

The η/η' system: Masses and matrix elements

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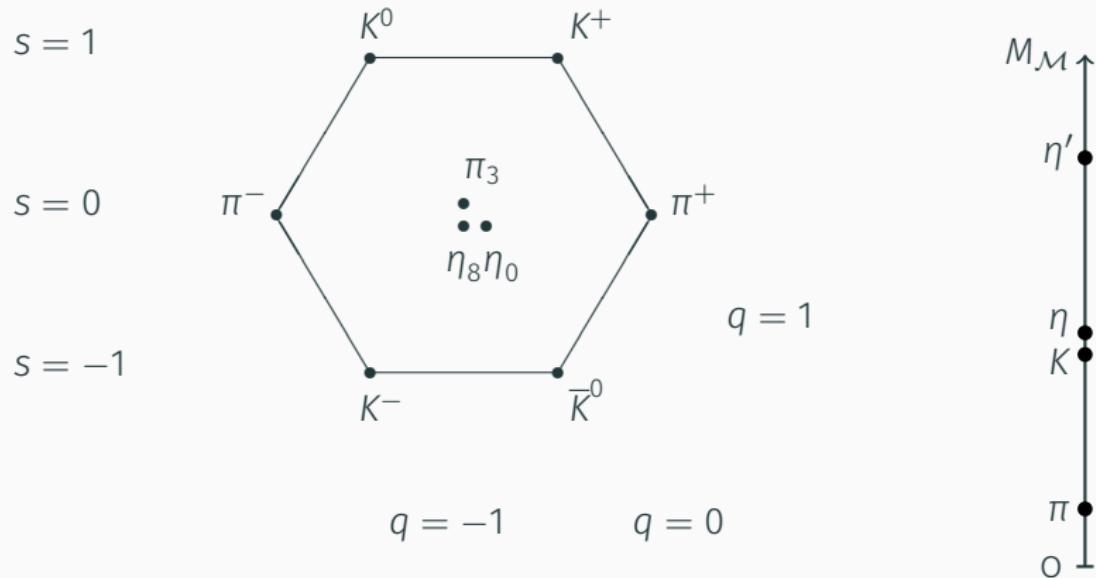
with Gunnar Bali, Sara Collins, Vladimir Braun, Andreas Schäfer

August 15, 2022

Numerical Challenges in Lattice QCD, Meinerzhagen

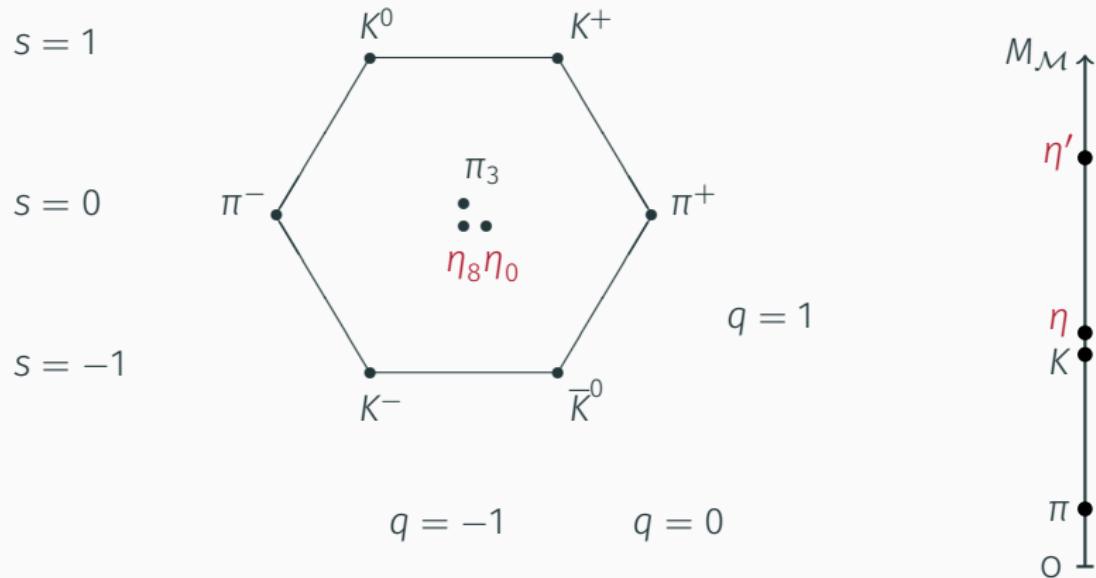


QUARK MODEL



$$SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$$

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THE QCD VACUUM: AXIAL SYMMETRY BREAKING

- η_0 becomes heavy compared to the octet mesons due to anomalous breaking of $U_A(1)$ axial symmetry:

$$\partial_\mu \widehat{A}^{a\mu} = (\overline{\psi} \gamma_5 \widehat{M, t^a} \psi) + \delta^{a0} \sqrt{2N_f} \widehat{\omega}, \quad a = 0, \dots, 8$$

- $SU(3)$ flavour symmetry for $m_s = m_\ell$:

$$\eta = \eta_8 = \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}}, \quad \eta' = \eta_0 = \frac{u\bar{u} + d\bar{d} + s\bar{s}}{\sqrt{3}}$$

- Away from flavour symmetric limit, η and η' states are not flavour eigenstates and

$$\eta \neq \eta_8, \quad \eta' \neq \eta_0$$

- Four independent decay constants defined in the singlet/octet ($a = 0, 8$) basis:

$$\langle 0 | \widehat{A}_\mu^a | \mathcal{M} \rangle = i p_\mu F_\mathcal{M}^a,$$

MOTIVATION

- Masses well-known from experiment, but important benchmark quantity
- Decay constants required for many phenomenological applications, e.g. transition form factors $F_{\gamma\gamma^*\rightarrow\eta^{(\prime)}}$ at large Q^2 .
- Decay constants have never before been obtained from first principles and without model assumptions, see, e.g.,
ETMC (arxiv:1710.07986) for a lattice determination relating them to the pseudoscalar matrix elements
- Check of NLO large- N_c $U(3)$ ChPT using two mass trajectories and including the physical point
- Renormalization scale dependence of the singlet decay constants have frequently been ignored in the literature
In the \overline{MS} scheme: $\gamma_A^0 \neq 0$

OUTLINE

Lattice setup and efficient trace estimation

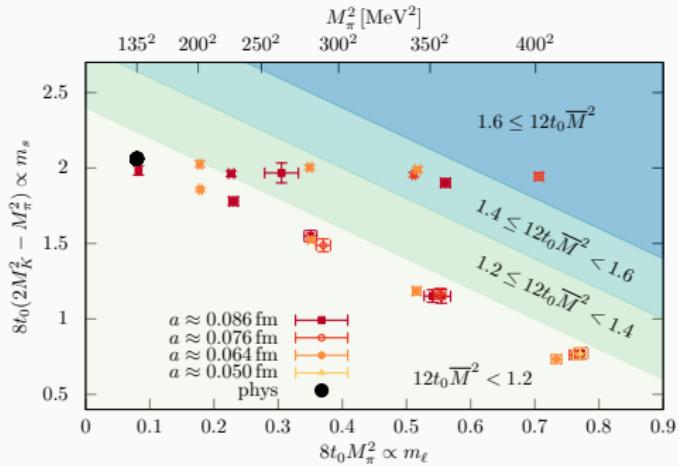
Fitting to matrices of correlators

Physical point results for masses and decay constants

Gluonic matrix elements, topological susceptibility and singlet Ward identity

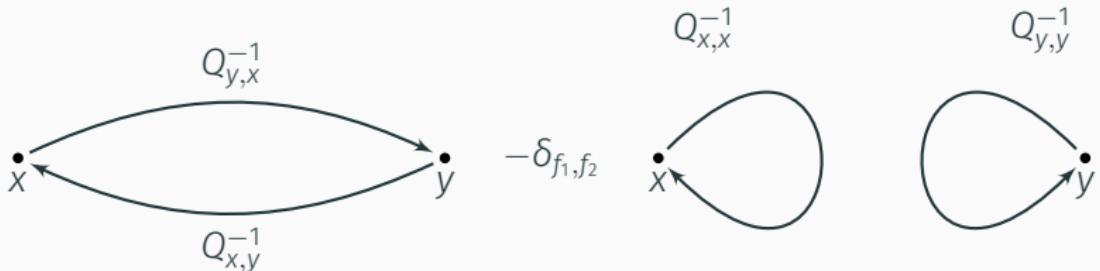
Lattice setup and efficient trace estimation

CLS ENSEMBLES AND ACTION



- analysed 21 CLS ensembles
- four lattice spacings
 $0.049 \text{ fm} \leq a \leq 0.086 \text{ fm}$
- $N_f = 2 + 1$ non-perturbatively improved Wilson-Clover fermions, tree-level improved gauge action
- two mass trajectories leading to (and including) the physical point:
 - $\text{tr}M = \text{const}$
 - $\hat{m}_s = \hat{m}_s^{\text{ph}}$

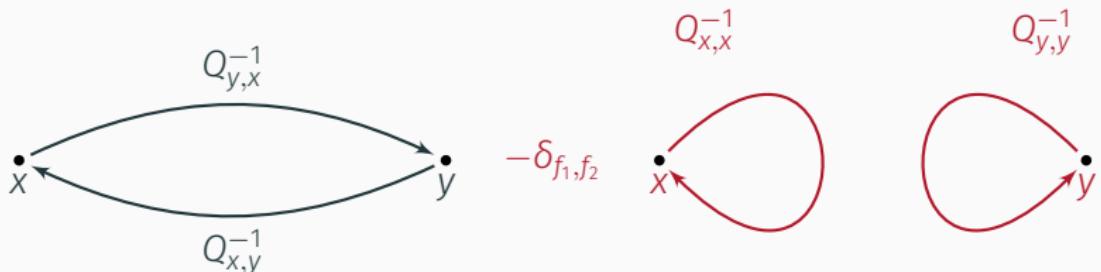
STOCHASTIC ESTIMATION OF DISCONNECTED LOOPS



Wick contractions of mesons ($Q = \gamma_5 D$):

$$\begin{aligned}\langle q_{f_1}(y)\gamma_5\bar{q}_{f_2}(y)\bar{q}_{f_1}(x)\gamma_5q_{f_2}(x) \rangle &= \boxed{q_{f_1}(y)\gamma_5\bar{q}_{f_2}(y)}\boxed{\bar{q}_{f_1}(x)\gamma_5q_{f_2}(x)} \\ &= Q_{f_1,f_1}^{-1}(y,x)Q_{f_2,f_2}^{-1}(x,y) - \delta_{f_1,f_2}Q_{f_1,f_2}^{-1}(y,y)Q_{f_1,f_2}^{-1}(x,x)\end{aligned}$$

