

$$x^A = (x^\mu, (L/c)\dot{x}^\mu) \tag{1}$$

where x^μ is the four spacetime dimensions, $\dot{x}^\mu = \frac{dx^\mu}{ds}$ is the four velocity, $A = 0, ..., 7$, $\mu = 0, ..., 3$, L is the minimal length. L may be defined according to GUP as a minimal uncertainty of positions $L = \hbar\sqrt{\beta}$ [8], one can consider the value of minimal length to be the Planck length $L = \sqrt{(\hbar G/c^3)}$.

The deformed line element in eight dimensions manifold M_8 [9, 10] is,

$$d\tilde{s}^2 = g_{AB} dx^A dx^B \tag{2}$$

where g_{AB} is a result of outer product as the following $g_{AB} = g_{\mu\nu} \otimes g_{\mu\nu}$. In Eq.(2), substitute dx^A, and dx^B by Eq.(1),

$$d\tilde{s}^2 = \left(1 + L g_{\mu\nu} \frac{dx^\mu}{ds}\frac{d\dot{x}^\nu}{ds} + L g_{\mu\nu} \frac{d\dot{x}^\mu}{ds}\frac{dx^\nu}{ds} + L^2 g_{\mu\nu} \frac{d\dot{x}^\mu}{ds}\frac{d\dot{x}^\nu}{ds}\right) ds^2 \tag{3}$$

where $c = 1$, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is the classical line element,

$$d\tilde{s}^2 = ds^2 + L^2 g_{\mu\nu} \ddot{x}^\mu \ddot{x}^\nu ds^2 \tag{4}$$

where $\ddot{x}^\mu = \frac{d\dot{x}^\mu}{ds}$ is the acceleration, μ, ν are dummy indices, and $\vec{\dot{x}}.\vec{\dot{x}} = -1$, then $\vec{\dot{x}}.\vec{\ddot{x}} = 0$,

$$d\tilde{s}^2 = (1 + L^2 \ddot{x}^2) ds^2 \tag{5}$$

where $\ddot{x}^2 = g_{\mu\nu} \ddot{x}^\mu \ddot{x}^\nu$. The deformed line element in four dimensions spacetime, as a projection from eight dimensions into four dimensions, will be

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu \tag{6}$$

where $\tilde{g}_{\mu\nu}$ is an assumed deformed metric tensor, which will be calculated by equating Eqs.(5), and (6),

$$\tilde{g}_{\mu\nu} = (1 + L^2 \ddot{x}^2) g_{\mu\nu} \tag{7}$$

where $\ddot{x}^2 = g_{\alpha\beta} \ddot{x}^\alpha \ddot{x}^\beta$, β, α are dummy indices, μ, and ν are free indices.

For flat spacetime,

$$\tilde{\eta}_{\mu\nu} = (1 + L^2 \ddot{x}^2) \eta_{\mu\nu} \tag{8}$$

The correction factor of deformed metric tensor can be redefined by the maximal acceleration A_{max}, where $A_{max} = (c^2/L) = \sqrt{(c^7/\hbar G)}$ [11], the deformed metric tensor will be

$$\tilde{g}_{\mu\nu} = \left(1 + \frac{1}{A_{max}^2} \ddot{x}^2\right) g_{\mu\nu} \tag{9}$$

where $c = 1$.

III. Minimal measurable length

The parameter β introduced in section II emerged from the assumption that a minimal length uncertainty that was predicted in various theories of quantum gravity, string theory, for instance, as a consequence of the gravitational fields on the uncertainty principle suggests that [8],

$$\Delta x \, \Delta p \geq \frac{\hbar}{2} \left[1 + \beta \left(\Delta p\right)^2 + \beta \langle p \rangle^2\right], \tag{10}$$

