



Covariant Hamiltonians for F(R)-Gravity

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- Lagrangian $\mathcal{L}(\phi, \partial\phi, x) \rightarrow$
Legendre transformation $\partial\phi \leftrightarrow p \rightarrow$
Hamiltonian $\mathcal{H}(\phi, p, x)$

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- Conventional theory: Momentum $p^A = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_A}$
- Covariant theory: Multi-momentum $p^{iA} = \frac{\partial\mathcal{L}}{\partial(\partial_i\dot{\phi}_A)}$
- Covariant Hamiltonian $\mathcal{H} = p^{iA}\partial_i\phi_A - \mathcal{L}$

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- $\partial_d p^{cA} = ??$ Gauge freedom for multi-momenta

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Einstein frame

- Jordan frame $\mathcal{L}^J = \frac{1}{16\pi} \sqrt{-g} [F(B) + F'(B)(R - B)]$

- Weyl transformation

$$\tilde{g}^{ij} = F' g^{ij} \implies \tilde{R} = \frac{R}{F'} - \frac{3}{F'} \square \ln F' - \frac{3}{2F'} \partial_i \ln F' g^{ij} \partial_j \ln F'$$

- B -terms transformation $V = \frac{F'B - F}{F'}, \quad \tilde{\phi} = \sqrt{\frac{3}{2}} \ln F'$

- Einstein frame

$$\mathcal{L}^E = \frac{1}{16\pi} \sqrt{-g} \left[\tilde{R} - \tilde{g}^{ij} \partial_i \tilde{\phi} \partial_j \tilde{\phi} - V(\tilde{\phi}) \right] + \frac{\sqrt{6}}{16\pi} \partial_i \left(\sqrt{-\tilde{g}} \tilde{g}^{ij} \partial_j \tilde{\phi} \right)$$

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$$\mathcal{L}_{quad}^J = \frac{\sqrt{-g}}{16\pi} \left[F' g^{ab} \left(\Gamma_{bd}^c \Gamma_{ac}^d - \Gamma_{dc}^d \Gamma_{ab}^c \right) - F'' \partial_c B \left(\Gamma_{ab}^c g^{ab} - \Gamma_{ab}^a g^{cb} \right) + F - F' B \right],$$

$$\mathcal{L}_{quad}^E = \frac{\sqrt{-\tilde{g}}}{16\pi} \left[\tilde{g}^{ab} \left(\tilde{\Gamma}_{bd}^c \tilde{\Gamma}_{ac}^d - \tilde{\Gamma}_{dc}^d \tilde{\Gamma}_{ab}^c - \partial_a \tilde{\phi} \partial_b \tilde{\phi} \right) - V \right]$$

New variable f^{ab}

- Momentum $M^{abc} = \frac{\partial \mathcal{L}}{\partial(\partial_c g_{ab})} = \text{complicated}$
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$$\begin{aligned}\mathcal{L}_{quad}^J &= \frac{1}{16\pi} \left[F' f^{ab} \left(\Gamma_{bd}^c \Gamma_{ac}^d - \Gamma_{dc}^d \Gamma_{ab}^c \right) - \right. \\ &\quad \left. - F'' \partial_c B \left(\Gamma_{ab}^c f^{ab} - \Gamma_{ab}^a f^{cb} \right) + \sqrt{-f} (F - F' B) \right] , \\ \mathcal{L}_{quad}^E &= \frac{1}{16\pi} \left[\tilde{f}^{ab} \left(\tilde{\Gamma}_{bd}^c \tilde{\Gamma}_{ac}^d - \tilde{\Gamma}_{dc}^d \tilde{\Gamma}_{ab}^c - \partial_a \tilde{\phi} \partial_b \tilde{\phi} \right) - \sqrt{-f} V \right]\end{aligned}$$