STOCHASTIC ESTIMATION OF DISCONNECTED LOOPS



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→ disconnected loops arise

STOCHASTIC ESTIMATION OF DISCONNECTED LOOPS

Momentum-projected loop, applying a smearing kernel φ^s :

$$L_s^{\Gamma,f}(\vec{p}, t) = a^3 \sum_{\vec{x}, \vec{y}, \vec{z}} \text{tr} \left(e^{-i\vec{p} \cdot \vec{x}} \varphi^s(x, y) \Gamma D_f^{-1}(y, z) \varphi^s(z, x) \right),$$

Stochastic estimation is required for the inversion of D_f :

The N_{stoch} linear systems $D | s_i \rangle = | \eta_i \rangle$ are solved on random sources
 $\eta_{ixa} \in (\mathbb{Z}_2 + i\mathbb{Z}_2)/\sqrt{2}$

$$D_f^{-1} = \frac{1}{N_{stoch}} \sum_i^{N_{stoch}} | s_i \rangle \langle \eta_i | + \mathcal{O}\left(\frac{1}{\sqrt{N_{stoch}}}\right)$$

NOISE REDUCTION TECHNIQUES

Time dilution [Bernardson et al.,1993; Viehoff et al.,1998; O'Cais et al.,2005](#)

- put random sources at every 4th time slice
- set source at (open) boundaries to zero

Hopping parameter expansion [Thron et al.,1998; Michael et al., 2000; Bali et al., 2005](#)

- use locality of the Wilson Dirac operator and expand in small κ
- using two and four applications for the pseudoscalar and axialvector loops, respectively

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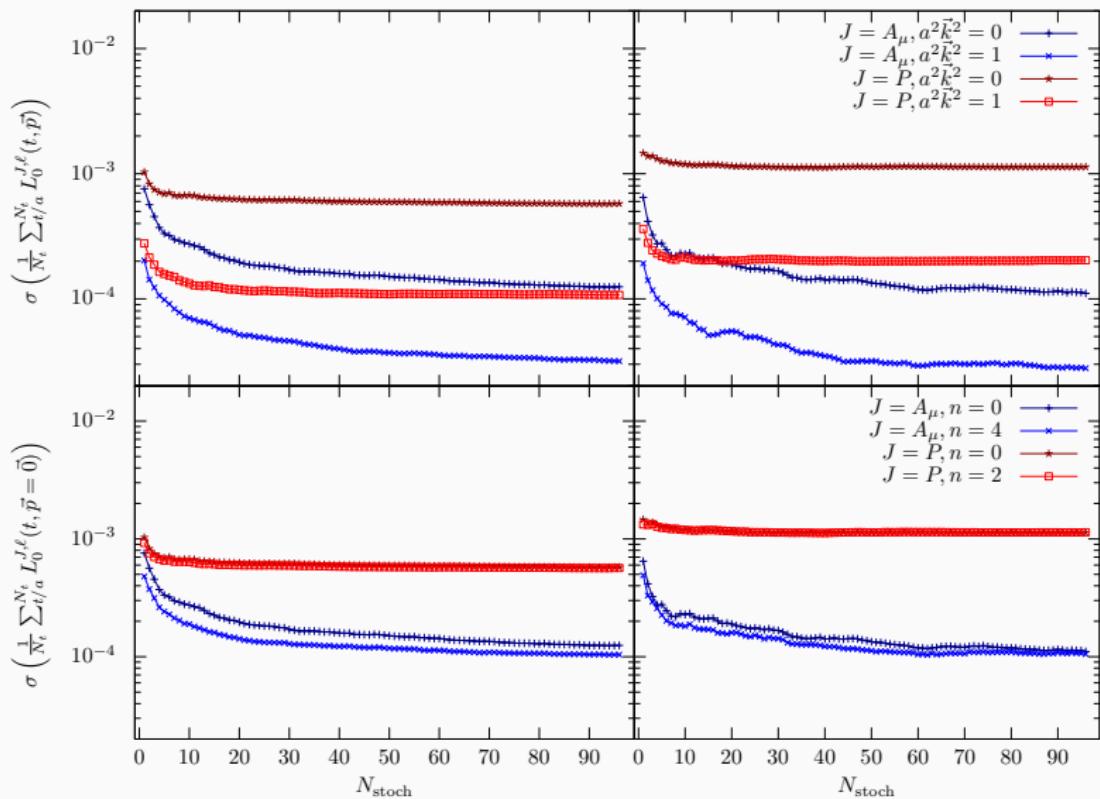
Hopping parameter expansion [Thron et al.,1998; Michael et al., 2000; Bali et al., 2005](#)

- use locality of the Wilson Dirac operator and expand in small κ
 - using two and four applications for the pseudoscalar and axialvector loops, respectively
- + efficient solvers and comparably expensive setup / DD- α -AMG
- Smearing is a significant fraction of the total cost
 - renders many approximate methods (eigenmode deflation, TSM, AMA,...) inefficient for our case

STOCHASTIC ESTIMATION OF LOOPS

H101

D150



Fitting to matrices of correlators

MATRICES OF CORRELATORS

- Construct N bases from n biquark fields:

$$b_i(t, \vec{p}) = \sum_{j=0}^{n-1} B_{ij} \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} (\bar{q}_j \gamma_5 q_j)(x),$$

where $B \in \mathbb{R}^{N \times n}$ matrix that defines a basis and subscripts are superindices defining flavour and smearing.

- matrix of correlators:

$$C(t) = \frac{1}{N_t} \sum_{t'=0}^{N_t-1} \begin{pmatrix} \langle b_1(t'+t) | b_1(t') \rangle & \cdots & \langle b_1(t'+t) | b_N(t') \rangle \\ \vdots & \ddots & \vdots \\ \langle b_N(t'+t) | b_1(t') \rangle & \cdots & \langle b_N(t'+t) | b_N(t') \rangle \end{pmatrix}.$$

- Note: In infinite statistics, C is a **real, positive semidefinite and symmetric** matrix, containing both connected and disconnected pieces.

- Usually: solve GEVP to diagonalize C
- Instead: decompose correlator in terms of orthogonal (mass) eigenstates:

$$\begin{aligned} C(t)_{ij} &= \sum_n \frac{1}{2E_n} \langle b_i | n \rangle \langle n | b_j \rangle \exp(-E_n t) \\ &= [Z D(t) Z^T]_{ij}, \end{aligned}$$

where

$$Z_{in} = \frac{1}{\sqrt{2E_n}} \langle b_i | n \rangle, \quad \text{and} \quad D(t) = \text{diag}_{n=0}(\exp(-E_n t))$$

- Note that Z does *not* depend on t .
- Need to truncate the sum: $Z : N \times \infty \rightarrow Z : N \times N_{\text{st}}$
- There is no need for Z to be quadratic, may fit more or fewer states N_{st} than available bases N .

Derivative trick

Replacing correlators with their derivatives removes any constant shifts in the correlator and decreases autocorrelations (similar to [Takashi \(arxiv:hep-lat/0701005\)](#), [Feng et al. \(arxiv:0909.3255\)](#)):

$$C(t) = ZD(t)Z^T \mapsto \partial_t C(t) = -ZED(t)Z^T$$

Combine data with dispersion relation

Signals typically better at $p^2 > 0$: Combine several correlators in a joint fit, using the dispersion relation:

$$E(p) = \sqrt{m^2 + p^2}, \quad p_i = \frac{2\pi k_i}{L}$$

EXCITED STATES



- η' signal deteriorates at short Euclidean times
- plethora of states nearby
- coverage of excited states depends on the (octet or singlet) nature of these states and the overlap with the chosen basis
- need to include these states in the fit \rightarrow need at least a 3×3 matrix
- typical fit windows $t \in [0.35, 1]$ fm
- fit functions contain multi-exponentials:

$$C_{ij}(t) = \sum_{n=0}^{N-1} Z_{in} D_{nn} Z_{jn} = \sum_{n=0}^{N-1} Z_{in} Z_{jn} \exp(-E_n t)$$

GENERALIZED EFFECTIVE MASSES

- Excited states become visible when looking at

$$(\partial_t C(t))C^{-1} = -ZEZ^{-1} + \mathcal{O}(\exp(-(E_{N_{\text{st}}} - E_{N_{\text{st}}-1})t))$$

