

Temporal factorization of the Wilson fermion determinant and multi-level integration

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Numerical Challenges in QCD
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- ▶ Temporal factorization of fermion determinants in LQFTs:

Step 1: dimensional reduction

Step 2: projection to canonical sectors

Step 3: temporal factorization

- ▶ Application to QCD

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- ▶ Application to QCD
- ▶ Application to the Hubbard model

Introduction and motivation

- Consider the grand-canonical partition function **at finite μ** ,

$$\begin{aligned} Z_{\text{GC}}(\mu) &= \int \mathcal{D}\mathcal{U} e^{-S_b[\mathcal{U}]} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-\bar{\psi} M[\mathcal{U}; \mu] \psi} \\ &= \int \mathcal{D}\mathcal{U} e^{-S_b[\mathcal{U}]} \det M[\mathcal{U}; \mu] \end{aligned}$$

where $\det M[\mathcal{U}; \mu]$ is highly non-local in \mathcal{U} , difficult to calculate. . .

- In the Hamiltonian formulation one has

$$\begin{aligned} Z_{\text{GC}}(\mu) &= \text{Tr} [e^{-\mathcal{H}(\mu)/T}] = \text{Tr} \prod_t \mathcal{T}_t(\mu) \\ &= \sum_N e^{-N\mu/T} \cdot Z_C(N) \end{aligned}$$

where $Z_C(N) = \text{Tr} \prod_t \mathcal{T}_t^{(N)}$.

Step 1: dimensional reduction

- ▶ The fermion matrix $M[\mathcal{U}; \mu]$ has generic (temporal) structure

$$M = \begin{pmatrix} B_0 & e^{+\mu} C'_0 & 0 & \dots & \pm e^{-\mu} C_{L_t-1} \\ e^{-\mu} C_0 & B_1 & e^{+\mu} C'_1 & & 0 \\ 0 & e^{-\mu} C_1 & B_2 & \ddots & \vdots \\ \vdots & & \ddots & & \\ \pm e^{+\mu} C'_{L_t-1} & 0 & & B_{L_t-2} & e^{+\mu} C'_{L_t-2} \\ & & & e^{-\mu} C_{L_t-2} & B_{L_t-1} \end{pmatrix}$$

for which the determinant can be reduced to

$$\det M[\mathcal{U}; \mu] = \prod_t \det \tilde{B}_t \cdot \det (1 \mp e^{\mu L_t} \cdot \mathcal{T})$$

where $\mathcal{T} = \mathcal{T}_0 \cdot \dots \cdot \mathcal{T}_{L_t-1}$ and $\mathcal{T}_t = \mathcal{T}_t[B_t, C_t, C'_t]$.

- ▶ $M[\mathcal{U}; \mu]$ is $(L \cdot L_t) \times (L \cdot L_t)$, while \mathcal{T} is $L \times L$.

Fugacity expansion and canonical determinants

- Fugacity expansion

$$\det M[\mathcal{U}; \mu] = \sum_N e^{-N \cdot \mu / T} \cdot \det_N M[\mathcal{U}]$$

yields the canonical determinants

$$\det_N M[\mathcal{U}] = \sum_J \det \mathcal{T}^{JJ}[\mathcal{U}] = \text{Tr} \left[\prod_t \mathcal{T}_t^{(N)} \right],$$

where $\det \mathcal{T}^{JJ}$ is the **principal minor** of order N .

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Key object from step 1:

$$\mathcal{T}[\mathcal{U}] \equiv \prod_{t=0}^{L_t-1} \mathcal{T}_t \quad \Leftrightarrow \quad \text{product of **spatial** matrices}$$

Fugacity expansion and canonical determinants

Key step 2:

$$\mathcal{T}[\mathcal{U}] \equiv \prod_{t=0}^{L_t-1} \mathcal{T}_t \quad \Rightarrow \quad \det_N M[\mathcal{U}] = \sum_J \det \mathcal{T}^{J\bar{J}}[\mathcal{U}]$$

Fugacity expansion and canonical determinants

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- Fugacity expansion:

$$\det M[\mathcal{U}; \mu] \propto \det \left(e^{-\mu/T} + \mathcal{T}[\mathcal{U}] \right) = \sum_N e^{-N \cdot \mu/T} \cdot \det_N M[\mathcal{U}]$$