(Multi-)Momenta

Two kinds of momenta $N_{ab}^c = \frac{\partial \mathcal{L}}{\partial(\partial_c f^{ab})}$, $p^a = \frac{\partial \mathcal{L}}{\partial(\partial_a B)}$

$$\begin{aligned} N_{ab}^c &= \frac{F'}{16\pi} \left[-\Gamma_{ab}^c + \frac{1}{2} \left(\Gamma_{ak}^k \delta_b^c + \Gamma_{bk}^k \delta_a^c \right) \right] + \\ &\quad + \frac{F''}{32\pi} (\partial_a B \delta_b^c + \partial_b B \delta_a^c + \partial_g B f^{gc} f_{ab}) , \\ \tilde{N}_{ab}^c &= \frac{F'}{16\pi} \left[-\tilde{\Gamma}_{ab}^c + \frac{1}{2} \left(\tilde{\Gamma}_{ak}^k \delta_b^c + \tilde{\Gamma}_{bk}^k \delta_a^c \right) \right] , \\ p^c &= -\frac{F''}{16\pi} \left(f^{bd} \Gamma_{bd}^c - f^{bc} \Gamma_{bd}^d \right) , \\ \tilde{p}^c &= -\frac{1}{8\pi} \tilde{f}^{cb} \partial_b \tilde{\phi} . \end{aligned}$$

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$$\begin{aligned}\mathcal{H}' &= \frac{16\pi}{F'} f^{ab} \left(N_{bd}^c N_{ca}^d - \frac{1}{3} N_{ca}^c N_{bd}^d \right) - \\ &\quad - \frac{8\pi F'}{3} f_{ab} \left(\frac{p^a}{F''} - \frac{N_{mn}^a}{F'} f^{mn} \right) \left(\frac{p^b}{F''} - \frac{N_{rs}^b}{F'} f^{rs} \right) - \\ &\quad - \frac{\sqrt{-f}}{16\pi} (F - F' B) , \\ \widetilde{\mathcal{H}}^E &= 16\pi \widetilde{f}^{ab} \left(\widetilde{N}_{bd}^c \widetilde{N}_{ca}^d - \frac{1}{3} \widetilde{N}_{ca}^c \widetilde{N}_{bd}^d \right) - 4\pi \widetilde{f}_{ab} \widetilde{p}^a \widetilde{p}^b - \frac{\sqrt{-\widetilde{f}}}{16\pi} V .\end{aligned}$$

Canonical Transformation

■ Canonical Transformation

$$(f, N) \rightarrow (\tilde{f}, \tilde{N}) \Big| \frac{\partial \mathcal{H}}{\partial f^{ab}} = -\partial_c N_{ab}^c, \quad \frac{\partial \mathcal{H}}{\partial N_{ab}^c} = \partial_c f^{ab}$$

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■ Principle of least action

$$\delta \int_M \left(N_{ab}^c \partial_c f^{ab} + p^c \partial_c B - \mathcal{H}^J \right) dx^4 = 0,$$

$$\delta \int_M \left(\tilde{N}_{ab}^c \partial_c \tilde{f}^{ab} + \tilde{p}^c \partial_c \tilde{\phi} - \mathcal{H}^E + \partial_a G_1^a(f, B, \tilde{f}, \tilde{\phi}, x) \right) dx^4 = 0,$$

where

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- Conventional Invariants: Poisson brackets, Lagrange brackets
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Generating Function of Canonical Transformation

- Four types of generating function bound with Legendre Transformation

$$G_2^a(f, B, \tilde{N}, \tilde{p}) = G_1^a + \tilde{f}^{ik} \tilde{N}_{ik}^a + \tilde{\phi} \tilde{p}^a$$

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- Conditions from Hamilton's Principle

$$\frac{\partial G_2^c}{\partial \tilde{p}^a} = \delta_a^c \tilde{\phi}, \quad \frac{\partial G_2^c}{\partial \tilde{N}_{ab}^d} = \delta_d^c \tilde{f}^{ab}, \quad \frac{\partial G_2^c}{\partial f^{ab}} = N_{ab}^c, \quad \frac{\partial G_2^c}{\partial B} = p^c$$

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- Generating function of Jordan \leftrightarrow Einstein transformation

$$G_2^c = F' \tilde{N}_{ab}^c f^{ab} + \sqrt{\frac{3}{2}} \ln F' \tilde{p}^c$$

Summary

- Using covariant Hamiltonian theory, we are able to achieve Hamiltonian description of relativistic fields without loss of covariance.
- Such covariant Hamiltonian can be constructed for $F(R)$ gravity.
- It can be constructed for Jordan frame as well as for Einstein frame
- Jordan frame Hamiltonian and Einstein frame Hamiltonian are bound with canonical transformation.

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