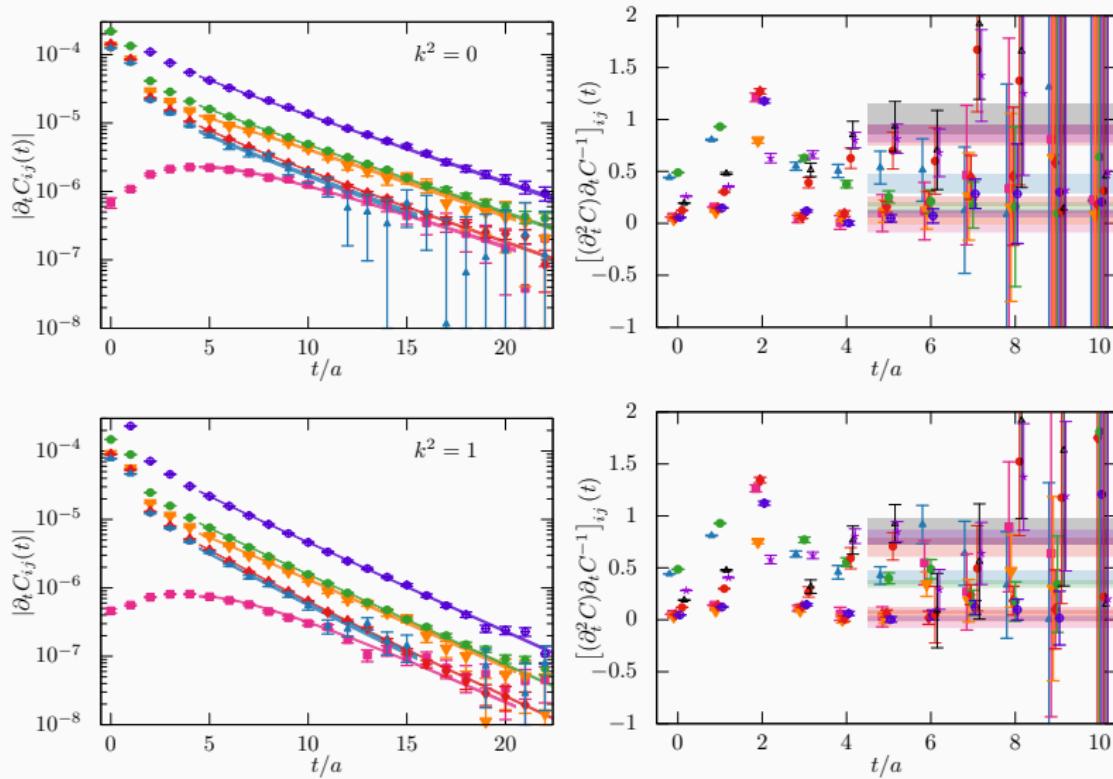
- leading term ZEZ^{-1} does not depend on time
- generalization of simple effective masses (for single correlators):

$$(\partial_t C(t))C^{-1} = \partial_t \log(C(t))$$

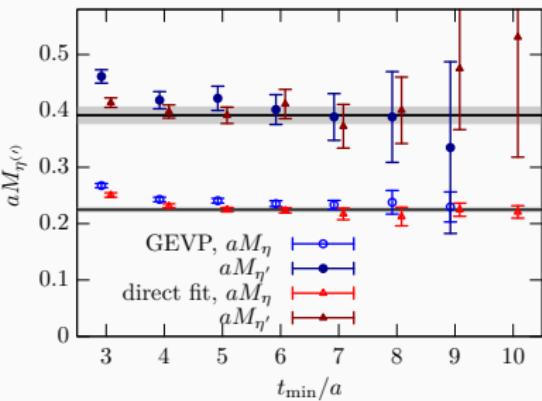
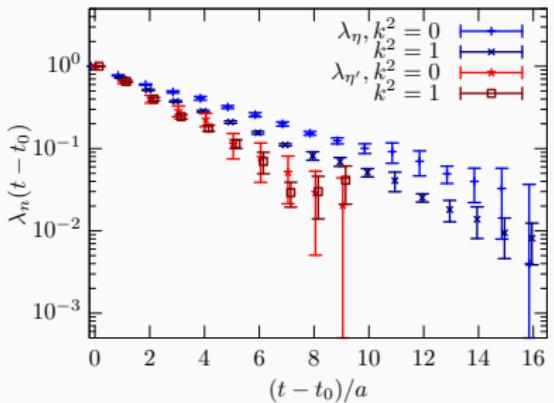
- can be included in a (joint) fit and helps constraining amplitudes (Z)
- leading term unchanged when using higher derivatives

$$(\partial_t^2 C(t))(\partial_t C)^{-1} = -ZEZ^{-1} + \mathcal{O}(\exp(-(E_{N_{\text{st}}} - E_{N_{\text{st}}-1})t))$$

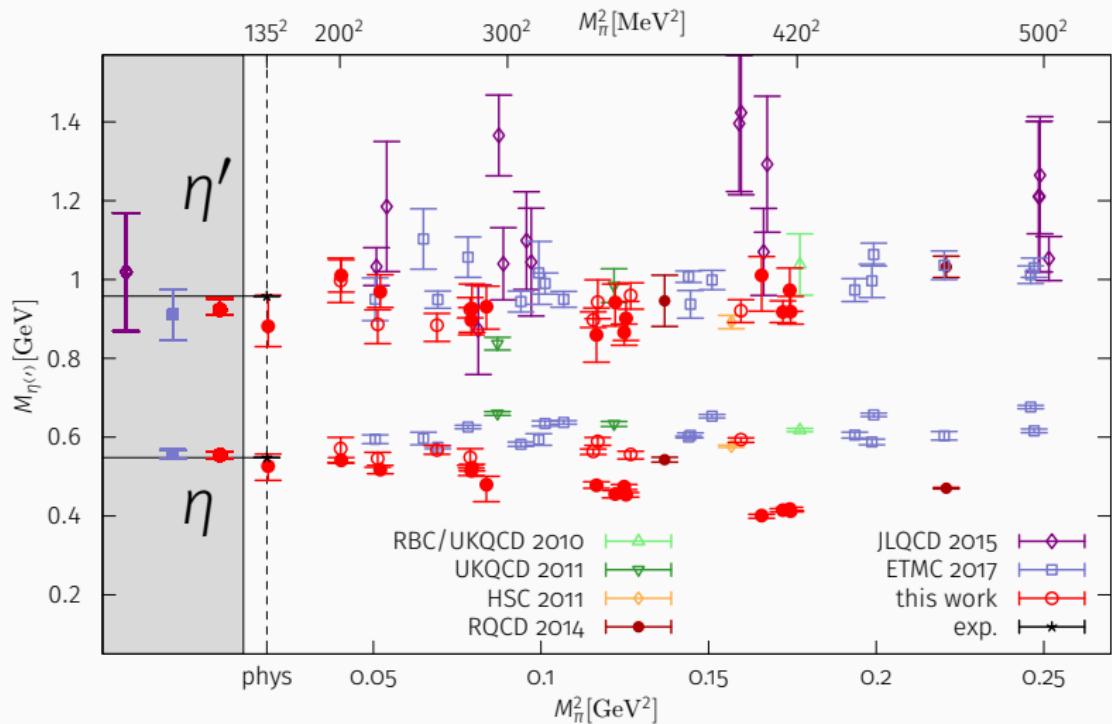
FITTING (EXAMPLE: H105, $M_\pi \approx 280$ MEV)



GEVP AND FITTING METHOD



Physical point results for masses and decay constants



$$f_O(a, \overline{M}^2, \delta M^2) =$$

$f_O^{\text{cont}}(\overline{M}^2, \delta M^2 | F, L_5, L_8, M_0^2, \Lambda_1, \Lambda_2)$ Large- N_c ChPT continuum

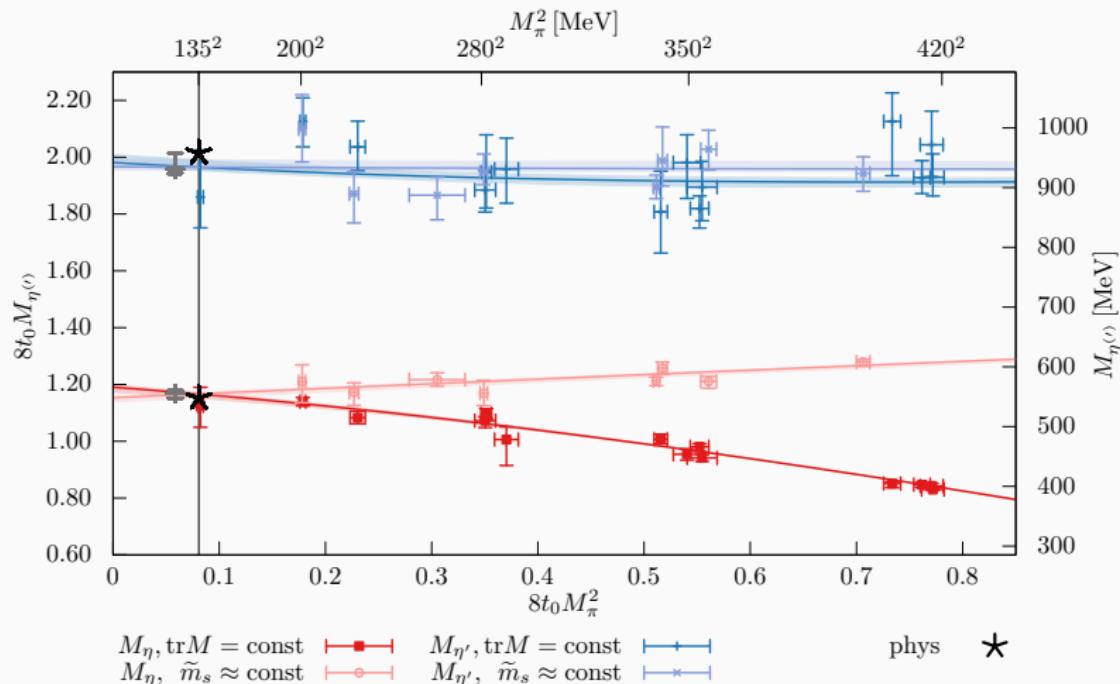
$\times h_O^{(1)}(a, am_\ell, am_s | f_A^l, d_A^l, \tilde{d}_A^l, \delta c_A^l)$ $\mathcal{O}(a)$ improvement

$\times h_O^{(2)}(a^2/t_0^*, a^2\overline{M}^2, a^2\delta M^2 | l_O, m_O, n_O)$ $\mathcal{O}(a^2)$ terms