Fugacity expansion and canonical determinants

Key step 2:

$$\mathcal{T}[\mathcal{U}] \equiv \prod_{t=0}^{L_t-1} \mathcal{T}_t \quad \Rightarrow \quad \det_N M[\mathcal{U}] = \sum_J \det \mathcal{T}^{\mathcal{Y}\mathcal{Y}}[\mathcal{U}]$$

- ▶ Fugacity expansion:

$$\det M[\mathcal{U}; \mu] \propto \det \left(e^{-\mu/T} + \mathcal{T}[\mathcal{U}] \right) = \sum_N e^{-N \cdot \mu/T} \cdot \det_N M[\mathcal{U}]$$

- ▶ Coefficients given by the elementary symmetric functions S_k of order k of $\{\tau_i\}$:

$$\det_N M[\mathcal{U}] = S_{L-N}(\mathcal{T})$$

where

$$S_k(\mathcal{T}) \equiv S_k(\{\tau_i\}) = \sum_{1 \leq i_1 < \dots < i_k \leq L} \prod_{j=1}^k \tau_{i_j} = \sum_{|J|=k} \det \mathcal{T}^{\mathcal{Y}\mathcal{Y}}.$$

Canonical determinants

$$\sum_J \det \mathcal{T}^{JJ}[\mathcal{U}] = \text{Tr} \left[\prod_t \mathcal{T}_t^{(N)} \right]$$

- ▶ States are labeled by index sets $J \subset \{1, \dots, L\}$, $|J| = N$
 - ▶ number of states grows exponentially with L at half-filling

$$N_{\text{states}} = \binom{L}{N} = N_{\text{principal minors}}$$

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 - ▶ treat index set J as dynamical degree of freedom

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$$N_{\text{states}} = \binom{L}{N} = N_{\text{principal minors}}$$

- ▶ Efficient stochastic evaluation of \sum_J :
 - ▶ treat index set J as dynamical degree of freedom
 - ▶ update $J \rightarrow J'$ using Fisher-Yates reshuffling and

$$p_{J \rightarrow J'} = \min[1, A_{J \rightarrow J'}] \quad \text{with} \quad A_{J \rightarrow J'} = \left| \frac{\det \mathcal{T}^{Y'Y'}}{\det \mathcal{T}^{YY}} \right|.$$

Transfer matrices and factorization

- Use Cauchy-Binet formula

$$\det(A \cdot B)^{\lambda\kappa} = \sum_J \det A^{\lambda J} \cdot \det B^{J\kappa}$$

to factorize into product of transfer matrices

- Transfer matrices in sector N are hence given by

$$\det \mathcal{T}^{\lambda\lambda} = \det(\mathcal{T}_0 \cdot \dots \cdot \mathcal{T}_{L_t-1})^{\lambda\lambda} = (\mathcal{T}_0)_{Jl} \cdot (\mathcal{T}_1)_{lK} \cdot \dots \cdot (\mathcal{T}_{L_t-1})_{LJ}$$

with $(\mathcal{T}_t)_{lK} = \det \tilde{B}_t \cdot \det \mathcal{T}_t^{\lambda\kappa}$ and implicit sums $\{J, l, K, \dots\}$.

- Finally, we have

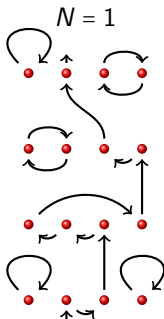
$$\det_N M[\mathcal{U}] = \prod_t \det \tilde{B}_t \cdot \sum_{\{J_t\}} \prod_t \det \mathcal{T}_t^{\lambda_{t-1}\lambda_t}$$

where $|J_t| = N$ and $J_{L_t} = J_0$.

