Transverse Linear Beam Dynamics

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The CERN Accelerator School

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Transverse Linear Beam Optics

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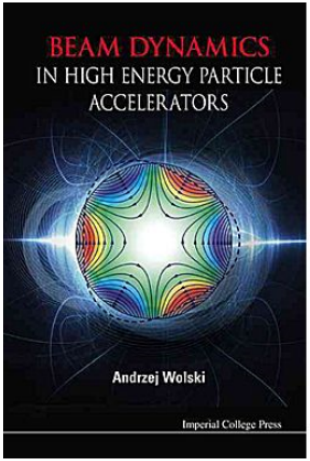
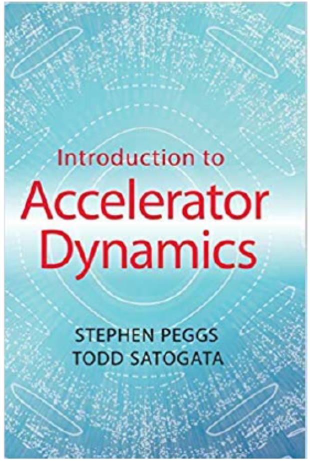


Recommended reading:

- **A. Wolski**, *Beam Dynamics in high energy particle accelerators*, Imperial College Press, ISBN 978-1-78326-277-9
- **S. Peggs**, **T. Satogata**, *Introduction to Accelerator Dynamics*, Cambridge University Press, ISBN 978-1107132849
- **CAS proceedings** and references therein

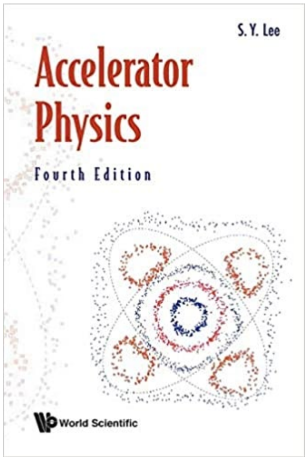


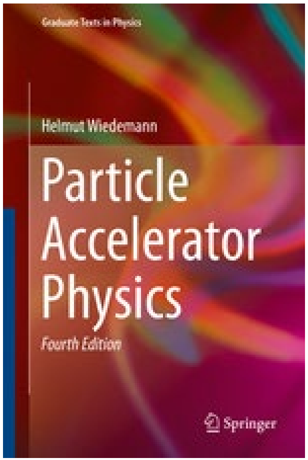
1. Introduction

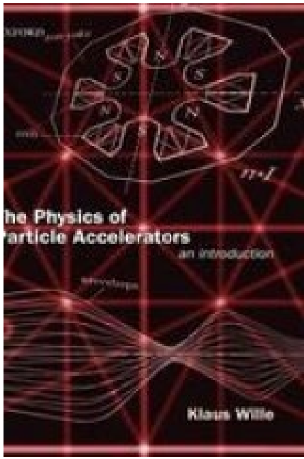
1.1. Additional Literature

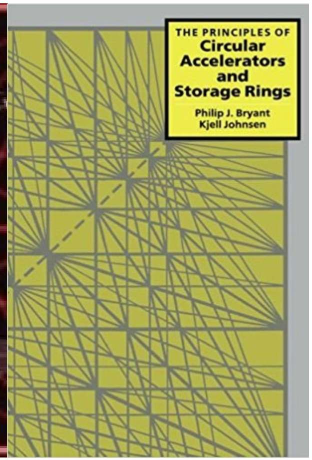
- S.Y. Lee: ***Accelerator Physics***,
  4th edition, World Scientific, New Yersey 2018, ISBN 978-981-4374-94-1
- Bryant / Johnson: ***The Principles of Circular Accelerators and Storage Rings***,
  Cambridge University Press, Cambridge 2005, ISBN 978-0-521-61969-1
- Edwards / Syphers: ***An Introduction to the Physics of High Energy Accelerators***,
  John Wiley & Sons, New York 1992, ISBN 978-0-471-55163-8
- K. Wille: ***The physics of particle accelerators***,
  Oxford Univ. Press 2005, Oxford, ISBN 0-19-850550-7
- H. Wiedemann: ***Particle Accelerator Physics***,
  4th edition, Springer 2015, Berlin, ISBN 978-3-319-18316-9
- Chao / Tigner: ***Handbook of Accelerator Physics and Engineering*****,**
  2nd edition, World Scientific, Singapore 2013, ISBN 987-4417-17-4

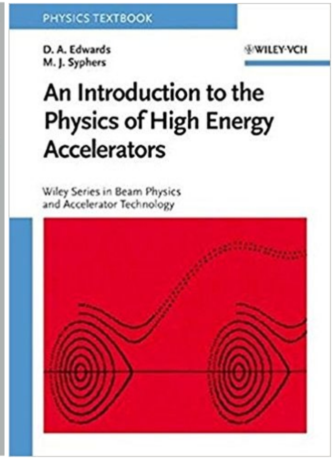
- F. Hinterberger: ***Physik der Teilchenbeschleuniger und Ionenoptik***,
  2. Ausgabe, Springer 2008, Berlin, ISBN 978-3-540-75281-3
- K. Wille: ***Physik der Teilchenbeschleuniger und Synchrotronstrahlungsquellen***,
  2. überarb. und erw. Ausgabe, Teubner 1996, Stuttgart, ISBN 978-3-519-13087-1
- Rossbach / Schmüser: ***Basic Course on Accelerator Optics***,
  CAS 5th general accelerator physics course CERN 94-01











1.2. Bending radius and beam rigidity

Particle guidance and focusing based on beam deflection by Lorentz force

$$\vec{F} = q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Ultra-relativistic particles move with speed very close to speed of light!

Impact of magnetic fields is enhanced by enormous factor:

$$v \approx c \qquad \Rightarrow \qquad B = 1\ \text{Tesla} \quad \leftrightarrow \quad E = 3 \cdot 10^8\ \text{V/m}$$

Only magnetic fields are used for beam deflection!

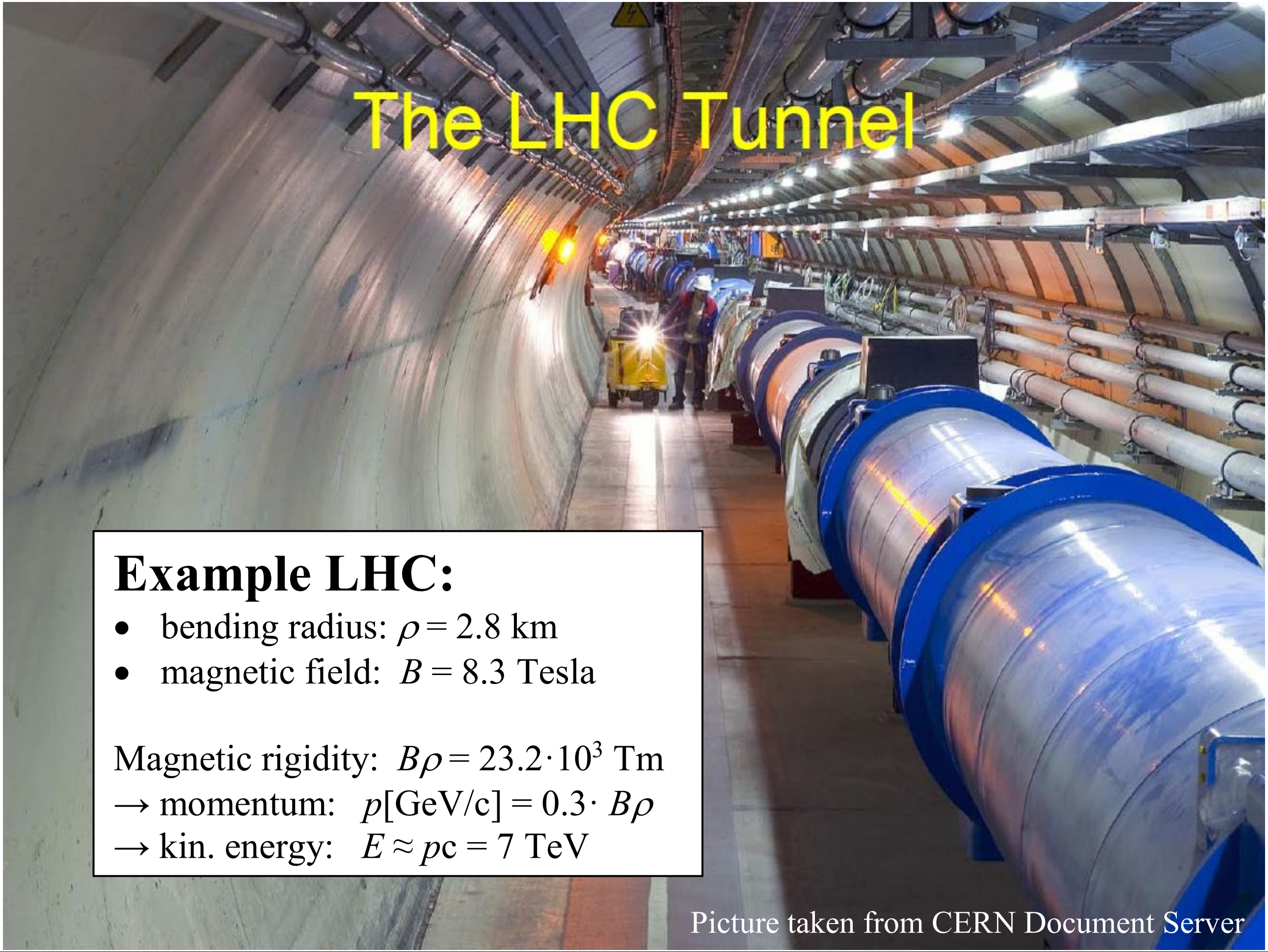
Bending radius from balance of forces $\left(m = \gamma_r m_0\right)$:

$$\vec{B} \perp \vec{v}: \qquad m\frac{v^2}{\rho} = q \cdot v \cdot B \qquad p = mv = q\rho B$$

Leads to the definition of the **magnetic rigidity** $\boldsymbol{B\rho}$!

In circular accelerators, the magnetic rigidity defines the momentum of the beam:

$$\frac{p}{q} = B\rho = 1\ \text{Tm} \quad \hat{=} \qquad p = 0.3\,\frac{\text{GeV}}{\text{c}}$$

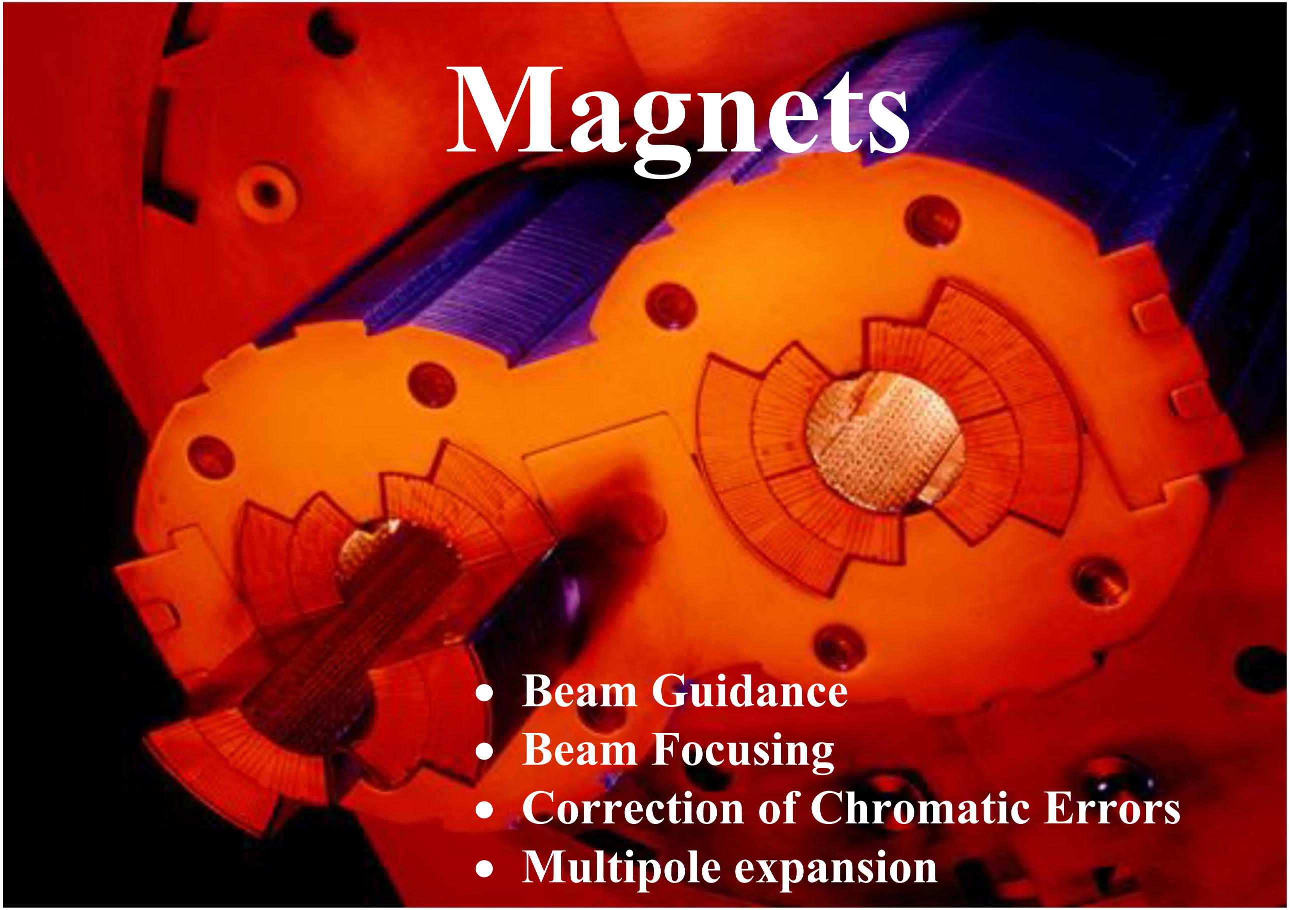


Example LHC:

- bending radius: $\rho = 2.8$ km
- magnetic field: $B = 8.3$ Tesla

Magnetic rigidity: $B\rho = 23.2 \cdot 10^3$ Tm
$\rightarrow$ momentum: $p[\mathrm{GeV/c}] = 0.3 \cdot B\rho$
$\rightarrow$ kin. energy: $E \approx pc = 7$ TeV

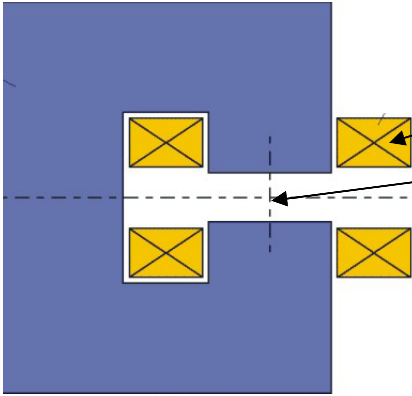
Picture taken from CERN Document Server



Picture taken from https://www.lhc-closer.es/taking_a_closer_look_at_lhc/0.magnetic_dipoles

2. Magnets

2.1. General remarks on the calculation of magnetic fields



Maxwell's Equations:

$\vec{\nabla} \times \vec{H} = \vec{j}$ (coils)

$\vec{\nabla} \times \vec{H} = 0$ (gap) $\longrightarrow$ $\vec{H} = -\vec{\nabla}\Phi$!!!

nc Magnets:

Φ = const. defines the pole's contour!

Magn. Induction from $\vec{B} = \mu_0 \, \mu_r \, \vec{H}$

Taylor expansion of the vertical magnetic field:

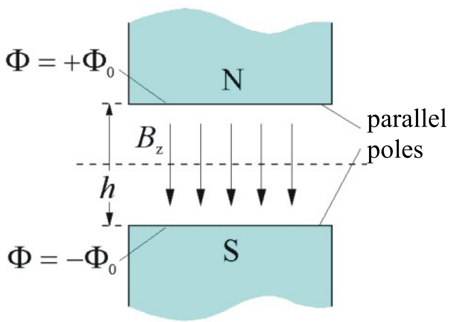
$$B_y(x,y) = \underbrace{B_y(0,y)}_{\text{Dipoles}} + \underbrace{x \cdot \frac{\partial B_y}{\partial x}(0,y)}_{\text{Quadrupoles}} + \underbrace{x^2 \cdot \frac{\partial^2 B_y}{\partial x^2}(0,y)}_{\text{Sextupoles}} + \ldots$$

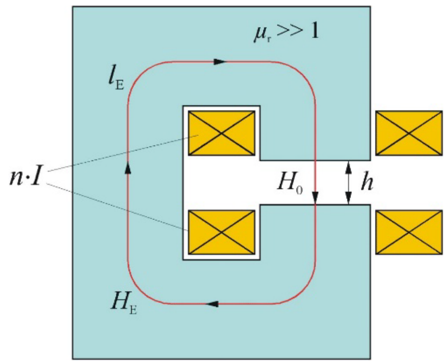
2.2. Particle beam guidance

Deflection of particles $\rightarrow$ homogenous field: $\vec{B} = B_0 \cdot \hat{e}_y = \text{const.}$

Corresponding magnetic potential: $\Phi(x,y) = -B_0 \cdot y$

defining the pole's profile to be flat and parallel: Dipole Magnets!



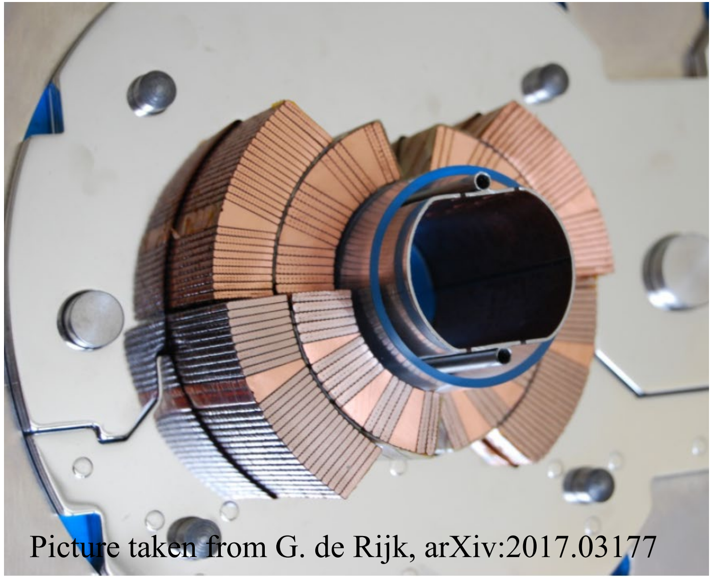


$$n \cdot I = \oint \vec{H} \cdot \mathrm{d}\vec{s} = \int_{gap} \vec{H}_0 \cdot \mathrm{d}\vec{s} + \int_{yoke} H_E \cdot \mathrm{d}\vec{s}\,, \qquad \mu_r \cdot |H_E| = |H_0| \quad \Rightarrow \quad |H_0| \gg |H_E|$$

$$B_0 = \mu_0 \frac{n \cdot I}{h}, \qquad \textbf{Curvature:} \; \boxed{\kappa = \frac{1}{\rho} = \frac{q}{p} B_0 = \frac{q\,\mu_0}{p} \frac{n \cdot I}{h}}, \quad [\kappa] = \mathrm{m}^{-1}$$

Dipole Magnets:

Picture taken from CERN Document Server



Picture taken from G. de Rijk, arXiv:2017.03177

Iron dominated:
field determined by
geometry of poles
$\rightarrow$ 2 flat poles

Superconducting:
field determined by
geometry of coils
$\rightarrow j(\phi) \sim \cos\phi$

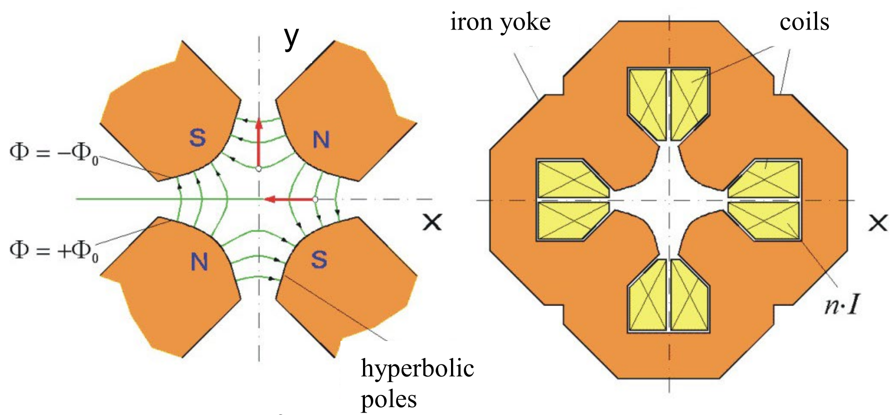
2.3. Particle beam focusing

Restoring force, linearly increasing with increasing distance from the axis:

$$B_y = -g\cdot x, \quad B_x = -g\cdot y \quad \text{with} \quad g = -\frac{\partial B_y}{\partial x} = -\frac{\partial B_x}{\partial y} = \text{const.}$$

Corresponding potential: $\Phi(x,y) = g\cdot x\cdot y\,,$ solves $\vec{\nabla}\cdot\vec{B} = -\Delta\Phi = 0$

defining the pole's profile to four hyperbolic poles: **Quadrupole Magnets**!



$$y(x) = \pm\frac{\Phi_0}{g\cdot x} = \pm\frac{a^2}{2x} \text{ at a distance } a = \sqrt{2\Phi_0/g} \text{ from the axis.}$$

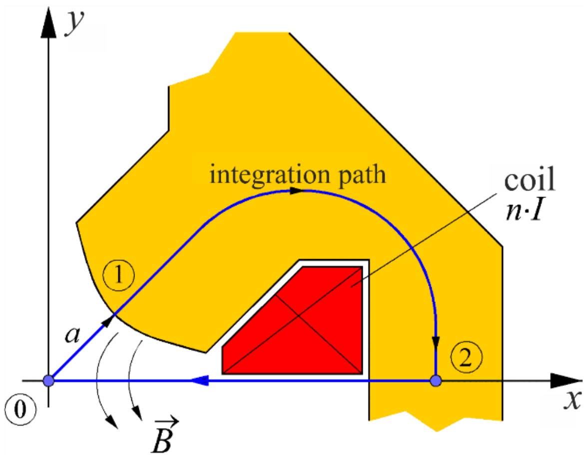
The "restoring" force acting on the particles is

$$\vec{F} = q \cdot \left(\vec{\mathrm{v}} \times \vec{B} \right) = q\mathrm{v}g \cdot \left(x\,\hat{e}_x - y\,\hat{e}_y \right)$$

A quadrupole magnet is therefore focusing only in one plane and defocusing in the other; depending on the sign of *g*.

The g-parameter may be related to the current of the coils by evaluating the closed loop integral

$$n \cdot I = \oint \vec{H} \cdot \mathrm{d}\vec{s} = \int_0^1 \vec{H}_0 \cdot \mathrm{d}\vec{s} + \int_1^2 \vec{H}_E \cdot \mathrm{d}\vec{s} + \int_2^0 \vec{H}_0 \cdot \mathrm{d}\vec{s} \approx \int_0^1 \vec{H}_0 \cdot \mathrm{d}\vec{s}\,,$$



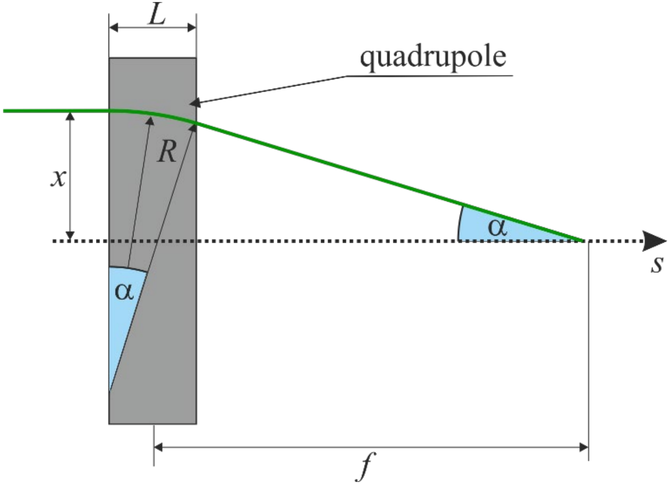
One obtains with $\vec{H} \cdot \mathrm{d}\vec{s} = \dfrac{g}{\mu_0} r \cdot \mathrm{d}r$:

$$g = \frac{2 \cdot \mu_0 \cdot n \cdot I}{a^2},$$ normalized:

Quadrupole Strength

$$\boxed{k = \frac{q}{p} g = \frac{2q\,\mu_0}{p} \frac{n \cdot I}{a^2}}, \quad \left[k \right] = \mathrm{m}^{-2}$$

The **focal length** of a thin quadrupole magnet of length L can be derived from the deflection angle α of the particles beam and its relation to the quadrupole strength k,



$$\tan\alpha = \frac{x}{f}$$

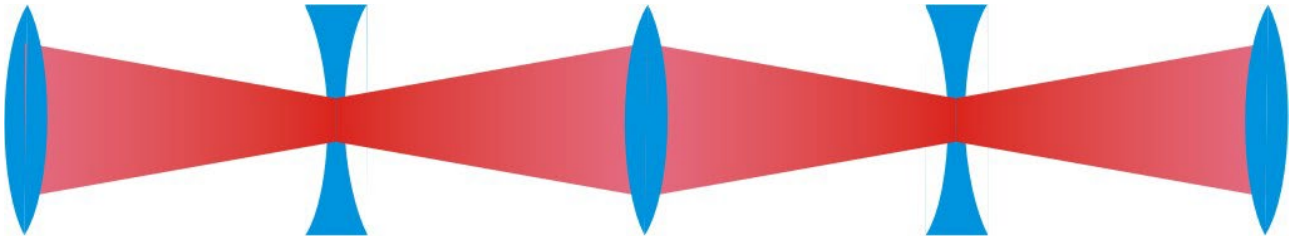
$$\tan\alpha = \frac{L}{R} = L \cdot \frac{q}{p} B_y = -\frac{q}{p} g\, x\, L = -x\, k\, L$$

→ Gives a better understanding of the quadrupole strength: $\boxed{\frac{1}{f_x} = -k \cdot L, \quad \frac{1}{f_y} = k \cdot L}$

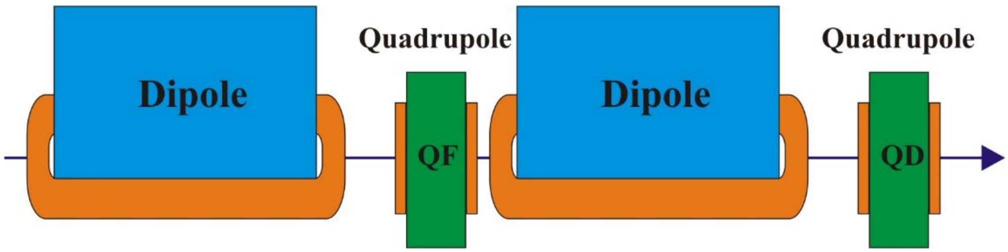
Here we have assumed the length L to be short compared to the focal length f such that R does not change significantly within the quadrupole magnetic field.