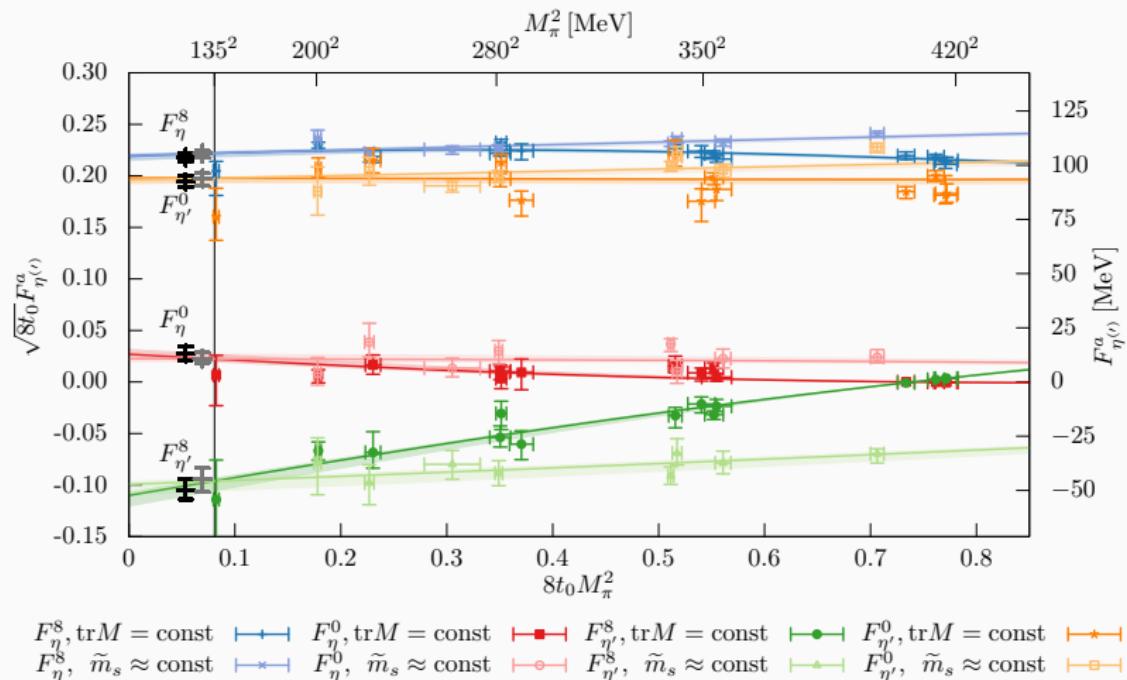
where $O \in \{M_\eta, M_{\eta'}, F_\eta^8, F_\eta^0, F_{\eta'}^8, F_{\eta'}^0\}$ and $h_{M_\eta}^{(1)} = h_{M_{\eta'}}^{(1)} = 1$

- Fix numerically irrelevant lattice spacing terms to zero
- $\mathcal{O}(a)$ improvement for decay constants: d_A (singlet) and f_A (octet) seem to be particularly important
- Combined, fully correlated fit gives $\chi^2/N_{\text{df}} \approx 179/122 \approx 1.47$

PHYSICAL POINT RESULTS: MASSES

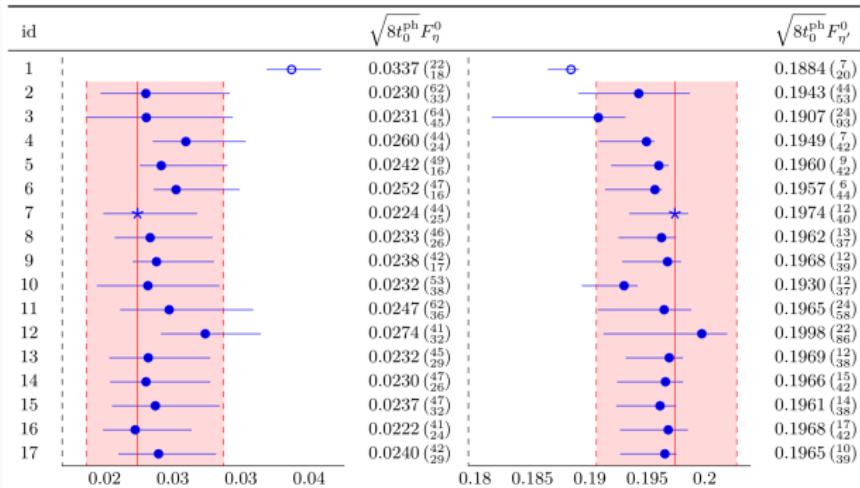


PHYSICAL POINT RESULTS: DECAY CONSTANTS



SYSTEMATICS

- Volume: only large volumes: $L_s^3 > (2.2 \text{ fm})^3 \gg R_\eta^3 \approx R_\pi^3$
Bernstein (arxiv:1511.03242) and typically $L_s M_\pi > 4$
- Lattice spacing: vary parametrization of discretization effects
- NLO large- N_c ChPT: impose cutoffs on the average (non-singlet) pseudoscalar mass: $\overline{M}^2 \leq \overline{M}_{\max}^2$, $12t_0 \overline{M}_{\max}^2 \in \{1.2, 1.4, 1.6\}$
- Renormalization: matching to PT done at $\mu \in \{a^{-1}/2, a^{-1}, 2a^{-1}\}$



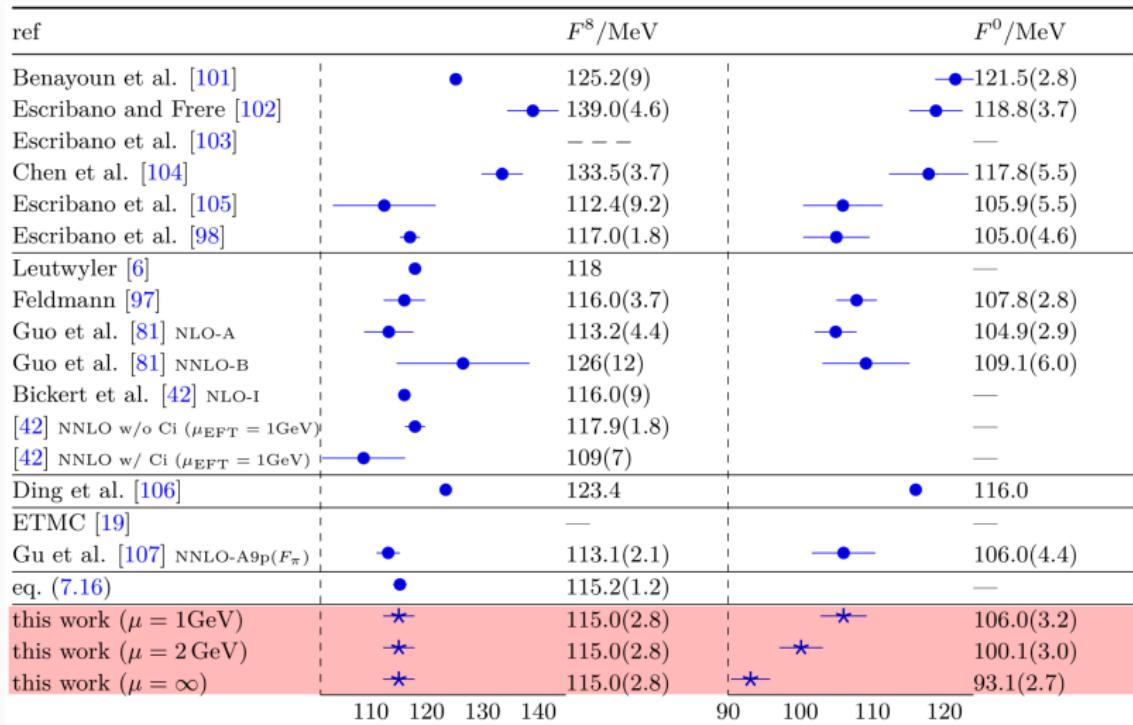
PHYSICAL POINT RESULTS

- Scale setting using $(8t_0^{\text{ph}})^{-1/2} = 475(6) \text{ MeV}$
Bruno et al. (arxiv:1608.08900)
- Masses agree with experiment ($M_{\eta}^{\text{phys}} = 547.9 \text{ MeV}$ and
 $M_{\eta'}^{\text{phys}} = 957.8 \text{ MeV}$, [PDG](#))

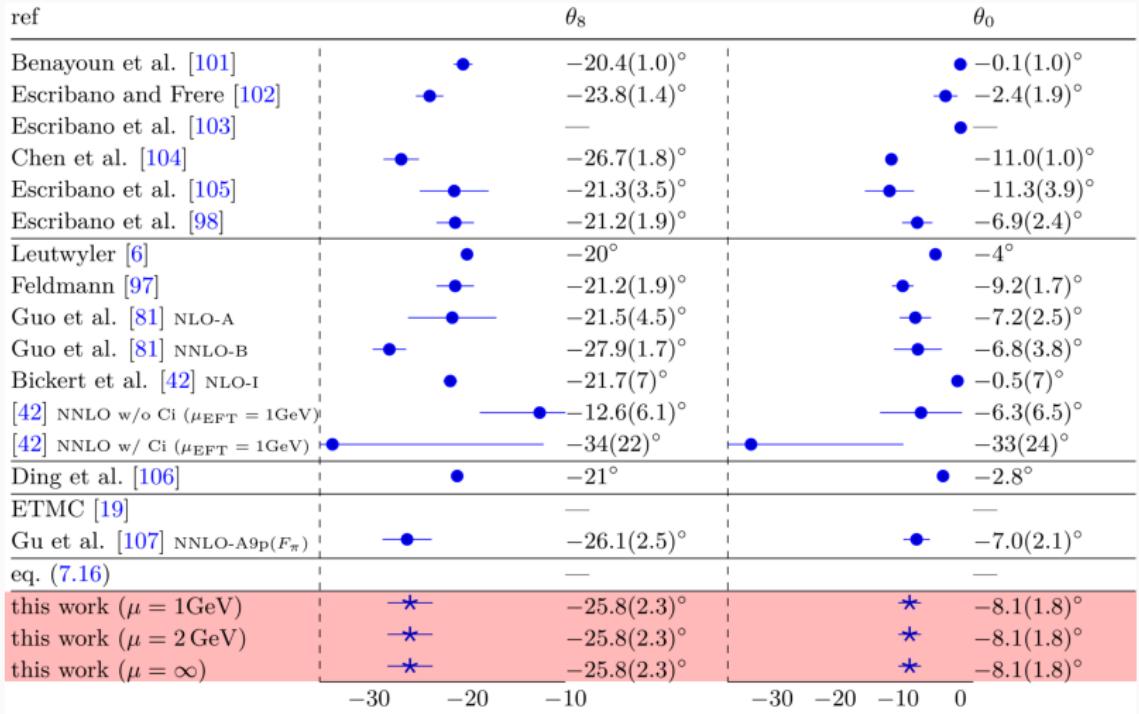
$$M_{\eta} = 554.7 \left(\frac{4.0}{6.6}\right)_{\text{stat}} \left(\frac{2.4}{2.7}\right)_{\text{syst}} (7.0)_{t_0}, \text{ MeV}$$

$$M_{\eta'} = 929.9 \left(\frac{12.9}{6.0}\right)_{\text{stat}} \left(\frac{22.9}{3.3}\right)_{\text{syst}} (11.7)_{t_0} \text{ MeV},$$

