A first taste of Non-Linear Beam Dynamics

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Disclaimer

- These lectures are largely based on the lectures of **A. Wolski** (University of Liverpool) from the CAS 2016 on “Introduction to Accelerator Physics” at Budapest, and on the lectures of **Y. Papaphilippou** on “A first taste of Non-Linear Beam Dynamics” from the CAS 2019 on “Introduction to Accelerator Physics” at Vysoké Tatry.

Purpose of the lecture

- Introducing aspects of non-linear dynamics
  - **Mathematical tools** for modelling nonlinear dynamics
    - Power series (Taylor) maps and symplectic maps
  - Effects of **nonlinear perturbations**
    - Resonances, tune shifts, dynamic aperture
  - **Analysis** methods
    - Normal forms, frequency map analysis

- Employ two types of accelerator systems for illustrating methods and tools
  - **Bunch compressor** (single-pass system)
  - **Storage ring** (multi-turn system)

Aim of the 2nd Lecture

- Describe some of the **phenomena** associated with nonlinearities in periodic beamlines (such as storage rings)
- Explain significance of **symplectic maps**, and describe some of the challenges in calculating and applying symplectic maps
- Outline some of the **analysis methods** that can be used to characterise nonlinear beam dynamics in periodic beamlines.

Example of a periodic system: a simple storage ring

A simple storage ring

- As example, consider the transverse dynamics in a **simple storage ring**, assuming:
  - The storage ring is constructed from some number of **identical cells** consisting of dipoles, quadrupoles and sextupoles.
  - The **phase advance** per cell can be tuned from close to zero, up to about $0.5 \times 2\pi$.
  - There is **one sextupole per cell**, which is located at a point where the horizontal beta function is 1 m, and the alpha function is zero.
  - Usually, storage rings will contain (at least) two sextupoles per cell, to correct horizontal and vertical chromaticity. To keep things simple, we will use only one sextupole per cell.

Linear dynamics in a storage ring

- The **chromaticity**, and hence the sextupole strength, will normally be a **function** of the **phase advance**
- To investigate the nonlinear effects of sextupoles, we shall keep the **sextupole strength** $k_2 L$ **fixed**, and **change** only the **phase advance**
- We can assume that the **map** from **one sextupole** to the **next** is **linear**, and corresponds to a **rotation** in phase space through an angle equal to the phase advance:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto \begin{pmatrix} \cos\mu_x & \sin\mu_x \\ -\sin\mu_x & \cos\mu_x \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

- Again to keep things simple, we shall consider only **horizontal motion**, and assume that the vertical co-ordinate $y = 0$
- In the “thin lens” approximation, the **deflection** of a particle passing through the sextupole of length L is

$$\Delta p_x = -\int \frac{B_y}{B\rho} ds = -\frac{1}{2} k_2 L x^2$$

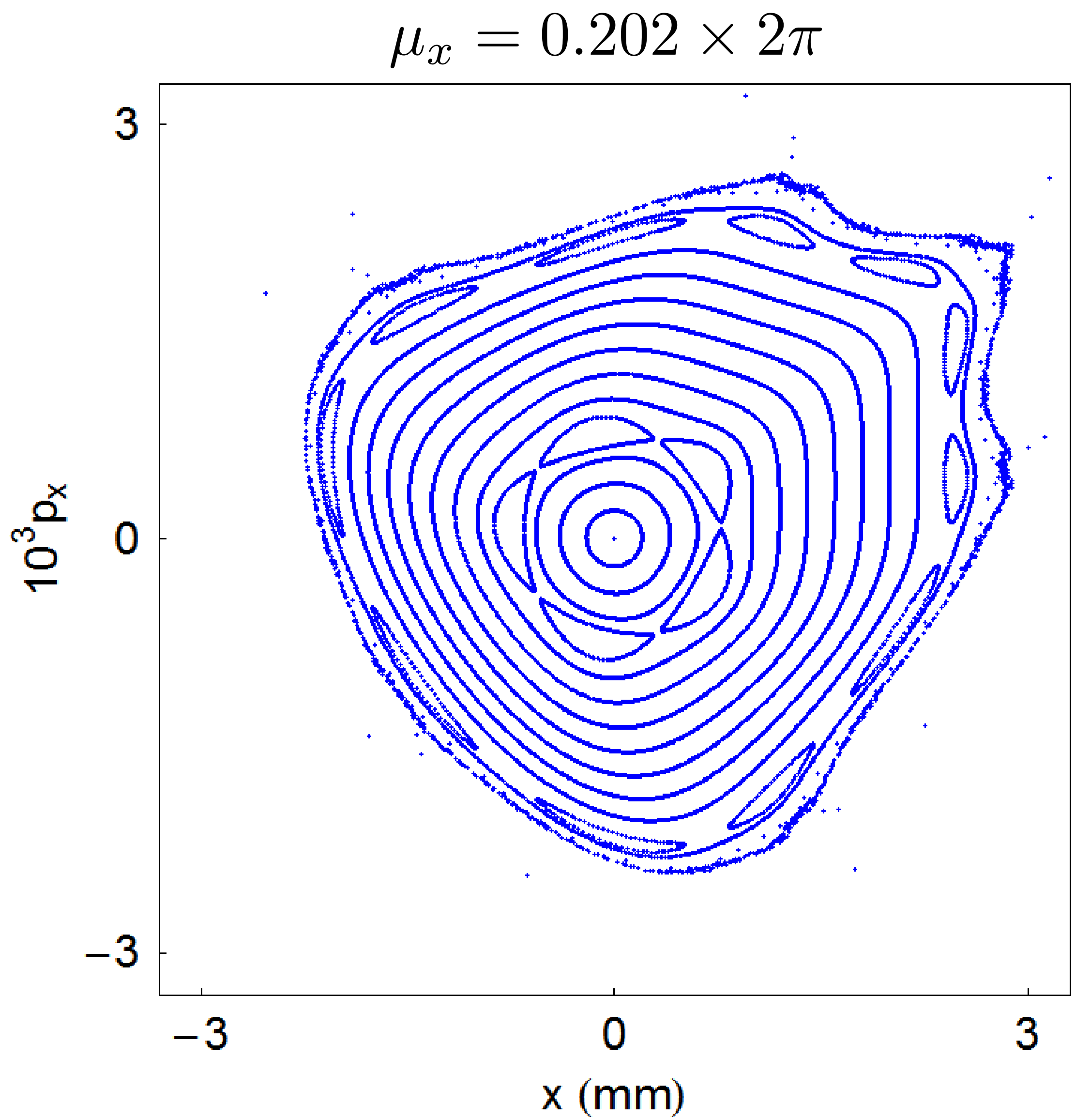
Nonlinear transfer map: sextupole

- The map for a particle moving through a short **sextupole** can be represented by a **"kick"** in the horizontal momentum:

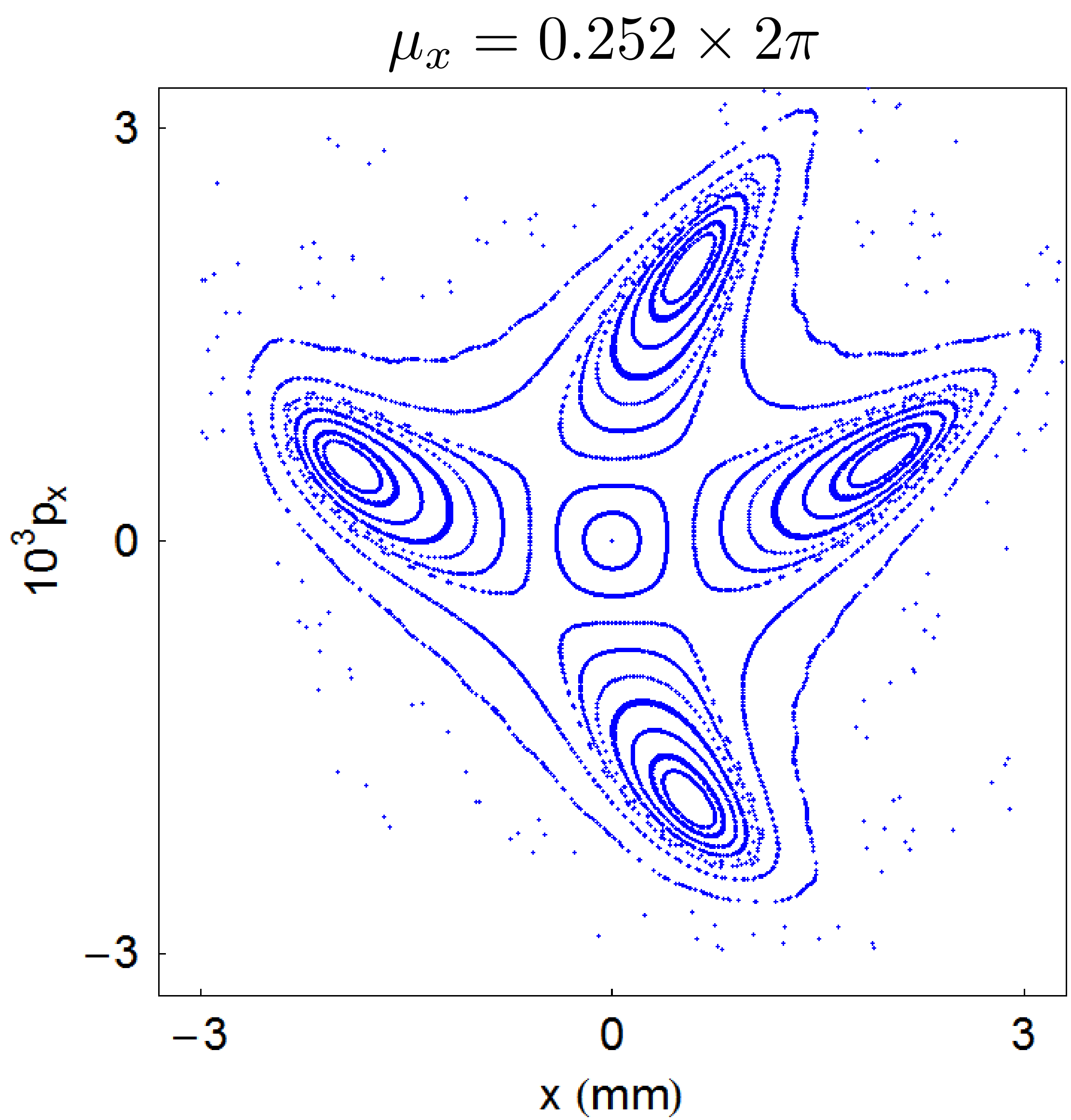
$$\begin{aligned} x &\mapsto x \,, \\ p_x &\mapsto p_x - \frac{1}{2} k_2 L x^2 \end{aligned}$$

- Let us choose a fixed value $k_2 L = -600\ \mathrm{m}^{-2}$, and look at the effects of the maps for different phase advances.
- For each case, we construct a **phase space portrait** by plotting the values of the dynamical variables after repeated application of the map (rotation + sextupole) for a range of initial conditions.
- First, let us look at the phase space portraits for a range of phase advances from **0.2 × 2π** to **0.5 × 2π**