Strong Focusing:

Light optics:



Magnet optics:



Strong focusing
or
AG focusing

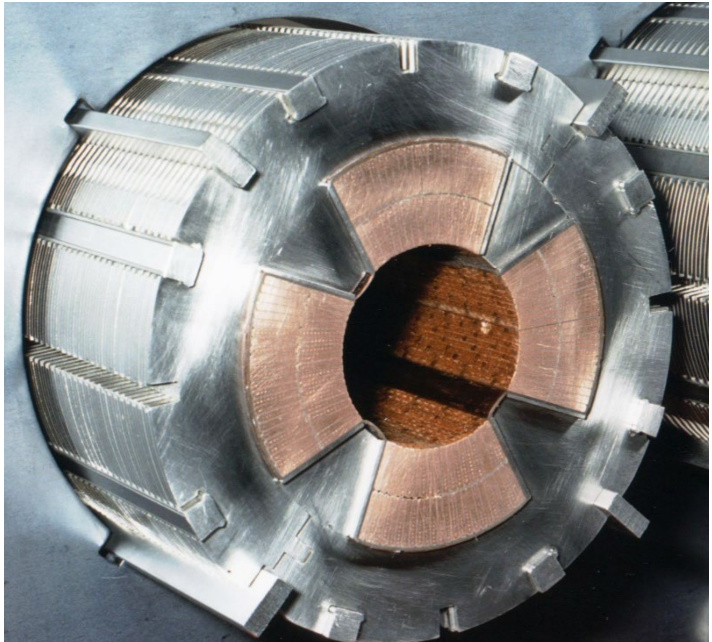
Simplest way:

FODO lattice

Detailed discussion later!

Quadrupole magnets:

Picture taken from https://cds.cern.ch/record/1333874/plots



Picture taken from https://cds.cern.ch/record/1333874/plots

Iron dominated:
field determined by geometry of poles
→ 4 hyperbolic poles

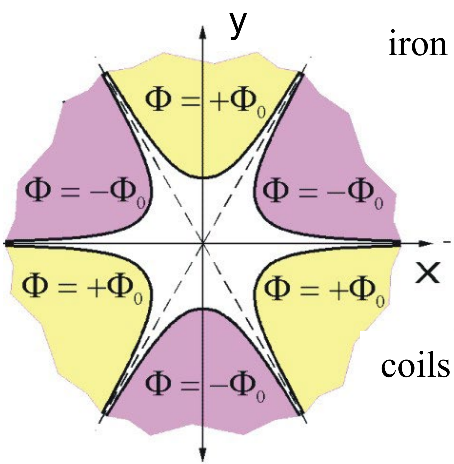
Superconducting:
field determined by geometry of coils
$\rightarrow j(\phi) \sim \cos 2\phi$

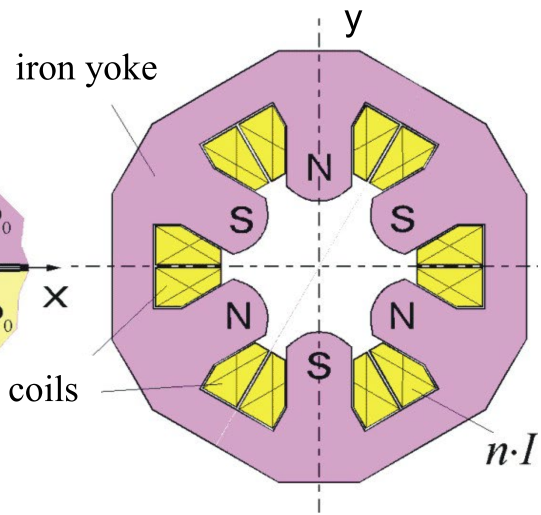
2.4. Correction of chromatic errors

Quadratic increase of magnetic fields increasing distance from the axis:

$$B_y = \frac{1}{2} g' \cdot \left(x^2 - y^2 \right) \quad \text{with} \quad g' = \frac{\partial^2 B_y}{\partial x^2} = \text{const.}$$

Corresponding potential: $\Phi(x,y) = \frac{1}{6} g' \left(y^3 - 3x^2 y \right)$, solves $\vec{\nabla} \cdot \vec{B} = -\Delta\Phi = 0$





Sextupole Magnets

Six poles, profile

$$x(y) = \pm \sqrt{\frac{y^2}{3} \pm \frac{2\Phi_0}{g' y}}$$

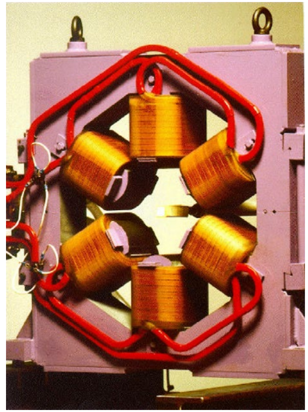
or using the aperture

$$a = \sqrt[3]{6\Phi_0 / g'}$$

$$x(y) = \pm \sqrt{\frac{y^2}{3} \pm \frac{a^3}{3y}}$$

The *g'* parameter may be related to the current of the coils in the well-known manner:

$$g' = \frac{\partial^2 B_y}{\partial x^2} = 6\,\mu_0 \frac{n\,I}{a^3}$$



and we obtain for the transverse magnetic fields:

$$\boxed{B_x(x,y) = -\frac{\partial \Phi}{\partial x} = g'\,x\,y \quad \text{and} \quad B_y(x,y) = -\frac{\partial \Phi}{\partial y} = \frac{1}{2} g' \left(x^2 - y^2\right)}$$

We will therefore expect a coupling of particles motion in the horizontal and vertical plane due to the *y*-dependence of the vertical field.

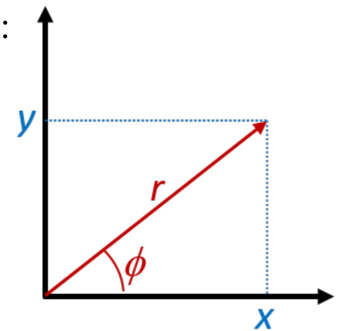
Normalizing *g'* to the particles momentum, we obtain the sextupole strength

$$\boxed{m = \frac{q}{p} g' = \frac{6\,q\,\mu_0}{p} \frac{n\,I}{a^3}}, \quad [m] = \mathrm{m}^{-3}$$

A simple understanding of the action of a sextupole will be given later!

2.5. Multipole expansion

General treatment by multipole expansion, e.g. in polar coordinates:



$$B_r(r,\phi) = B_0 \sum_{n=1}^{\infty} \left(\frac{r}{R_{ref}}\right)^{n-1} \cdot \left(b_n \sin(n\phi) - a_n \cos(n\phi)\right)$$

$$B_\phi(r,\phi) = B_0 \sum_{n=1}^{\infty} \left(\frac{r}{R_{ref}}\right)^{n-1} \cdot \left(a_n \sin(n\phi) + b_n \cos(n\phi)\right)$$

Contribution of multipole n: $$|B|_n = \sqrt{B_{r,n}^2 + B_{\phi,n}^2} = B_0 \left(\frac{r}{R_{ref}}\right)^{n-1} \sqrt{a_n^2 + b_n^2}$$

Generally: 2n-pole has $2\pi/n$ symmetry, $|B|_n$ scales with r^{n-1}.

$n = 1$ dipole magnet

$n = 2$ quadrupole magnet

$n = 3$ sextupole magnet

$n = 4$ octupole magnet

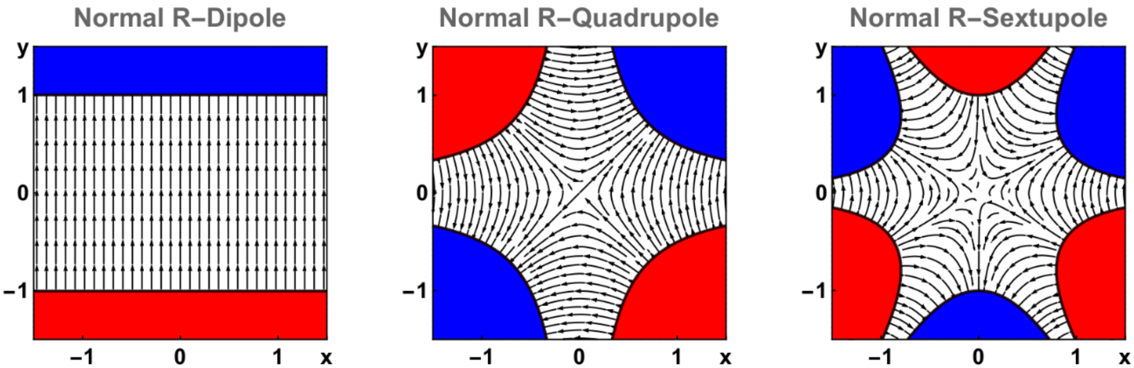
$n = 5$ decapole magnet

Classification:

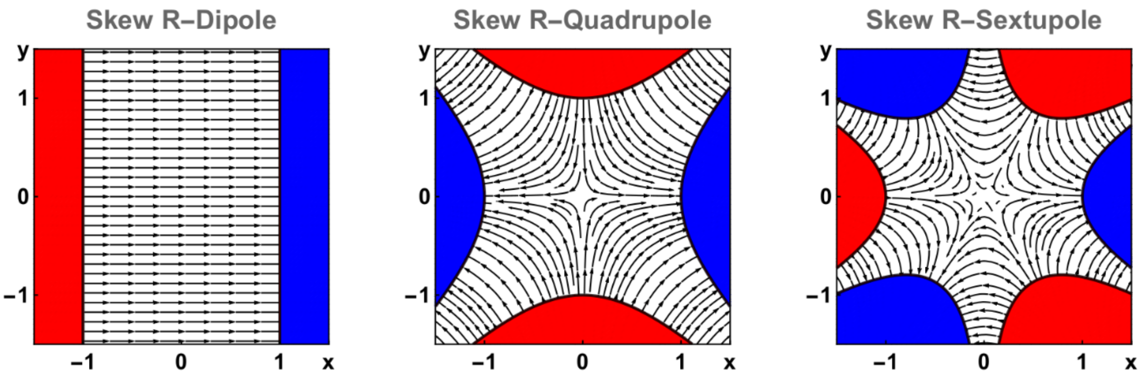
$b_n \neq 0$: "upright" magnets

$a_n \neq 0$: "skew" magnets, rotated by $\pi/2n$

Normal or upright magnets:



Skew or rotated magnets:

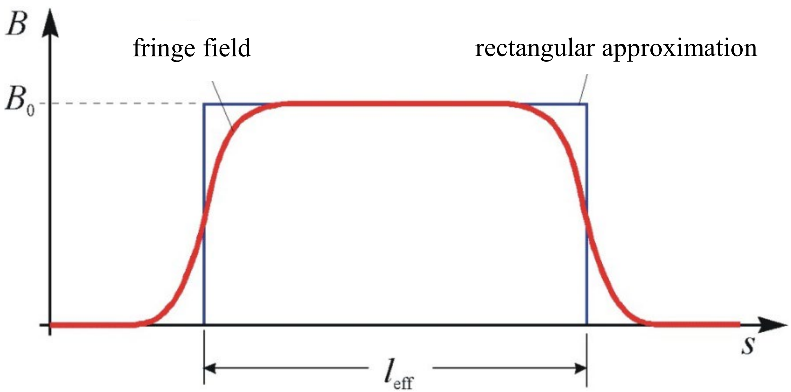


Taken from Zolkin, Timofey, Phys.Rev.Accel.Beams **20** (2017) no.4, 043501

2.6. Effective field length

The assumption of a constant field distribution along the longitudinal axis $(\partial\vec{B}/\partial s = 0)$ is not valid in general due to the fringing fields at the end of the magnets. In order to simplify the calculation of the optics of particle accelerators, an effective field length l_{eff} of each magnet is usually defined, calculated from the path-integral

$$\boxed{\int_{-\infty}^{\infty} \vec{B} \cdot \mathrm{d}\vec{s} = \vec{B}_0 \cdot l_{eff}}$$



and approximating the real longitudinal field by a rectangular shaped profile.

Note: l_{eff} differs from the length L of the iron poles, in almost all cases $l_{\mathrm{eff}} > L$.

Linear Beam Optics

- Geometric Optics
- Equation of Motion
- Matrix Formalism
- Beams and Trace Space

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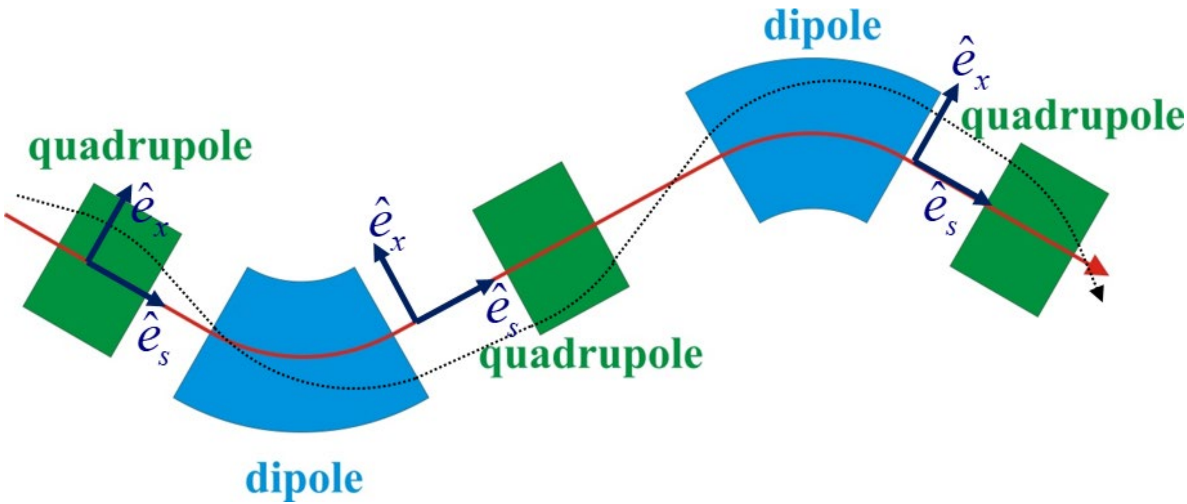
3. Linear Beam Optics

3.1. coordinate system following the design orbit

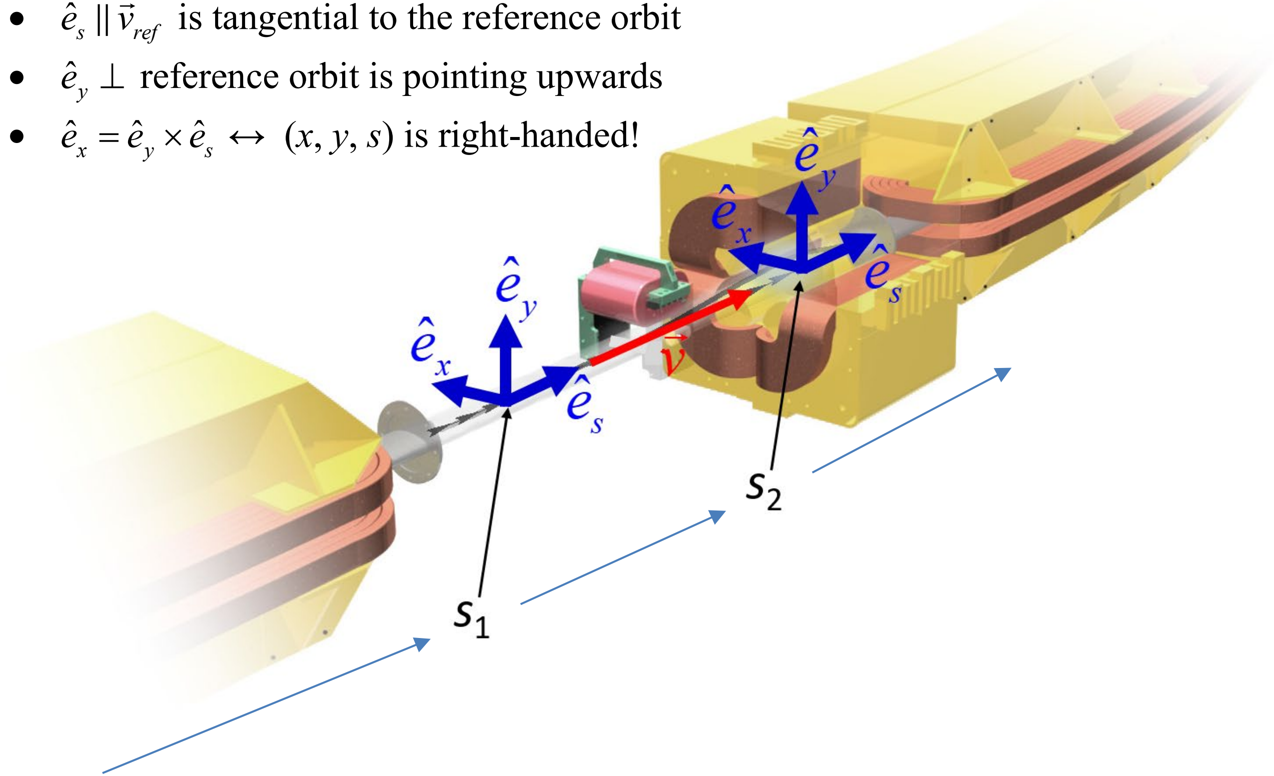
Reference path = path of a particle moving on the design path (reference orbit):



Use coordinate system fixed to reference orbit ("following the orbit"), changing its orientation! Horizontal position and angle of a particle given by **displacements** $\boldsymbol{x, x'}$.

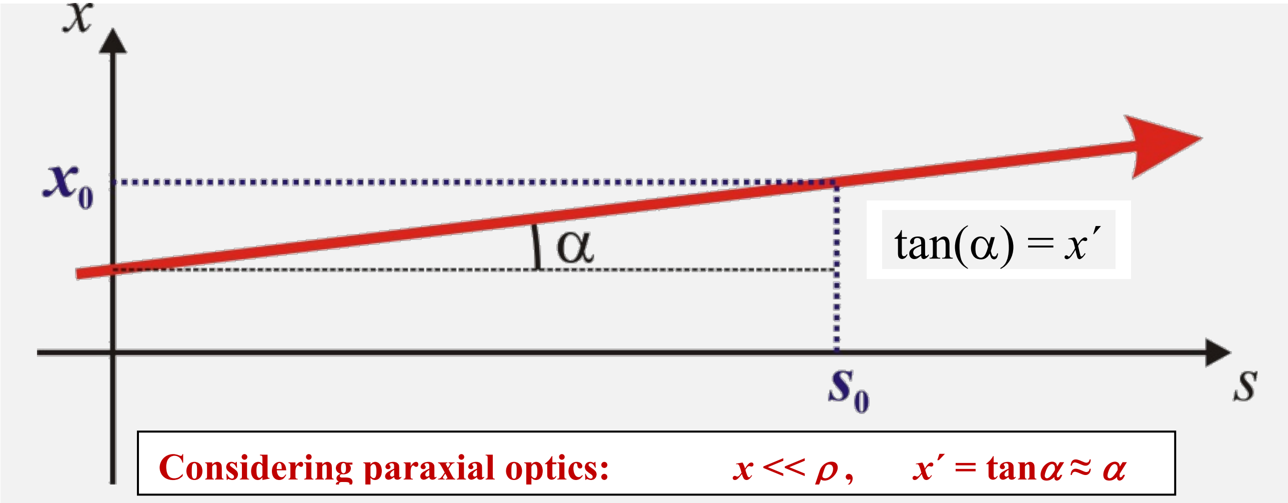
Coordinate unit vectors:

- $\hat{e}_s \parallel \vec{v}_{ref}$ is tangential to the reference orbit
- $\hat{e}_y \perp$ reference orbit is pointing upwards
- $\hat{e}_x = \hat{e}_y \times \hat{e}_s \leftrightarrow (x, y, s)$ is right-handed!



longitudinal coordinate s is measured along the design orbit

3.2. A quick and simple first approach using geometric optics

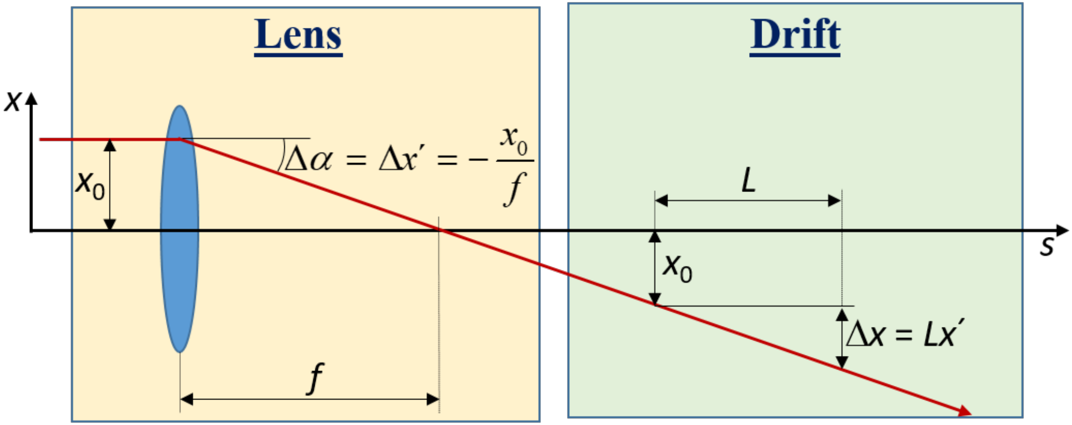


Impact of magnets in a very rough approximation:

dipole magnet: drift of length L_D

quadrupole magnet: thin lens with focal lengths $f_x = -\dfrac{1}{kL_Q}, \; f_y = \dfrac{1}{kL_Q}$

Particle positions in horizontal / vertical planes are changed by matrices:



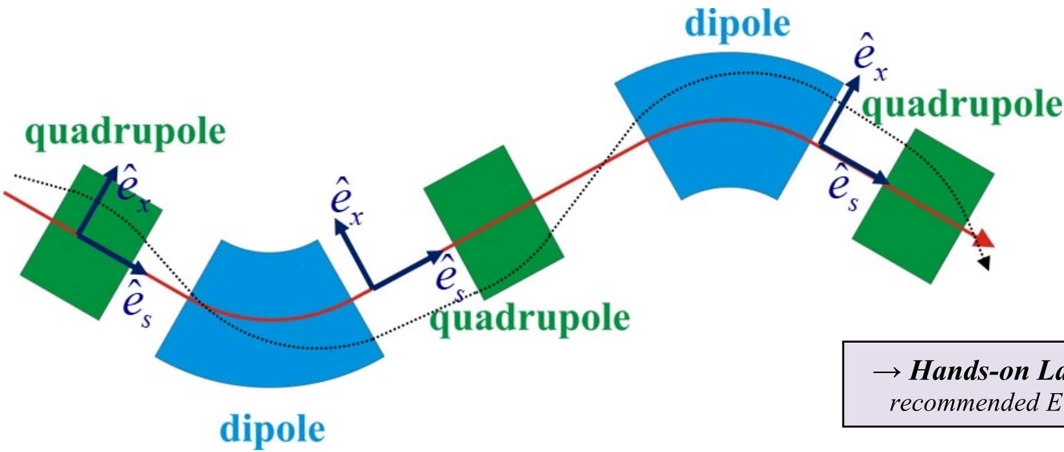
$$x = x_0 \qquad x' = x_0' - x_0/f$$

$$x = x_0 + L \cdot x_0' \qquad x' = x_0'$$

Plane	Position	Drift	Dipole	Quadrupole
horizontal	$\vec{x}(s) = \begin{pmatrix} x \\ x' \end{pmatrix}$	$\mathbf{M}_d = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$	$\mathbf{M}_D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$	$\mathbf{M}_Q = \begin{pmatrix} 1 & 0 \\ \pm 1/f & 1 \end{pmatrix}$

Plane	Position	Drift	Dipole	Quadrupole
vertical	$\vec{y}(s) = \begin{pmatrix} y \\ y' \end{pmatrix}$	$\mathbf{M}_d = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$	$\mathbf{M}_D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$	$\mathbf{M}_Q = \begin{pmatrix} 1 & 0 \\ \mp 1/f & 1 \end{pmatrix}$

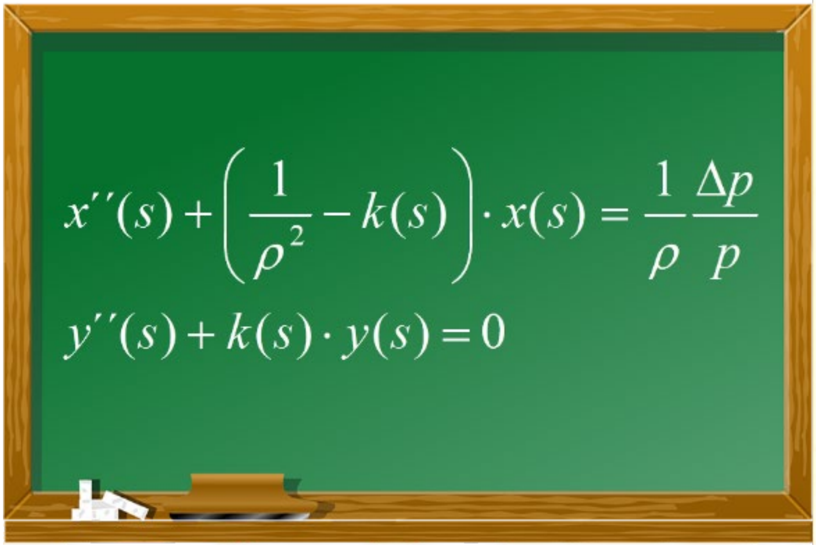
Calculation of single particle trajectories by matrix multiplication, e.g.:



→ *Hands-on Lattice Calculation*
recommended E1-E5, optional E6

$$\vec{x} = \underbrace{\mathbf{M}_d \bullet \mathbf{M}_Q \bullet \mathbf{M}_d \bullet \mathbf{M}_D \bullet \mathbf{M}_d \bullet \mathbf{M}_Q \bullet \mathbf{M}_d \bullet \mathbf{M}_D \bullet \mathbf{M}_d \bullet \mathbf{M}_Q \bullet \mathbf{M}_d}_{= \text{Transfer Matrix } \mathbf{M}} \bullet \vec{x}_0$$

More precise description → matrices have to be derived from equations of motion!