PHYSICAL POINT RESULTS



PHYSICAL POINT RESULTS



Gluonic matrix elements, topological susceptibility and singlet Ward identity

$$\partial_\mu \widehat{A}_\mu^a = (\overline{\psi} \gamma_5 \widehat{\{M, t^a\}} \psi) + \sqrt{2N_f} \delta^{a0} \widehat{\omega},$$

where $M = \text{diag}(m_\ell, m_\ell, m_s)$. For $a = 8, 0$ this leads to

$$\partial_\mu \widehat{A}_\mu^8 = \frac{2}{3} (\widehat{m}_\ell + 2\widehat{m}_s) \widehat{P}^8 - \frac{2\sqrt{2}}{3} \delta \widehat{m} \widehat{P}^0,$$

$$\partial_\mu \widehat{A}_\mu^0 = \frac{2}{3} (2\widehat{m}_\ell + \widehat{m}_s) \widehat{P}^0 - \frac{2\sqrt{2}}{3} \delta \widehat{m} \widehat{P}^8 + \sqrt{6} \widehat{\omega},$$

$$\partial_\mu \widehat{A}_\mu^a = (\overline{\psi} \gamma_5 \widehat{\{M, t^a\}} \psi) + \sqrt{2N_f} \delta^{a0} \widehat{\omega},$$

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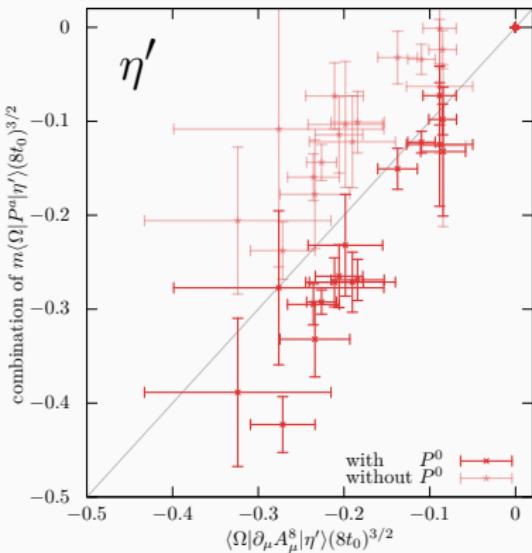
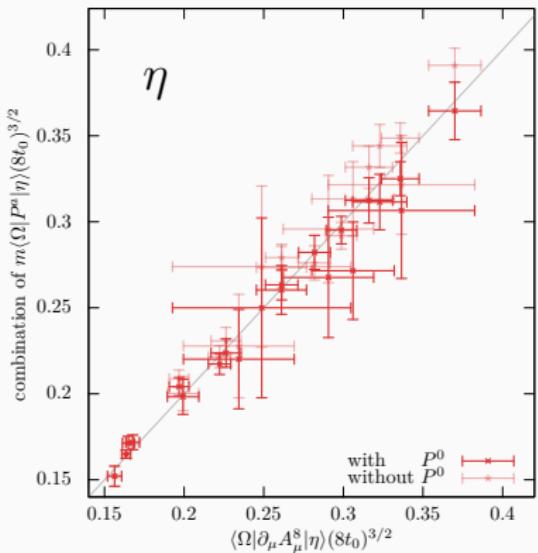
$$\partial_\mu \widehat{A}_\mu^0 = \frac{2}{3} (2\widehat{m}_\ell + \widehat{m}_s) \widehat{P}^0 - \frac{2\sqrt{2}}{3} \delta \widehat{m} \widehat{P}^8 + \sqrt{6} \widehat{\omega},$$

Renormalization of $\widehat{\omega}$ makes things complicated.

Strategy:

1. check octet Ward identity
2. Compute $a_{\eta^{(\prime)}} = \langle \Omega | 2\widehat{\omega} | \eta^{(\prime)} \rangle$ from AV and PS matrix elements
3. Extract $a_{\eta^{(\prime)}}$ directly from gluonic correlators and check with 2).

CHECK OF THE OCTET AWI

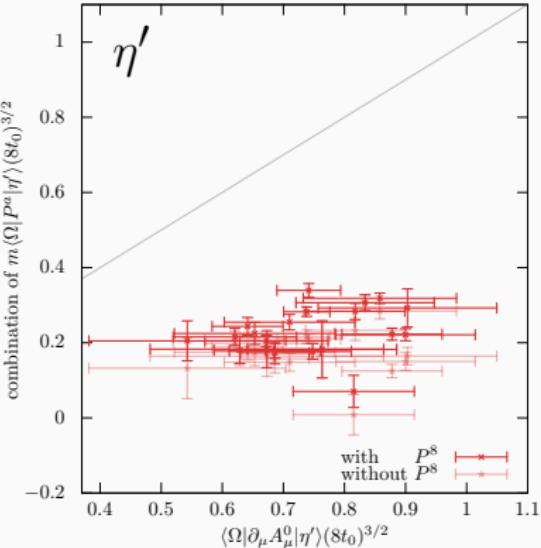
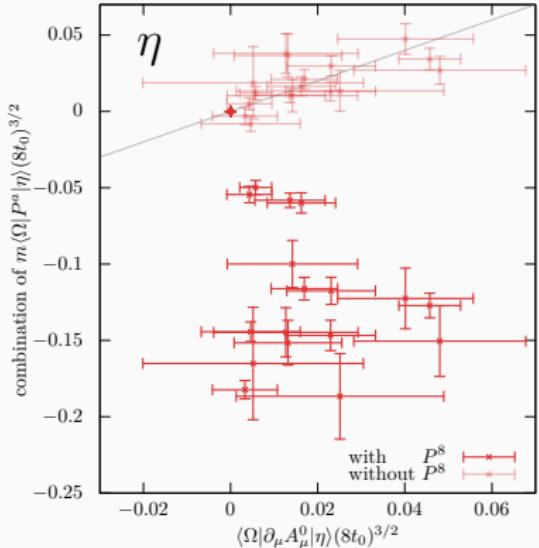


$$\partial_\mu \langle \Omega | A_\mu^8 | \mathcal{M} \rangle = \frac{2}{3} (\tilde{m}_\ell + 2\tilde{m}_s) \langle \Omega | P^8 | \mathcal{M} \rangle - \frac{2\sqrt{2}}{3} \delta \tilde{m} \, r_P \langle \Omega | P^0 | \mathcal{M} \rangle,$$

\tilde{m} bare AWI quark masses,

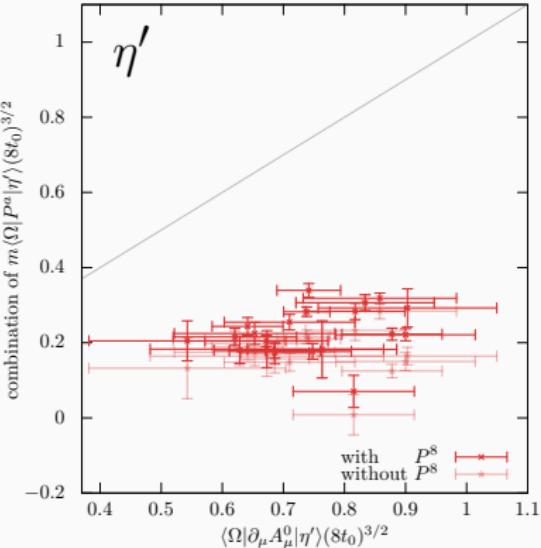
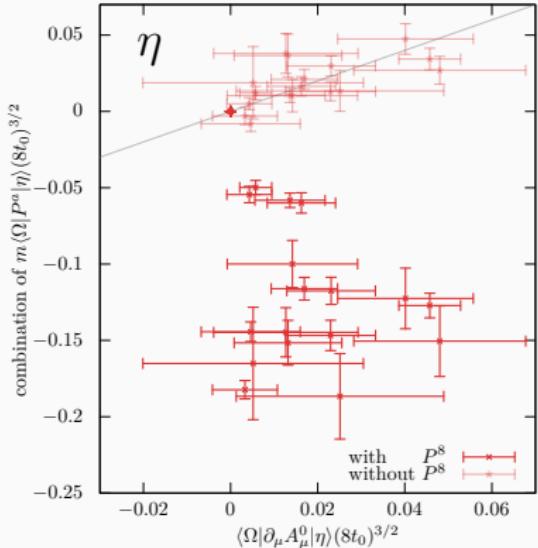
$r_P = 1 + \mathcal{O}(g^6)$ singlet-to-nonsinglet pseudoscalar renormalization

GLUONIC CONTRIBUTIONS TO THE SINGLET AWI?



$$\partial_\mu \langle \Omega | A_\mu^0 | \mathcal{M} \rangle \neq 2\bar{m} r_P \langle \Omega | P^0 | \mathcal{M} \rangle - \frac{2\sqrt{2}}{3} \delta \bar{m} \langle \Omega | P^8 | \mathcal{M} \rangle,$$

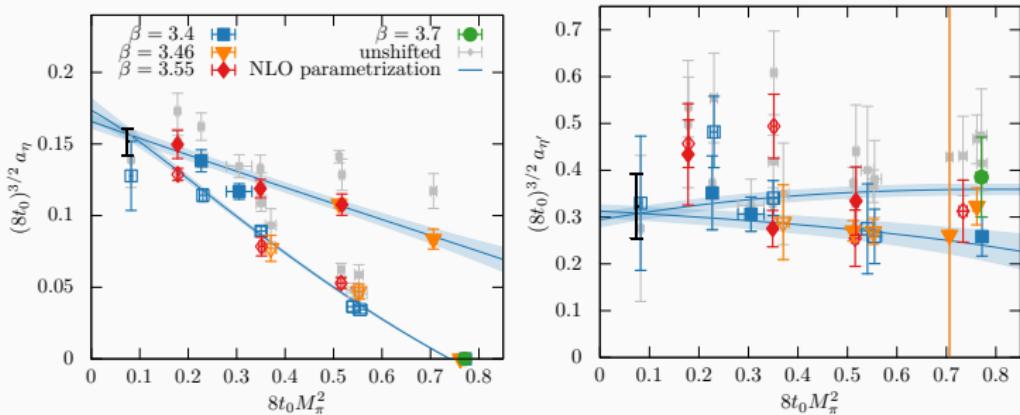
GLUONIC CONTRIBUTIONS TO THE SINGLET AWI?