Factorization and fermion bags

Key step 3:

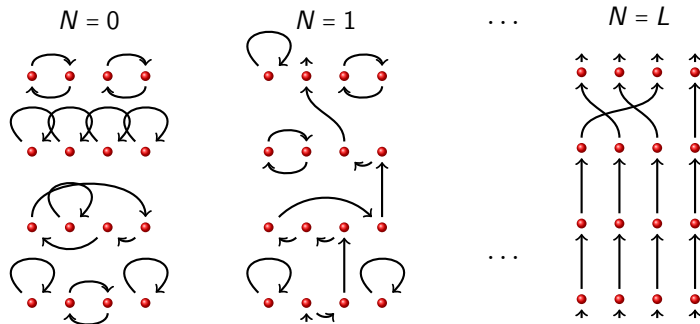
$$\sum_J \det \mathcal{T}^{\chi\chi} = \sum_J \det(\mathcal{T}_0 \cdots \mathcal{T}_{L_t-1})^{\chi\chi} = (\mathcal{T}_0)_{JI} \cdot (\mathcal{T}_1)_{IK} \cdots (\mathcal{T}_{L_t-1})_{LJ}$$



Factorization and fermion bags

Key step 3:

$$\sum_J \det \mathcal{T}^{\chi\chi} = \sum_J \det(\mathcal{T}_0 \cdots \mathcal{T}_{L_t-1})^{\chi\chi} = (\mathcal{T}_0)_{JI} \cdot (\mathcal{T}_1)_{IK} \cdots (\mathcal{T}_{L_t-1})_{LJ}$$



Dimensional reduction of QCD – Step 1

- Consider the **Wilson fermion matrix** for a single quark with chemical potential μ :

$$M_{\pm}(\mu) = \begin{pmatrix} B_0 & P_+ A_0^+ & & & \pm P_- A_{L_t-1}^- \\ P_- A_0^- & B_1 & P_+ A_1^+ & & \\ & P_- A_1^- & B_2 & \ddots & \\ & & \ddots & \ddots & \\ \pm P_+ A_{L_t-1}^+ & & & P_- & P_+ A_{L_t-2}^+ \\ & & & & B_{L_t-1} \end{pmatrix}$$

- temporal hoppings are

$$A_t^+ = e^{+\mu} \cdot \mathbb{I}_{4 \times 4} \otimes \mathcal{U}_t = (A_t^-)^{-1}$$

- Dirac projectors $P_{\pm} = \frac{1}{2}(\mathbb{I} \mp \Gamma_4)$
- B_t are **(spatial) Wilson Dirac operators** on time-slice t
- all blocks are $(4 \cdot N_c \cdot L_s^3 \times 4 \cdot N_c \cdot L_s^3)$ -matrices

Dimensional reduction of QCD – Step 1

- ▶ Reduced Wilson fermion determinant is given by

$$\det M_{p,a}(\mu) \propto \prod_t \det Q_t^+ \cdot \det [\mathbb{I} \pm e^{+\mu L_t} \mathcal{T}]$$

where \mathcal{T} is the product of spatial matrices given by

$$\mathcal{T} = \prod_t Q_t^+ \cdot \mathcal{U}_t \cdot (Q_{t+1}^-)^{-1} \equiv \prod_t \mathcal{T}_t$$

$$Q_t^\pm = B_t P_\mp + P_\pm, \quad B_t = \begin{pmatrix} D_t & C_t \\ -C_t & D_t \end{pmatrix}$$

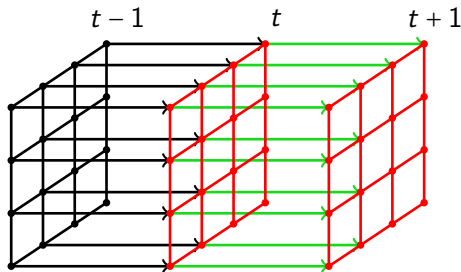
and

$$Q_t^+ = \begin{pmatrix} 1 & C_t \\ 0 & D_t \end{pmatrix}, \quad (Q_t^-)^{-1} = \begin{pmatrix} D_t^{-1} & 0 \\ C_t \cdot D_t^{-1} & 1 \end{pmatrix}.$$

Structure of building blocks

- Product of spatial matrices:

$$\mathcal{T} = \prod_t Q_t^+ \cdot \mathcal{U}_t \cdot (Q_{t+1}^-)^{-1} \quad \text{or} \quad \mathcal{T} = \prod_t \mathcal{U}_{t-1}^- \cdot (Q_t^-)^{-1} \cdot Q_t^+ \cdot \mathcal{U}_t^+$$