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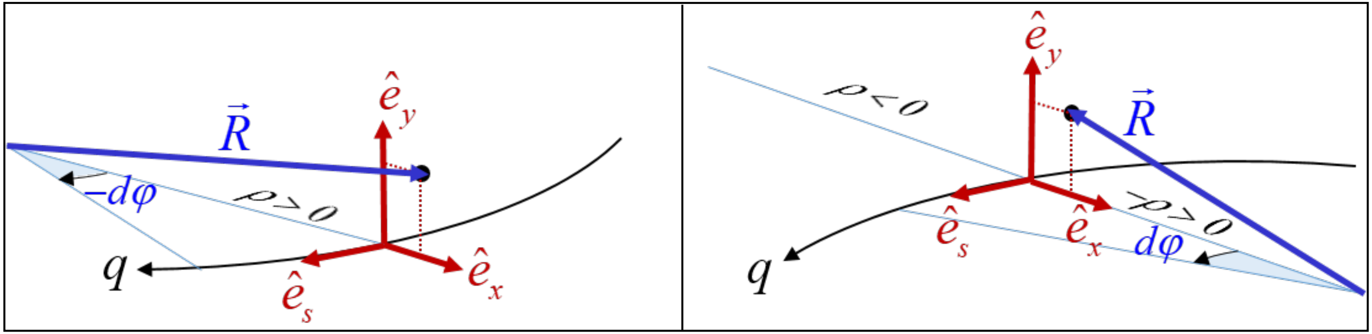
Taken from https:/www.wikipedia.org/wiki/Datei:Lämpel.jpg

Derivation of the equations of motion

3.3. Equations of motion in a reference system following the design orbit

Orthogonal, right-handed coordinate system (x, y, s) that follows a reference particle traveling along its ideal path (design orbit) **and changes its orientation with *s***:

positive curvature $1/\rho > 0$	negative curvature $1/\rho < 0$



We will concentrate on ideal orbits laying within the horizontal plane and on a local orbit with positive curvature ($\rho > 0$), therefore

$$\boxed{\vec{R} = (\rho + x) \cdot \hat{e}_x + y \cdot \hat{e}_y}$$

and $\hat{e}_y = \text{const.}$ pointing always upwards!

The case $\rho < 0$ can be treated fully analog and is left as an exercise ☺!

We use polar coordinates for the horizontal plane:

$$\hat{e}_x = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \qquad \hat{e}_\varphi = -\hat{e}_s = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

An increase ds along the reference path will lead to a rotation of $\hat{e}_x$ and $\hat{e}_y$ by

$$\mathrm{d}\varphi = -\frac{1}{\rho}\mathrm{d}s$$

and using

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}s} = -\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\varphi}$$

we get for the derivative of the unit vectors with respect to s

$$\boxed{\hat{e}_x{}' \equiv \frac{\mathrm{d}}{\mathrm{d}s}\hat{e}_x = +\frac{1}{\rho}\hat{e}_s, \qquad \hat{e}_s{}' \equiv \frac{\mathrm{d}}{\mathrm{d}s}\hat{e}_s = -\frac{1}{\rho}\hat{e}_x}$$

Changing orientation of coordinate vectors $\hat{e}_x(s)$, $\hat{e}_s(s)$

Now we proceed with several fundamental approximations:

1) All particles are moving with individual, but constant longitudinal velocities:

$$\dot{s} = v_s = \text{const.} \quad \gg \quad |v_x|, |v_y|$$

2) The curvature of the orbit is varying "slowly", the derivatives can be neglected:

$$\rho' = \rho'' = 0$$

3) The design orbit is in the (x, s) plane, the vertical coordinate is independent of s:

$$\hat{e}_y' = 0$$

4) The transverse displacements are small compared to the bending radius of the orbit (paraxial optics):

$$|x|, |y| \quad \ll \quad |\rho|$$

5) There exist no coupling of the transverse to the longitudinal motion. We will neglect all longitudinal components and put

$$R_s = R_s' = R_s'' = 0$$

For reasons of simplicity, we set $r(s) = \rho + x(s)$ and stop indicating the s dependence, thus getting for $\vec{R}$ and its derivatives

$$\vec{R} = r \cdot \hat{e}_x + y \cdot \hat{e}_y$$

$$\vec{R}' = x' \cdot \hat{e}_x + \cancel{\frac{r}{\rho} \hat{e}_s} + y' \cdot \hat{e}_y$$

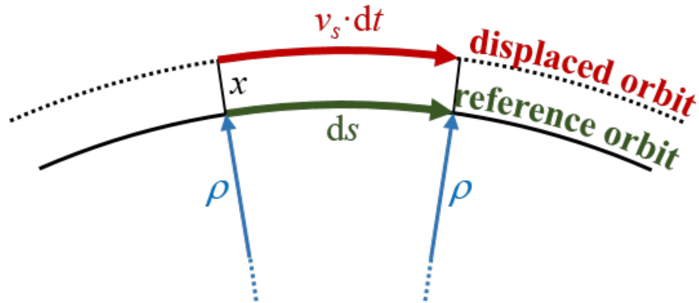
$$\vec{R}'' = \left(x'' - \frac{r}{\rho^2} \right) \cdot \hat{e}_x + \cancel{2 \frac{x'}{\rho^2} \hat{e}_s} + y'' \cdot \hat{e}_y$$

The change of the transverse position vector is caused by the external Lorentz force leading to a momentum change:

$$\dot{\vec{p}} = \frac{\mathrm{d}}{\mathrm{d}t} \vec{p} = \gamma_r m_0 \frac{\mathrm{d}^2}{\mathrm{d}t^2} \vec{R} = \gamma_r m_0 \ddot{\vec{R}} = q\left(\vec{v} \times \vec{B} \right)$$

→ Transformation of derivatives with respect to time $\frac{\mathrm{d}}{\mathrm{d}t}$ ("dot") to derivatives with respect to longitudinal position $\frac{\mathrm{d}}{\mathrm{d}s}$ ("prime") required!

From simple geometric considerations we obtain



Particle needs time dt *to travel a distance on displaced orbit which corresponds to* ds *on reference orbit*

$$\mathrm{d}s = v_s \cdot \mathrm{d}t \cdot \left(\frac{\rho}{\rho + x} \right) = \frac{\rho}{r} v_s \mathrm{d}t \qquad \rightarrow \qquad \boxed{\frac{\mathrm{d}}{\mathrm{d}t} = v_s \frac{\rho}{r} \frac{\mathrm{d}}{\mathrm{d}s}}$$

and therewith get

$$\ddot{\vec{R}} = \left(v_s \frac{\rho}{r} \right)^2 \vec{R}'' = \left(v_s \frac{\rho}{r} \right)^2 \left[\left(x'' - \frac{r}{\rho^2} \right) \cdot \hat{e}_x + y'' \cdot \hat{e}_y \right]$$

Using the assumption $v_s \gg |v_x|, |v_y|$ the Lorentz force can be simplified to

$$q\left(\vec{v} \times \vec{B}\right) = q\left(-v_s B_y \hat{e}_x + v_s B_x \hat{e}_y + \left(B_y v_x - B_x v_y \right) \right) \approx q v_s \left(-B_y \hat{e}_x + B_x \hat{e}_y \right),$$

Combining all, we obtain the equations of motion (p is the momentum of the particle)

$$\left(x'' - \frac{r}{\rho^2}\right)\cdot \hat{e}_x + y''\cdot\hat{e}_y = \frac{q}{p}\left(\frac{x+\rho}{\rho}\right)^2\left(-B_y\hat{e}_x + B_x\hat{e}_y\right)$$

We restrict the magnetic field components to the first two normal multipoles

$$B_x = -\frac{p_0}{q}k_y, \qquad B_y = \frac{p_0}{q}\left(\frac{1}{\rho} - k_x\right)$$

Important: The multipole strengths have been normalized to the momentum p_0 of the reference particle, the reference momentum!

Assuming a small momentum deviation $\Delta p = p - p_0$ of the considered non-reference particle we approximately can set

$$\frac{1}{p} \approx \frac{1}{p_0} + \Delta p \left.\frac{\partial(1/p)}{\partial p}\right|_{p=p_0} = \frac{1}{p_0}\left(1 - \frac{\Delta p}{p_0}\right)$$

Inserting this we obtain

$$x'' = \frac{x}{\rho^2} + \frac{1}{\rho} - \left(1 - \frac{\Delta p}{p_0}\right)\left(1 + \frac{x}{\rho}\right)^2 \left(\frac{1}{\rho} - kx\right)$$

$$y'' = -\left(1 - \frac{\Delta p}{p_0}\right)\left(1 + \frac{x}{\rho}\right)^2 ky$$

Neglecting all nonlinear terms in x, y, and $\Delta p / p_0$, we again obtain the

Linear Equations of Motion:

$$x''(s) + \left(\frac{1}{\rho^2(s)} - k(s)\right) \cdot x(s) = \frac{1}{\rho(s)} \frac{\Delta p}{p}$$

$$y''(s) + k(s) \cdot y(s) = 0$$

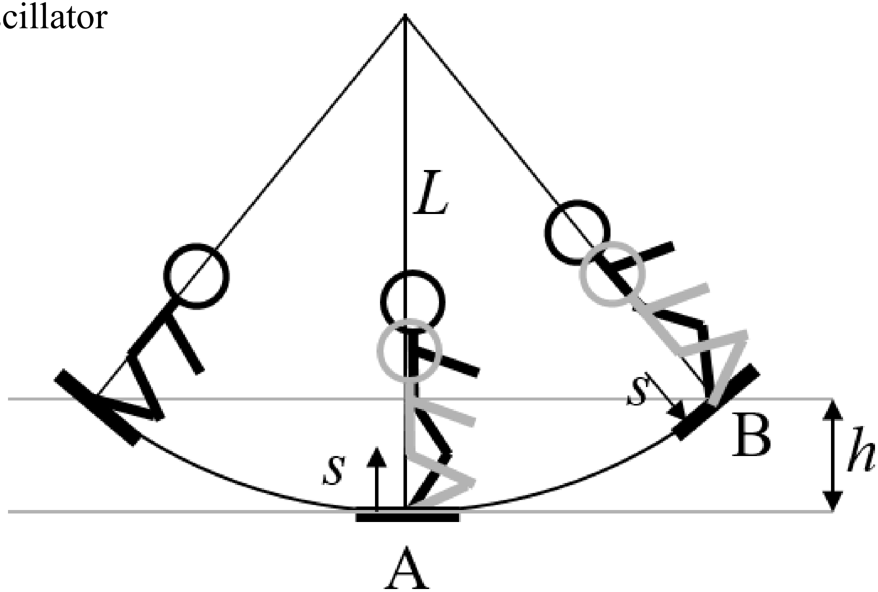
These are often referred to the Hill's equation which we will do only in case of periodicity / application to periodic cells or circular accelerators.

For the following, we will first look at monochromatic beams and set $\Delta p/p = 0$!

Remember:

Can be driven resonantly like a child's swing

↔ Parametric oscillator



$$\boxed{\omega = \sqrt{\frac{g}{l}}} \quad \rightarrow \quad \ddot{\varphi} + \omega^2(t) \cdot \varphi = 0$$

3.4. Matrix formalism

We will characterize a particles state by a vector built from its relative coordinates:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} \text{horizontal displacement} \\ \text{horizontal angular displacement} \\ \text{vertical displacement} \\ \text{vertical angular displacement} \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} x \\ x' \end{matrix}}\right\} \text{hor. trace space} \\ \left.\vphantom{\begin{matrix} y \\ y' \end{matrix}}\right\} \text{vert. trace space} \end{array}$$

and use the matrix formalism to describe particles trajectories: $\vec{X} = \mathbf{M} \cdot \vec{X}_0$. In case of upright magnets there will be no coupling of the transverse planes and we can generally write (and treat the 2 trace spaces separately!):

$$\mathbf{M} = \begin{pmatrix} r_{11} & r_{12} & 0 & 0 \\ r_{21} & r_{22} & 0 & 0 \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & r_{43} & r_{44} \end{pmatrix} = \begin{pmatrix} \mathbf{M}_x & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \mathbf{M}_y \end{pmatrix}$$

Next, we have to derive the matrices for drift, dipole and quadrupole magnets.

3.4.1. Drift space

$1/\rho(s) = k(s) = 0$ gives $\qquad x'(s) = x_0' = \text{const.},\; y'(s) = y_0' = \text{const.}$

Thus we get:

$$\mathbf{M}_{drift} = \begin{pmatrix} \boxed{\begin{matrix} 1 & L \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & L \\ 0 & 1 \end{matrix}} \end{pmatrix}$$

3.4.2. Dipole magnets

Constant bending radius: $k = 0$. Homogeneous solution (case $\Delta p/p = 0$):

$$x_h(s) = a \cdot \cos\frac{s}{\rho} + b \cdot \sin\frac{s}{\rho}$$

The integration constants *a*, *b* are derived from the boundary conditions at $s = 0$

$$x(s=0) = a = x_0, \quad x'(s=0) = \frac{b}{\rho} = x_0',$$

and by defining the bending angle $\varphi = L/\rho$ of the dipole magnet, we obtain :

$$x(L) = x_0 \cdot \cos\varphi + \rho \cdot x_0{}' \cdot \sin\varphi$$
$$y(L) = y_0 + \rho \cdot \varphi \cdot y_0{}'$$

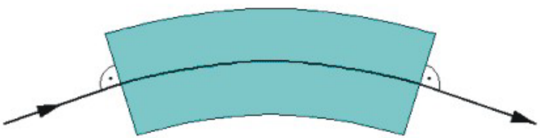
$$\mathbf{M}_{dipole} = \begin{pmatrix} \cos\varphi & \rho\sin\varphi & 0 & 0 \\ -1/\rho \cdot \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & \rho\varphi \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

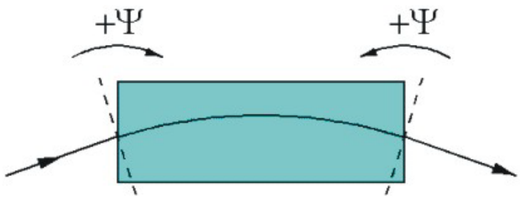
→ ***Hands-on Lattice Calculation***
recommended E31-E32

A sector magnet is therefore focusing in the horizontal plane.

Sector / rectangular dipole magnets and edge focusing:

Sector Magnet





Rectangular Magnet

The focusing / defocusing effect of the fringe fields (edge focusing) depends on the entrance (exit) angle ψ and may again be described by a linear transformation matrix

$$\mathbf{M}_\psi = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ \tan\psi/\rho & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & 0 \\ -\tan\psi/\rho & 1 \end{matrix}} \end{pmatrix}$$

We finally obtain with $\psi = \varphi/2$ and $\mathbf{M}_{rect} = \mathbf{M}_\psi \cdot \mathbf{M}_{dipole} \cdot \mathbf{M}_\psi$

$$\mathbf{M}_{rect} = \begin{pmatrix} \boxed{\begin{matrix} 1 & \rho\sin\varphi \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1-\rho\varphi/f & \rho\varphi \\ \rho\varphi/f^2 - 2/f & 1-\rho\varphi/f \end{matrix}} \end{pmatrix}$$

where we have defined the focal length $f \approx \rho/\tan\psi$ caused by edge (de)focusing.

A rectangular dipole magnet is therefore focusing in the vertical plane.
It acts like a drift space in the horizontal plane!

3.4.3. Quadrupole magnets

Assuming a pure quadrupole magnet we set the bending term $1/\rho = 0$. The solution of the equation of motion depends on the sign of the quadrupole strength *k*. For $k < 0$ we get the solution of a quadrupole magnet, which is horizontal focusing and vertical defocusing (the case $k > 0$ can be treated completely analog):

$$x(s) = a \cdot \cos\left(\sqrt{|k|} \cdot s\right) + b \cdot \sin\left(\sqrt{|k|} \cdot s\right)$$
$$y(s) = c \cdot \cosh\left(\sqrt{|k|} \cdot s\right) + d \cdot \sinh\left(\sqrt{|k|} \cdot s\right)$$

The integration constants *a, b, c, d* are derived from the boundary conditions at $s = 0$:

$$x(s=0) \;=\; a \;=\; x_0, \qquad x'(s=0) \;=\; b \;=\; x_0'$$
$$y(s=0) \;=\; c \;=\; y_0, \qquad y'(s=0) \;=\; d \;=\; y_0'$$

Substituting and building the first derivative, we obtain the transformation matrices for a **horizontal focusing (FQ) and a horizontal defocusing (DQ) quadrupole**, where we put $\Omega = \sqrt{|k|} \cdot L$ with the quadrupole length *L* and focal length $1/f = kL$.

QF ($k < 0$):

$$\mathbf{M}_{QF} = \begin{pmatrix} \boxed{\begin{matrix} \cos\Omega & \frac{1}{\sqrt{|k|}}\sin\Omega \\ -\sqrt{|k|}\sin\Omega & \cos\Omega \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} \cosh\Omega & \frac{1}{\sqrt{|k|}}\sinh\Omega \\ \sqrt{|k|}\sinh\Omega & \cosh\Omega \end{matrix}} \end{pmatrix} \xrightarrow{L\to 0} \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & 0 \\ \frac{1}{f} & 1 \end{matrix}} \end{pmatrix}$$

→ ***Hands-on Lattice Calculation***
recommended E30

QD ($k > 0$):

$$\mathbf{M}_{QD} = \begin{pmatrix} \boxed{\begin{matrix} \cosh\Omega & \frac{1}{\sqrt{|k|}}\sinh\Omega \\ \sqrt{|k|}\sinh\Omega & \cosh\Omega \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} \cos\Omega & \frac{1}{\sqrt{|k|}}\sin\Omega \\ -\sqrt{|k|}\sin\Omega & \cos\Omega \end{matrix}} \end{pmatrix} \xrightarrow{L\to 0} \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ \frac{1}{f} & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{matrix}} \end{pmatrix}$$

3.4.4. General properties of the transfer matrices

Each 4x4 transfer matrix consists of two 2x2 matrices

$$\mathbf{M}_{4x4} = \begin{pmatrix} \mathbf{M}_x & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \mathbf{M}_y \end{pmatrix}$$

The elements of the 2x2 matrices can be expresses by the fundamental solutions $C(s)$, $S(s)$, which from $x(s) = x_0 \cdot C(s) + x_0{'} \cdot S(s)$ satisfy $C(0) = S'(0) = 1,\ C'(0) = S(0) = 0$:

$$\mathbf{M}(s_0, s) = \begin{pmatrix} C(s-s_0) & S(s-s_0) \\ C'(s-s_0) & S'(s-s_0) \end{pmatrix} \quad \rightarrow \quad \mathbf{M}(s_0, s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

The determinant for $s = s_0$ is =1 and in general represented by the Wronskian

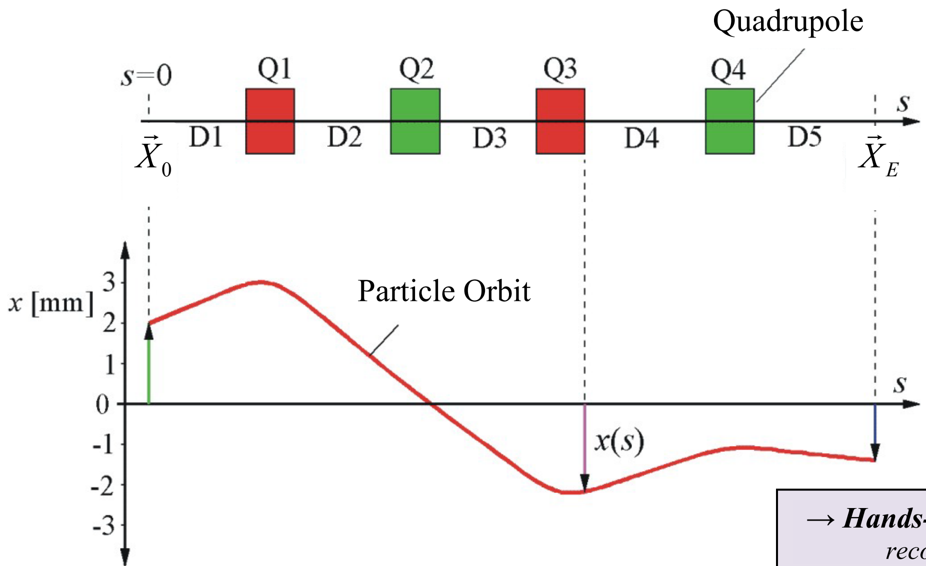
$$\det(\mathbf{M}) = CS' - SC'$$

Since C and S are solutions of the equation of motion, the derivative vanishes and

$$\frac{\mathrm{d}}{\mathrm{d}s}\det(\mathbf{M}) = CS'' - SC'' = 0 \quad \rightarrow \quad \boxed{\det(\mathbf{M}) = 1}$$

3.4.5. Particle orbits in a system of magnets

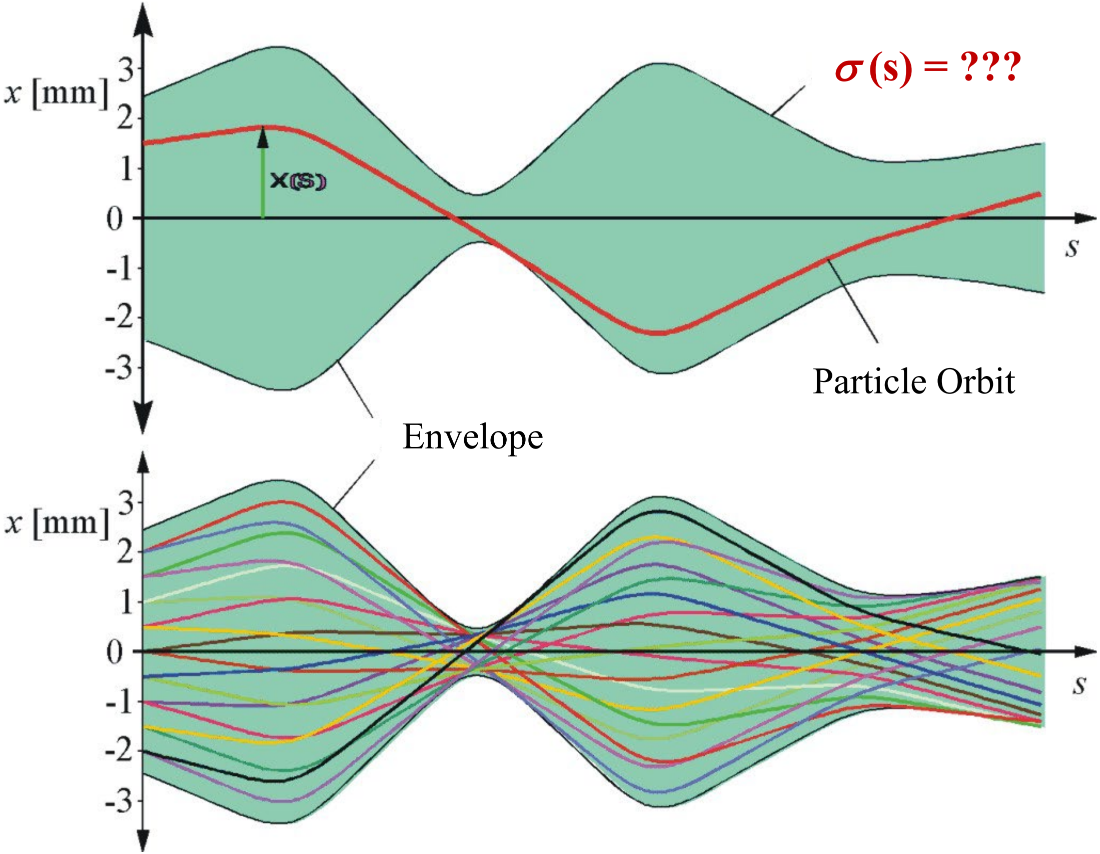
With the derived matrixes particle trajectories may be calculated for any given arbitrary beam transport line by cutting this beam line into smaller uniform pieces so that k = const. and ρ = const. in each of these pieces:



→ ***Hands-on Lattice Calculation***
recommended E7-E11

$$\boxed{\vec{X}_E = \mathbf{M}_{D5} \cdot \mathbf{M}_{Q4} \cdot \mathbf{M}_{D4} \cdot \mathbf{M}_{Q3} \cdot \mathbf{M}_{D3} \cdot \mathbf{M}_{Q2} \cdot \mathbf{M}_{D2} \cdot \mathbf{M}_{Q1} \cdot \mathbf{M}_{D1} \cdot \vec{X}_0}$$

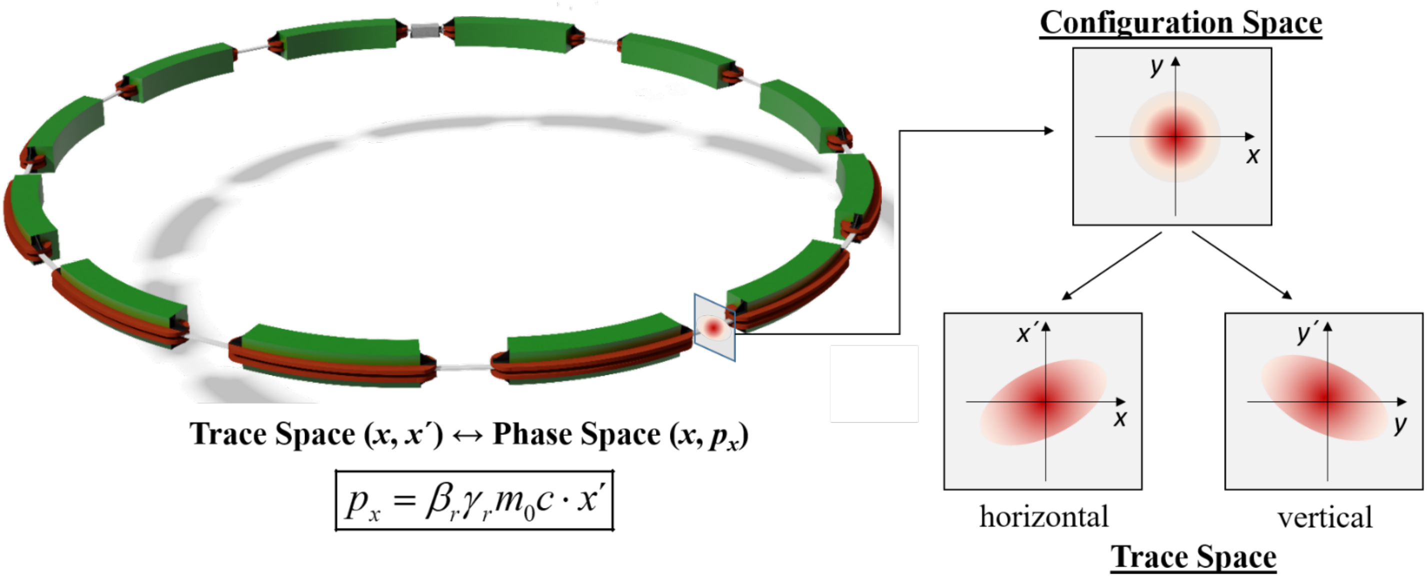
but:



σ (s) = ???
x [mm]
3
2
1
0
-1
-2
-3
x(s)
s
Particle Orbit
Envelope
x [mm]
3
2
1
0
-1
-2
-3
s

3.5. Particle beams and trace space

Configuration space ↔ trace space ↔ phase space



Trace Space (x, x') ↔ Phase Space (x, p_x)

$$\boxed{p_x = \beta_r \gamma_r m_0 c \cdot x'}$$

Famous theorem of Liouville:

The phase space distribution function describing the density of possible states around a phase space point is invariant under conservative forces"!