$$\partial_\mu \langle \Omega | A_\mu^0 | \mathcal{M} \rangle \neq 2\bar{m} r_P \langle \Omega | P^0 | \mathcal{M} \rangle - \frac{2\sqrt{2}}{3} \delta \bar{m} \langle \Omega | P^8 | \mathcal{M} \rangle,$$

→ Sizable gluonic contribution missing: $\sqrt{\frac{3}{2}} \langle \Omega | 2\hat{\omega} | \mathcal{M} \rangle$

GLUONIC MATRIX ELEMENTS FROM THE SINGLET AWI



- Continuum fit parametrization: NLO large- N_c ChPT (using priors from the masses and decay constant fits)

- Three additional parameters for lattice spacing effects

- Combined, fully correlated $\chi^2/N_{\text{df}} \approx 34/31$

$$a_\eta(\mu = 2 \text{ GeV}) = 0.01700 \left(\frac{40}{69} \right)_{\text{stat}} \left(\frac{48}{66} \right)_{\text{syst}} \left(\frac{66}{t_0} \right) \text{ GeV}^3,$$

$$a_{\eta'}(\mu = 2 \text{ GeV}) = 0.0381 \left(\frac{18}{17} \right)_{\text{stat}} \left(\frac{80}{17} \right)_{\text{syst}} \left(\frac{17}{t_0} \right) \text{ GeV}^3.$$

Systematic error = difference to the NLO prediction using the set of LECs from the masses / decay constants fit.

COMPARISONS WITH LITERATURE

ref		a_η/GeV^3		$a_{\eta'}/\text{GeV}^3$
Novikov et al. [116]	•	0.021	•	0.035
Feldmann [97]	•	0.023	•	0.058
Beneke and Neubert [9]	•	0.022(2)	•	0.057(2)
Cheng et al. [118]	•	0.026(28)	•	0.054(57)
Singh [117]	•	0.0220(50)	•	0.037(10)
Qin et al. [119]	•	0.016	•	0.051
Ding et al. [106]	•	0.024	•	0.051
this work at $\mu = 1\text{GeV}$	*	0.0172(10)	*	0.0424(84)
this work at $\mu = 2\text{GeV}$	*	0.0170(10)	*	0.0381(84)
this work at $\mu = \infty$	*	0.0168(10)	*	0.0330(83)

0.00 0.01 0.02 0.03 0.00 0.02 0.04 0.06 0.08

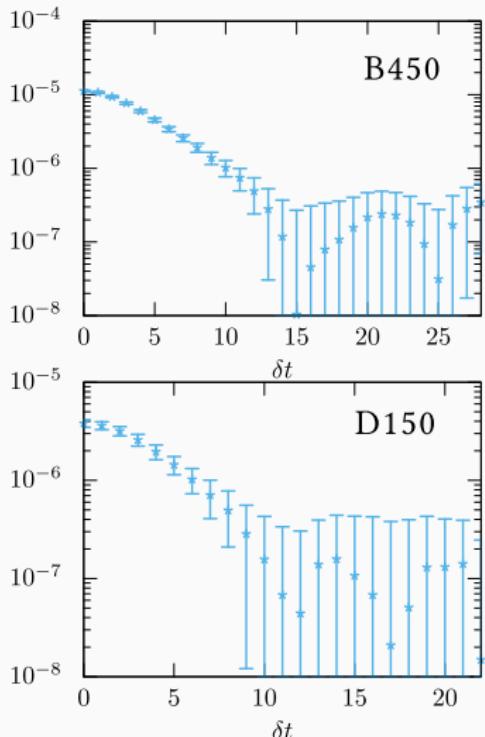
If decays $J/\psi \rightarrow \eta^{(\prime)}\gamma$ are dominated by the anomaly, then Novikov et al.

$$R(J/\psi) = \frac{\Gamma[J/\psi \rightarrow \eta'\gamma]}{\Gamma[J/\psi \rightarrow \eta\gamma]} = \frac{a_{\eta'}^2}{a_\eta^2} \left(\frac{k_{\eta'}}{k_\eta} \right)^3$$

$$R(J/\psi, \mu = 2\text{ GeV}) = 5.03(19)_{45}\text{stat}(1.94)\text{syst} \Leftrightarrow R(J/\psi) = 4.74(13) \text{ (PDG)}$$

GLUONIC MATRIX ELEMENTS FROM THE SINGLET AWI

- Topological charge density:
 $\omega(x) = -\frac{1}{32\pi^2} \text{tr} F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a$
- Fit matrix elements
 $\langle \Omega | 2\omega | \eta^{(\prime)} \rangle$ directly
- Decent signals at Wilson flowtime $t \approx t_0^*$
 $(\sqrt{8t_0^*} \approx 0.413 \text{ fm})$
- Complicated renormalization
(in the \overline{MS} scheme):
 $\hat{\omega}(\mu) = Z_\omega \omega + Z_{\omega A}(\mu) \partial_\mu A_\mu^0$
- no mixing with $a^{-1}P^0$ after Wilson flow



$$\langle \Omega | \omega(\delta t) | P^\ell(0) \rangle$$

RENORMALIZATION (I)

$$\hat{\omega}(\mu) = Z_{\omega}\omega + Z_{\omega A}(\mu)\partial_{\mu}A_{\mu}^0$$

$$Z_{\omega} = 1??$$

- Expected if extracted after gradient flow

Del Debbio, Pica (arxiv:hep-lat/0309145)

- Topological susceptibility:

$$\hat{\tau} = \frac{1}{V} \sum_{x,y} \langle \hat{\omega}(x) \hat{\omega}(y) \rangle = Z_{\omega}^2 \frac{\langle Q^2 \rangle}{V}$$

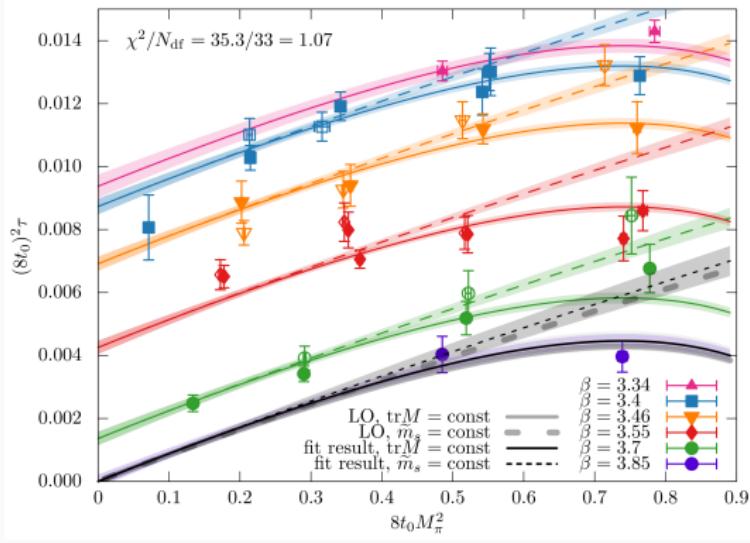
- LO expectation for $N_f = 2 + 1$:

$$\tau = \frac{1}{2} F^2 \left(\frac{1}{2M_K^2 - M_\pi^2} + \frac{2}{M_\pi^2} \right)^{-1}$$

RENORMALIZATION (I)

$$\hat{\omega}(\mu) = Z_w \omega + Z_{\omega A}(\mu) \partial_\mu A_\mu^0$$

$Z_\omega = 1??$



- fit to 37 CLS ensembles
- sizable lattice spacing effects explained by three parameters ($\propto a^n / (t_0^*)^{n/2}$, $n = 2, 3, 4$)
- $\frac{\sqrt{8t_0^\chi} F}{Z_w} = 0.190(13)$ (this fit) vs. $\sqrt{8t_0^\chi} F = 0.1866(48)$ ($\eta^{(')}$ fit)

RENORMALIZATION (II)

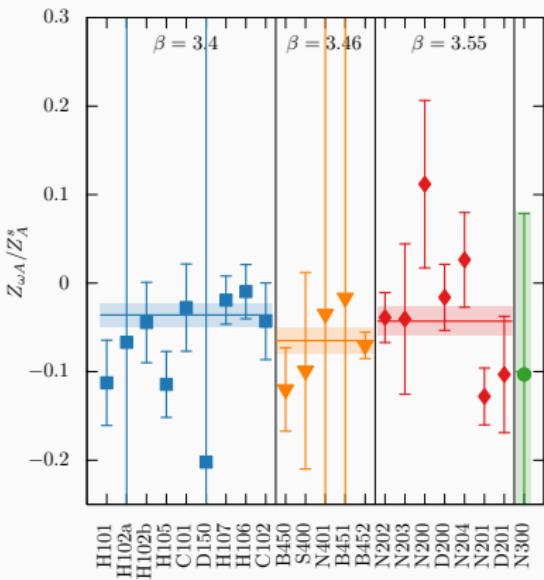
$$\hat{\omega}(\mu) = Z_{\omega}\omega + Z_{\omega A}(\mu)\partial_{\mu}A_{\mu}^0$$

$Z_{\omega A}$ unknown, one value per β : Use fermionic determination as additional input