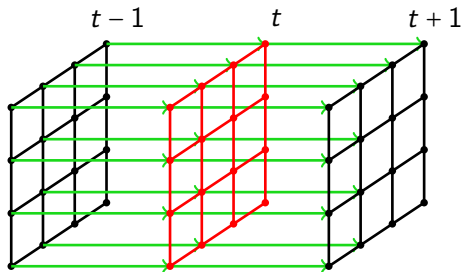


$$Q_t^+ \cdot \mathcal{U}_t \cdot (Q_{t+1}^-)^{-1}$$

Structure of building blocks

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$$\mathcal{T} = \prod_t Q_t^+ \cdot \mathcal{U}_t \cdot (Q_{t+1}^-)^{-1} \quad \text{or} \quad \mathcal{T} = \prod_t \mathcal{U}_{t-1}^- \cdot (Q_t^-)^{-1} \cdot Q_t^+ \cdot \mathcal{U}_t^+$$



$$\mathcal{U}_{t-1}^- \cdot (Q_t^-)^{-1} \cdot Q_t^+ \cdot \mathcal{U}_t^+$$

Structure of building blocks

- Several ways to rewrite spatial matrices:

$$\tilde{\mathcal{T}}_t \equiv (Q_t^-)^{-1} \cdot Q_t^+ = \begin{pmatrix} 1 & 0 \\ C_t & 1 \end{pmatrix} \begin{pmatrix} D_t^{-1} & 0 \\ 0 & D_t \end{pmatrix} \begin{pmatrix} 1 & C_t \\ 0 & 1 \end{pmatrix}$$

- spectral property and determinant:

$$\lambda \leftrightarrow \frac{1}{\lambda^*}, \quad \det(Q_t^-)^{-1} \cdot Q_t^+ = 1$$

- Relation to 1d scattering matrix $\tilde{\mathcal{S}}_t$:

$$\tilde{\mathcal{T}}_t = \begin{pmatrix} D_t^{-1} & D_t^{-1} \cdot C_t \\ C_t \cdot D_t^{-1} & D_t + C_t \cdot D_t^{-1} \cdot C_t \end{pmatrix} \Leftrightarrow \tilde{\mathcal{S}}_t = \begin{pmatrix} C_t & D_t \\ D_t & -C_t \end{pmatrix}.$$

Canonical projection and factorization – Step 2+3

Step 2: Canonical projection of QCD

$$\det M_{N_q} = \prod_t \det Q_t^+ \cdot \sum_A \det \mathcal{T}^{\cancel{A}\cancel{A}}$$

- sum is over all index sets $A \in \{1, 2, \dots, 2N_q^{\max}\}$ of size

$$|A| = N_q^{\max} + N_q, \quad N_q^{\max} = 2 \cdot N_c \cdot L_s^3$$

- i.e., the trace over the minor matrix of rank N_q of \mathcal{T}

Step 3: Factorization of QCD determinant

$$\det M_{N_q} = \prod_t \det Q_t^+ \cdot \prod_t M\left((Q_t^-)^{-1}\right)_{\cancel{A_t}\cancel{B_t}} M(Q_t^+)_{B_t\cancel{C_t}} M(\mathcal{U}_t)_{\cancel{C_t}\cancel{A_{t+1}}}$$

Relation between quark and baryon number in QCD

Step 3: Factorization of QCD determinant

$$\det M_{N_q} = \prod_t \det Q_t^+ \cdot \prod_t M((Q_t^-)^{-1})_{A_t \setminus B_t} M(Q_t^+)_{B_t \setminus C_t} M(U_t)_{C_t \setminus A_{t+1}}$$

- Consider $\mathbb{Z}(N_c)$ -transformation by $z_k = e^{2\pi i \cdot k / N_c} \in \mathbb{Z}(N_c)$:

$$U_t \rightarrow U'_t = z_k \cdot U_t \quad \text{at one fixed } t.$$

- As a consequence we have

$$\det M_{N_q} \rightarrow \det M'_{N_q} = \prod_t \det Q_t^+ \cdot \sum_A \det(z_k \cdot \mathcal{T})_{A \setminus A} = z_k^{-N_q} \cdot \det M_{N_q}$$

and **summing over** z_k therefore yields

$$\det M_{N_q} = 0 \quad \text{for } N_q \neq 0 \bmod N_c.$$

Properties of minor matrices

Step 3: Factorization of QCD determinant

$$\det M_{N_q} = \prod_t \det Q_t^+ \cdot \prod_t M((Q_t^-)^{-1})_{A_t B_t} M(Q_t^+)_{B_t C_t} M(U_t)_{C_t A_{t+1}}$$

- Note that

$$M(Q^{-1})_{A_t B_t} = (-1)^{p(A,B)} \frac{\tilde{M}(Q)_{BA}}{\det Q}, \quad \det Q_t^+ = \det Q_t^-$$

avoids inverting Q_t^- .