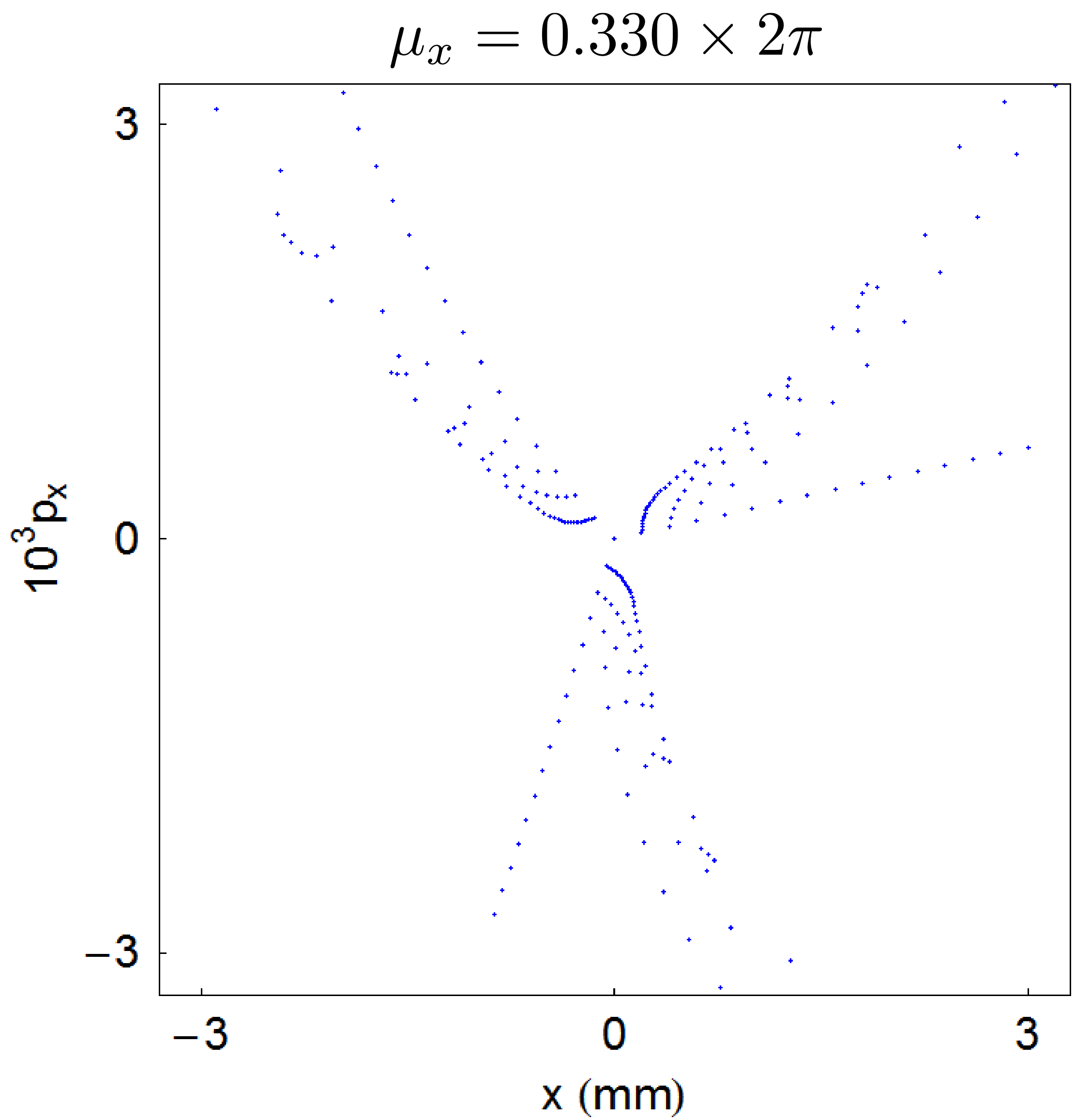
Example of a simple storage ring

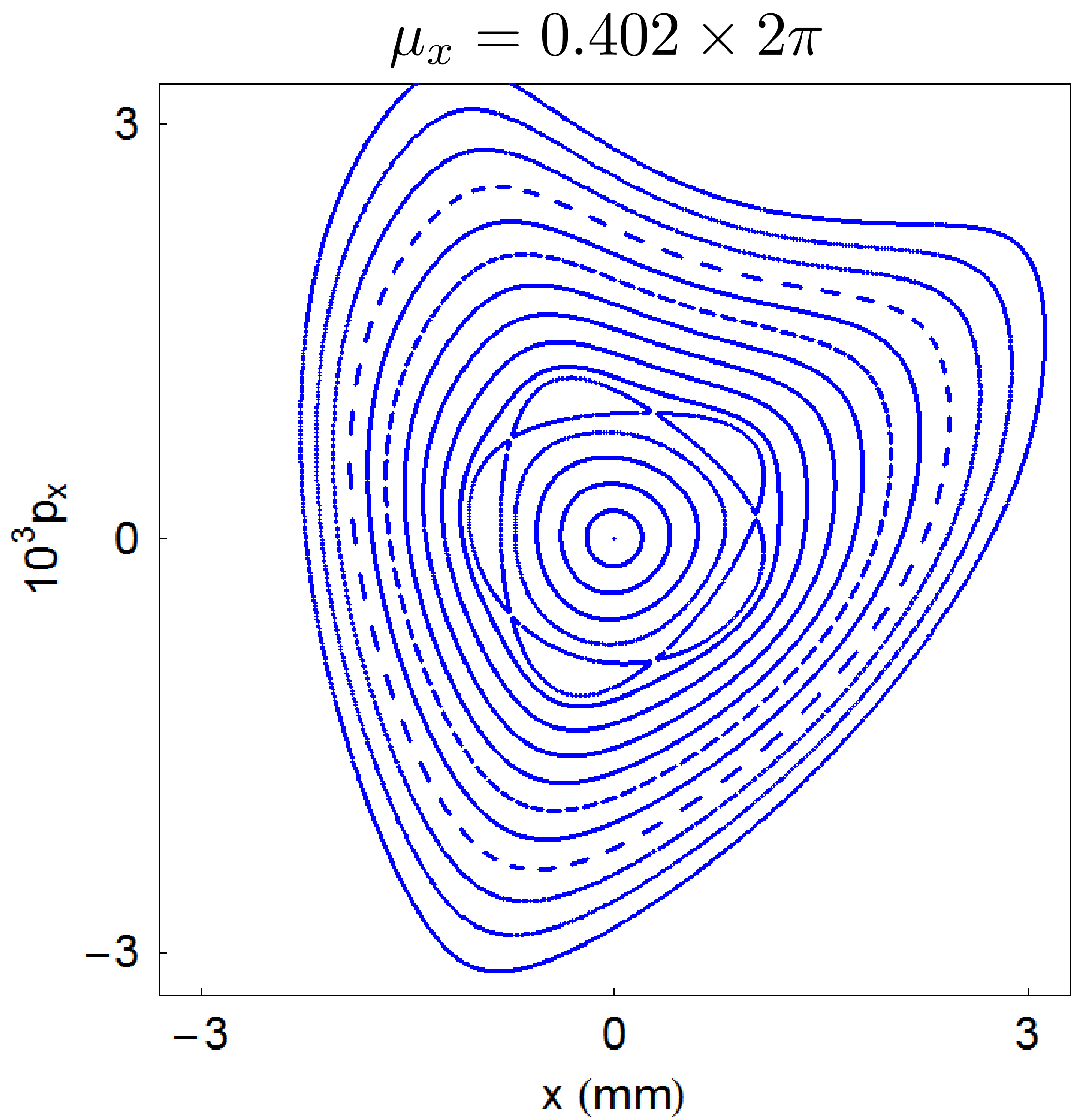


Example of a simple storage ring

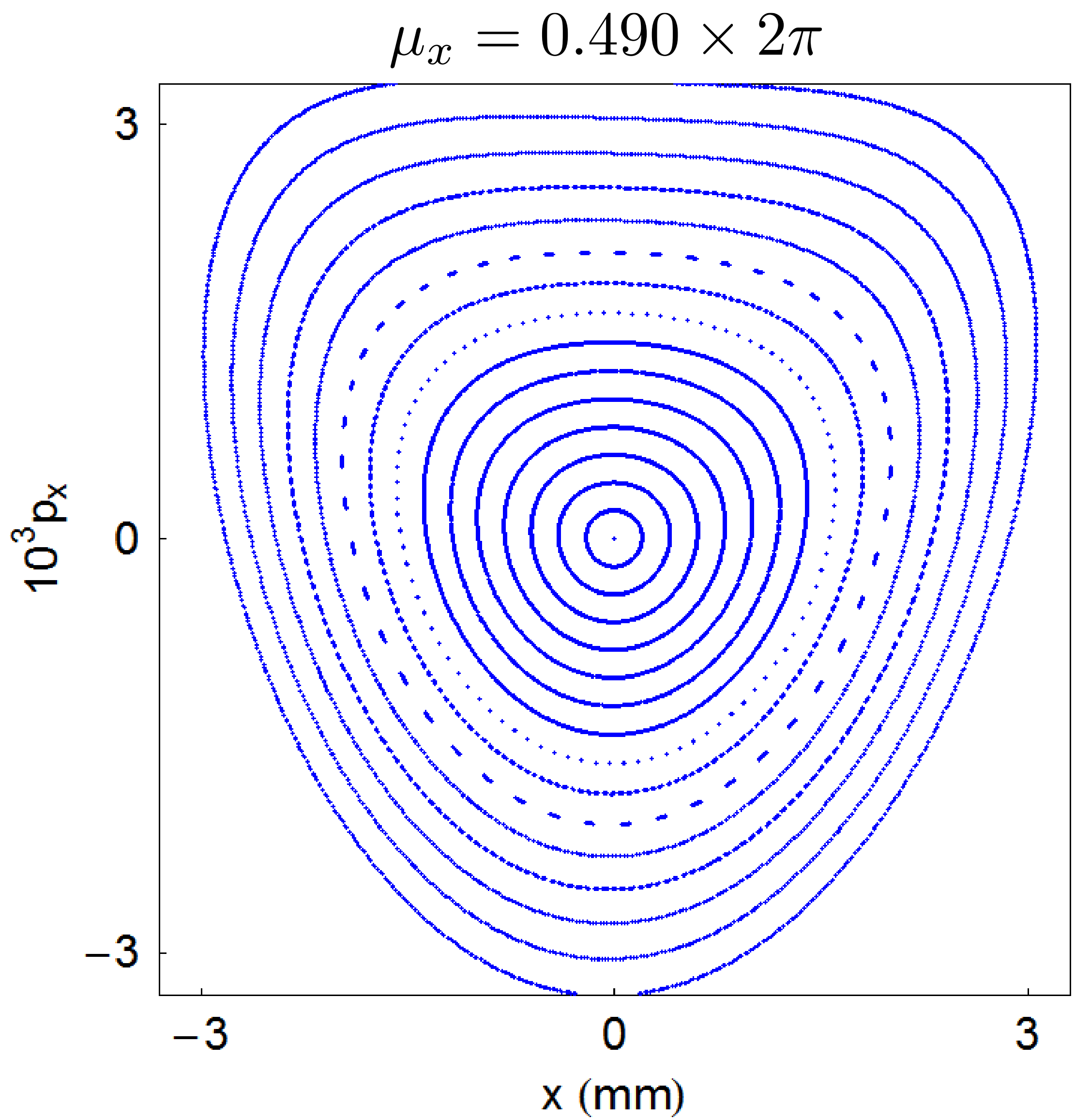


Example of a simple storage ring




$\mu_x = 0.402 \times 2\pi$
3
0
−3
$10^3 p_x$
−3
0
3
x (mm)

Example of a simple storage ring



Some observations

- **There are some interesting features in these phase space portraits to which it is worth drawing attention:**
  - For small amplitudes (small x and p_x), **particles trace out closed loops around the origin**: this is what we expect for a linear map
  - As the **amplitude is increased**, "**islands**" appear in phase space: the phase advance (for the linear map) is often close to m/p where m is an integer and p is the number of islands
  - Sometimes, a larger number of islands appears at larger amplitude
  - Usually, there is a **closed curve that divides a region of stable motion from a region of unstable motion**. Outside that curve, the amplitude of particles increases without limit as the map is repeatedly applied
  - **The area of the stable region depends strongly on the phase advance**: for a phase advance close to 2π/3, it appears that the stable region almost vanishes altogether
  - As **the phase advance is increased towards π, the stable area becomes large**, and distortions from the linear ellipse become small

Effect of phase advance on nonlinear dynamics

Effect of phase advance