where $\langle p \rangle$ is the momentum expectation value and Δx and Δp, respectively, represent the length and momentum uncertainties. The GUP parameter, $\beta = \beta_0 G/(c^3 \hbar)$, with β_0 is a dimensionless parameter to be determined from recent cosmological observations [12, 13] introduces the consequences of gravity to the uncertainty principle, the fundamental theory of quantum mechanics. The commutation relation between length and momentum operators,

$$[\hat{x}, \hat{p}] \; = \; i\hbar \left(1 + \beta \hat{p}^2\right). \tag{11}$$

The minimum uncertainty of position Δx_{min} for all values of expectation values of momentum $\langle p \rangle$ will be

$$\Delta x_{min}(\langle p \rangle) = \hbar \sqrt{\beta} \sqrt{1 + \beta \langle p \rangle^2} \tag{12}$$

then the absolute minimum uncertainty of position is at $\langle p \rangle^2 = 0$,

$$\Delta x_0 = \hbar \sqrt{\beta} \tag{13}$$

One can consider the value of Δx_0 as the possible minimal length according to the GUP, which represents the effect of gravitational field on the QM, then the minimal length will be

$$L = \hbar \sqrt{\beta} \tag{14}$$

The minimal length may be assumed as a fundamental physical quantity obtained from a combination of fundamental physical quantities, gravitational constant (G) from gravity, reduced Planck constant ($\hbar$) from quantum mechanics, and speed of light (c) from spacial relativity [14]. The minimal length will be

$$\begin{aligned} L &= l_p \\ &= \sqrt{\frac{\hbar G}{c^3}} \end{aligned} \tag{15}$$

where l_p is called Planck length.

The existence of maximal acceleration can be assumed by the existence of minimal length as a combination of fundamental physical quantities [11],

$$A_{max} = \frac{c^2}{l_p} = \sqrt{\frac{c^7}{\hbar G}} \quad (16)$$

According to the GUP definition of minimal length stated in Eq. (14), the maximal acceleration will be

$$A_{max} = \frac{c^2}{L} = \sqrt{\frac{c^4}{\hbar^2 \beta}} \quad (17)$$

IV. The deformation of affine connection in Riemannian manifold

The minimal length approach, suggests deformation of the metric tensor as follows

- For curved space,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + L^2 \ddot{x}^2 \, g_{\mu\nu} = g_{\mu\nu} + q_{\mu\nu}. \quad (18)$$

  With Eq. (14), the $q_{\mu\nu}$ can be suggested as GUP contributed part, which reads

$$q_{\mu\nu} = \beta \hbar^2 \ddot{x}^2 \, g_{\mu\nu} \quad (19)$$

- For flat space,

$$\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} + \beta \hbar^2 \ddot{x}^2 \, \eta_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (20)$$

  where $h_{\mu\nu} = \beta \hbar^2 \ddot{x}^2 \, \eta_{\mu\nu}$.

Both $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ share common properties. They turn the covariant tensor into contravariant tensor and vice versa. The symmetry property of deformed metric tensor is

$$\tilde{g}_{\mu\nu} = \frac{1}{2}(\tilde{g}_{\mu\nu} + \tilde{g}_{\nu\mu}), \quad (21)$$

the L.H.S of Eq. (21) will be

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + L^2 \ddot{x}^2 g_{\mu\nu}, \quad (22)$$

and the R.H.S of Eq. (21) is

$$\frac{1}{2}(\tilde{g}_{\mu\nu}+\tilde{g}_{\nu\mu}) = \frac{1}{2}(g_{\mu\nu}+L^2\ddot{x}^2 g_{\mu\nu}+g_{\nu\mu}+L^2\ddot{x}^2 g_{\nu\mu}) \tag{23}$$

we know that $g_{\mu\nu}$ in classical GR is symmetric in its indices, then we will replace $g_{\nu\mu}$ in third and fourth terms by $g_{\mu\nu}$,

$$\frac{1}{2}(\tilde{g}_{\mu\nu}+\tilde{g}_{\nu\mu}) = \frac{1}{2}(g_{\mu\nu}+L^2\ddot{x}^2 g_{\mu\nu}+g_{\mu\nu}+L^2\ddot{x}^2 g_{\mu\nu}) \tag{24}$$