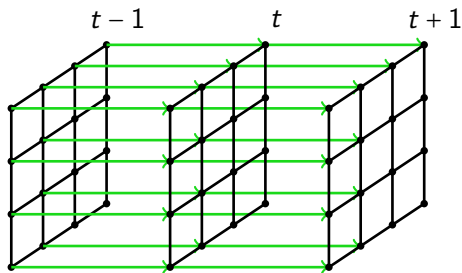
- Consider temporal gauge link $W_t = \mathbb{I}_{4 \times 4} \otimes U_4(\bar{x}, t)$ at \bar{x} :

$$M(W_t)_{C_t A_{t+1}} = 0 \quad \text{if } |c_t| \neq |a_t|$$

for $c_t \in C_t$ and $a_{t+1} \in A_{t+1}$.

Multi-level integration schemes

- Temporal gauge links in \mathcal{U}_t are completely decoupled:



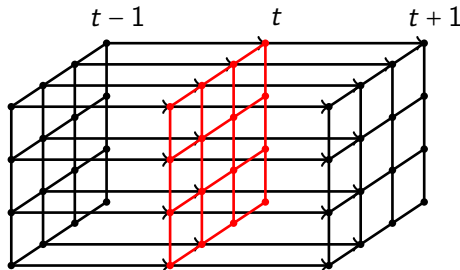
$$M(\mathcal{U}_{t-1})_{\mathcal{C}_{t-1} \mathcal{A}_t} \cdot M((Q_t^-)^{-1})_{\mathcal{A}_t \mathcal{B}_t} \cdot M(Q_t^+)_{\mathcal{B}_t \mathcal{C}_t} \cdot M(\mathcal{U}_t)_{\mathcal{C}_t \mathcal{A}_{t+1}}$$

- spatial matrix \mathcal{U}_t is block diagonal:

$\Rightarrow M(\mathcal{U}_t)$ trivial to calculate!

Multi-level integration schemes

- Spatial gauge links in Q_t^\pm coupled through temporal plaquettes only:



$$M(U_{t-1})_{\zeta_{t-1} \lambda_t} \cdot M((Q_t^-)^{-1})_{\lambda_t \beta_t} \cdot M(Q_t^+)_{\beta_t \zeta_t} \cdot M(U_t)_{\zeta_t \lambda_{t+1}}$$

- spatial matrices Q_t^\pm can be treated together:

$$M((Q_t^-)^{-1})_{\lambda_t \beta_t} \cdot M(Q_t^+)_{\beta_t \zeta_t} = M((Q_t^-)^{-1} \cdot Q_t^+)_{\lambda_t \zeta_t}$$

Correlation functions

- ▶ Source and sink operators \mathcal{S} and $\overline{\mathcal{S}}$:
 - ▶ remove or re-add indices from/to the index set,
 - ▶ potentially change quark number N_q , e.g.,

$$\dots \cdot \mathcal{T}_{t-1}^{(N_q)} \cdot \mathcal{S}_{N_q \rightarrow N_q+3} \cdot \mathcal{T}_t^{(N_q+3)} \cdot \dots \cdot \mathcal{T}_{t'}^{(N_q+3)} \cdot \overline{\mathcal{S}}_{N_q+3 \rightarrow N_q} \cdot \mathcal{T}_{t'+1}^{(N_q)} \cdot \dots$$

- ▶ vacuum sector corresponds to $N_q = 0$
- ▶ Natural to construct **improved estimators**:
 - ▶ simulate directly the correlation function at $C(t' - t)$,
 - ▶ measure $C(t' + 1 - t)$ relative to $C(t' - t)$

$$\langle C(t' + 1 - t) \rangle_{C(t'-t)} \sim e^{-aE}$$

$$\text{from additional insertion } \mathcal{T}_{t'+1}^{(N_q)} \rightarrow \mathcal{T}_{t'+1}^{(N_q+3)}$$