→ The phase space area covered by the beam remains constant!

3.5.1. Beam emittance

Beam = statistical set of points in trace space!

Consider e.g. horizontal trace space, intensity distribution in x, x'.

Choose origin of the coordinate axes $\hat{e}_x$ and $\hat{e}_{x'}$ at the barycentre of the points:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i = 0, \qquad \bar{x'} = \frac{1}{N}\sum_{i=1}^{N} x_i' = 0$$

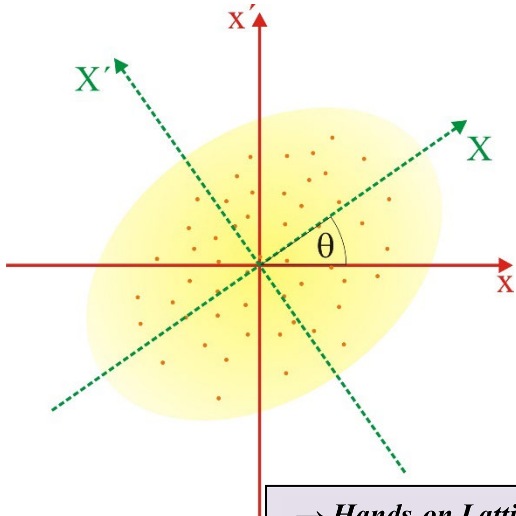
for a centered beam (will be assumed)!

Interested in variances (rms spread):

$$\sigma_x^{\ 2} = \frac{1}{N}\sum_{i=1}^{N} x_i^{\ 2}, \qquad \sigma_{x'}^{\ 2} = \frac{1}{N}\sum_{i=1}^{N} x_i'^{\,2}$$

System (X, X') which is rotated by θ:

$$\frac{\partial \sigma_X^{\ 2}}{\partial \theta} = \frac{\partial \sigma_{X'}^{\ 2}}{\partial \theta} = 0$$



→ *Hands-on Lattice Calculation*
recommended E12 - E15

We will define the spread of the distribution, which is called the **emittance ε_x**, by

$$\boxed{\varepsilon_x = \sigma_X \cdot \sigma_X{}' = \sqrt{\overline{x^2} \cdot \overline{x'^2} - \overline{x\,x'}^2}}$$

It is important to note that this is a statistical definition of ε!
More general, ε will be defined over the area $\pi\varepsilon = \int \mathrm{d}x \cdot \mathrm{d}x'$!

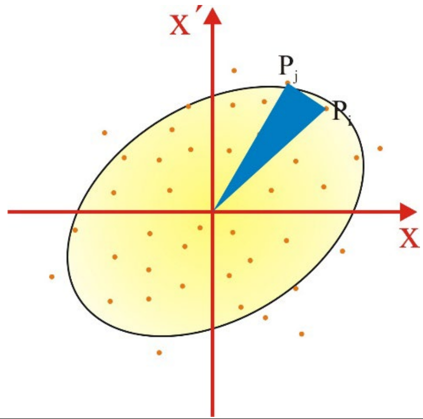
The emittance can be considered as a statistical mean area:

$$\varepsilon_x = \frac{1}{N}\sqrt{\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(x_i\,x_j{}' - x_j\,x_i{}'\right)^2} = \frac{1}{N}\sqrt{2\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}{}^2}$$

(remember: $2A_\Delta = \left|\vec{a}\times\vec{b}\right| \overset{a_3=b_3=0}{=} \left|a_1 b_2 - a_2 b_1\right|$)

where A_{ij} is the area of the triangle $0P_iP_j$

and ε is a measure of the spread of the points around their barycentre.



The area of the "rms"-envelope-ellipse is just π times the emittance ε

$$A = \pi\, a\, b = \pi\, \sigma_X\, \sigma_{X'} = \pi\, \varepsilon_x$$

and its equation with respect to the axes ***X*** and ***X′*** is

$$\boxed{\frac{X^2}{\sigma_X{}^2} + \frac{X'^2}{\sigma_{X'}{}^2} = 1 \qquad \Leftrightarrow \qquad X^2 \cdot \sigma_{X'}{}^2 + X'^2 \cdot \sigma_X{}^2 = \varepsilon_x^2}$$

3.5.2. Twiss parameters

By an inverse rotation of angle $-\theta$ in trace space we obtain

$$\boxed{\varepsilon_x^2 = x^2 \cdot \sigma_{x'}^2 - 2\, x\, x' \cdot \overline{x\, x'} + x'^2 \cdot \sigma_x^2 = x^2 \cdot \sigma_{x'}^2 - 2\, x\, x' \cdot r\, \sigma_x\, \sigma_{x'} + x'^2 \cdot \sigma_x^2}$$

where we have defined the correlation coefficient

$$r = \frac{\overline{x\, x'}}{\sqrt{\overline{x^2} \cdot \overline{x'^2}}}$$

It is more or less obvious, that such a correlation term must exist in general.

We may define the so-called **Twiss-parameters α_x, β_x, and γ_x** such that

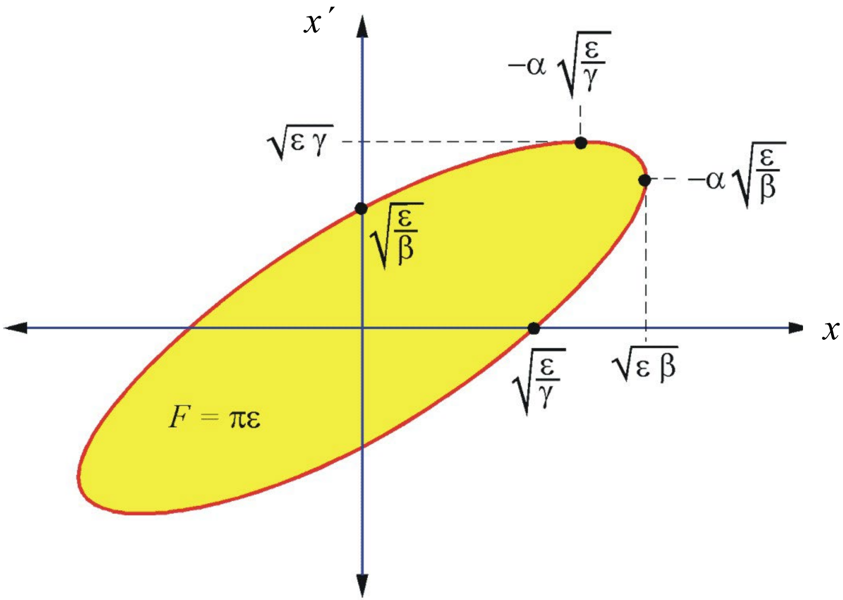
$$\begin{aligned} \sigma_x &= \sqrt{\overline{x^2}} = \sqrt{\beta_x \varepsilon_x} \\ \sigma_{x'} &= \sqrt{\overline{x'^2}} = \sqrt{\gamma_x \varepsilon_x} \\ r\,\sigma_x \sigma_{x'} &= \overline{x x'} = -\alpha_x \varepsilon_x \end{aligned}$$

and the equation of the envelope-ellipse reads in the "conventional" form:

$$\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 = \varepsilon_x$$

All the above derived equations appear in identical form for the vertical plane, x has only to be replaced by y. In the following, we will skip the index x for reason of simplicity. Please note, that this doesn't imply that emittances and corresponding Twiss parameters are equal in both planes – they are not!

The meaning of the Twiss-parameters can be read off from the graphical representation of the envelope-ellipse:



- $\sqrt{\beta}$ represents the r.m.s. beam-envelope per unit emittance,
- $\sqrt{\gamma}$ represents the r.m.s. beam divergence per unit emittance,
- α is proportional to the correlation between x and x'.

3.5.3. Beta functions

In the following, we will continue to concentrate on the situation where $\Delta p / p = 0$.

With $K_x(s) = 1/\rho^2(s) - k(s)$ and $K_y(s) = k(s)$ the equations of motion read

$$x''(s) + K_x(s) \cdot x(s) = 0, \qquad y''(s) + K_y(s) \cdot y(s) = 0$$

They describe a transverse oscillation with position dependent amplitude and phase, which is called **betatron oscillation**. Both transverse planes can be treated similar! We will therefore concentrate on x and try to solve this equation, making the Ansatz

$$x(s) = A \cdot w_x(s) \cdot \cos\left(\mu_x(s) + \varphi_0\right)$$

(A and φ_0 are integration constants, we will skip the index x from now on) and obtain:

$$\left[w'' - w \cdot \mu'^2 + K \cdot u\right] \cdot \cos\left(\mu + \varphi_0\right) - \left[2 \cdot w' \cdot \mu' + w \cdot \mu''\right] \sin\left(\mu + \varphi_0\right) = 0$$

This relation is valid for any given phase $\mu(s)$ at any given position s, therefore

$$w'' - w \cdot \mu'^2 + K \cdot w = 0$$
$$2 \cdot w' \cdot \mu' + w \cdot \mu'' = 0$$

By integration of the second equation we obtain

$$\mu(s) = \int_0^s \frac{\mathrm{d}\,\tilde{s}}{w^2(\tilde{s})}$$

and by using this relation $w'' - \frac{1}{w^3} + K \cdot w = 0$.

With the definition of the beta function $\beta(s) := w^2(s)$ we derive for the amplitude and phase of the oscillation:

$$\boxed{\begin{gathered} x(s) \;=\; A \cdot \sqrt{\beta(s)} \cdot \cos\left(\mu(s) + \varphi_0\right) \\ \mu(s) = \int_0^s \frac{\mathrm{d}\,\tilde{s}}{\beta(\tilde{s})} \end{gathered}}$$

Building the first derivative and defining $\alpha(s) := -\frac{\beta'(s)}{2}$, we obtain

$$\boxed{x'(s) = -\frac{A}{\sqrt{\beta(s)}}\left\{\alpha(s) \cdot \cos\left(\mu(s) + \varphi_0\right) + \sin\left(\mu(s) + \varphi_0\right)\right\}}$$

The equation for x can be transformed to

$$\cos^2(\mu+\varphi_0)=\frac{x^2}{A^2\cdot\beta},$$

which can be used in combination with the equation for x′ to obtain

$$\sin^2(\mu+\varphi_0)=\left(\frac{\sqrt{\beta}}{A}\cdot x'+\frac{\alpha}{A\sqrt{\beta}}\cdot x\right)^2$$

Using $\cos^2+\sin^2=1$ we derive

$$\frac{x^2}{\beta(s)}+\left(\frac{\alpha(s)}{\sqrt{\beta(s)}}\cdot x+\sqrt{\beta(s)}\cdot x'\right)^2=A^2$$

which can be transformed by defining $\gamma(s):=\frac{1+\alpha^2(s)}{\beta(s)}$ to:

$$\boxed{\gamma x^2+2\alpha x x'+\beta x'^2 = A^2, \qquad \text{where} \quad \frac{1}{\beta(s)}=\mu'(s), \quad \alpha=-\frac{\beta'}{2}, \quad \gamma=\frac{1+\alpha^2}{\beta}}$$

Are the newly defined functions α, β, γ identical with the Twiss Parameters?

Each particle is represented by its individual A_i and φ_i.

But all particles are described by the same optical functions $\alpha, \beta, \gamma, \mu$!

Let's check and calculate the second statistical moments:

$$\sigma_x^2 = \overline{x^2} = \overline{A_i^2 \beta \cos^2\left(\mu + \varphi_{0,i}\right)} = \boxed{\frac{1}{2}\overline{A_i^2}\beta} \quad \rightarrow \quad \sigma_x^2 = \varepsilon\beta$$

$$\sigma_{x'}^2 = \overline{x'^2} = \overline{\frac{A_i^2}{\beta}\left\{\alpha^2\cos^2(\ldots) + \sin^2(\ldots) + 2\alpha\cos(\ldots)\sin(\ldots)\right\}} = \boxed{\frac{1}{2}\overline{A_i^2}\gamma} \quad \rightarrow \quad \sigma_{x'}^2 = \varepsilon\gamma$$

$$\overline{xx'} = \overline{-A_i^2\left\{\alpha\cos^2\left(\mu + \varphi_{0,i}\right) + \cos\left(\mu + \varphi_{0,i}\right)\sin\left(\mu + \varphi_{0,i}\right)\right\}} = \boxed{-\frac{1}{2}\overline{A_i^2}\alpha} \quad \rightarrow \quad \overline{xx'} = -\varepsilon\alpha$$

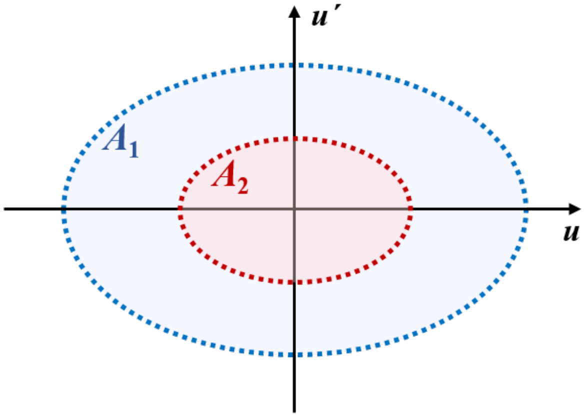
$$\varepsilon^2 = \overline{x^2}\,\overline{x'^2} - \overline{xx'}^2 = \overline{\frac{1}{2}A_i^2\left(\beta\gamma - \alpha^2\right)} = \boxed{\frac{1}{2}\overline{A_i^2}}$$

→ Indeed – they are!!!

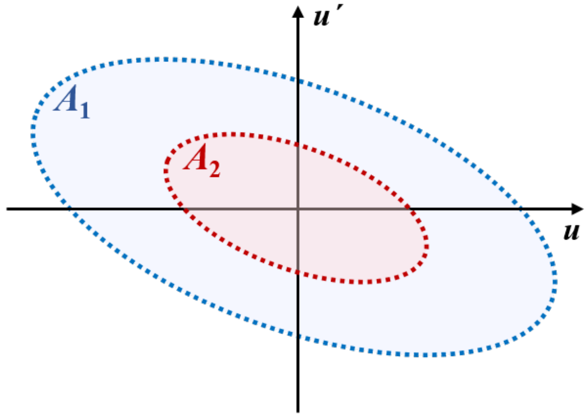
Note:

Each particle will stay on its own ellipse, which will enclose a constant area in phase space A. The amplitude factor ***A* represents the Courant Snyder invariant**! The shape of the ellipse is determined by the Twiss parameters α, β, γ and will change along the magneto-optics system, its area will stay always constant (Rem.: in case of conservative forces and no acceleration). The shape (not the size) of all single particle ellipses are determined by the same Twiss parameters!

→ ***Hands-on Lattice Calculation***
recommended E18-E21



longitudinal position s_1



longitudinal position s_2

3.5.4. Transformation in trace space

According to Liouville's theorem, all particles enclosed by an envelope ellipse will stay within that ellipse. The transformation of the horizontal and vertical ellipse parameters along the beam line may be derived from the transport matrixes in the horizontal and vertical plane. Starting at $s=0$, we have for a particle on this ellipse

$$\boxed{\gamma_0 x_0{}^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2 = A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2}$$

Any particle trajectory starting at $s=0$ transforms to $s\neq 0$ by

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

which gives for the transformed ellipse equation via

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \underbrace{\frac{1}{CS'-C'S} \cdot \begin{pmatrix} S'(s) & -S(s) \\ -C'(s) & C(s) \end{pmatrix}}_{=\mathbf{M}^{-1}} \cdot \begin{pmatrix} x \\ x' \end{pmatrix} \overset{|\mathbf{M}|=1}{=} \begin{pmatrix} S'x - Sx' \\ -C'x + Cx' \end{pmatrix}$$

and $\quad x_0{}^2 = S'^2 x^2 - 2SS'xx' + S^2 x'^2, \quad x_0'^2 = C'^2 x^2 - 2CC'xx' + C^2 x'^2, \quad x_0 x_0' = \ldots$

$$\underbrace{\left(S'^2\cdot\gamma_0 - 2\,S'C'\cdot\alpha_0 + C'^2\cdot\beta_0\right)}_{=\gamma}\cdot x^2 + 2\underbrace{\left(-S\,S'\cdot\gamma_0 + \left(S'C + S\,C'\right)\cdot\alpha_0 - C\,C'\cdot\beta_0\right)}_{=\alpha}\cdot x\,x'$$

$$+\underbrace{\left(S^2\cdot\gamma_0 - 2\,S\,C\cdot\alpha_0 + C^2\cdot\beta_0\right)}_{=\beta}\cdot x'^2 = \varepsilon$$

This gives the transformation of the beam parameters in matrix formulation

$$\boxed{\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix} = \begin{pmatrix} C^2 & -2\,S\,C & S^2\\ -C\,C' & S'C + S\,C' & -S\,S'\\ C'^2 & -2\,S'C' & S'^2\end{pmatrix}\cdot\begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix}}$$

Another useful relation may be obtained by defining the Beta matrix **B**

$$\mathbf{B} \equiv \begin{pmatrix}\beta & -\alpha\\ -\alpha & \gamma\end{pmatrix}, \qquad \det(\mathbf{B}) = \beta\,\gamma - \alpha^2 = 1, \qquad \varepsilon\cdot\mathbf{B} = \begin{pmatrix}\sigma_x^2 & \overline{xx'}\\ \overline{xx'} & \sigma_{x'}^2\end{pmatrix} \equiv \mathbf{\Sigma}$$

The equation of the envelope-ellipse can be transformed to:

$$\varepsilon = {}^T\vec{x}_0\cdot\mathbf{B}_0^{-1}\cdot\vec{x}_0 = {}^T\vec{x}_1\cdot\mathbf{B}_1^{-1}\cdot\vec{x}_1$$

where the inverse of the Beta matrix is

$$\mathbf{B}^{-1} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}$$

and displacement-vector $\vec{X}$ transforms according to

$$\vec{x}_1 = \mathbf{M} \cdot \vec{x}_0, \qquad {}^T\vec{x}_1 = {}^T\!\left(\mathbf{M} \cdot \vec{x}_0\right) = {}^T\vec{x}_0 \cdot {}^T\mathbf{M}$$

By inserting $\mathbf{1} = \mathbf{M}^{-1} \cdot \mathbf{M}$, we obtain:

$$\begin{aligned} \varepsilon &= {}^T\vec{x}_0 \cdot {}^T\mathbf{M} \cdot {}^T\mathbf{M}^{-1} \cdot \mathbf{B}_0^{-1} \cdot \mathbf{M}^{-1} \cdot \mathbf{M} \cdot \vec{x}_0 \\ &= {}^T\!\left(\mathbf{M} \cdot \vec{x}_0\right) \cdot \left({}^T\mathbf{M}^{-1} \cdot \mathbf{B}_0^{-1} \cdot \mathbf{M}^{-1} \right) \cdot \left(\mathbf{M} \cdot \vec{x}_0\right) \\ &= {}^T\vec{x}_1 \cdot \left(\mathbf{M} \cdot \mathbf{B}_0 \cdot {}^T\mathbf{M}\right)^{-1} \cdot \vec{x}_1 \end{aligned}$$

and we can read off the transformation of the Beta matrix:

$$\boxed{\mathbf{B}_1 = \mathbf{M} \cdot \mathbf{B}_0 \cdot {}^T\mathbf{M}}$$

This can e.g. be used to derive the beta-function around a symmetry-point of a trans-fer-line where $\alpha = 0$ in a simple way:

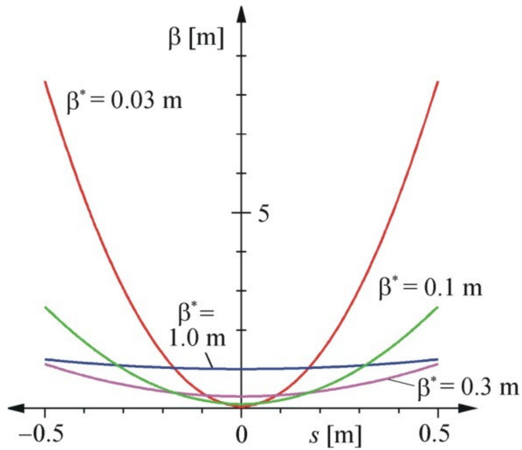
$$\mathbf{B}_1(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \beta_{sym} & 0 \\ 0 & 1/\beta_{sym} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_{sym} + \dfrac{s^2}{\beta_{sym}} & \dfrac{s}{\beta_{sym}} \\ \dfrac{s}{\beta_{sym}} & \dfrac{1}{\beta_{sym}} \end{pmatrix}$$

This gives the relations for the beam parameters around a symmetry-point:

$$\beta(s) = \beta_{sym} + \frac{s^2}{\beta_{sym}}$$

$$\alpha(s) = -\frac{s}{\beta_{sym}}$$

$$\gamma(s) = \frac{1}{\beta_{sym}}$$



The corresponding beam size scales with

$$\sigma_x = \sqrt{\varepsilon \cdot \beta(s)} \;\; !$$

Remember: $\sigma_x = \sqrt{\varepsilon \cdot \beta(s)}$, $\sigma_x{}' = \sqrt{\varepsilon \cdot \gamma(s)}$, and therewith:

$$\sigma(s) = \sigma_0 \cdot \sqrt{1 + \left(\frac{s}{\beta_0}\right)^2}, \qquad \sigma'(s) = \frac{\varepsilon}{\sigma_0} = \text{const.}$$

To obtain further insights, we will compare the particle's beam with a Gaussian light beam (TEM_{00}), characterized by its waste radius w(s) and Rayleigh length z_R, in which w is doubled. From diffraction theory, we know (from diffraction integrals):

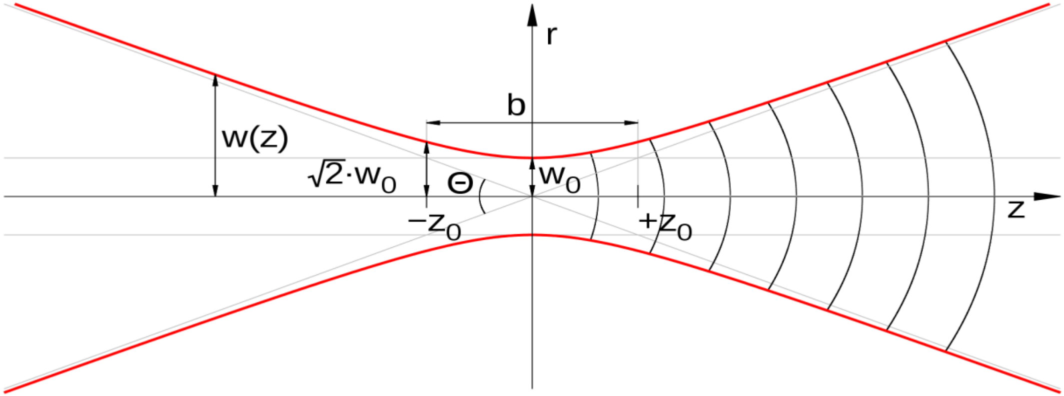
$$w(s) = w_0 \cdot \sqrt{1 + \left(\frac{s}{z_R}\right)^2}, \quad z_R = \frac{\pi w_0^{\,2}}{\lambda}, \quad \theta_{\max} = \frac{\lambda}{\pi w_0}, \quad I(x,y) = I_{\max} \cdot \left(\frac{w_0}{w}\right)^2 \cdot e^{\frac{-2\left(x^2+y^2\right)}{w^2}}$$

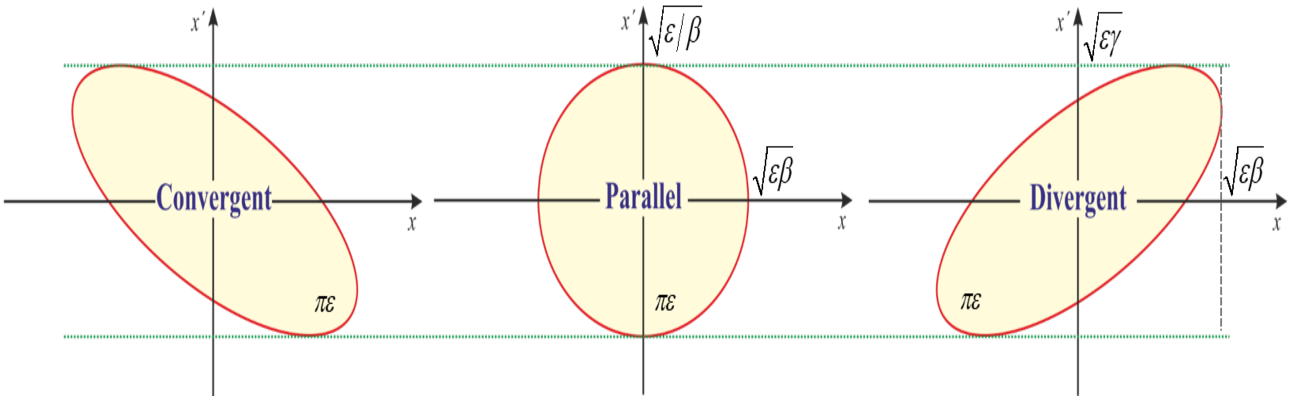
This indicates:

$$\beta_0 = z_R = \frac{\pi w_0^2}{\lambda}, \quad \text{and from} \quad \sigma_x^{\,2} = \frac{\iint x^2 I(x,y) \cdot \mathrm{d}x\,\mathrm{d}y}{\iint I(x,y) \cdot \mathrm{d}x\,\mathrm{d}y} = \frac{w^2}{4}$$

and replacing $w = 2\sigma = 2\sqrt{\varepsilon\beta}$ we obtain the important relation:

$$\Rightarrow \qquad 4\pi \cdot \varepsilon = \lambda$$


r
b
w(z)
√2·w0
Θ
w0
−z0
+z0
z


x′
x
Convergent
πε
x′
√ε/β
√εβ
Parallel
x
πε
x′
√εγ
√εβ
Divergent
x
πε

The transformation matrix **M** can be derived also from the Twiss parameters. With

$$x(s) = \sqrt{\varepsilon\beta}\cos\left(\mu+\varphi_0\right) = \sqrt{\varepsilon}\cdot\sqrt{\beta}\cdot\left\{\cos\mu\cdot\cos\varphi_0 - \sin\mu\cdot\sin\varphi_0\right\}$$

$$x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta}}\cdot\left\{\alpha\cdot\left[\cos\mu\cdot\cos\varphi_0 - \sin\mu\cdot\sin\varphi_0\right] - \sin\mu\cdot\cos\varphi_0 + \cos\mu\cdot\sin\varphi_0\right\}$$

and the starting conditions $x(0) = x_0$, $x'(0) = x_0'$, $\mu(0) = 0$, which transform to

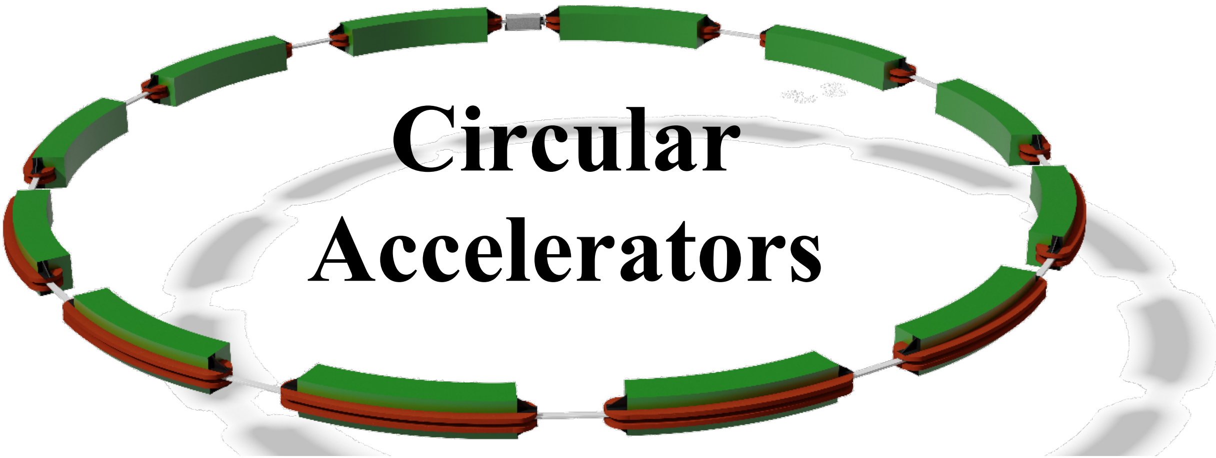
$$\cos\varphi_0 = x_0 \Big/ \sqrt{\varepsilon\,\beta_0}$$

$$\sin\varphi_0 = -1\Big/\sqrt{\varepsilon}\left(x_0'\sqrt{\beta_0} + \alpha_0\,x_0\Big/\sqrt{\beta_0}\right)$$

we obtain:

$$\boxed{\mathbf{M}(s_0,s) = \begin{pmatrix} \dfrac{\sqrt{\beta}}{\sqrt{\beta_0}}\left(\cos\mu + \alpha_0\sin\mu\right) & \sqrt{\beta\,\beta_0}\sin\mu \\ \dfrac{\alpha_0-\alpha}{\sqrt{\beta\,\beta_0}}\cos\mu - \dfrac{1+\alpha\,\alpha_0}{\sqrt{\beta\,\beta_0}}\sin\mu & \dfrac{\sqrt{\beta_0}}{\sqrt{\beta}}\left(\cos\mu - \alpha\sin\mu\right) \end{pmatrix}}$$


Picture taken from https://commons.wikimedia.org/wiki/File:Swiss_Light_Source,_PSI_-_panoramio.jpg



Circular Accelerators

4. Circular Accelerators

4.1. Weak focusing

Beam stability: transverse focusing in both planes!