$$\frac{1}{2}(\tilde{g}_{\mu\nu}+\tilde{g}_{\nu\mu}) = \frac{1}{2}(2g_{\mu\nu}+2L^2\ddot{x}^2 g_{\mu\nu}) \tag{25}$$

$$\frac{1}{2}(\tilde{g}_{\mu\nu}+\tilde{g}_{\nu\mu}) = g_{\mu\nu}+L^2\ddot{x}^2 g_{\mu\nu} \tag{26}$$

we find that Eq. (22) and Eq. (26) are equal, then R.H.S = L.H.S of Eq. (21). The deformed metric tensor is symmetric under interchange of its indices μ and ν.

For the affine connection, Eq. (A3), with the deformed metric tensor in curved and flat spacetime $\tilde{g}_{\mu\nu}$, Eq. (18) and $\tilde{\eta}_{\mu\nu}$, Eq. (20), the partial derivatives are obviously commutative, as well. The deformed metric tensor is compatible, see appendix B. Thus, the deformation of the affine connection, Eq. (A3), can be expressed as

$$\tilde{\Gamma}^{\gamma}_{\beta\mu} = \frac{1}{2}\tilde{g}^{\alpha\gamma}(\tilde{g}_{\alpha\beta,\mu}+\tilde{g}_{\alpha\mu,\beta}-\tilde{g}_{\beta\mu,\alpha}). \tag{27}$$

For curved space, by substituting Eqs. (18), and (B.9) into Eq. (27), we get

$$\tilde{\Gamma}^{\gamma}_{\beta\mu} \;=\; \frac{1+2L^2\ddot{x}^2}{1+L^2\ddot{x}^2}\,\frac{1}{2}\,g^{\alpha\gamma}\left(g_{\alpha\beta,\mu}+g_{\alpha\mu,\beta}-g_{\beta\mu,\alpha}\right) = \frac{1+2L^2\ddot{x}^2}{1+L^2\ddot{x}^2}\Gamma^{\gamma}_{\beta\mu}. \tag{28}$$

where $\tilde{g}^{\alpha\gamma} = \dfrac{g^{\alpha\gamma}}{(1+L^2\ddot{x}^2)}$.

It is obvious that vanishing $L^2\ddot{x}^2$ straightforwardly retrieves the undeformed affine connection $\Gamma^{\gamma}_{\beta\mu}$. This is also the case, at vanishing L^2, no minimal length uncertainty, and/or at vanishing $\ddot{x}^2$, cancellation of the GUP effect on GR. We have shown in Eq. (18) that both deformation ingredients are interdependent. The parameterization of the four–coordinates on M in eight–coordinates on TM emerges spacelike four-acceleration $\ddot{x}^2$ and creates additional geometric structure. Eq. (28) reveals that the deformation of the affine connection is exclusively localized in its coefficient: while undeformed $\Gamma^{\gamma}_{\beta\mu}$ possesses unity as a coefficient, its deformed version gets the coefficient $(1+2L^2\ddot{x}^2)/(1+L^2\ddot{x}^2)$. This means that the affine (linear) connection preserves, on one hand, its geometric nature as in GR, for instance. On the other hand, the deformation via additional curvature on higher–dimensional manifold, especially at the energy scale, in which $L^2\ddot{x}^2$ becomes significant.