- An important observation is that the **effect** of the sextupole in the periodic cell **depends strongly** on the **phase advance** across the cell
- We can start to understand the significance of the phase advance by considering **two special cases**:
  - Phase advance equal to an **integer** times 2π
  - Phase advance equal to a **half integer** times 2π

Integer tune

- Let us consider first a **phase advance** equal to an **integer** times 2π. In that case, the linear part of the map is just the identity

$$\begin{aligned} x &\mapsto x\ , \\ p_x &\mapsto p_x \end{aligned}$$

- The **combined effect** of the **linear map** and the **sextupole kick** is:

$$\begin{aligned} x &\mapsto x\ , \\ p_x &\mapsto p_x - \frac{1}{2} k_2 L x^2 \end{aligned}$$

- Clearly, the **horizontal momentum** will **increase** without limit
- There are **no stable regions** of phase space, apart from $x = 0$

Half-Integer tune

- Now consider what happens if the phase advance of a cell is a **half integer** times 2π, so the linear part of the map is just a rotation through π
- If a **particle** starts at the entrance of a sextupole with $x = x_0$ and $p_x = p_{x0}$, then at the **exit** of that sextupole:

$$\begin{aligned} x_1 &= x_0 \ , \\ p_{x1} &= p_{x0} - \frac{1}{2} k_2 L x_0^2 \end{aligned}$$

- Then, after passing to the **entrance** of the **next sextupole**, the co-ordinates will be:

$$\begin{aligned} x_2 &= \cos(\pi) x_1 = -x_1 = -x_0 \ , \\ p_{x2} &= \cos(\pi) p_{x1} = -p_{x1} = -p_{x0} + \frac{1}{2} k_2 L x_0^2 \end{aligned}$$