Equation of motion:

$$x''(s) + \underbrace{\left(\frac{1}{\rho^2(s)} - k(s) \right)}_{>0} \cdot x(s) = 0$$

$$y''(s) + \overbrace{k(s)} \cdot y(s) = 0$$

Idea: horizontally defocusing k is overcompensated by geometrical focusing!

$$0 < k = -\frac{q}{p} \frac{\partial B_y}{\partial x} < \frac{1}{\rho^2}$$

With $p = q\rho B_0$, where B_0 defines the bending field at the design orbit, one obtains

$$\boxed{0 < n = -\frac{\rho}{B_0} \frac{\partial B_y}{\partial x} < 1 \qquad \text{(Steenbeck 1924)}}$$

where we have defined the field index n to

$$n = k\rho^2 = -\frac{\rho}{B_0}\frac{\partial B_y}{\partial \rho} \qquad \rightarrow \qquad B_y(x) = B_0 \cdot \left(\frac{x+\rho}{\rho}\right)^{-n}$$

Thus, a circular accelerator like a synchrotron has to be made of dipole magnets with radially decreasing bending field strength fulling the above derived weak focusing condition.

Particles will oscillate around the reference trajectory with the spatial frequency

$$\omega_x = \sqrt{\frac{1}{\rho^2} - k} = \frac{\sqrt{1-n}}{\rho}, \qquad \omega_y = \sqrt{k} = \frac{\sqrt{n}}{\rho}$$

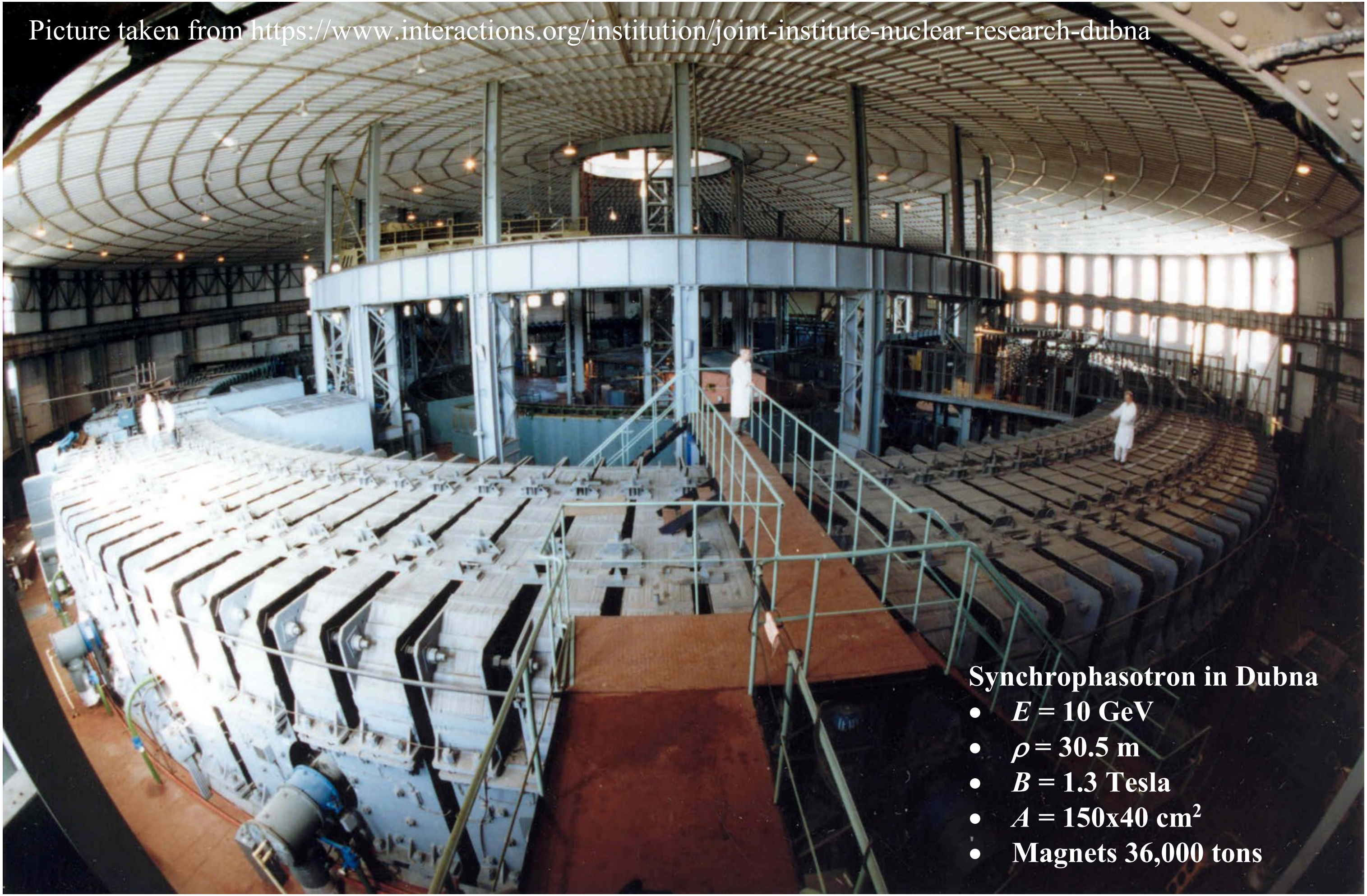
The number Q of oscillations per turn of length $L = 2\pi\rho$ will then be

$$Q_x = \frac{1}{2\pi}\oint \frac{\mathrm{d}s}{\beta_x} = \sqrt{1-n} < 1, \qquad Q_y = \frac{1}{2\pi}\oint \frac{\mathrm{d}s}{\beta_y} = \sqrt{n} < 1$$

Problem:

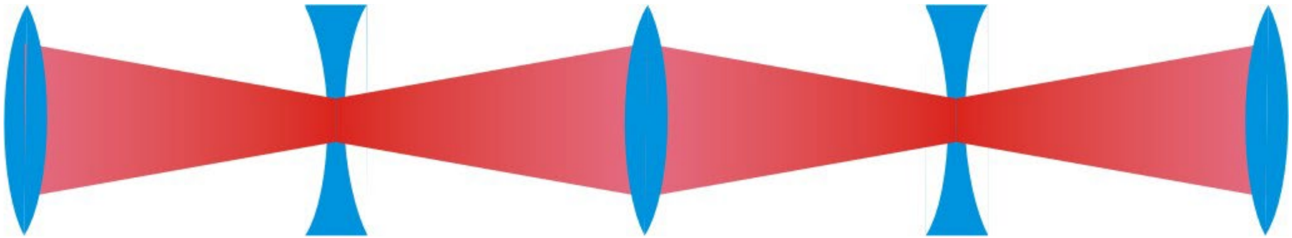
We derive for the constant beta functions $\beta_{x,y} > \rho$

$\rightarrow$ beam size $\sigma = \sqrt{\varepsilon\beta}$ will increase remarkably with increasing radius!


Picture taken from https://www.interactions.org/institution/joint-institute-nuclear-research-dubna
Synchrophasotron in Dubna
• E = 10 GeV
• ρ = 30.5 m
• B = 1.3 Tesla
• A = 150x40 cm^2
• Magnets 36,000 tons

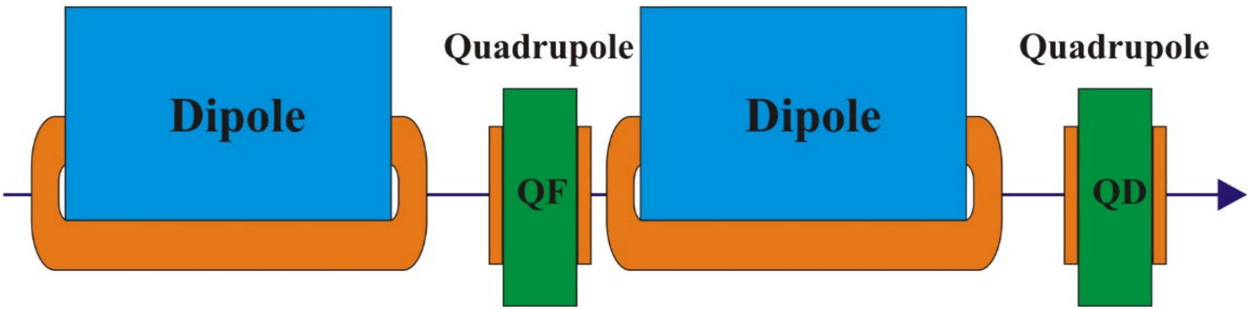
4.2. Strong focusing

Focusing in both planes possible in case of alternating gradient – well know from light optics:



Magnet optics:

Simplest configuration: FODO lattice, periodic arrangement of identical structures




FODO
DESY II →
← DESY I
Not-Aus
Picture taken from https://desy2.desy.de/

4.2.1. Stability criterion

If $\mathbf{M}(L)$ is the transformation matrix for one periodic cell we will have for *N* cells:

$$\mathbf{M}(N \cdot L) = \left[\mathbf{M}\left(L\right)\right]^{N}$$

For a full lattice period, we take use of **Floquet's theorem**. Recalling the equations of motions

$$x''(s) + K_x(s) \cdot x(s) = 0 \qquad \text{with} \quad K_x(s) = 1/\rho^2(s) - k(s)$$
$$y''(s) + K_y(s) \cdot y(s) = 0 \qquad \text{with} \quad K_y(s) = k(s)$$

it states (Gaston Floquet, 1847 – 1920) for e.g. $x(s) = A\sqrt{\beta_x(s)}\cos\left(\mu_x(s) + \varphi_0\right)$

If *K*(s) is periodic, the amplitude function (and therefore β(s)) is periodic as well.

In this case we call the DGL **Hill's equation** (George William Hill 1838 – 1914).

Please note and take care:

Floquet's theorem doesn't state that μ(s) and therewith *x*(s), *y*(s) are periodic as well!

This would be an exception! (catastrophic, as we will see later)

Thus we recommend periodic boundary conditions $\beta = \beta_0$, $\alpha = \alpha_0$ and obtain, using the Twiss parameter representation of the transfer matrix:

$$\mathbf{M} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix}$$

This matrix was first derived by Twiss from general mathematics principles and is called the **Twiss matrix** (Richard Q. Twiss, 1920 – 2005).

We calculate its eigenvalues from

$$\det(\mathbf{M} - \lambda \cdot \mathbf{I}) = \lambda^2 - \mathrm{Tr}\{\mathbf{M}\} \cdot \lambda + \det(\mathbf{M}) = 0$$

With $\mathrm{Tr}\{\mathbf{M}\} = 2 \cdot \cos\mu$ and $\det(\mathbf{M}) = 1$ we obtain

$$\lambda_{1,2} = \cos\mu \pm i \sin\mu = e^{\pm i\mu}$$

We require that the eigenvalues remain finite thus requiring a real betatron phase μ.

This is guaranteed when $|\cos\mu| \leq 1$ and leads to the general stability condition:

$$|\mathrm{Tr}\{\mathbf{M}\}| = |r_{11} + r_{22}| \leq 2$$

And now comes the "clou": Rewriting the Twiss matrix using

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \qquad \mathbf{J}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}$$

it can be expressed by

$$\mathbf{M} = \mathbf{I} \cdot \cos\mu + \mathbf{J} \cdot \sin\mu$$

Similar to Moivre's formula we get for N equal periods

$$\mathbf{M}^N = \left(\mathbf{I} \cdot \cos\mu + \mathbf{J} \cdot \sin\mu\right)^N = \mathbf{I} \cdot \cos(N\mu) + \mathbf{J} \cdot \sin(N\mu)$$

and

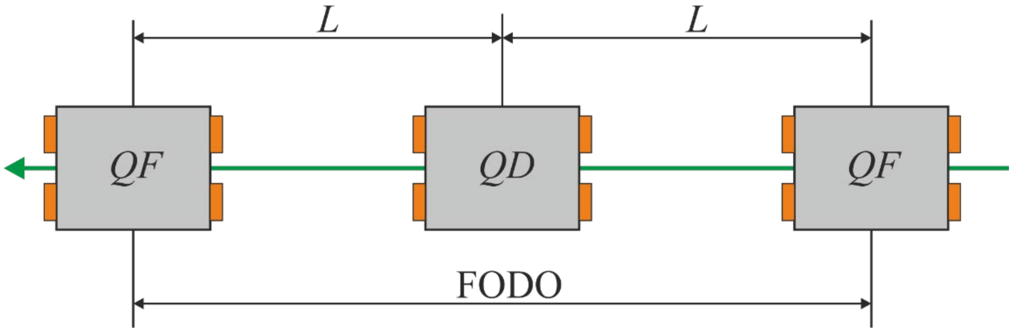
$$\left|\mathrm{Tr}\left\{\mathbf{M}^N\right\}\right| = \left|2 \cdot \cos(N\mu)\right| \leq 2$$

Conclusion:

In case of a real betatron phase advance μ, the beam size in a circular accelerator will remain finite (*the 100 Mio \$ question in the 50's!*). This can easily be proofed by calculating the trace of the one turn matrix: $\boxed{\left|\mathrm{Tr}\left\{\mathbf{M}\right\}\right| \leq 2}$

4.3. Periodic focusing systems

4.3.1. General FODO lattice



The FODO geometry can be expressed symbolically by the sequence

$$\underbrace{\tfrac{1}{2}\text{QF, D, }\tfrac{1}{2}\text{QD}}_{=\mathbf{M}_{-1/2}},\ \underbrace{\tfrac{1}{2}\text{QD, D, }\tfrac{1}{2}\text{QF}}_{=\mathbf{M}_{1/2}}$$

It is sufficient to use the thin lens approximation ($l_Q \ll f$). We will set the focal lengths to $f_2 = 2\,f_D$, $f_1 = 2\,f_F$, the drift length to L. Defining

$$\frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 \cdot f_2}$$

the transformation matrix of half a FODO cell is

$$\mathbf{M}_{1/2} = \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} = \begin{pmatrix} 1-L/f_1 & L \\ -1/f^* & 1-L/f_2 \end{pmatrix}$$

Multiplication with $\mathbf{M}_{-1/2}$ (for which f_1 has to be replaced by f_2 and v.v.) gives

$$\mathbf{M}_{\text{FODO}} = \begin{pmatrix} 1-2L/f^* & 2L\cdot(1-L/f_2) \\ -2/f^*\cdot(1-L/f_1) & 1-2L/f^* \end{pmatrix} \quad \text{and} \quad \left|\mathrm{Tr}\{\mathbf{M}\}\right| = \left|2-\frac{4L}{f^*}\right| \le 2$$

This is equivalent to $0 \le \dfrac{L}{f^*} \le 1$, and defining $u = \dfrac{L}{f_1}$, $v = \dfrac{L}{f_2}$ we get

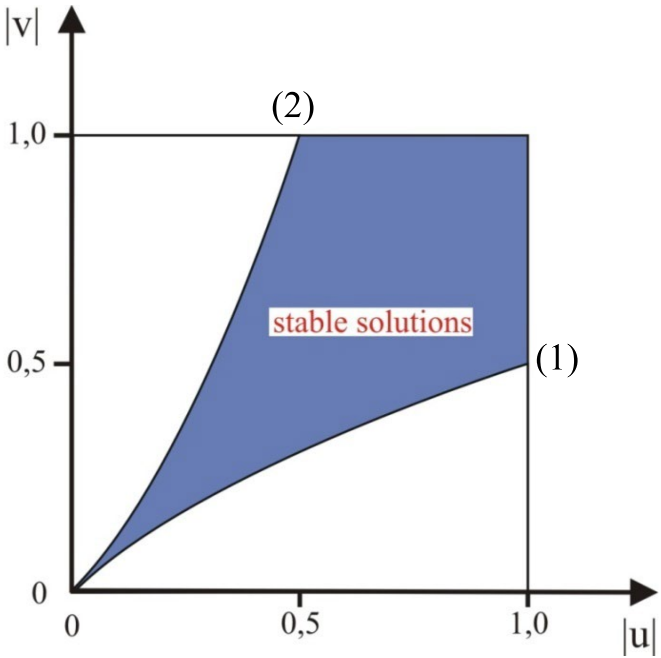
$$\boxed{0 \le u+v-u\cdot v \le 1}$$

from which we derive the boundaries of the stability region

$$|u| \le 1, \qquad |v| \ge \frac{|u|}{1+|u|} \qquad (1)$$

$$|v| \le 1, \qquad |v| \le \frac{|u|}{1-|u|} \qquad (2)$$

which gives the famous necktie diagram for thin lens approximation:



→ ***Hands-on Lattice Calculation***
recommended E22

<u>Remark:</u>

The average beta function is

$$\langle \beta \rangle = \frac{2L}{\sin \mu}$$

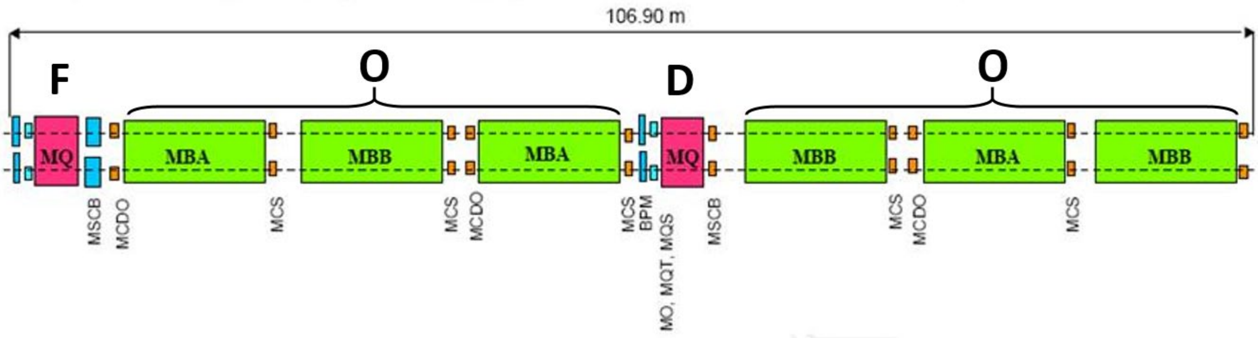
and minimized for

$$\mu = \frac{\pi}{2} = 90°$$

In the simple case of equal focusing strengths, we arrive at

$$|f_1| = |f_2| = 2f, \quad \frac{L}{f^*} = \left(\frac{L}{2f}\right)^2 \quad \rightarrow \quad \boxed{\left|\frac{L}{2f}\right| = \left|\frac{L_{\text{FODO}}}{4f}\right| \leq 1}$$

LHC: Lattice Design
the ARC 90° FoDo in both planes



equipped with additional corrector coils

MB: main dipole
MQ: main quadrupole
MQT: Trim quadrupole
MQS: Skew trim quadrupole
MO: Lattice octupole (Landau damping)
MSCB: Skew sextupole
Orbit corrector dipoles
MCS: Spool piece sextupole
MCDO: Spool piece 8 / 10 pole
BPM: Beam position monitor + diagnostics

Courtesy of Bernhard Holzer, CAS lectures

4.3.2. Periodic beta functions

Periodic solutions of a periodic lattice of period-length L will be

$$\beta(s_0 + L) = \beta(s_0) = \beta_0$$
$$\alpha(s_0 + L) = \alpha(s_0) = \alpha_0$$

Comparing the transfer matrix for one period with its Twiss parameter representation

$$\mathbf{M} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix}$$

we can determine the Twiss parameters at the symmetry points (where $\alpha = 0!$)

$$\boxed{\alpha_0 = 0, \qquad \beta_0 = \frac{r_{12}}{\sqrt{1 - r_{11}^2}}, \qquad \gamma_0 = \frac{1}{\beta_0} = \frac{-r_{21}}{\sqrt{1 - r_{11}^2}}, \qquad \cos\mu = r_{11}}$$

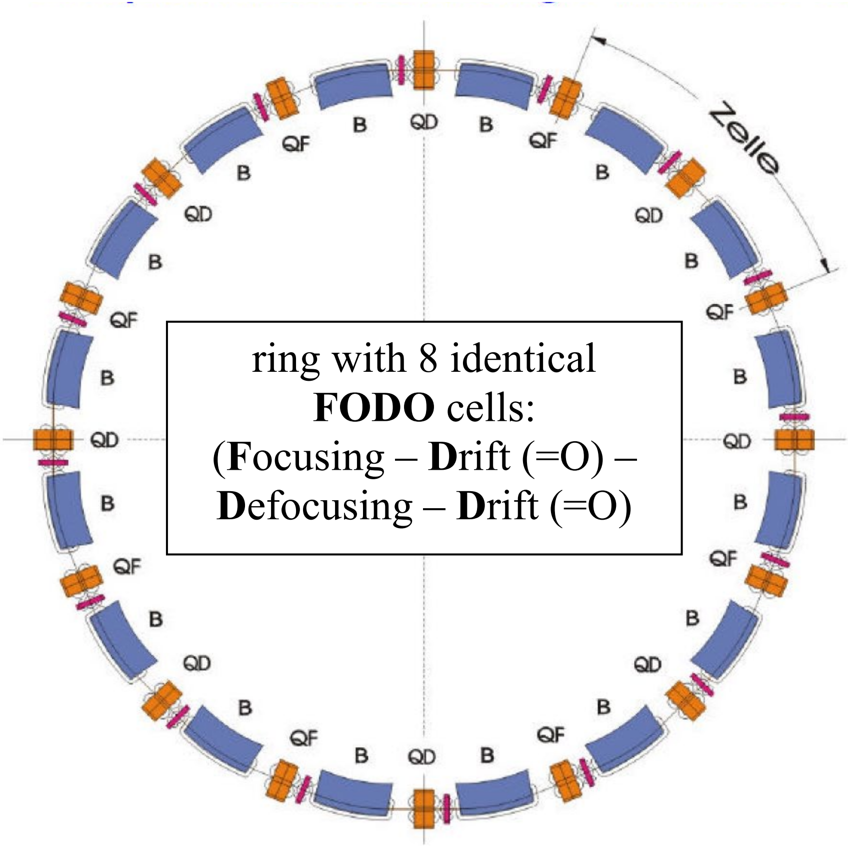
and transform them to any position s using e.g. the beta matrix formalism

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \mathbf{M}(s, s_0) \cdot \begin{pmatrix} \beta_0 & 0 \\ 0 & \gamma_0 \end{pmatrix} \cdot {}^T\mathbf{M}(s, s_0)$$

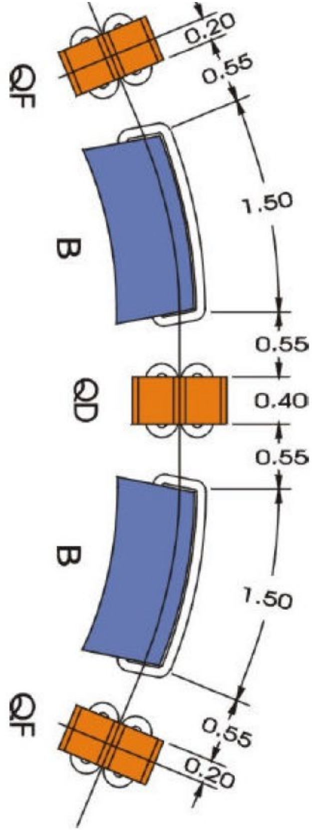
→ *Hands-on Lattice Calculation*
recommended E23-E26

thus revealing the development of β(s), α(s), γ(s).