V. Parallel transport on Riemannian manifold

In flat space, the covariant derivatives, where the vector components and the basis vectors are also differentiated, are just the ordinary derivatives. In curved space, the differentiation of the basis vectors can be expressed by the Christoffel symbols. In both flat and curved spaces, the covariant derivatives can be defined as the rates of change of the tangent vector fields (ordinary derivatives, for instance) with the normal component subtracted, i.e., parallel transport. Vanishing covariant derivatives of a vector $\vec{v} = v^\alpha e_\alpha$ means that $\vec{v}$ is parallel transported, i.e., keeping $\vec{v}$ as constant as possible,

$$\frac{d}{d\lambda} v^\alpha + \Gamma^\alpha_{\sigma\rho} \frac{dx^\sigma}{d\lambda} v^\rho = 0, \tag{29}$$

where the dependence of the parallel transport on the connection $\Gamma^\alpha_{\sigma\rho}$ is obvious. With the deformation, $\Gamma^\alpha_{\sigma\rho}$ is to be replaced by $\tilde{\Gamma}^\alpha_{\sigma\rho}$, Eq. (28), i.e., Eq. (29) can then be rewritten as

$$\frac{d}{d\lambda} v^\alpha + \frac{(1+2L^2\ddot{x}^2)}{(1+L^2\ddot{x}^2)} \Gamma^\alpha_{\sigma\rho} \frac{dx^\sigma}{d\lambda} v^\rho = 0 \tag{30}$$

$$\frac{d}{d\lambda} v^\alpha \;=\; -\frac{1}{1+L^2\ddot{x}^2} \Gamma^\alpha_{\sigma\rho} \frac{dx^\sigma}{d\lambda} v^\rho - \frac{2L^2\ddot{x}^2}{1+L^2\ddot{x}^2} \Gamma^\alpha_{\sigma\rho} \frac{dx^\sigma}{d\lambda} v^\rho. \tag{31}$$

The Fig. 1 draws the affine connection between the vector $\vec{v}$ parameterized in λ; $\vec{v}(\lambda)$ and its parallel transported counterpart, at $\lambda + d\lambda$; $\vec{v}(\lambda + d\lambda)$.

The parallel transport can define the curvature of a manifold by take the parallel transport of a vector over a closed loop then it will give the definition of curvature tensor [15, 16]. The equation of parallel transport of the vector v^α around a loop with the deformed connection will be

$$\delta v^\alpha = \delta a \delta b [\tilde{\Gamma}^\alpha_{\mu\sigma,\lambda} - \tilde{\Gamma}^\alpha_{\mu\lambda,\sigma} + \tilde{\Gamma}^\alpha_{\nu\lambda}\tilde{\Gamma}^\nu_{\mu\sigma} - \tilde{\Gamma}^\alpha_{\nu\sigma}\tilde{\Gamma}^\nu_{\mu\lambda}] v^\mu \tag{32}$$

where δv^α is the changing of v^α by the transport around the loop and $\delta a \delta b$ is the area of the loop. According to Eq. (32), the deformed curvature tensor is

$$\tilde{R}^\alpha_{\beta\mu\nu} = \tilde{\Gamma}^\alpha_{\mu\sigma,\lambda} - \tilde{\Gamma}^\alpha_{\mu\lambda,\sigma} + \tilde{\Gamma}^\alpha_{\nu\lambda}\tilde{\Gamma}^\nu_{\mu\sigma} - \tilde{\Gamma}^\alpha_{\nu\sigma}\tilde{\Gamma}^\nu_{\mu\lambda} \tag{33}$$

The more investigation and discussion of deformed Curvature tensor Eq. (33) will be in a future research.

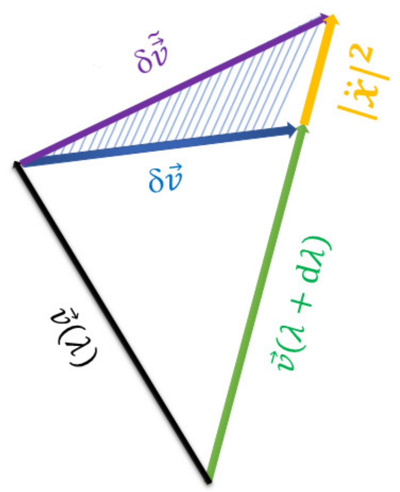

Fig. 1: In vector form, the affine connection and parallel transport are depicted.