Half-Integer tune

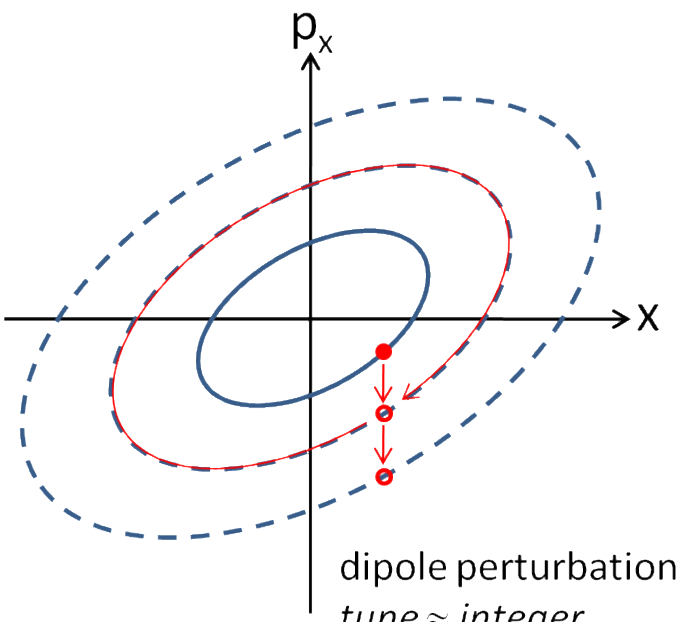
- Finally, on **passing** through the **second sextupole**:

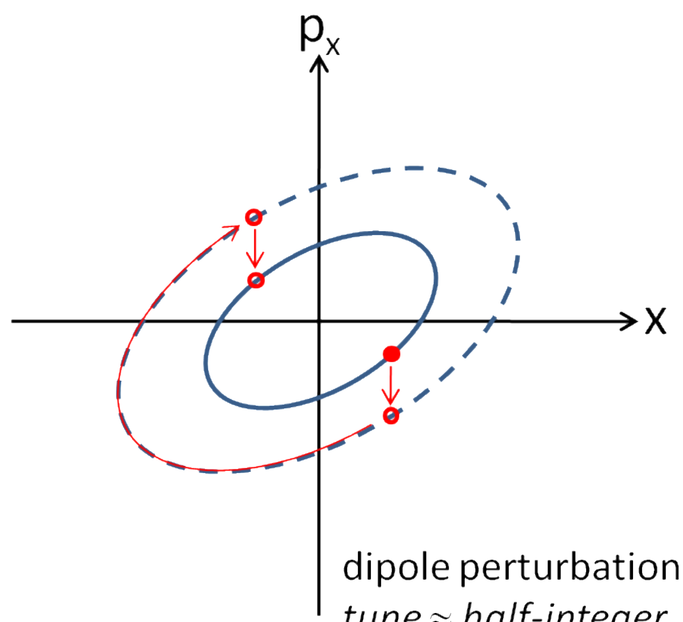
$$
\begin{aligned}
x_3 &= x_2 = -x_0 \ , \\
p_{x3} &= p_{x2} - \frac{1}{2} k_2 L x_2^2 = -p_{x0}
\end{aligned}
$$

- In other words, the **momentum kicks** from the two sextupoles **cancel** each other exactly
- The resulting map is a purely **linear phase space rotation** by π.
- In this situation, we expect the **motion** to be **stable** (and periodic), no matter what the amplitude

Impact of phase advance

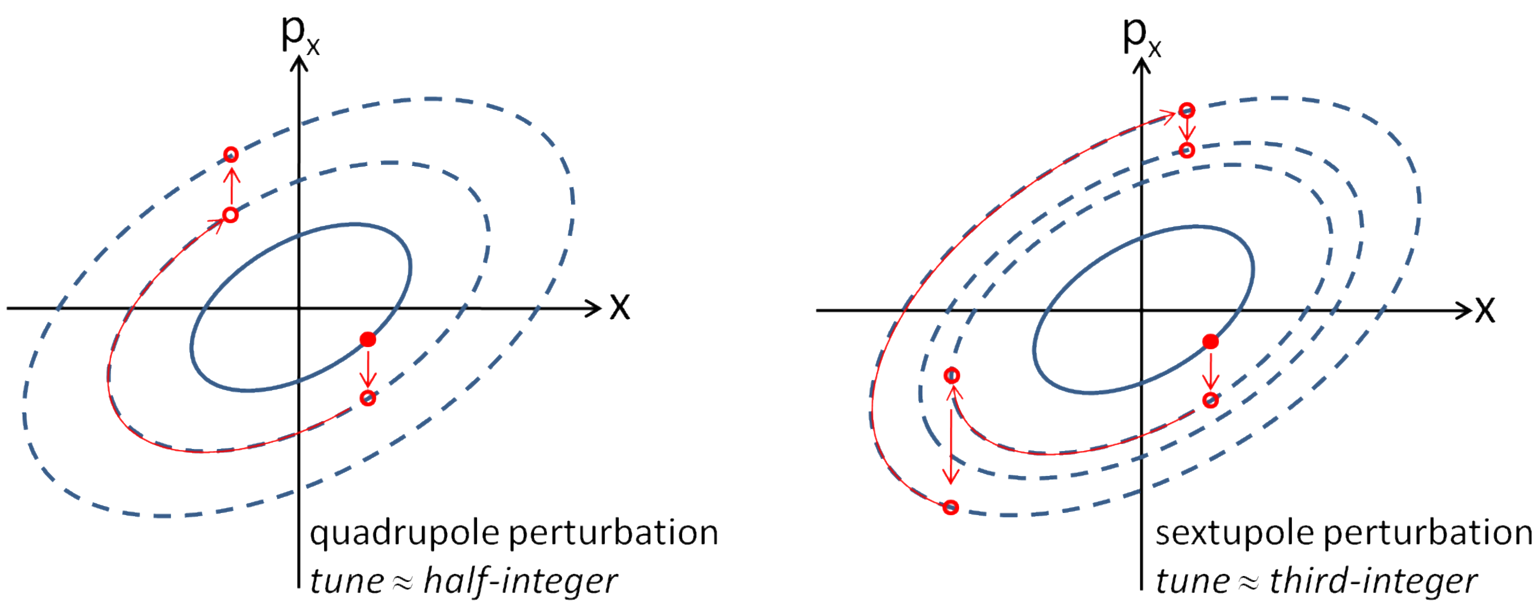
- The effect of the phase advance on the sextupole “kicks” is **similar** to the effect on **perturbations** arising from **dipole** and **quadrupole errors** in a storage ring
- In the case of **dipole errors**, the **kicks add up** if the phase advance is an **integer**, and **cancel** if the **phase advance** is a **half integer**





Impact of phase advance

- In the case of **quadrupole errors**, the **kicks add up** if the phase advance is a **half integer** times 2π
- **Higher-order multipoles** drive **higher-order resonances** but the effects are less easily illustrated on a phase space diagram