- ▶ **All spectral information is contained in $\langle \mathcal{T}_t^{(N_q)} \rangle$.**

Summary and outlook

- Complete temporal factorization of the Wilson fermion determinant:

$$\det M_{N_q} = \prod_t \det Q_t^+ \cdot \prod_t M\left((Q_t^-)^{-1}\right)_{\cancel{A_t} \cancel{B_t}} M(Q_t^+)_{\cancel{B_t} \cancel{C_t}} M(U_t)_{\cancel{C_t} \cancel{A_{t+1}}}$$

- works for **fixed quark numbers** N_q
- allows for **very flexible multi-level integration** schemes
- cf. [Gattringer et al, Giusti et al, Chandrasekharan et al]

Caveats: positivity? potential sign problem?

Q^\pm are strictly positive, $(\mathcal{T}_t)_{\cancel{B_t} \cancel{C_t}}$ not necessarily...

Hubbard model

- Consider the Hamiltonian for the Hubbard model

$$\mathcal{H}(\mu) = - \sum_{\langle x,y \rangle, \sigma} t_{\sigma} \hat{c}_{x,\sigma}^{\dagger} \hat{c}_{y,\sigma} + \sum_{x,\sigma} \mu_{\sigma} N_{x,\sigma} + U \sum_x N_{x,\uparrow} N_{x,\downarrow}$$

with particle number $N_{x,\sigma} = \hat{c}_{x,\sigma}^{\dagger} \hat{c}_{x,\sigma}$.

Hubbard model with Trotter

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with particle number $N_{x,\sigma} = \hat{c}_{x,\sigma}^{\dagger} \hat{c}_{x,\sigma}$.

- ▶ Trotter decomposition and coherent state representation yields

$$Z_{GC}(\mu) = \int \mathcal{D}\psi^{\dagger} \mathcal{D}\psi e^{-S[\psi^{\dagger}, \psi; \mu]}$$

with Euclidean action

$$S[\psi^{\dagger}, \psi; \mu] = \sum_{\sigma} \psi_{\sigma}^{\dagger} \nabla_t \psi_{\sigma} + H[\psi^{\dagger}, \psi; \mu].$$

Hubbard model with Trotter and Stratonovic

- Consider the Hamiltonian for the Hubbard model

$$\mathcal{H}(\mu) = - \sum_{\langle x,y \rangle, \sigma} t_{\sigma} \hat{c}_{x,\sigma}^{\dagger} \hat{c}_{y,\sigma} + \sum_{x,\sigma} \mu_{\sigma} N_{x,\sigma} + U \sum_x N_{x,\uparrow} N_{x,\downarrow}$$

with particle number $N_{x,\sigma} = \hat{c}_{x,\sigma}^{\dagger} \hat{c}_{x,\sigma}$.

- After a Hubbard-Stratonovich transformation we have

$$Z_{\text{GC}}(\mu) = \int \mathcal{D}\psi^{\dagger} \mathcal{D}\psi \mathcal{D}\phi \rho[\phi] e^{-\sum_{\sigma} S[\psi_{\sigma}^{\dagger}, \psi_{\sigma}, \phi; \mu_{\sigma}]}$$

with $S[\psi_{\sigma}^{\dagger}, \psi_{\sigma}, \phi; \mu_{\sigma}] = \psi_{\sigma}^{\dagger} M[\phi; \mu_{\sigma}] \psi_{\sigma}$, and hence

$$= \int \mathcal{D}\phi \rho[\phi] \prod_{\sigma} \det M[\phi; \mu_{\sigma}].$$

Fermion matrix and dimensional reduction

- The fermion matrix has the structure

$$M[\phi; \mu_\sigma] = \begin{pmatrix} B & 0 & \dots & \pm e^{\mu_\sigma} C(\phi_{N_t-1}) \\ -e^{\mu_\sigma} C(\phi_0) & B & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -e^{\mu_\sigma} C(\phi_{N_t-2}) & B \end{pmatrix}$$

for which the determinant can be reduced to

$$\det M[\phi; \mu_\sigma] = \det B^{N_t} \cdot \det (1 \mp e^{N_t \mu_\sigma} \mathcal{T}[\phi])$$

where $\mathcal{T}[\phi] = B^{-1} C(\phi_{N_t-1}) \cdot \dots \cdot B^{-1} C(\phi_0)$.