VI. Symmetry properties of deformed affine connection

The symmetric property of the affine connection depends on a) the symmetric property of the metric tensor, and b) the commutation of the partial derivatives. The deformed affine connection, Eq. (28), fulfills both conditions:

1. In any coordinate the deformed affine connection can be expressed in the deformed metric tensor and its derivatives,

$$\tilde{\Gamma}^{\gamma}_{\beta\mu} = \frac{1}{2}\tilde{g}^{\alpha\gamma}(\tilde{g}_{\alpha\beta,\mu} + \tilde{g}_{\alpha\mu,\beta} - \tilde{g}_{\beta\mu,\alpha}), \tag{34}$$

   where the deformed metric $\tilde{g}$ is symmetric, then the deformed affine connection is symmetric, as well.

2. The affine connection can be expressed as [17]

$$\tilde{\Gamma}^{\gamma}_{\beta\mu} = \frac{\partial x^{\gamma}}{\partial X^{\alpha}} \frac{\partial^2 X^{\alpha}}{\partial x^{\beta} \partial x^{\mu}}, \tag{35}$$

   where x^{λ} and X^{α} represent different coordinates in curved space, the commutation of the partial derivatives is still satisfied in the deformed affine connection, Eq. (28). This is also valid even when X^{α} is deformed to encounter the existence of a minimal length uncertainty.

Therefore, we conclude that the deformed affine connection is symmetric in its lower indices $\tilde{\Gamma}^{\gamma}_{(\beta\mu)}$, so that

$$\tilde{\Gamma}^{\gamma}_{(\beta\mu)} = \tilde{\Gamma}^{\gamma}_{\beta\mu} = \tilde{\Gamma}^{\gamma}_{\mu\beta}. \tag{36}$$

Also, because the deformed affine connection on Riemannian manifold is torsion–free, then

$$T^{\gamma}_{\beta\mu} = \tilde{\Gamma}^{\gamma}_{\beta\mu} - \tilde{\Gamma}^{\gamma}_{\mu\beta} = 2\tilde{\Gamma}^{\gamma}_{[\beta\mu]} = 0, \tag{37}$$

where $\tilde{\Gamma}^{\gamma}_{[\beta\mu]} = 0$.

VII. Conclusions and outlook

The minimal length approach emerging from noncommutative Heisenberg algebra, generalized uncertainty principle (GUP), is conjectured to integrate gravity in quantum mechanics through generalizing Heisenberg uncertainty principle to encounter impacts of gravitational fields [1, 2]. When applying this recipe on general relativity, the metric tensor becomes deformed, by gaining an additional term, which is related to the GUP parameter, undeformed metric tensor, and squared maximal four–acceleration $\ddot{x}^2$.

To achieve the deformation on spacetime monifold M, we have followed the same recipe for the curvature in relativistic eight–dimentional spacetime tangent bundle TM in Riemannian manifold. This is a differential manifold M equipped with tangent bundle manifold TM, in which the restriction on the nonquadratic length measure for vectors is relaxed. The local coordinates x^μ on M are combined with the tangent vectors $\dot{x}^\mu = dx^\mu/ds$ on TM.

The present script focuses on the possible deformation of the affine connection, which can exclusively be expressed in the metric tensor and its derivatives. We have discussed on its symmetric property and evaluated the dependence of a normalized parallel transported vector on the spacelike four–acceleration. This observation manifests that the minimal length uncertainty and the deformed recipe are significant, especially at the energy scale in which $L^2\ddot{x}^2$ becomes finite.

We conclude that the quantization of the affine connection is exclusively factorized in the coefficient $(1 + 2L^2\ddot{x}^2)/(1 + L^2\ddot{x}^2)$, which combines minimal length uncertainty (GUP effect), geometric structural, noncommutative algebraic, and gravitational ingredients. On one hand, this means that the affine connection preserves all properties of its undeformed counterpart, such as torsion–free and metric compatibility. On the other hand, its geometric nature as connecting nearby tangent spaces on a *smooth* manifold is also preserved on *discrete* spaces.