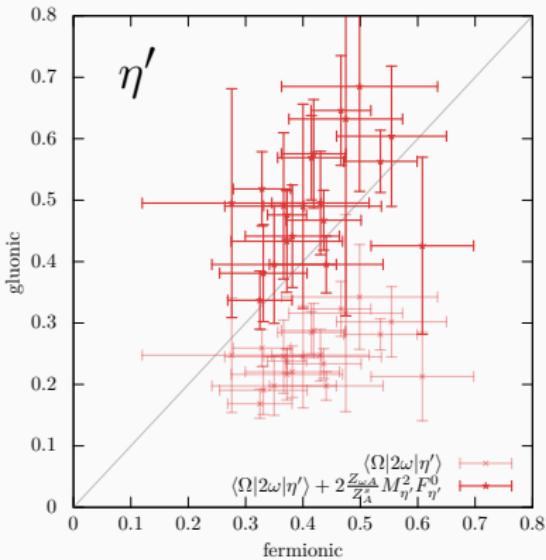
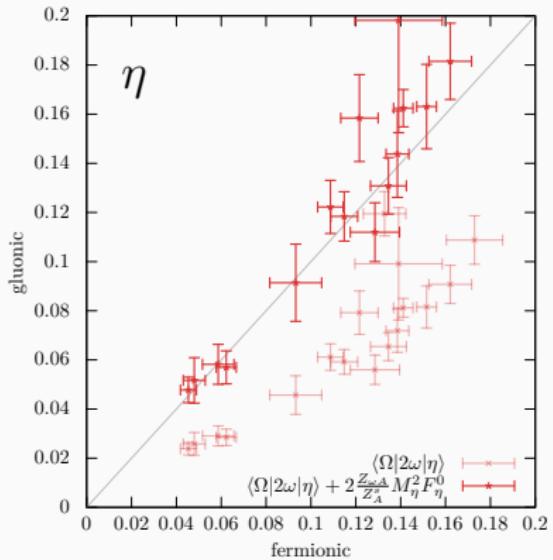
Solve $a_{\eta'} =$
 $Z_{\omega}\langle\Omega|2\omega|\eta'\rangle + 2\frac{Z_{\omega A}}{Z_A^s}\partial_{\mu}\langle\Omega|A_{\mu}^0|\eta'\rangle$:

$$\frac{Z_{\omega A}}{Z_A^s} = \frac{a_{\eta'} - Z_{\omega}\langle\Omega|2\omega|\eta'\rangle}{2\partial_{\mu}\langle\Omega|A_{\mu}^0|\eta'\rangle},$$

assuming $Z_{\omega} = 1$ and taking weighted averages for every lattice spacing



CHECK OF SINGLET AWI



Clear evidence for $Z_{\omega A} \neq 0!$

SUMMARY

- precise, ab initio determinations of η/η' masses and decay constants.
- Decay constants: first such determination directly from axialvector matrix elements, without resorting to large- N_c ChPT.
- Made possible by a combination of (quite basic) noise reduction techniques – and an efficient solver.
- Gluonic matrix elements: Fermionic and direct calculation enables first check of singlet AWI with Wilson fermions

More details: JHEP 08 (2021) 137 (arxiv:2106.05398)

Large- N_c ChPT

LARGE- N_c LIMIT

- Witten-Veneziano relation: in the t'Hooft limit (large- N_c , fixed N_f):

$$M_0^2 = \frac{2N_f}{F^2} \tau_0,$$

where τ_0 is the quenched topological susceptibility and F is the pseudoscalar decay constant in the chiral limit (normalization $F \approx 92$ MeV)

- Anomaly vanishes in the limit $N_c \rightarrow \infty$:

$$M_0 \xrightarrow{1/N_c \rightarrow 0} 0$$

- Large- N_c ChPT: Simultaneous expansion around small masses and $1/N_c$ with power counting

$$p = \mathcal{O}(\sqrt{\delta}), \quad m = \mathcal{O}(\delta), \quad 1/N_c = \mathcal{O}(\delta).$$

Hence, above: $F = \mathcal{O}(\delta^{-1/2})$ and $M_0^2 = \mathcal{O}(\delta)$

LARGE- N_c CHPT

- No mixing with π^0 in the isospin limit
- Mass matrix:

$$R \begin{pmatrix} \mu_8^2 & \mu_{80}^2 \\ \mu_{80}^2 & \mu_0^2 \end{pmatrix} R^\top = \begin{pmatrix} M_\eta^2 & 0 \\ 0 & M_{\eta'}^2 \end{pmatrix}, \quad \text{where} \quad R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

- To leading order:

$$\mu_8^2 = 2B_0(m_\ell + 2m_s) = \overline{M}^2 + \frac{1}{3}\delta M^2,$$

$$\mu_0^2 = 2B_0(m_\ell + m_s) + M_0^2 = \overline{M}^2 + M_0^2$$

$$\mu_{80}^2 = -\frac{2\sqrt{2}}{3}B_0(m_s - m_\ell) = -\frac{\sqrt{2}}{3}\delta M^2, \quad \tan 2\theta = -2\sqrt{2} \frac{\delta M^2}{3M_0^2 - \delta M^2},$$

- Convenient quark mass parametrization:

$$\overline{M}^2 = \frac{1}{3}(M_K^2 + M_\pi^2) \approx 2B_0\overline{m} \quad \text{and} \quad \delta M^2 = 2(M_K^2 - M_\pi^2) \approx 2B_0\delta m$$

NLO LOW ENERGY CONSTANTS

To NLO: six LECs $F, L_5, L_8, M_0^2, \Lambda_1, \Lambda_2$

- F is the non-singlet decay constant in the chiral limit
- L_5, L_8 defined as in standard SU(3) ChPT and can be converted *Kaiser et al. (arxiv:hep-ph/0007101)*
- Λ_1, Λ_2 and M_0^2 depend on the renormalization scale
- Many phenomenological studies:
 - scale dependence is ignored (fixed by experiment)
 - often assume $\Lambda_1 = \Lambda_2 = 0$ (OZI suppressed)
- EFT scale dependence only enters at NNLO

NNLO expressions available, too: *Bickert et al., Guo et al.*

DECAY CONSTANTS

Four independent decay constants defined in the singlet/octet ($a = 0, 8$) basis:

$$\langle 0 | \hat{A}_\mu^a | \mathcal{M} \rangle = i p_\mu F_\mathcal{M}^a,$$

Possible parametrization:

$$\begin{pmatrix} F_\eta^8 & F_\eta^0 \\ F_{\eta'}^8 & F_{\eta'}^0 \end{pmatrix} = \begin{pmatrix} F^8 \cos \theta_8 & -F^0 \sin \theta_0 \\ F^8 \sin \theta_8 & F^0 \cos \theta_0 \end{pmatrix}.$$

- Singlet and octet decay constants renormalize differently
- Non-singlet Z_A known non-perturbatively [Bulava et al.](#)
- Singlet: difference to non-singlet known perturbatively to two loops [Constantinou et al. \(arxiv:1610.06744\)](#)
- Non-vanishing anomalous dimension $\gamma_{A_\mu^0} \neq 0$
→ singlet decay constants depend on renormalization scale
- 1-loop coefficient $\gamma_{A_\mu^0}^{(0)} = 0$
→ finite renormalization at $\mu = \infty$.

FLAVOUR BASIS AND FELDMANN-KROLL-STECH SCHEME (I)

Exploiting flavour SU(3) suggests a different scheme ($q = \ell, s$):

$$\langle 0 | \hat{A}_\mu^q | \mathcal{M} \rangle = i\sqrt{2} p_\mu F_\mathcal{M}^q$$

related to the octet/singlet basis by a rotation,

$$\begin{pmatrix} F_\mathcal{M}^8 \\ F_\mathcal{M}^0 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} F_\mathcal{M}^\ell \\ F_\mathcal{M}^s \end{pmatrix}.$$

Again, in angle representation

$$F^\ell = \sqrt{(F_\eta^\ell)^2 + (F_{\eta'}^\ell)^2}, \quad F^s = \sqrt{(F_\eta^s)^2 + (F_{\eta'}^s)^2},$$

$$\tan \varphi_\ell = \frac{F_{\eta'}^\ell}{F_\eta^\ell}, \quad \tan \varphi_s = - \frac{F_\eta^s}{F_{\eta'}^s}.$$

Octet/Singlet basis: Only F^0 (or F_η^0 and $F_{\eta'}^0$) are renormalization scale dependent!

Flavour basis: All four decay constants are renormalization scale dependent!

FLAVOUR BASIS AND FELDMANN-KROLL-STECH SCHEME (II)

NLO large- N_c ChPT gives: *FKS* ([arxiv:hep-ph/9802409](https://arxiv.org/abs/hep-ph/9802409))

$$F^\ell(\mu)^2 = F_\pi^2 + \frac{2}{3} \Lambda_1(\mu) (2F_K^2 + F_\pi^2), \quad F^s(\mu)^2 = 2F_K^2 - F_\pi^2 - \frac{1}{3} \Lambda_1(\mu) (2F_K^2 + F_\pi^2)$$

- Scale dependence of singlet current translates to scale dependence of LECs, e.g.,

$$\mu \frac{d}{d\mu} \frac{F_0(\mu)}{\sqrt{1 + \Lambda_1(\mu)}} = 0.$$

- Λ_1 is OZI-suppressed. The FKS model is equivalent to setting $\Lambda_1 = \Lambda_2 = 0$.
- In effect, 4 decay constants reduce to three parameters in the flavour basis (two decay constants + one angle):

$$\frac{\sqrt{2}}{3} F_\pi^2 \Lambda_1 = F^\ell F^s \sin(\varphi_\ell - \varphi_s) \quad \rightsquigarrow \varphi := \varphi_\ell = \varphi_s$$

- Model renders singlet current scale independent

NLO EXPRESSIONS

Elements of the mass matrix:

$$(\mu_8^{\text{NLO}})^2 = (\mu_8^{\text{LO}})^2 + \frac{8}{3F^2} (2L_8 - L_5) \delta M^4,$$

$$(\mu_0^{\text{NLO}})^2 = (\mu_0^{\text{LO}})^2 + \frac{4}{3F^2} (2L_8 - L_5) \delta M^4 - \frac{8}{F^2} L_5 \bar{M}^2 M_0^2 - \tilde{\Lambda} \bar{M}^2 - \Lambda_1 M_0^2,$$

$$(\mu_{80}^{\text{NLO}})^2 = (\mu_{80}^{\text{LO}})^2 - \frac{4\sqrt{2}}{3F^2} (2L_8 - L_5) \delta M^4 + \frac{4\sqrt{2}}{3F^2} L_5 M_0^2 \delta M^2 + \frac{\sqrt{2}}{6} \tilde{\Lambda} \delta M^2,$$