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where $\mathcal{T}[\phi] = B^{-1} C(\phi_{N_t-1}) \cdot \dots \cdot B^{-1} C(\phi_0)$.

- Transfer matrices are hence given by

$$\begin{aligned} (\mathcal{T}_t)_{IK} &= \det B \cdot \det [B^{-1} \cdot C(\phi_t)]^{\Lambda_K} \\ &= \det B \cdot \det(B^{-1})^{\Lambda_I} \cdot \det C(\phi_t)^{\Lambda_K} \\ &= (-1)^{p(I,J)} \det B^{J_I} \cdot \det C(\phi_t)^{\Lambda_K}. \end{aligned}$$

Transfer matrices

- ▶ Since $C(\phi_t)$ can be chosen diagonal, we have

$$\det C(\phi_t)^{JK} = \delta_{JK} \prod_{x \in J} \phi_{x,t}$$

and the HS field can be integrated out site by site:

$$\int d\phi_{x,t} \rho(\phi_{x,t}) \phi_{x,t}^{\sum_{\sigma} \delta_{x \notin J^{\sigma}}} \equiv w_{x,t} = \begin{cases} w_2 & \text{if } x \notin J^{\uparrow}, x \notin J^{\downarrow} \\ w_1 & \text{else} \\ w_0 & \text{if } x \in J^{\uparrow}, x \in J^{\downarrow} \end{cases}$$

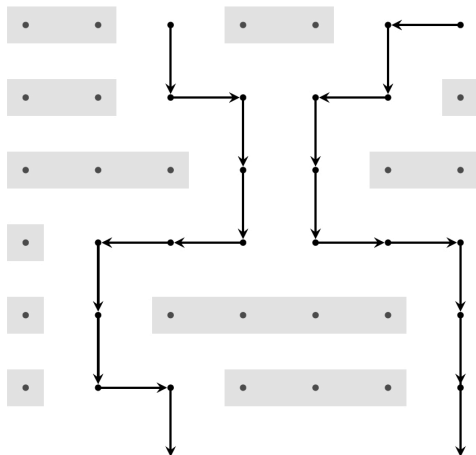
- ▶ Finally, with $\prod_x w_{x,t} \equiv W(\{J_t^{\sigma}\})$ we have

$$Z_C(\{N_{\sigma}\}) = \sum_{\{J_t^{\sigma}\}} \prod_t \left(\prod_{\sigma} \det B^{J_{t-1}^{\sigma} J_t^{\sigma}} \right) W(\{J_t^{\sigma}\}), \quad |J_t^{\sigma}| = N_{\sigma}$$

Relation to fermion loop formulation

$$Z_C(\{N_\sigma\}) = \sum_{\{J_t^\sigma\}} \prod_t \left(\prod_\sigma \det B^{J_{t-1}^\sigma J_t^\sigma} \right) W(\{J_t^\sigma\})$$

index sets J_t :



$\{3,6\}$

$\{4,5\}$

$\{4,5\}$

$\{2,7\}$

$\{2,7\}$

$\{3,7\}$

Relation to fermion bag formulation

- ▶ In $d = 1$ dimension the 'fermion bags' $\det B^{IJ}$ can be calculated analytically:



The diagram shows an equation between three terms. On the left is a light gray rectangle containing five black dots. This is followed by an equals sign. To the right of the equals sign is a purple rectangle containing four black dots, followed by a circular arrow pointing clockwise. This is followed by a plus sign. To the right of the plus sign is an orange rectangle containing three black dots, followed by a circular arrow pointing counter-clockwise.

and one can prove that

$$\det B^{IJ} \geq 0 \quad \text{for open b.c.}$$

⇒ there is no sign problem

- ▶ For periodic b.c. there is no sign problem either, because

$$Z_C^{\text{pbc}}(L_s \rightarrow \infty) = Z_C^{\text{obc}}(L_s \rightarrow \infty)$$