The deformation via additional curvature on higher–dimensional manifold likely reveals fine geometric structure, similar to radiation beam in classical and quantum mechanics.

We have studied how the deformed affine connection performs parallel transport of a tangent vector on Riemannian manifold.

A. Differential geometry and affine connection

An affine (linear) connection is defined as a geometric object connecting nearby tangent (curved) spaces, i.e., permitting differentiability of the tangent vector fields or assuring them a restrict dependence on the manifold in a fixed vector space [17]. This is a function assigning to each tangent vector and each vector field a covariant derivative or a new tangent vector. In differential geometry, the generic form of the affine connection was suggested as [18]

$$\Gamma^{\mu}_{\lambda\nu} = \left\{ {}^{\mu}_{\lambda\nu} \right\} + K^{\mu}_{\lambda\nu} + \frac{1}{2}(Q^{\mu}_{\lambda\nu.} + Q^{\mu}_{\nu\lambda.} - Q^{\mu}_{.\nu\lambda}) \tag{A1}$$

where dot in lower indices refers to the position of upper index, $\left\{ {}^{\mu}_{\lambda\nu} \right\}$ is the Christoffel symbol, and $Q_{\mu\nu\lambda} = -D_{\mu}(\Gamma)g_{\nu\lambda}$ is the covariant derivative of the metric tensor. $K^{\mu}_{\lambda\nu} = \frac{1}{2}(T^{\mu}_{.\lambda\,\nu} - T^{\mu}_{\lambda.\nu} - T^{\mu}_{\nu.\lambda})$ is contortion, and $T^{\mu}_{\lambda\nu} = \Gamma^{\mu}_{\lambda\nu} - \Gamma^{\mu}_{\nu\lambda} = 2\Gamma^{\mu}_{[\lambda\nu]}$ is the torsion. The latter represents the anti–symmetric part of the connection.

The theory of general relativity assumes metric compatibility of the connection, which implies linear independence of the partial derivative tangent vectors and accordingly leads to vanishing $D_{\mu}(\Gamma)g_{\nu\lambda}$. Also, the metric compatibility of the connection naturally arises, if the covariant derivative is tensor applying the Leibniz rule [19].

The metric compatibility means that a flat space can be found locally in a suitable frame (Minkowski space). In free falling frame, for example, $g_{\nu\lambda} = \eta_{\nu\lambda}$, then $D_{\mu}(\Gamma)g_{\nu\lambda}$ vanishes for $g_{\nu\lambda} = \eta_{\nu\lambda}$ [17]. In such a frame, the covariant derivative of a tensor is the same for all observers and frames, i.e., $D_{\mu}(\Gamma)g_{\nu\lambda} = 0$ [15, 18–20].

The theory of general relativity also assumes[1], that the affine geodesics matches with the metrical geodesics. The latter is obviously given by extremizing ds^2, the spacetime interval [3, 4]. For the torsion–free assumption, $K^{\mu}_{\lambda\nu}$ vanishes, entirely, the metric plays the role of the gravitational field potential, and the Riemann geometry is symmetric (also the energy–

[1] Other assumptions, i.e., nonsymmetric energy–momentum tensor or finite torque density, are also possible, e.g., Einstein–Cartan–Sciama–Kibble theory [21]

momentum tensor is symmetric). Then, the affine connection reduces to

$$\Gamma^{\mu}_{\lambda\nu} = \left\{ {\mu \atop \lambda\nu} \right\}. \tag{A2}$$

The assumption of symmetric connection coefficients leads to commutative partial derivatives, Eq. (35).

Under the conditions of the metric compatibility, the symmetry of the metric tensor indices, and the partial derivative commutation, there is one particular version of the connection coefficients (Levi–Civita connection). Then, the Christoffel symbols can be expressed as [15]

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} (g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}), \tag{A3}$$

where $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$.