Example: simple model toy ring (taken from Wille):

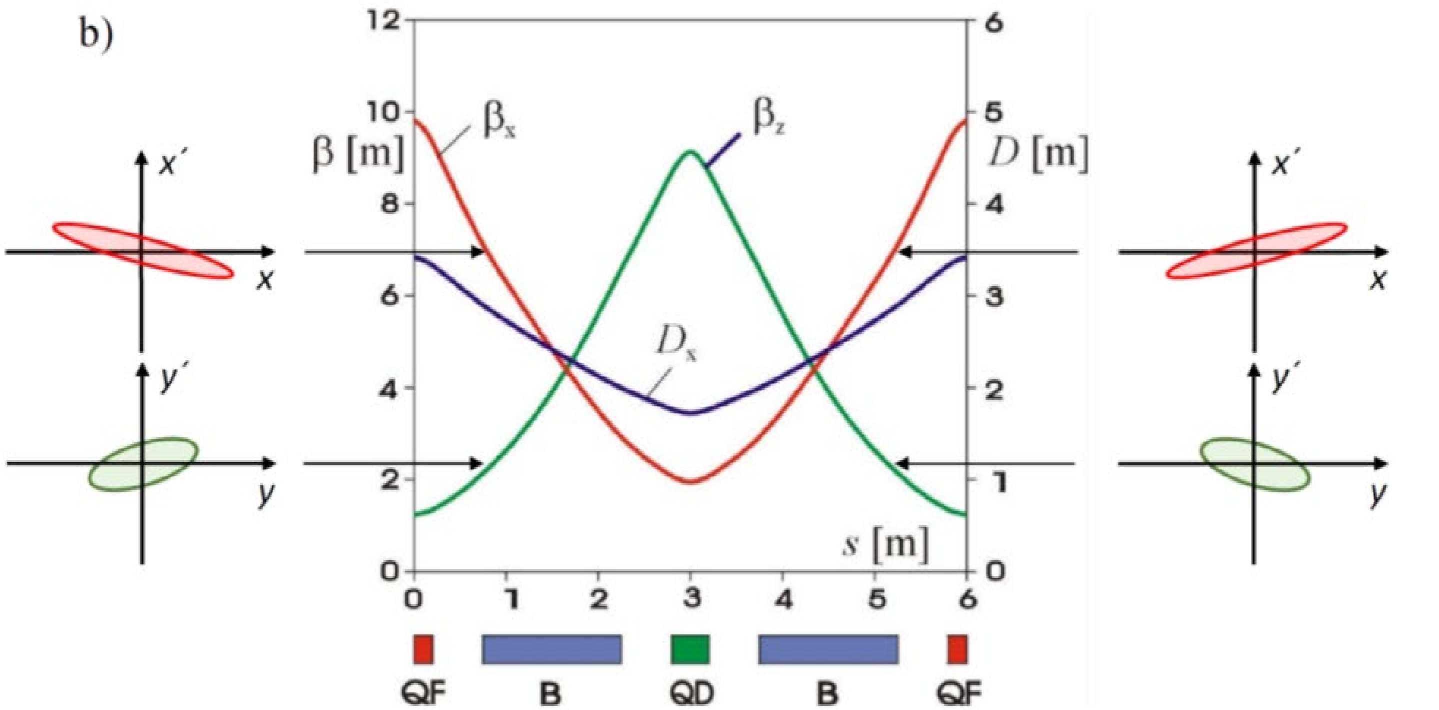


ring with 8 identical
FODO cells:
(**F**ocusing – **D**rift (=O) –
Defocusing – **D**rift (=O)



Choosing $|k_{QF}| = |k_{QD}| = 1.20\text{m}$, we can calculate the transfer matrix M and extract the Twiss parameters, obtaining:

→ Hands-on Lattice Calculation
recommended E32, E34, E39-40



4.4. Transverse beam dynamics

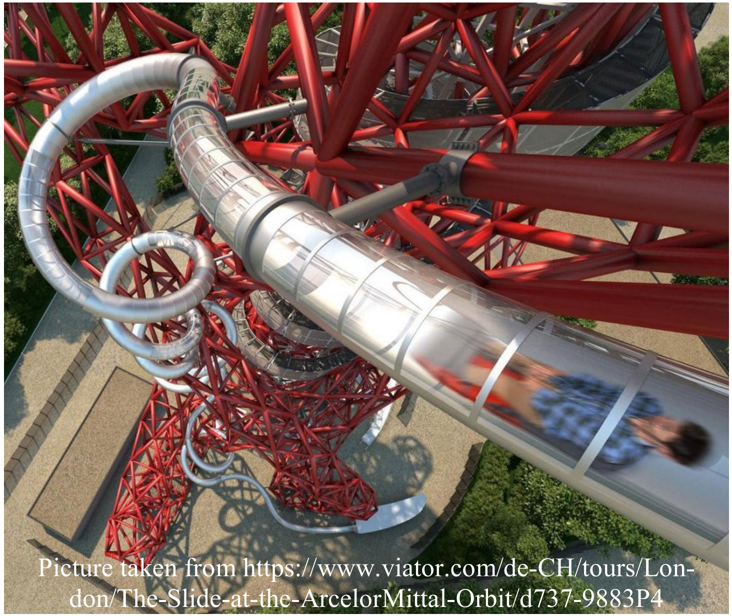
4.4.1. Closed orbit

Remember: In circular accelerators the amplitude function is periodic according to Floquet's theorem and reproduces itself after one turn.

This implies, that the charge center of the beam also moves on a closed trajectory, which is called the closed orbit!

The shape of the closed orbit is determined by the magnets and can – due to errors and misalignments – significantly deviate from the design orbit!

Dedicated steerer magnets (small dipoles), which have to be installed around the ring, are used to correct closed orbit deviations.



Picture taken from https://www.viator.com/de-CH/tours/London/The-Slide-at-the-ArcelorMittal-Orbit/d737-9883P4

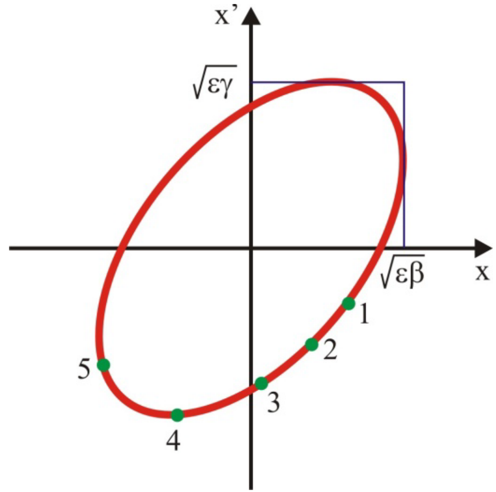
4.4.2. Betatron tune

The betatron tune Q is defined as the number of oscillations per revolution:

$$Q_{x,y} = \frac{\mu(L)}{2\pi} = \frac{1}{2\pi} \cdot \oint \frac{\mathrm{d}s}{\beta_{x,y}(s)}$$

→ ***Hands-on Lattice Calculation***
recommended E27

If one regards the phase space at an arbitrarily chosen point, a single particle moves on its phase space ellipse.
The points represents the parameters after 1,2, … 5 revolutions.



The betatron tune is one of the most important parameter in circular accelerators!

4.4.2. Filamentation

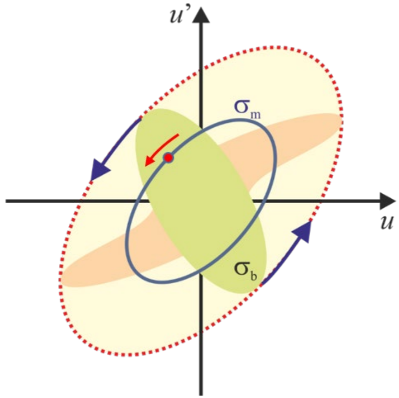
If the envelope ellipse σ_b of the beam is not matched to the ellipse σ_m of the periodic lattice, it will start to rotate with a phase advance per revolution of $2\pi Q$

<u>Beam matrix:</u>

$$\mathbf{\Sigma}_{\text{beam}} = \begin{pmatrix} \sigma_x^2 & \overline{xx'} \\ \overline{xx'} & \sigma_{x'}^2 \end{pmatrix}$$

$$\mathbf{\Sigma}_{\text{beam}} = \varepsilon \cdot \mathbf{B}_{\text{beam}}$$

$$\mathbf{B}_{\text{beam}} = \begin{pmatrix} \beta_{\text{b}} & -\alpha_{\text{b}} \\ -\alpha_{\text{b}} & \gamma_{\text{b}} \end{pmatrix}$$



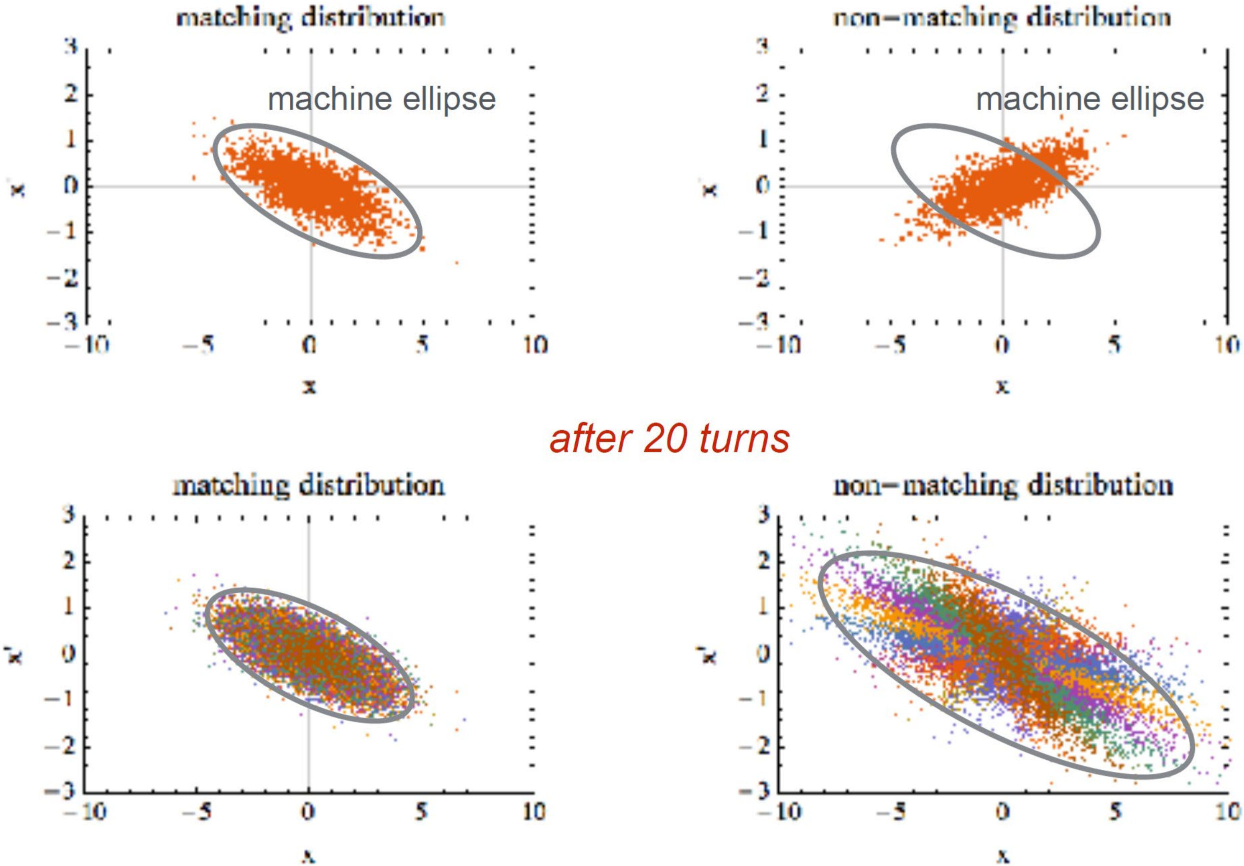
<u>Matching:</u>

$$\beta_{\text{b}} = \beta_{\text{m}}$$
$$\alpha_{\text{b}} = \alpha_{\text{m}}$$
$$\gamma_{\text{b}} = \gamma_{\text{m}}$$

→ ***Hands-on Lattice Calculation***
recommended E28

Due to effects of higher order the quadrupole strengths and therefore the phase advance depends on the amplitude (horizontal and vertical displacements). In case of mismatch, the beam phase space distribution starts to filament. After a large number of revolutions, the distribution may be surrounded by a large ellipse of the form of the lattice ellipse.

Example for an unmatched and matched beam (courtesy of B. Schmidt):



4.4.3. Normalized phase space and normal forms

It is useful to transform the oscillatory solution with varying amplitude and frequency to a solution which looks exactly like that of a harmonic oscillator. So far, we had:

$$x(s) = A\sqrt{\beta_x(s)} \cdot \cos\left(\mu_x(s) + \varphi_0\right)$$

$$x'(s) = -A\frac{\alpha(s)}{\sqrt{\beta_x(s)}}\cos\left(\mu_x(s) + \varphi_0\right) - \frac{A}{\sqrt{\beta_x(s)}}\sin\left(\mu_x(s) + \varphi_0\right)$$

We now introduce new coordinates $x_n(\mu)$ or $x_n(\psi)$ defined by:

$$\boxed{x_n = \frac{x(s)}{\sqrt{\beta_x(s)}}, \qquad \psi = \frac{\mu(s)}{Q}}$$

The angle ψ advances by 2π every revolution. It coincides with θ at each $\beta^{\max}$ and $\beta^{\min}$ location and does not depart very much from θ in between. We can as well use the set $x_n(\mu)$ which only differs by the different phase advance $2\pi Q$ per revolution. Here, we will continue to use the phase advance μ as argument.

In case of $x_n(\mu)$ we get the required transformation using $\mu' = 1/\beta$ and $\beta' = -2\alpha$

$$x_n(\mu) = \frac{1}{\sqrt{\beta}} x(s)$$

$$\dot{x}_n = \frac{\mathrm{d}}{\mathrm{d}\mu} x_n(\mu) = \frac{\mathrm{d}x_n}{\mathrm{d}s} \underbrace{\frac{\mathrm{d}s}{\mathrm{d}\mu}}_{=\beta} = \sqrt{\beta} \cdot x'(s) + \frac{\alpha}{\sqrt{\beta}} \cdot x(s) \quad \rightarrow \quad \underbrace{\begin{pmatrix} x_n \\ \dot{x}_n \end{pmatrix}}_{=\vec{x}_n} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}}_{=\mathbf{T}} \cdot \underbrace{\begin{pmatrix} x \\ x' \end{pmatrix}}_{=\vec{x}}$$

or in short from:

$$\vec{x}_n = \mathbf{T} \cdot \vec{x}, \qquad \vec{x} = \mathbf{T}^{-1} \cdot \vec{x}_n$$

with

$$\mathbf{T} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}, \qquad \mathbf{T}^{-1} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

Please note that the transformation matrix **T** is explicitly depending on the longitudinal position *s*, since the optical functions are explicitly dependent on *s* as well!

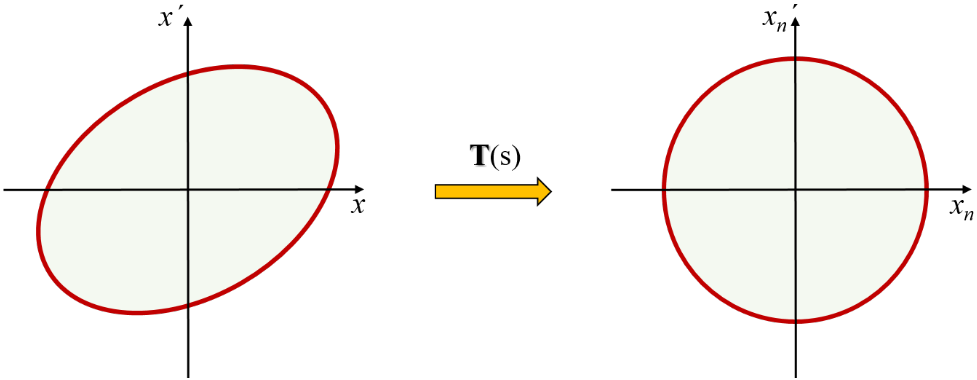
Using these normalized coordinates, the equation of motion is simplified to

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}\mu^2} + x_n = 0, \qquad \frac{\mathrm{d}^2 x_n}{\mathrm{d}\psi^2} + Q^2 x_n = 0$$

The ellipse equation transforms to

$$A = \gamma x^2 + 2\alpha x x' + \beta x'^2 = x_n^2 + \dot{x}_n{}^2$$

and thus the ellipse transforms to a circle (→ normalized phase space)



We have vanishing correlation and get for the variances

$$\langle x_n{}^2 \rangle = \langle \dot{x}_n{}^2 \rangle = \varepsilon, \qquad \langle x_n \cdot \dot{x}_n \rangle = 0$$

Looking at one turn in a circular accelerator, the one-turn matrix **M** is simplified to a simple rotation matrix. Using (x, x'), we obtained

$$\mathbf{M} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix}$$

Using (x_n, x_n'), the one-turn matrix transforms to **R**

$$\vec{x} = \mathbf{T}^{-1} \cdot \vec{x}_n = \mathbf{M} \cdot \vec{x}_0 = \mathbf{M} \cdot \left(\mathbf{T}^{-1} \cdot \vec{x}_{n,0} \right) \quad \rightarrow \quad \vec{x}_n = \underbrace{\mathbf{T} \circ \mathbf{M} \circ \mathbf{T}^{-1}}_{=\mathbf{R}} \cdot \vec{x}_{n,0}$$

and simplifies to a pure rotation matrix:

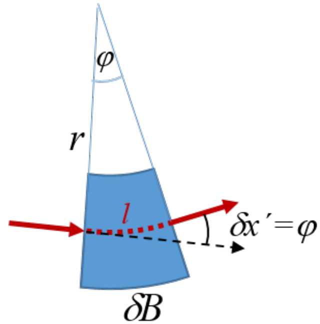
$$\mathbf{R} = \mathbf{T} \circ \mathbf{M} \circ \mathbf{T}^{-1} = \begin{pmatrix} \cos(2\pi Q) & \sin(2\pi Q) \\ -\sin(2\pi Q) & \cos(2\pi Q) \end{pmatrix}$$

In general, we can transform any quadratic (n x n) matrix **M** to its Jordan normal form **R**. From the transformation, we get a bunch of useful information (here α, β, γ, and the tune Q).

4.4.4. Closed orbit distortions

Let us assume a dipole field error produced by a short dipole which makes a constant angular kick in divergence (from $l = r \cdot \varphi \approx \frac{p}{q(\delta B)} \cdot \delta x' = \frac{\rho B}{\delta B} \cdot \delta x'$)

$$\delta x' = \frac{\delta(B l)}{B \rho}$$



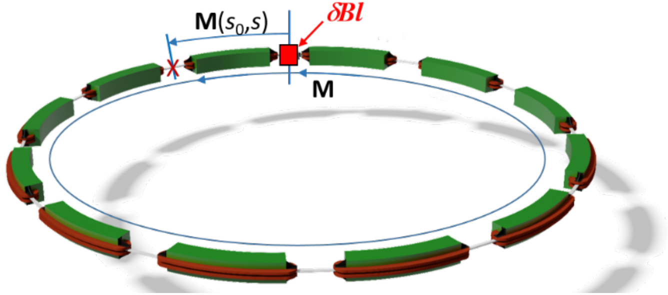
This perturbs the orbit trajectory which elsewhere obeys the unperturbed Hills differential equations

$$x''(s) + \left(\frac{1}{\rho^2(s)} - k(s)\right) \cdot x(s) = 0, \qquad y''(s) + k(s) \cdot y(s) = 0$$

Using matrix algebra, the displacement of the closed orbit at the position of the field error can be calculated from the displacement just before and after the kick element:

$$\begin{pmatrix} x_{c,0} \\ x'_{c,0} - \delta x' \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} x_{c,0} \\ x'_{c,0} \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix} \cdot \begin{pmatrix} x_{c,0} \\ x'_{c,0} \end{pmatrix}$$

which yields with $\mu = 2\pi Q$



$$x_{c,0} = \frac{\beta_0\,\delta x'}{2\sin(\pi Q)}\cos(\pi Q)$$

$$x'_{c,0} = \frac{\delta x'}{2\sin(\pi Q)}\left[\sin(\pi Q) - \alpha_0\cos(\pi Q)\right]$$

The closed orbit displacement $x_c(s)$ is calculated from $\vec{x}_c(s) = \mathbf{M}(s_0,s)\cdot\vec{x}_{c,0}$:

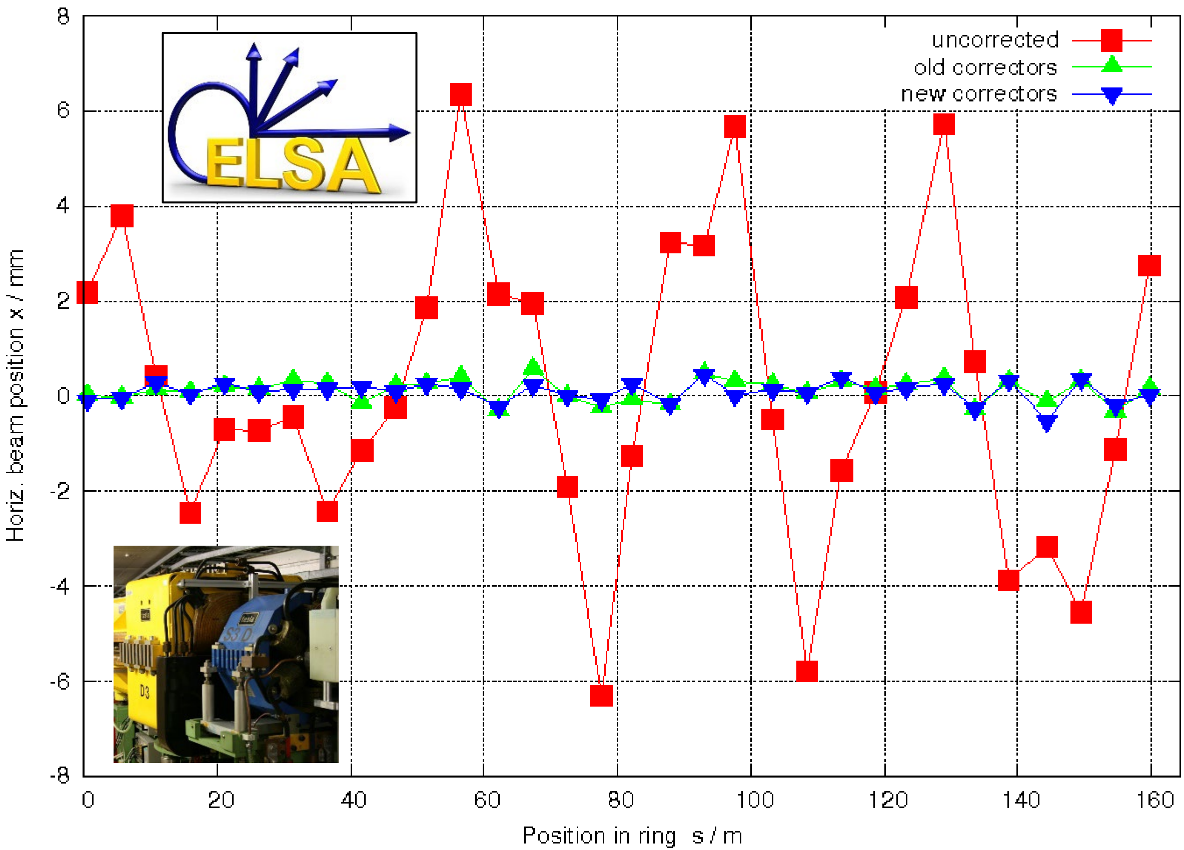
$$\begin{pmatrix} x_c(s) \\ x'_c(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\dfrac{\beta(s)}{\beta_0}}(\cos\mu + \alpha_0 \sin\mu) & \sqrt{\beta(s)\beta_0}\,\sin\mu \\ -\dfrac{1+\alpha(s)\alpha_0}{\sqrt{\beta(s)\beta_0}}\sin\mu + \dfrac{1-\alpha(s)\alpha_0}{\sqrt{\beta(s)\beta_0}}\cos\mu & \sqrt{\dfrac{\beta_0}{\beta(s)}}(\cos\mu - \alpha_0\sin\mu) \end{pmatrix} \cdot \begin{pmatrix} x_{c,0} \\ x'_{c,0} \end{pmatrix}$$

$$\rightarrow \quad \boxed{x_c(s) = \sqrt{\beta}\,\eta_0\cos(Q\psi) = \frac{\sqrt{\beta(s)\beta(s_0)}}{2\sin(\pi Q)}\frac{\delta(Bl)}{B\rho}\cdot\cos\left(\mu(s) - \mu(s_0) + Q\pi\right)}$$

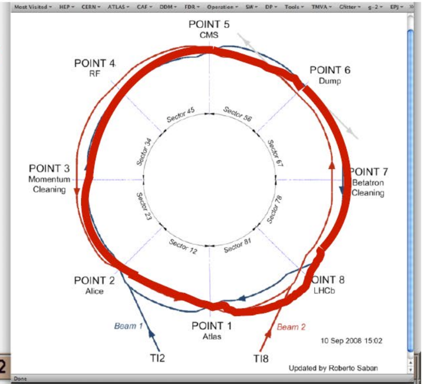
In case of a random distribution we have to integrate over the kick density $\frac{\mathrm{d}\delta(Bl)}{\mathrm{d}s}$

$$x_c(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)}\oint\sqrt{\beta(s_0)}\,\frac{1}{B\rho}\frac{\mathrm{d}\delta(Bl)}{\mathrm{d}s}(s_0)\cdot\cos\left(\mu(s) - \mu(s_0) + Q\pi\right)\cdot\mathrm{d}s_0$$

Closed orbit distortions: uncorrected and corrected



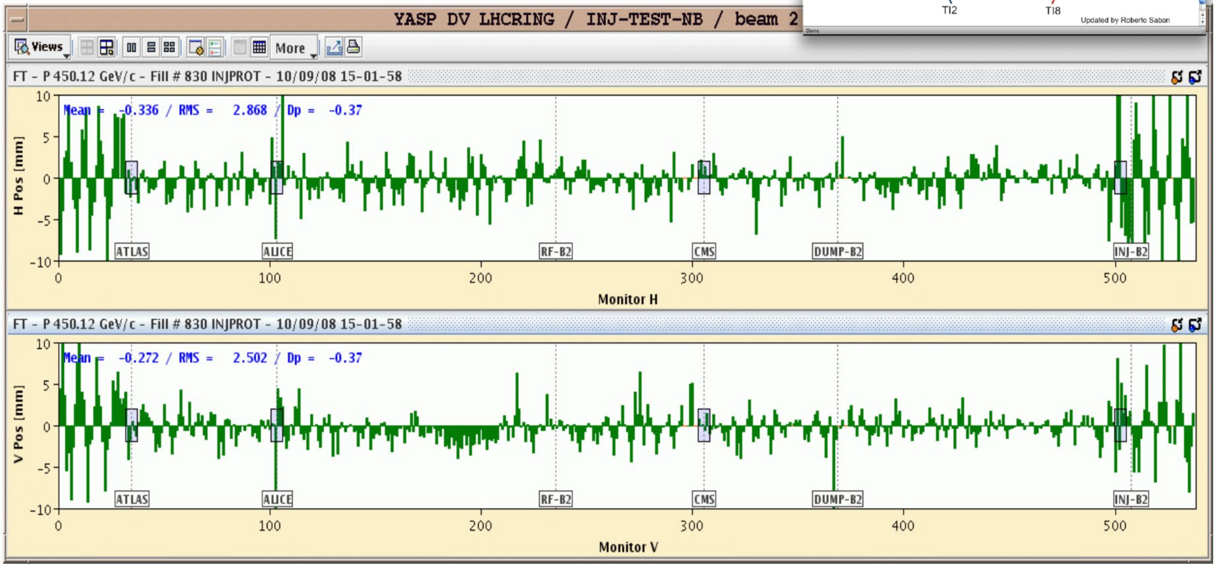
LHC Operation: Beam Commissioning



First turn steering "by sector:"

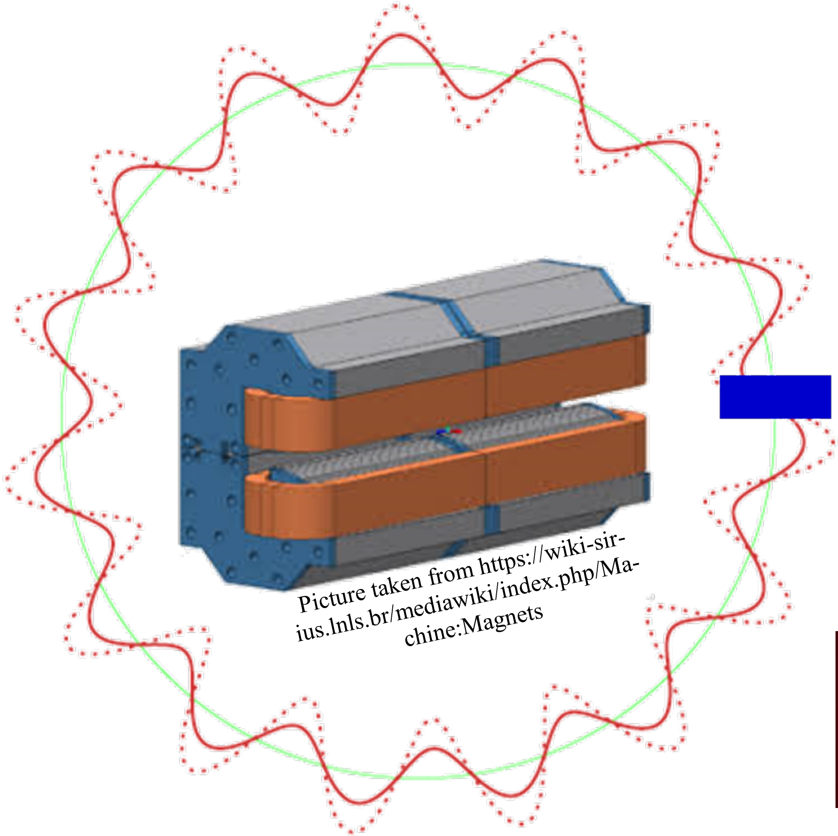
- One beam at the time
- Beam through 1 sector (1/8 ring),

correct trajectory, open collimator and move on.