Resonances

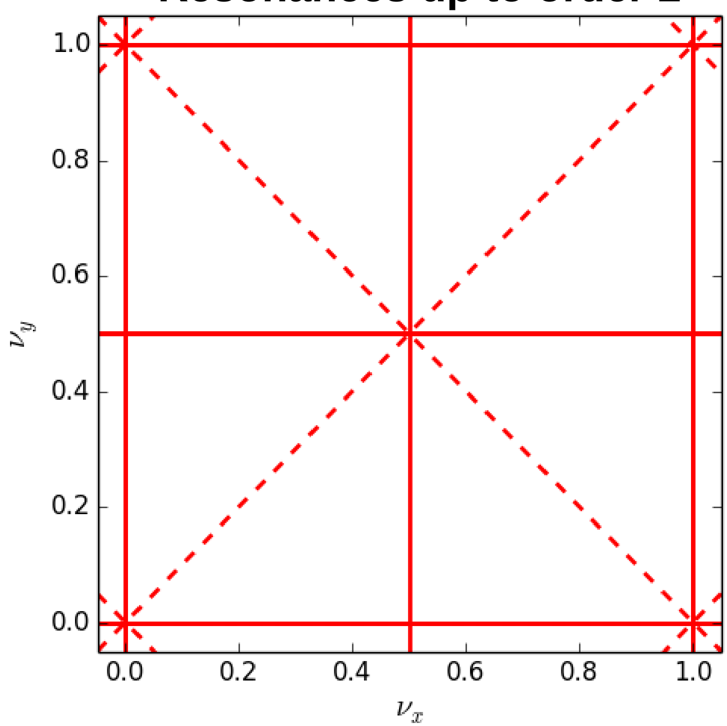
Resonances

- If we include vertical as well as horizontal motion, then we find that **resonances** occur when the tunes satisfy

$$\boxed{m_x \nu_x + m_y \nu_y = \ell}$$

where m_x, m_y and l are integers; resonance is of **order** $|m_x| + |m_y|$

Resonances up to order 2



normal resonances (= even m_y)

skew resonances (= odd m_y)

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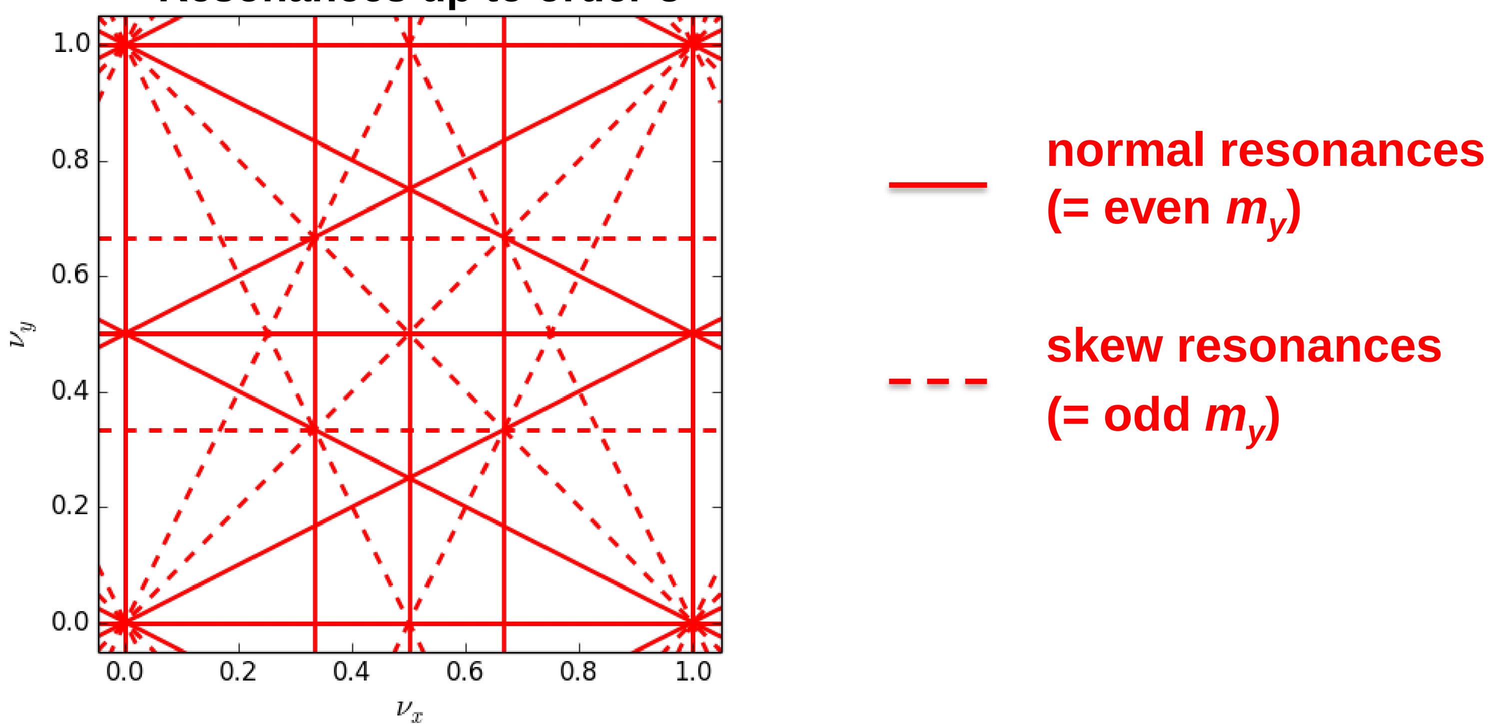

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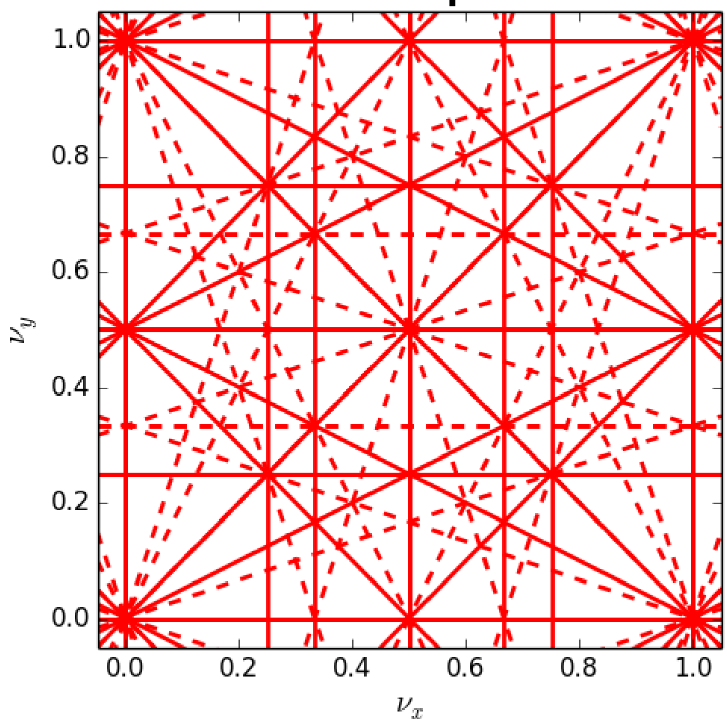
$$\boxed{m_x \nu_x + m_y \nu_y = \ell}$$

where m_x, m_y and l are integers; resonance is of **order $|m_x| + |m_y|$**

Resonances up to order 4



normal resonances (= even m_y)

skew resonances (= odd m_y)