The decay constants are given by

$$F_\eta^8 = F \left[\cos \theta + \frac{4L_5}{3F^2} \left(3 \cos \theta \bar{M}^2 + (\sqrt{2} \sin \theta + \cos \theta) \delta M^2 \right) \right],$$

$$F_{\eta'}^8 = F \left[\sin \theta + \frac{4L_5}{3F^2} \left(3 \sin \theta \bar{M}^2 + (\sin \theta - \sqrt{2} \cos \theta) \delta M^2 \right) \right],$$

$$F_\eta^0 = -F \left[\sin \theta \left(1 + \frac{\Lambda_1}{2} \right) + \frac{4L_5}{3F^2} \left(3 \sin \theta \bar{M}^2 + \sqrt{2} \cos \theta \delta M^2 \right) \right],$$

$$F_{\eta'}^0 = F \left[\cos \theta \left(1 + \frac{\Lambda_1}{2} \right) + \frac{4L_5}{3F^2} \left(3 \cos \theta \bar{M}^2 - \sqrt{2} \sin \theta \delta M^2 \right) \right],$$

LECs, their scale dependence and validity of approximations

RESULTS: LARGE- N_c CHPT LOW ENERGY CONSTANTS TO NLO

Low energy constants (all errors added in quadrature):

$$M_0(\mu = 2 \text{ GeV}) = 818(27) \text{ MeV}, \quad F = 87.7(2.8) \text{ MeV},$$

$$\Lambda_1(\mu = 2 \text{ GeV}) = -0.13(5), \quad L_5 = 1.66(23) \cdot 10^{-3},$$

$$\Lambda_2(\mu = 2 \text{ GeV}) = 0.19(10), \quad L_8 = 1.08(13) \cdot 10^{-3}.$$

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Scale independent combinations:

$$M_0/\sqrt{1 + \Lambda_1} = 877(22) \text{ MeV},$$

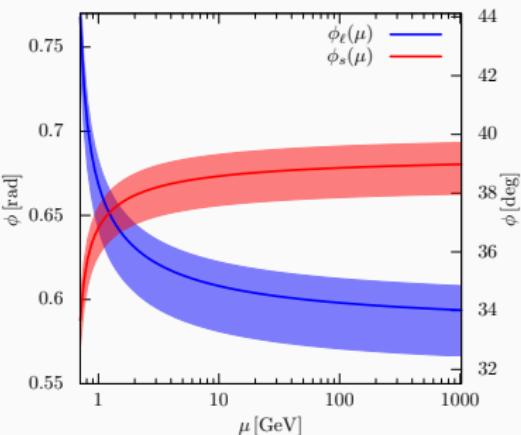
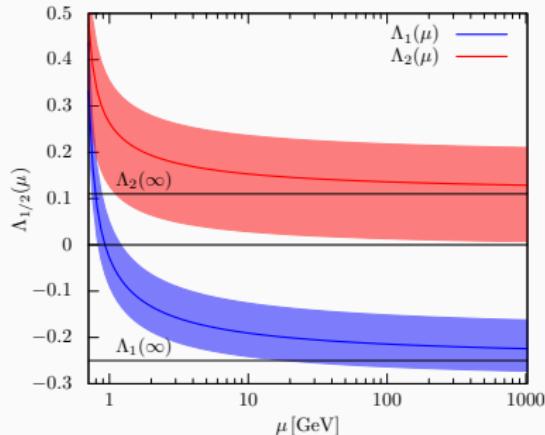
$$\tilde{\Lambda} = \Lambda_1 - 2\Lambda_2 = -0.46(19)$$

QCD SCALE DEPENDENCE OF LECS

ref		Λ_1		$F^0/\sqrt{1+\Lambda_1}/\text{MeV}$
Benayoun et al. [101]		0.20(4)		110.9(4.8)
Escribano and Frere [102]		0.34(10)		102.6(8.2)
Escribano et al. [103]		0		—
Chen et al. [104]		—		—
Escribano et al. [105]		0		—
Escribano et al. [98]		0.01(13)		105(11)
Leutwyler [6]		—	•	101
Feldmann [97]		0.0(3)	•	107.8(2.8)
Guo et al. [81] NLO-A		0.02(8)	•	102.8(7.0)
Guo et al. [81] NNLO-B		-0.04(14)	•	111(14)
Bickert et al. [42] NLO-I		—	•	104.1(0)
[42] NNLO w/o Ci ($\mu_{\text{EFT}} = 1\text{GeV}$)		0	•	79.2(9)
[42] NNLO w/ Ci ($\mu_{\text{EFT}} = 1\text{GeV}$)		0	•	76.4(9)
Ding et al. [106]		—	—	—
ETMC [19]		0	—	—
Gu et al. [107] NNLO-A9p(F_π)		0.24(21)	•	95(20)
eq. (7.16)		—	•	104.3(1.1)
this work ($\mu = 1\text{GeV}$)	*	-0.03(5)	*	107.3(2.2)
this work ($\mu = 2\text{ GeV}$)	*	-0.13(5)	*	107.3(2.2)
this work ($\mu = \infty$)	*	-0.25(5)	*	107.3(2.2)

-0.2 0 0.2 0.4 0.6 80 90 100 110 120

QCD SCALE DEPENDENCE OF LECS AND THE FKS MODEL



Reminder:

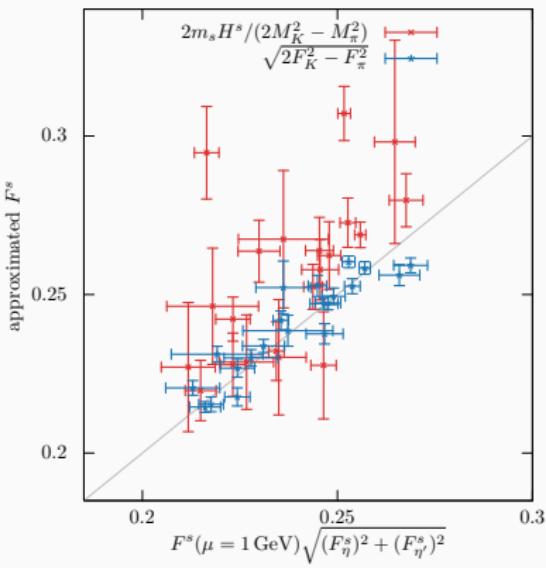
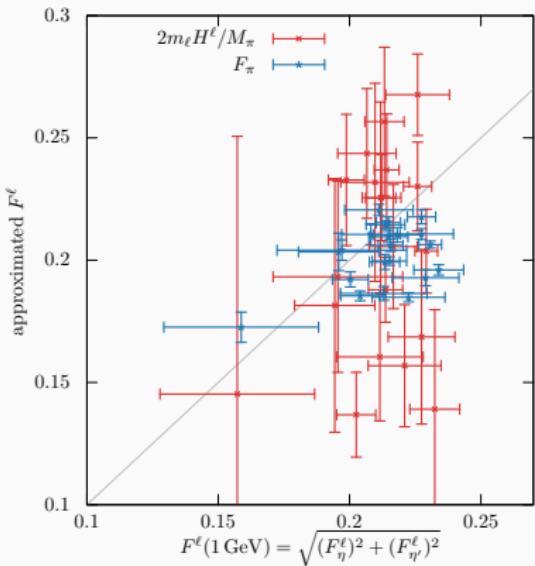
$$\frac{\sqrt{2}}{3} F_\pi^2 \Lambda_1 = F^\ell F^s \sin(\varphi_\ell - \varphi_s)$$

- Approximation seems justified at low scales
- extent of validity of approximation previously unknown (but observation that $\varphi_\ell \approx \varphi_s$)

Equivalent at NLO in large- N_c ChPT and employing FKS
 $(\Lambda_1 = 0)$ **Feldmann**:

$$\begin{aligned} F^\ell &\sim F_\pi & \sim \frac{2m_\ell H^\ell}{M_\pi^2} \\ F^s &\sim \sqrt{2F_K^2 - F_\pi^2} & \sim \frac{2m_s H^s}{2M_K^2 - M_\pi^2} \\ \varphi_{PS} &\sim \arcsin \sqrt{\frac{(M_{\eta'}^2 - (2M_K^2 - M_\pi^2))(M_\eta^2 - M_\pi^2)}{(M_{\eta'}^2 - M_\eta^2)(2M_K^2 - 2M_\pi^2)}} & \sim \arctan \sqrt{-\frac{H_\eta^\ell H_{\eta'}^s}{H_\eta^\ell H_{\eta'}^s}} \end{aligned}$$

where $H_M^q = \sqrt{2}\langle \Omega | P^q | \mathcal{M} \rangle$ and $H^q = \sqrt{(H_\eta^q)^2 + (H_{\eta'}^q)^2}$ are pseudoscalar matrix elements.



- SU(3) part of large- N_c ChPT seems to work well
- FKS scheme works well at small energies, where $\Lambda_1 = 0$
- at large scales and for the singlet decay constant, $\Lambda_1 = 0$ implies a 13 % error

$\mathcal{O}(a)$ IMPROVEMENT OF FLAVOUR DIAGONAL MATRIX ELEMENTS

Bhattacharya et al. (arxiv:hep-lat/0511014):

non-singlet:

$$\widehat{\text{tr } \lambda \mathcal{O}} = Z_0 \left[(1 + a\tilde{b}_O \text{tr } M) \text{tr}(\lambda \mathcal{O})^I + \frac{a}{2} b_O \text{tr}(\{\lambda, M\} \mathcal{O}) + a f_O \text{tr}(\lambda M) \text{tr } \mathcal{O} \right]$$

singlet: $\widehat{\text{tr } \mathcal{O}} = Z_0^s \left[(1 + a\tilde{d}_O \text{tr } M) \text{tr } \mathcal{O}^I + a d_O \text{tr } M \mathcal{O} \right],$

where, e.g., $\mathcal{O} = \bar{\psi} \gamma_\mu \gamma_5 \psi$ and $\lambda = t^8$ Gell-Mann matrix

- Octet acquires singlet contribution at finite lattice spacing
- \tilde{b}_A, b_A, c_A known from [Bali et al. \(arxiv:1607.07090\)](#) and [Bulava et al. \(arxiv:1502.04999\)](#)
- unknown improvement coefficients $d_A, \tilde{d}_A, f_A, c_A^s$ are parameterized for the fits:

$$\begin{aligned} f_A(g^2) &= f_A^l g^6, & d_A(g^2) &= b_A(g^2) + d_A^l g^4, \\ \tilde{d}_A(g^2) &= \tilde{d}_A^l g^4, & c_A^s(g^2) &= c_A + \delta c_A^l g^4, \end{aligned}$$

→ 4 fit parameters for full $\mathcal{O}(a)$ improvement of all decay constants