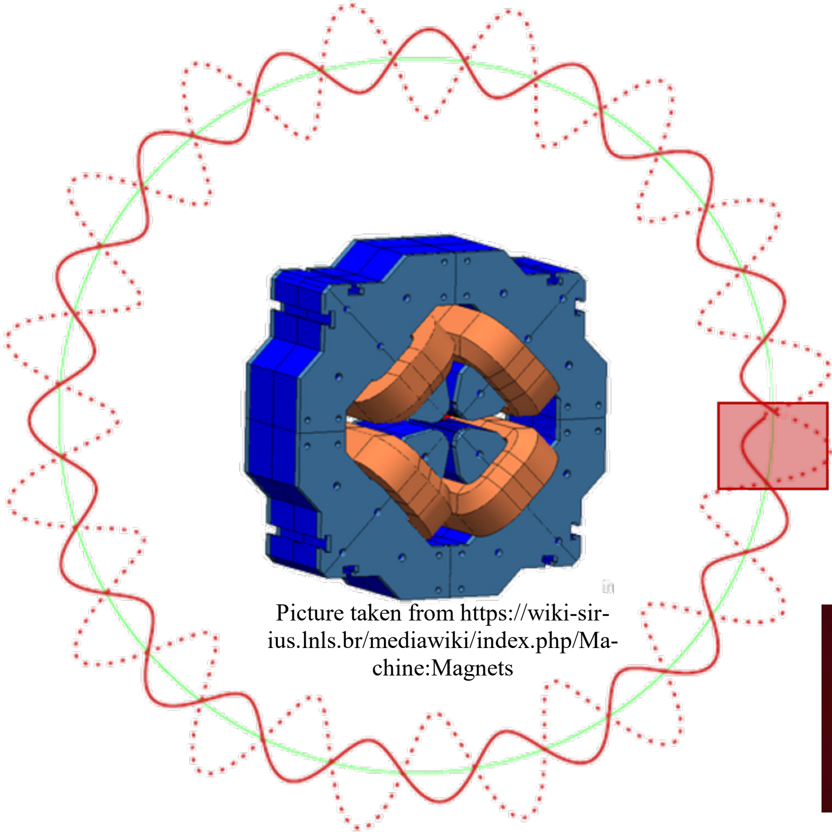


Courtesy of Bernhard Holzer, CAS lectures

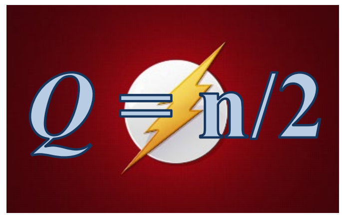
Dipole error and integer tune:



Quadrupole error and half integer tune:



Picture taken from https://wiki-sirius.lnls.br/mediawiki/index.php/Machine:Magnets



4.4.5. Gradient errors

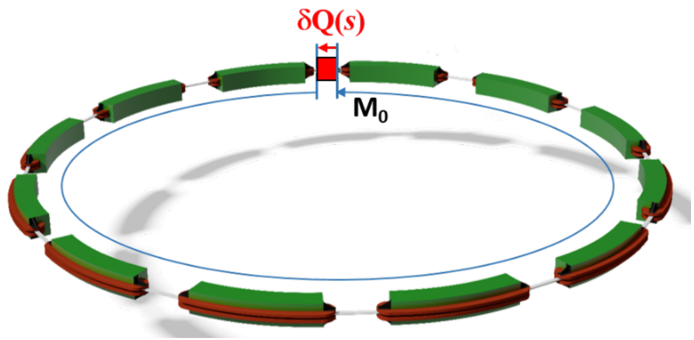
Consider a small gradient error which affects a quadrupole at position s in the lattice of a circular accelerator. Translated to matrix algebra, we have to multiply a perturbation matrix (where we have used the capital K and $K > 0$ means focusing in the plane considered)

$$\boldsymbol{\delta}\mathbf{Q}(s) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\delta K(s)\cdot \mathrm{d}s & 1 \end{pmatrix}$$

Take care:
$\delta K_x = -\delta k$
$\delta K_y = +\delta k$

with the **unperturbed matrix** for one circle

staring at s (where $\alpha(s)=\alpha_0,\ \beta(s)=\beta_0,\ \gamma(s)=\gamma_0$)



$$\mathbf{M}_0 = \begin{pmatrix} \cos\mu_0 + \alpha_0 \sin\mu_0 & \beta_0 \sin\mu_0 \\ -\gamma_0 \sin\mu_0 & \cos\mu_0 - \alpha_0\sin\mu_0 \end{pmatrix}$$

giving:

$$\tilde{\mathbf{M}}_0(s) = \boldsymbol{\delta}\mathbf{Q}(s)\cdot\mathbf{M}_0$$

$$= \begin{pmatrix} \cos\mu_0 + \alpha_0\sin\mu_0 & \beta_0\sin\mu_0 \\ -\delta K\,\mathrm{d}s\left(\cos\mu_0 + \alpha_0\sin\mu_0\right) - \gamma_0\sin\mu_0 & -\delta K\,\mathrm{d}s\,\beta_0\sin\mu_0 + \cos\mu_0 - \alpha_0\sin\mu_0 \end{pmatrix}$$

From $\frac{1}{2}\mathrm{Tr}\left\{\tilde{\mathbf{M}}_0\right\} = \cos\mu = \cos(\mu_0 + \Delta\mu) \approx \cos\mu_0 - \Delta\mu \cdot \sin\mu_0$ we get:

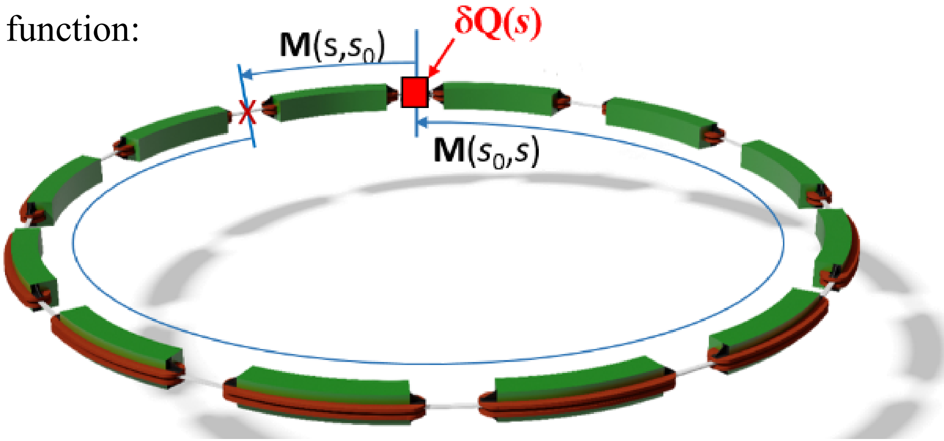
$$\frac{1}{2}\mathrm{Tr}\left\{\tilde{\mathbf{M}}_0\right\} = \cos\mu_0 - \frac{1}{2}\sin\mu_0\,\beta_0\,\delta K\,\mathrm{d}s$$

$$\rightarrow \quad 2\pi\Delta Q = \Delta\mu = \frac{1}{2}\beta(s)\,\delta K(s)\,\mathrm{d}s$$

Integrating over the length of the quadrupole perturbation, one obtains

$$\boxed{\Delta Q = \frac{1}{4\pi}\oint \beta(s)\,\delta K(s)\,\mathrm{d}s}$$

Effect on beta function:



A gradient error will not influence the closed orbit but the betatron function of the lattice. In order to calculate the betatron amplitude modulation, we have to determine the single turn transport matrix starting at a given observer position s, introducing a small gradient perturbation at position s_0:

$$\tilde{\mathbf{M}}_s = \mathbf{M}(s,s_0)\cdot\boldsymbol{\delta}\mathbf{Q}(s_0)\cdot\mathbf{M}(s_0,s) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ -\delta K\,\mathrm{d}s_0 & 1 \end{pmatrix}\cdot\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

It is only necessary to evaluate the element $\tilde{r}_{12}$ which is

$$\tilde{r}_{12} = b_{11}a_{12} + b_{12}\left(-\delta K\,\mathrm{d}s\cdot a_{12} + a_{22}\right) = r_{12} - \delta K\,\mathrm{d}s_0\cdot a_{12}b_{12}$$

where r_{12} from the unperturbed matrix found by putting $\delta K\,\mathrm{d}s_0 = 0$. Thus the variation in the r_{12} term due to the perturbation is

$$\begin{aligned}\Delta\left[\beta(s)\sin\left(2\pi Q_0\right)\right] &= -\delta K\,\mathrm{d}s_0\,\beta(s)\,\beta(s_0)\cdot\sin\left(\mu(s)-\mu(s_0)\right)\cdot\sin\left(\mu(s_0)-\mu(s)\right) \\ &= -\delta K\,\mathrm{d}s_0\,\beta(s)\,\beta(s_0)\cdot\sin\left(\mu(s)-\mu(s_0)\right)\cdot\sin\left[2\pi Q_0 - \left(\mu(s)-\mu(s_0)\right)\right]\end{aligned}$$

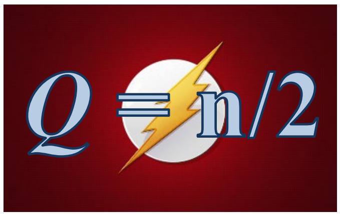
Using $\sin\alpha\cdot\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$ the left-hand and right-hand sides can be expanded to give

$$\Delta\beta(s)\sin(2\pi Q_0) + \beta(s)\cdot\underbrace{2\pi\Delta Q}_{\uparrow}\cdot\cos(2\pi Q_0) =$$

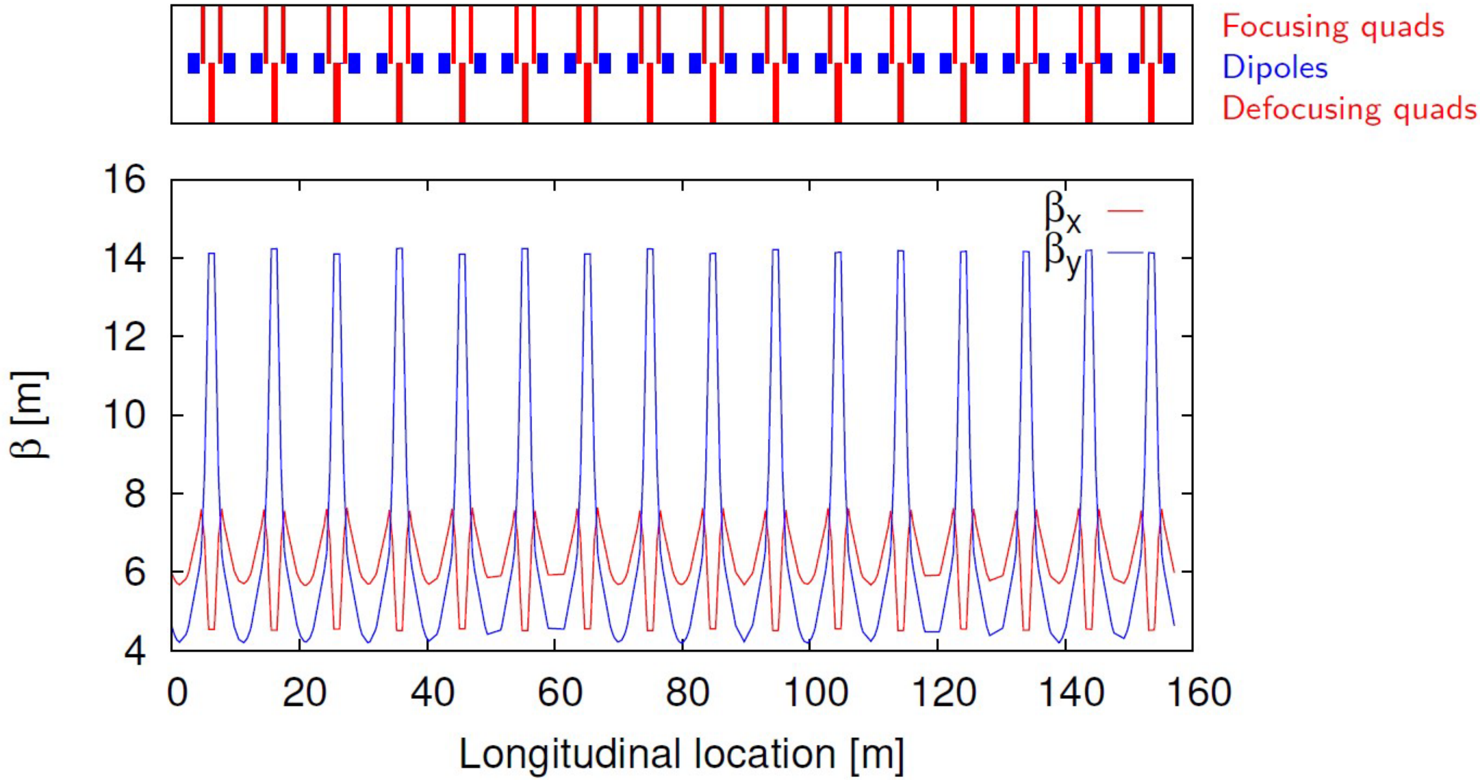
$$\overbrace{\frac{1}{2}\delta K\,ds_0\,\beta(s_0)}^{\downarrow}\beta(s)\left\{\cos(2\pi Q_0) - \cos\left[2\left(\mu(s)-\mu(s_0)-\pi Q_0\right)\right]\right\}$$

This leaves the final expression for the betatron amplitude modulation (the so called **beta-beating**):

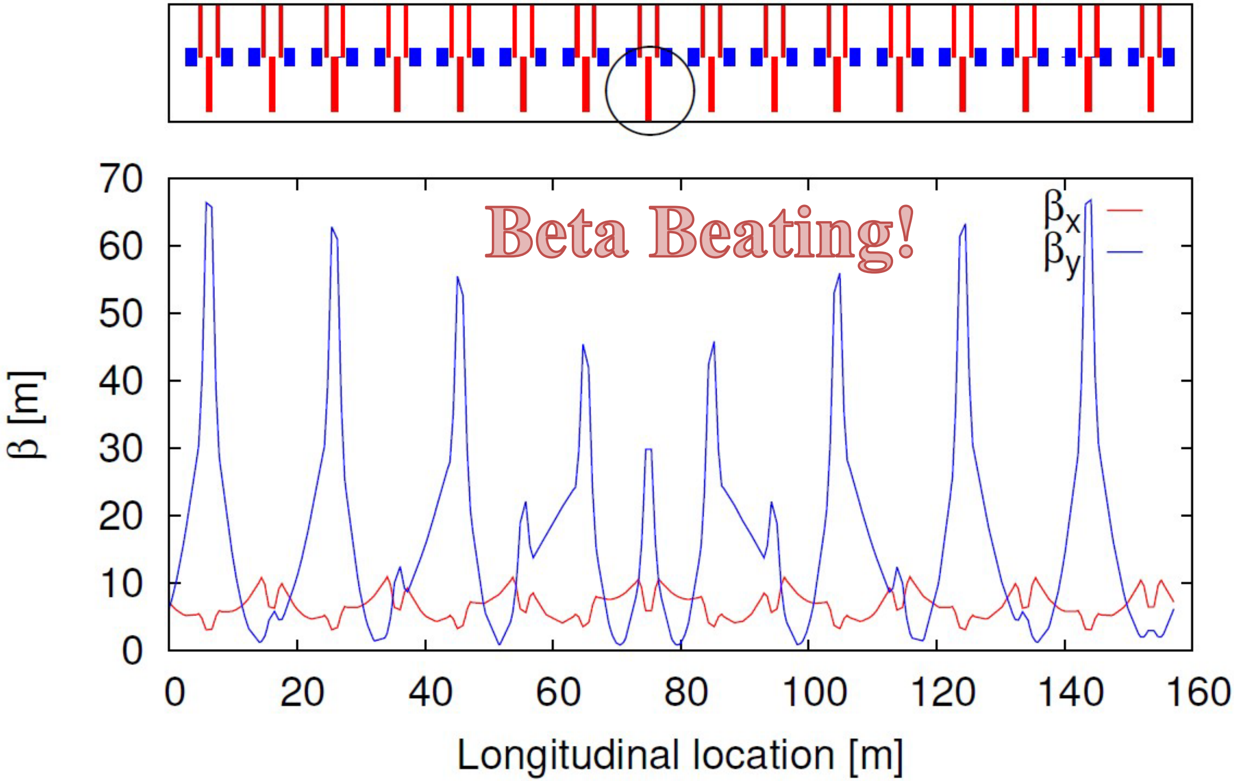
$$\boxed{\Delta\beta(s) = \frac{\beta(s)}{2\sin(2\pi Q_0)}\cdot\oint_s \delta K(s_0)\,\beta(s_0)\cos\left[2\left(\mu(s)-\mu(s_0)-\pi Q_0\right)\right]\cdot \mathrm{d}s_0}$$



Ideal World:


Focusing quads
Dipoles
Defocusing quads
β_x
β_y
β [m]
16
14
12
10
8
6
4
0
20
40
60
80
100
120
140
160
Longitudinal location [m]

Single Quadrupole Error:



4.4.6. Optical resonances

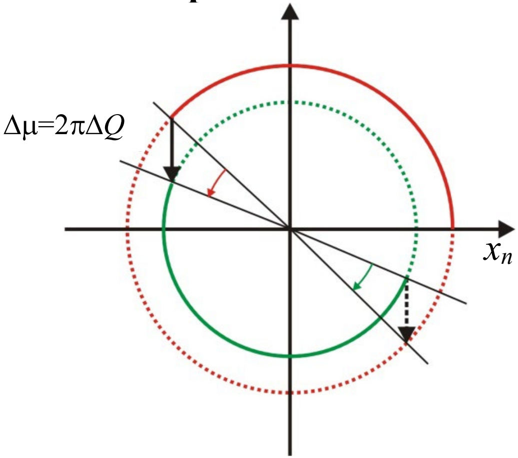
Impact of field errors in circular accelerators:

- Dipole errors will cause close orbit displacements which will grow indefinitely when the tune approaches an integer value.
- Gradient errors will produce an average tune shift ΔQ. They will not affect the closed orbit but change the beta function. The beam size will grow indefinitely when the tune approaches half integer values.

These phenomena are called ***optical resonances***. Due to the turn by turn modulation of the tune, there exist regions of instability called **stop bands** around the resonance conditions. The width of these stop bands are given by the tune modulation amplitude.

These effects can be studied best when regarding the normalized phase space, where the particles' ellipses transform to circles. Since we are interested in the impact of the error, we select the trace space at the position of the faulty element:

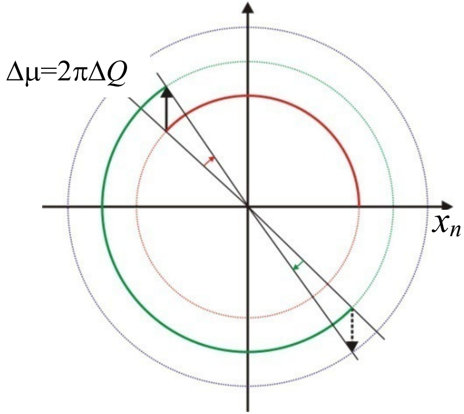
Dipole Errors:



No average tune shift

Tune modulation amplitude $\mathrm{d}Q$

Gradient Errors:



Average tune shift $\Delta Q = \dfrac{1}{4\pi}\beta\,\delta(Kl)$

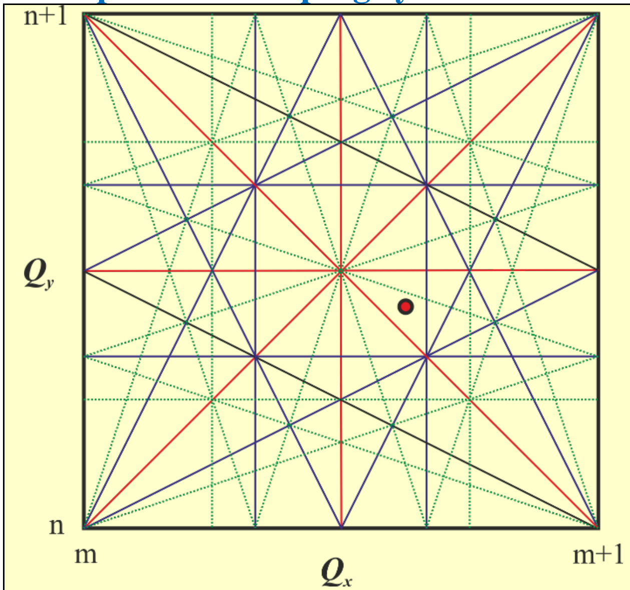
Tune modulation amplitude $\mathrm{d}Q = \Delta Q$

Any particle whose unperturbed *Q* lies in the stop band width d*Q* will lock into resonance and is lost.

We may generalize and give a list of resonances and their driving multipoles:

resonance type	**driving multipole**
integer resonance: $Q = n$	dipole errors
half-integer resonance $2 \cdot Q = n$	quadrupole errors
third-integer resonance $3 \cdot Q = n$	sextupole errors

Example: fast ramping synchrotron ↔ only “lower” orders are important (≤ 5)

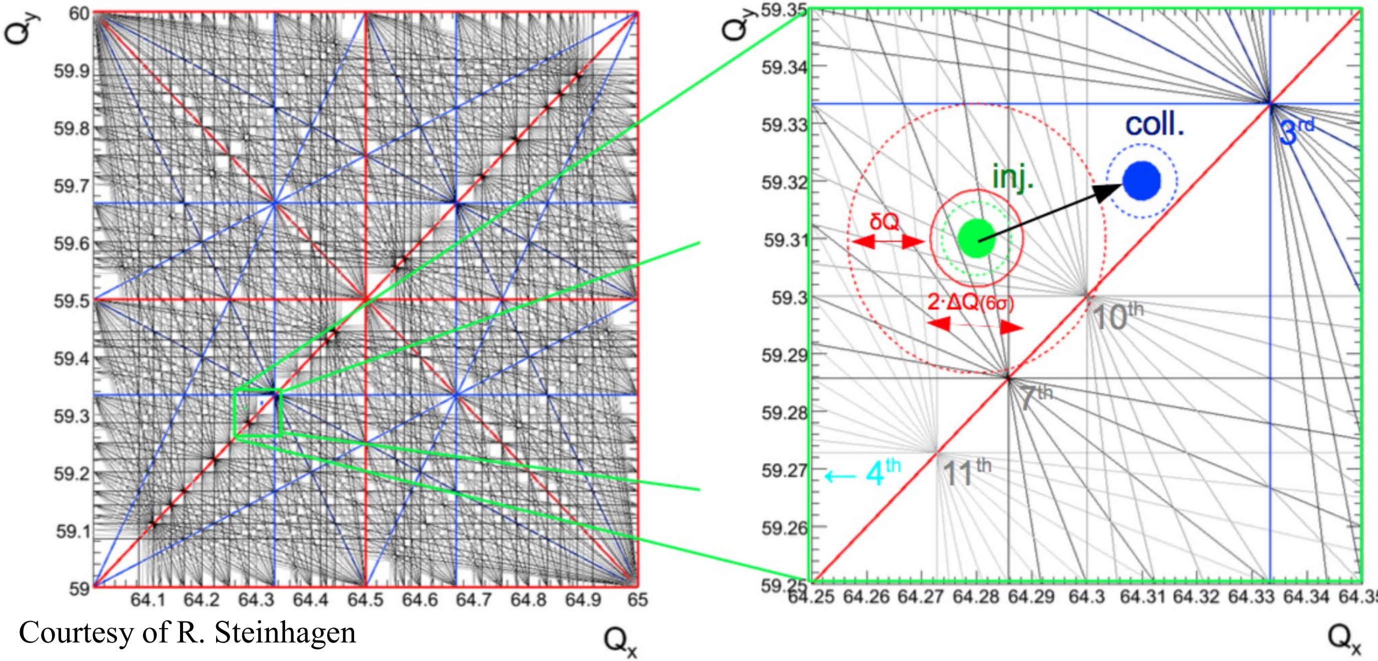


Due to betatron coupling, perturbations may depend on the betatron amplitude in both planes. These coupling terms lead to the generalized resonance condition

$$\boxed{m \cdot Q_x + n \cdot Q_y = N}$$

where $m+n$ indicates the **order** of the resonance. The working point chosen is indicated by a red filled circle.