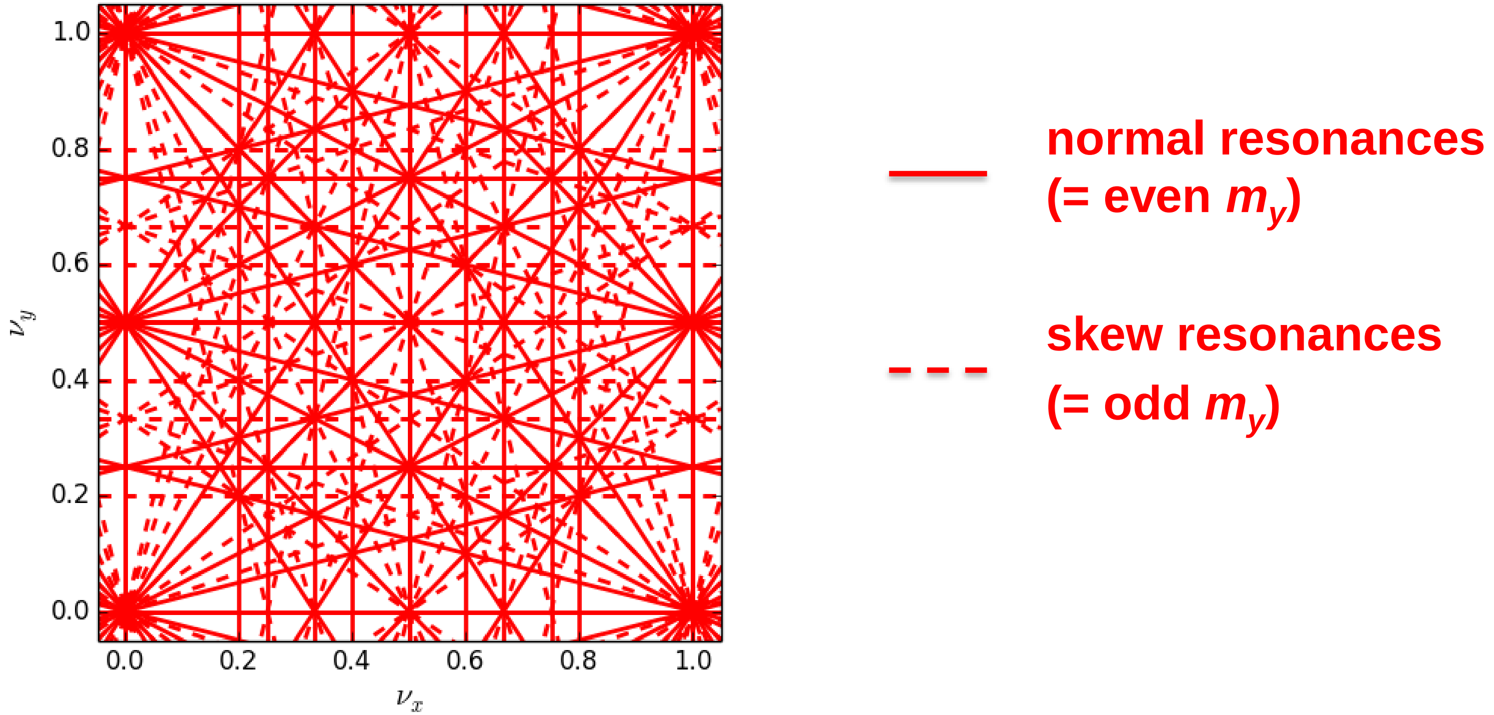
Resonances

- If we include vertical as well as horizontal motion, then we find that **resonances** occur when the tunes satisfy

$$\boxed{m_x \nu_x + m_y \nu_y = \ell}$$

where m_x, m_y and l are integers; resonance is of **order** $|m_x| + |m_y|$

Resonances up to order 5



Resonances

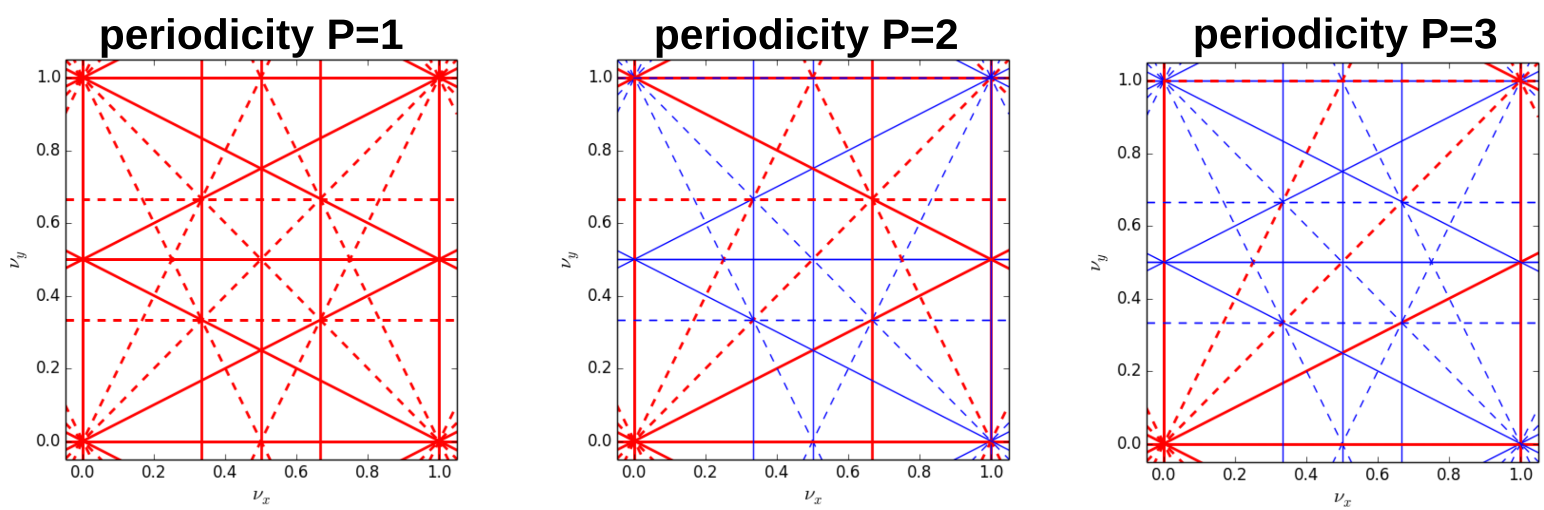
- **Resonances** are associated with **chaotic motion** for particles in storage rings
- However, the number of **resonance lines** in tune space is **infinite**: any point in tune space will be close to a resonance of some order
- This observation raises two questions:
  - How do we know what the real effect of any given resonance line will be?
  - How can we design a storage ring to minimise the adverse effects of resonances?