B. The metric tensor compatibility

The covariant derivative of deformed metric tensor can be defined as a partial derivative in free falling frame (Minkowski space),

$$\nabla_\sigma \tilde{g}_{\mu\nu} = \partial_\sigma \tilde{g}_{\mu\nu}, \tag{B.1}$$

$$\partial_\sigma \tilde{g}_{\mu\nu} = \left(1 + L^2 \ddot{x}^2\right) \partial_\sigma g_{\mu\nu} + g_{\mu\nu} L^2 (\ddot{x}^2)_{,\sigma} \tag{B.2}$$

where $\tilde{g}_{\mu\nu} = (1 + L^2\ddot{x}^2) g_{\mu\nu}$, and L is a constant. Use the following definition $\ddot{x}^2 = g_{\mu\nu}\ddot{x}^\mu \ddot{x}^\nu$, then Eq. (B.2) will be

$$\begin{aligned} \partial_\sigma \tilde{g}_{\mu\nu} = & \left(1 + L^2 \ddot{x}^2\right) \partial_\sigma g_{\mu\nu} \\ & + L^2 (g_{\mu\nu,\sigma} \ddot{x}^\mu \ddot{x}^\nu + g_{\mu\nu} \ddot{x}^\mu_{,\sigma} \ddot{x}^\nu + g_{\mu\nu} \ddot{x}^\mu \ddot{x}^\nu_{,\sigma}) g_{\mu\nu} \end{aligned} \tag{B.3}$$

The derivative of $\ddot{x}^\mu$ with respect to the space-time coordinates,

$$\ddot{x}^\mu_{,\sigma} = \frac{\partial}{\partial x^\sigma} \frac{\partial^2 x^\mu}{\partial s^2} \tag{B.4}$$

by using the commutation property of partial derivatives,

$$\ddot{x}^\mu_{,\sigma} = \frac{\partial^2}{\partial s^2} \frac{\partial x^\mu}{\partial x^\sigma} \tag{B.5}$$

$$\ddot{x}^\mu_{,\sigma} = \frac{\partial^2}{\partial s^2} \delta^\mu_\sigma = 0 \tag{B.6}$$

where $\delta^{\mu}_{\sigma} = \dfrac{\partial x^{\mu}}{\partial x^{\sigma}}$ [15]. Also, the same thing for $\ddot{x}^{\nu}_{,\sigma}$,

$$\ddot{x}^{\nu}_{,\sigma} = 0 \tag{B.7}$$

Substitute $\ddot{x}^{\mu}_{,\sigma}$ and $\ddot{x}^{\nu}_{,\sigma}$ in Eq. (B.3) by Eqs. (B.6) and (B.7) respectively,

$$\partial_{\sigma}\tilde{g}_{\mu\nu} = \left(1 + L^2\ddot{x}^2\right) g_{\mu\nu,\sigma} + \left(g_{\mu\nu,\sigma}\ddot{x}^{\mu}\ddot{x}^{\nu}\right) L^2 g_{\mu\nu}, \tag{B.8}$$

take $g_{\mu\nu,\sigma}$ as a common factor,

$$\partial_{\sigma}\tilde{g}_{\mu\nu} = \left(1 + 2L^2\ddot{x}^2\right) g_{\mu\nu,\sigma} \tag{B.9}$$

The metric tensor $g_{\mu\nu}$ in free falling frame is Minkowski metric tensor $\eta_{\mu\nu}$, then Eq. (B.9) will be

$$\partial_{\sigma}\tilde{\eta}_{\mu\nu} = \left(1 + 2L^2\ddot{x}^2\right) \eta_{\mu\nu,\sigma} = 0 \tag{B.10}$$

where $\eta_{\mu\nu,\sigma} = 0$. According to Eq. (B.10), the covariant derivative of deformed metric tensor is vanishing in free falling frame, then the covariant derivative will vanish for all frames,

$$\nabla_{\sigma}\tilde{g}_{\mu\nu} = 0 \tag{B.11}$$

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