Example: LHC ↔ long store needs to consider "higher" orders (> 10) as well



Courtesy of R. Steinhagen

Tune stability requirements: $\Delta Q < 0.001$ vs exp. Drifts ~ 0.06

Note: need to stay much further of resonances due to finite tune width (chromaticity, momentum spread), space charge, beam-beam, etc., and finite width of stop bands.

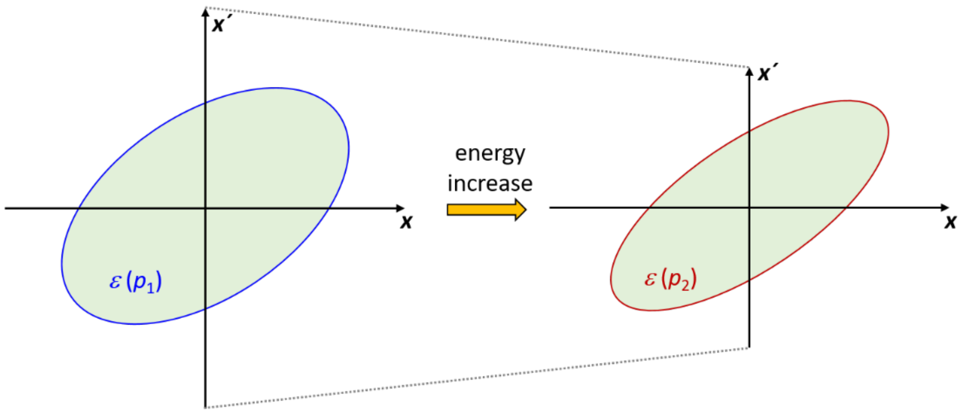
4.5. Beam dynamics with acceleration

Phase space in accelerator physics $\neq$ phase space in classical mechanics:

coordinates x, x' $\leftrightarrow$ canonical coordinates x, p_x

$$p_x = m \cdot \dot{x} = m \cdot \dot{s} \cdot x' \approx p_0 \cdot x' = \beta_r \gamma_r \cdot m_0 c \cdot x' \quad \rightarrow \quad \boxed{\beta_r \gamma_r \cdot x' = const.}$$

Beam acceleration (momentum increase) causes compression of x' axis and therewith decrease of the beam emittance, which is called **adiabatic damping**:



→ Define normalized emittance, which is conserved: $\varepsilon_n = \beta_r \gamma_r \cdot \varepsilon$

5. Dynamics with Off Momentum Particles

We will come back to the equation of motion, now explicitly treating the momentum dependent right hand side, depending on the relative momentum deviation $\delta = \Delta p / p_0$

$$x''(s) + \left(\frac{1}{\rho^2(s)} - k(s) \right) \cdot x(s) = \frac{1}{\rho(s)} \delta$$

$$y''(s) + k(s) \cdot y(s) = 0$$

Since the dynamics of off momentum particles is only affected in the horizontal plane, we will restrict the treatment to 1D including the momentum dependence.

5.1 Dispersion and dispersion functions

A particular solution for a non-vanishing $\delta = \Delta p / p$ is $x_{ih}(s) = \rho \cdot \delta$. Recalling the solution of the homogenous equation, this gives:

$$\boxed{x(s) = x_h(s) + x_{ih}(s) = a \cdot \cos\left(s/\rho \right) + b \cdot \sin\left(s/\rho \right) + \rho \cdot \delta}$$

The integration constants a, b are again derived from the boundary conditions at $s = 0$, but now the inhomogeneous solution has to be included:

$$x(s=0) = a + \rho \cdot \delta = x_0, \quad x'(s=0) = \frac{b}{\rho} = x_0{}',$$

and by defining the bending angle $\varphi = L/\rho$ of the dipole magnet, we obtain :

$$\boxed{\begin{aligned} x(L) &= x_0 \cdot \cos\varphi + \rho \cdot x_0{}' \cdot \sin\varphi + \rho(1-\cos\varphi) \cdot \delta \\ x'(L) &= -x_0/\rho \cdot \sin\varphi + x_0{}' \cdot \cos\varphi + \sin\varphi \cdot \delta \end{aligned}}$$

This can be easily implemented in the matrix formalism by adding a 3rd component to the particle's position vector dealing with the actual relative momentum deviation compared to the reference particle:

$$\vec{X} = \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} \qquad \mathrm{M}_{\text{dipole}} = \begin{pmatrix} \cos\varphi & \rho\sin\varphi & \rho(1-\cos\varphi) \\ -1/\rho\sin\varphi & \cos\varphi & \sin\varphi \\ 0 & 0 & 1 \end{pmatrix}$$

First neglecting the dependence of the quadrupole strength k on the actual particle's momentum, the quadrupole transfer matrices remain "unchanged":

$$M_{QF} = \begin{pmatrix} \boxed{\begin{matrix} \cos\Omega & \sqrt{|k|}\sin\Omega \\ -1/\sqrt{|k|}\sin\Omega & \cos\Omega \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & 1 \end{pmatrix} \qquad M_{QD} = \begin{pmatrix} \boxed{\begin{matrix} \cosh\Omega & \sqrt{|k|}\sinh\Omega \\ 1/\sqrt{|k|}\sinh\Omega & \cosh\Omega \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & 1 \end{pmatrix}$$

→ ***Hands-on Lattice Calculation***
recommended E33

Important:

Whereas a quadrupole magnet will not directly cause an impact on the particle's trajectory, **a dipole magnet creates a (horizontal) dispersion**:

$$D_{\text{dip}} = r_{13} = \rho(1-\cos\varphi), \qquad D'_{\text{dip}} = r_{23} = \sin\varphi$$

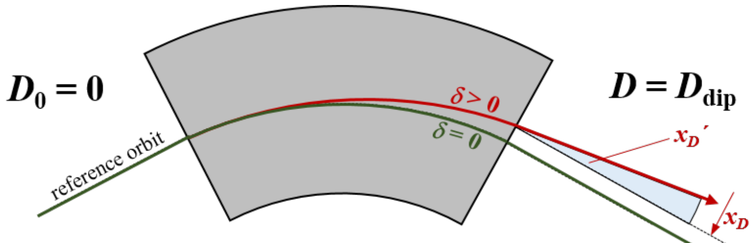
The dispersion represents the offset due to a relative momentum deviation $\Delta p/p = 1$.

In general, we have: $x(s) = x_h(s) + x_D(s) = x(s) + D(s)\cdot\delta$

Here, ***D*(s) is the dispersion function**, a solution of the equation of motion for $\delta = 1$.

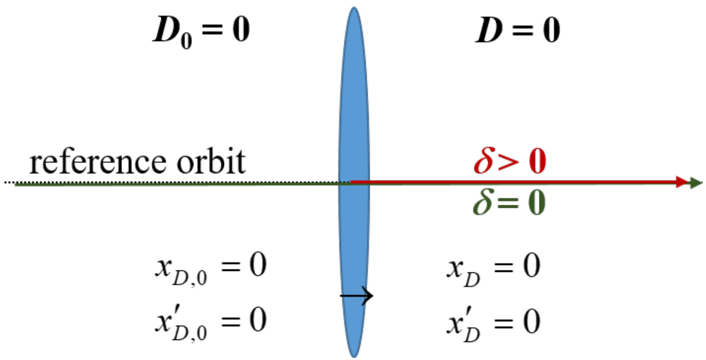
The dispersion D indicates how much reference trajectory is displaced for $\delta = 1$.

A dipole magnet will create dispersion → dispersion orbit x_D:



$$x_{D,0} = 0 \qquad x_D = D_{\text{dip}} \cdot \delta$$
$$x'_{D,0} = 0 \quad \rightarrow \quad x'_D = D'_{\text{dip}} \cdot \delta$$

A quadrupole magnet will not create any dispersion:



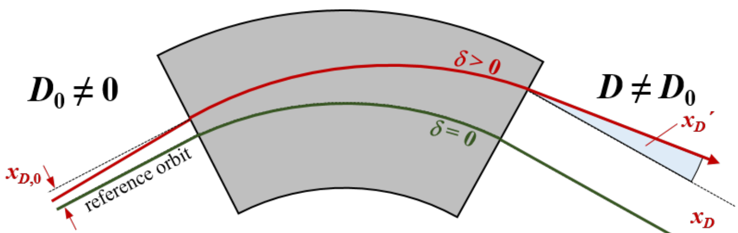
$$x_{D,0} = 0 \qquad x_D = 0$$
$$x'_{D,0} = 0 \quad \rightarrow \quad x'_D = 0$$

But now take care:

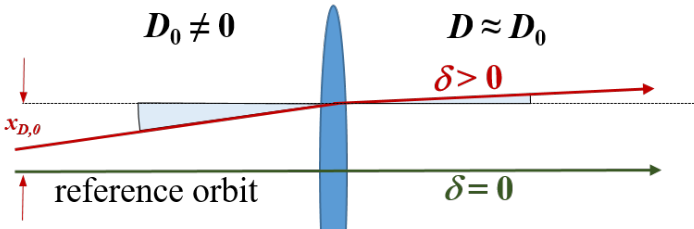
> Due to $x(s) = x_h(s) + x_D(s)$, we will observe a change of the **dispersion orbit** $x_D(s)$
> when passing a dipole magnet or a quadrupole magnet!!

Both dipole and quadrupole magnets will modify an existing dispersion according to

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \boxed{\begin{matrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{matrix}} & \begin{matrix} D_{\text{dip}} \\ D'_{\text{dip}} \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & 1 \end{pmatrix}}_{\mathbf{M}_{\text{dipole}}} \cdot \begin{pmatrix} D_0 \\ D_0{}' \\ 1 \end{pmatrix} \qquad \begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \boxed{\begin{matrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & 1 \end{pmatrix}}_{\mathbf{M}_{\text{quadrupole}}} \cdot \begin{pmatrix} D_0 \\ D_0{}' \\ 1 \end{pmatrix}$$



$$x_D = r_{11}\, x_{D,0} + r_{12}\, x'_{D,0} + D_{\text{dip}}\delta$$
$$x'_D = r_{21}\, x_{D,0} + r_{22}\, x'_{D,0} + D'_{\text{dip}}\delta$$



$$x_D = r_{11}\, x_{D,0} + r_{12}\, x'_{D,0}$$
$$x'_D = r_{21}\, x_{D,0} + r_{22}\, x'_{D,0}$$

5.2 Dispersion in circular accelerators

In a periodic lattice, the dispersion function has – as well as the beta function – to fulfill periodic boundary conditions:

$$D(s_0 + L) = D(s_0)$$

Thus the dispersion function can obtained from applying the 3x3 transport matrix $\mathbf{M}_3$ for a full period

$$\begin{pmatrix} D_0 \\ D_0{}' \\ 1 \end{pmatrix} = \mathbf{M}_3 \cdot \begin{pmatrix} D_0 \\ D_0{}' \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} D_0 \\ D_0{}' \\ 1 \end{pmatrix}$$

yielding:

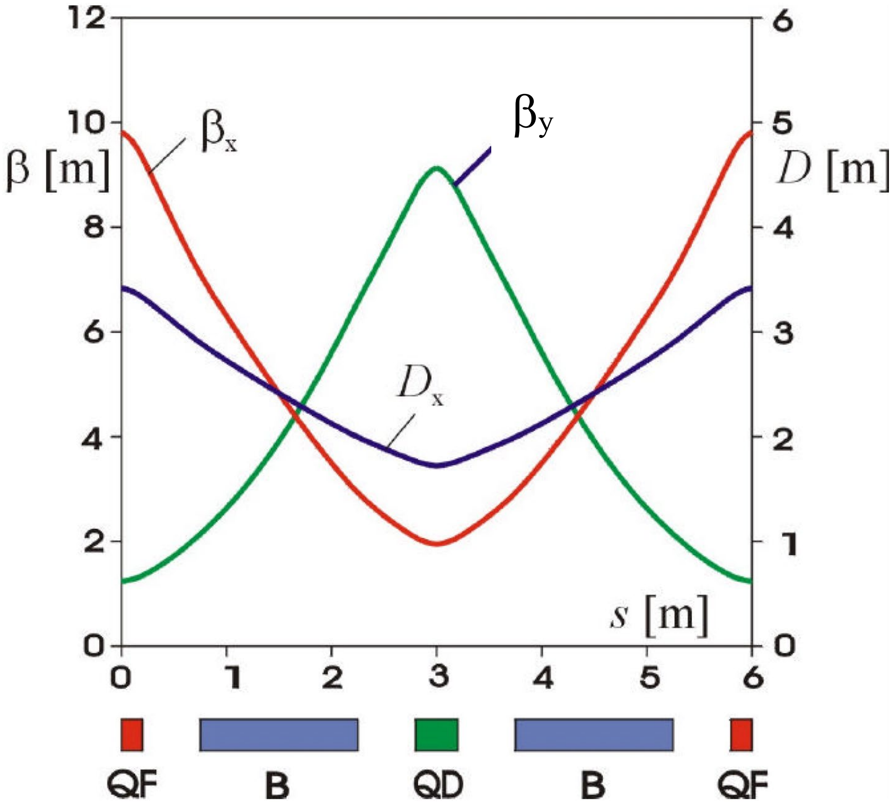
$$D_0 = \frac{r_{13}\left(1 - r_{22}\right) + r_{12} r_{23}}{2 - r_{11} - r_{22}} = \frac{r_{13}\left(1 - r_{22}\right) + r_{12} r_{23}}{2\left(1 - \cos\mu\right)}$$

$$D_0' = \frac{r_{13}\left(1 - r_{11}\right) + r_{21} r_{13}}{2 - r_{11} - r_{22}} = \frac{r_{13}\left(1 - r_{11}\right) + r_{21} r_{13}}{2\left(1 - \cos\mu\right)}$$

which for a symmetry point, where $D_0{}' = 0$, simplifies to

$$D_0{}^{\text{sym}} = \frac{r_{13}}{1 - r_{11}}$$

Applying this to our model toy synchrotron, we can derive the dispersion function which is plotted in blue:



Please note that the total beam width is given by

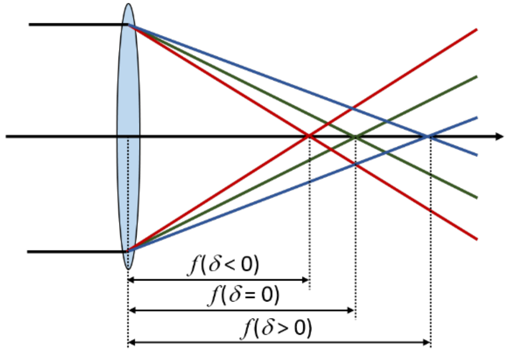
$$\sigma_x = \sqrt{\varepsilon_x \beta_x + \left(D_x \delta\right)^2} \;!$$

→ ***Hands-on Lattice Calculation***
recommended E34-38

5.3. Chromaticity

The variation of tunes is called **chromaticity** and is defined by the factor ξ in

$$\boxed{\Delta Q_{x,y} = \xi_{x,y} \cdot \frac{\Delta p}{p_0}}$$



We distinguish between natural chromaticity created by the chromatic aberration of quadrupole magnets and perturbations derived from non-linear perturbations in the particles trajectories (e.g. produced by sextupole magnets).

<u>Natural Chromaticity:</u>

The quadrupole strength scales with the particles' momenta:

$$k = k_0 + \Delta k = \frac{q}{p_0 + \Delta p} g \approx \frac{q}{p_0} g - \frac{q}{p_0^2} \Delta p \cdot g$$

Where k_0 represents the quadrupole strength experienced by a particle with p_0.

This results in a quadrupole error Δk for off momentum particles:

$$\Delta k = -k_0 \cdot \Delta p / p_0 = \begin{cases} -\delta K, & \text{horizontal plane} \\ +\delta K, & \text{vertical plane} \end{cases}$$

Inserting this in the perturbation formula for the tune, we find:

$$\boxed{\begin{aligned} \Delta Q_x &= +\frac{1}{4\pi} \oint \beta_x(\tilde{s}) \cdot k_0(\tilde{s}) \cdot \mathrm{d}\tilde{s} \cdot \delta \quad \rightarrow \quad \xi_x = +\frac{1}{4\pi} \oint \beta_x(\tilde{s}) \cdot k_0(\tilde{s}) \cdot \mathrm{d}\tilde{s} \\ \Delta Q_y &= -\frac{1}{4\pi} \oint \beta_y(\tilde{s}) \cdot k_0(\tilde{s}) \cdot \mathrm{d}\tilde{s} \cdot \delta \quad \rightarrow \quad \xi_y = -\frac{1}{4\pi} \oint \beta_y(\tilde{s}) \cdot k_0(\tilde{s}) \cdot \mathrm{d}\tilde{s} \end{aligned}}$$

Since the horizontal beta function is larger in focusing quadrupole (where $k_0 < 0$) than in a defocusing quadrupole, the natural chromaticity is always negative. The same argument holds for the vertical chromaticity as well being always negative.

$$\downarrow$$

The horizontal and vertical natural chromaticities are always negative.

Stronger focusing leads to larger natural chromaticities.

Big circ. accelerators will have a larger natural chromaticity than smaller rings.

Chromaticity produced by sextupoles:

A beam of particles moving on a dispersion orbit through a sextupole magnet is "focused" by the nonlinear field due to horizontal displacement $x = D \cdot \frac{\Delta p}{p_0}$. We can derived a position dependent focusing strength from

$$\frac{q}{p_0}\vec{B}_{\text{sext}} = m_0\, x\, y\, \hat{e}_x + \frac{1}{2} m_0 \left(x^2 - y^2\right) \hat{e}_y$$

giving a dispersion dependent δK_x and δK_z to:

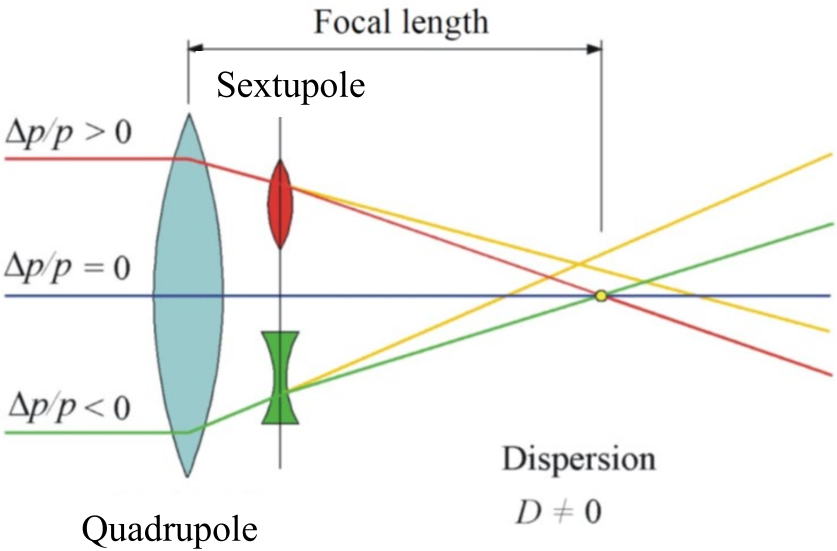
$$\delta K_x = -\delta k = \frac{q}{p_0} \cdot \frac{\partial B_{\text{sext},y}}{\partial x} = m_0 \cdot x = m_0 \cdot D \cdot \delta$$

$$\delta K_y = +\delta k = -\frac{q}{p_0} \cdot \frac{\partial B_{\text{sext},x}}{\partial y} = -m \cdot x = -m_0 \cdot D \cdot \delta$$

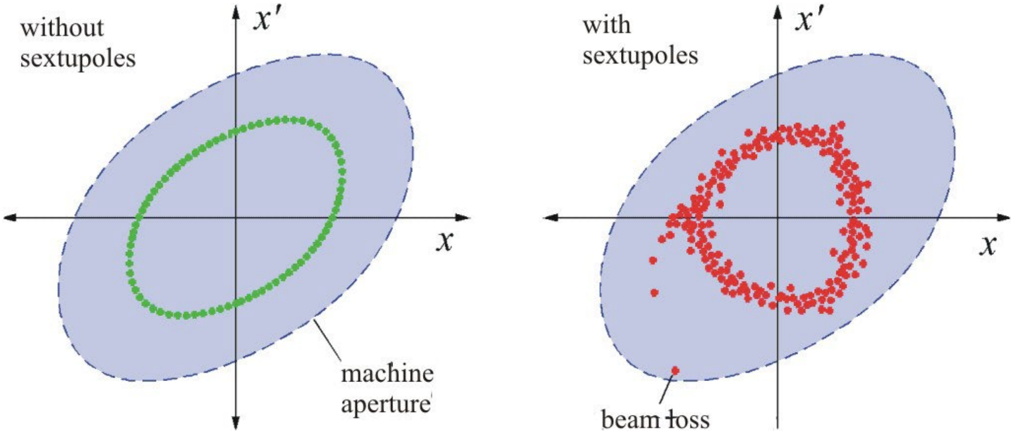
This adds to the natural chromaticity and gives in total:

$$\boxed{\xi_{x,y} = \pm \frac{1}{4\pi} \int \beta_{x,y}(\tilde{s}) \cdot \left[k_0(\tilde{s}) + m_0(\tilde{s})\, D(\tilde{s})\right] \cdot \mathrm{d}\tilde{s}}$$

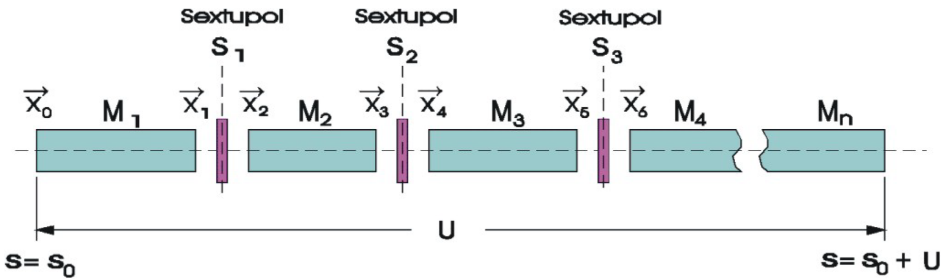
In order to avoid a large tune spread, chromaticity has to be corrected by the use of additional sextupole magnets right after focusing and defocusing quadrupoles where the horizontal dispersion does not vanish:



This correction will have an influence on the stability of the beam and the maximum aperture given by nonlinear effects (so called dynamic aperture):



The dynamic aperture can be calculated from a tracking of the particles orbit through the accelerator where the nonlinear effect of sextupole magnets has to be treated as step by step correction in linear beam matrix optics:



The orbit vector is transformed from s_0 to s_1 by matrix transformation

$$\vec{X}_1 = \mathbf{M}_1 \cdot \vec{X}_0$$

A sextupole of length l will produce an angular kick in the horizontal and vertical orbit of

$$\Delta x_1' = \frac{1}{2} m l \cdot \left(x_1^2 - y_1^2 \right)$$

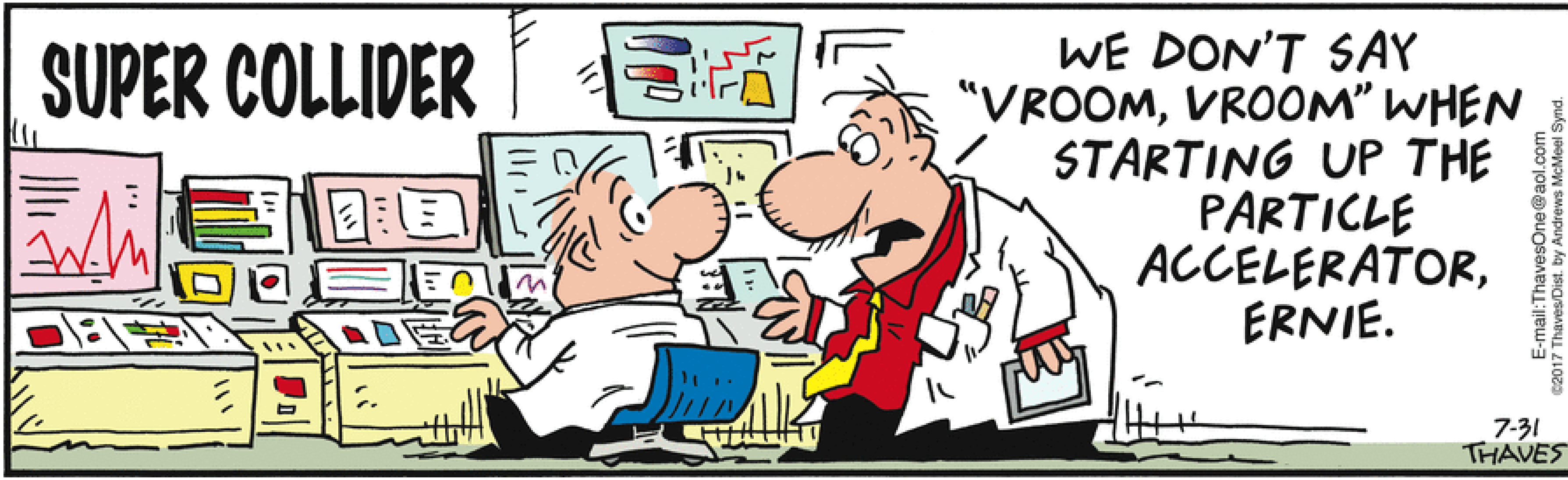
$$\Delta y_1' = m l \cdot x_1 y_1$$

which gives an orbit vector right after the sextupole of

$$\vec{X}_2 = \begin{pmatrix} x_1 \\ x_1' + \Delta x_1' \\ y_1 \\ y_1' + \Delta y_1' \end{pmatrix}$$

By this method a randomly chosen distribution of start vectors $\vec{X}_0$ is tracked through the accelerator for many revolutions and the resulting dynamic aperture is derived from the phase space representation.

The End!



Thanks for listening!

Enjoy the coming lectures

and hands-ON calculations !