Resonance cancellation by periodicity

- By imposing a **periodicity *P*** in the lattice (i.e. building a machine from *P* identical cells), the resonance condition becomes

$$m_x \frac{\nu_x}{P} + m_y \frac{\nu_y}{P} = l \quad \rightarrow \quad \boxed{m_x \nu_x + m_y \nu_y = P\ell}$$

  - … the resonance condition needs to be satisfied by each cell, as conceptually there is no difference between passing one cell P turns or passing a lattice consisting of P identical cells only once
- Resonances for which *l* is integer → systematic
- If *l* is NOT integer the resonance cancels → non-systematic

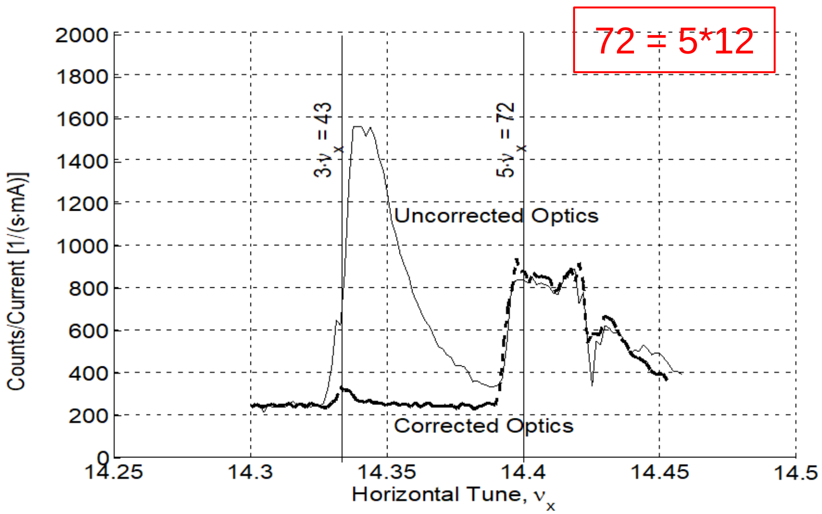


solid lines: normal resonances   dashed lines: skew resonances

Real life example for periodicity: ALS

Advanced Light Source, design lattice periodicity: 12

Measurement of beam loss as function of tune



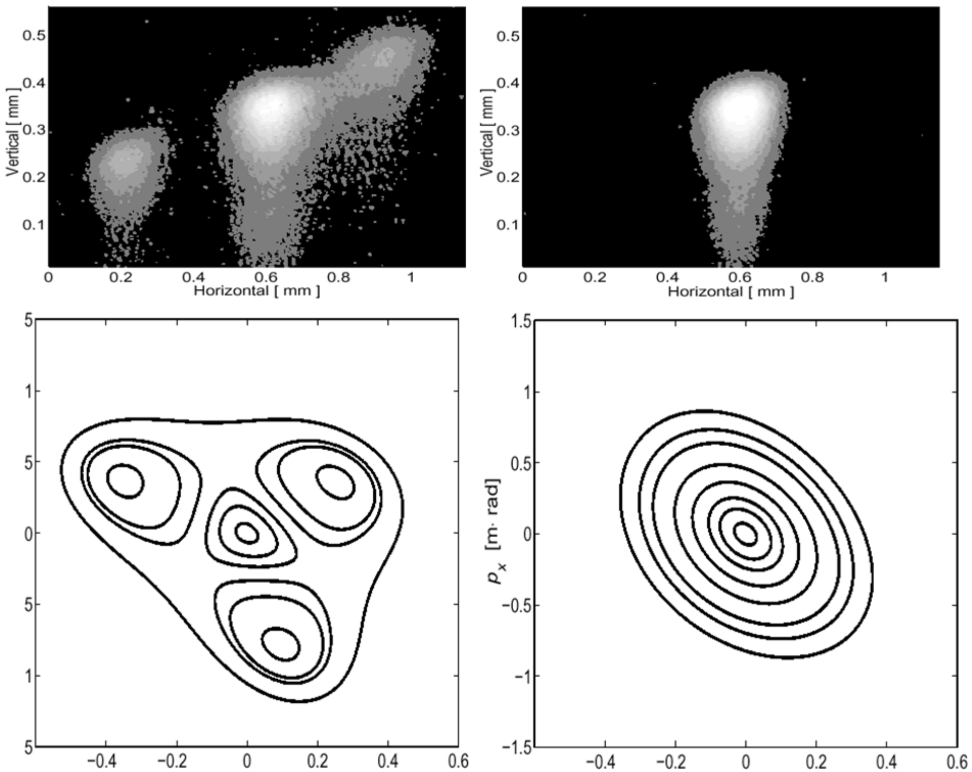
Beta beating

Before optics correction: ~30%

After optics correction: <1%

Synchrotron light beam spot

Uncorrected optics **Corrected optics**



Simulated phase space

D. Robin, C. Steier, J. Safranek, W. Decking, "Enhanced performance of the ALS through periodicity restoration of the lattice", proc. EPAC 2000.

Non-linear map representation

Taylor maps

- For any dynamical variable x_j the **Taylor map** up to **3rd order** can be written as

$$x_j^{\text{new}} = \sum_{k=1}^{6} R_{jk} x_k + \sum_{k=1}^{6}\sum_{l=1}^{6} T_{jkl} x_k x_l + \sum_{k=1}^{6}\sum_{l=1}^{6}\sum_{m=1}^{6} U_{jklm} x_k x_l x_m$$

- **Taylor series** provide a convenient way of systematically representing **transfer maps** for **beamline components**, or sections of beamline
- The **main drawback** of Taylor series is that in general, transfer maps can only be represented exactly by series with an **infinite number** of **terms**
- In practice, we have to **truncate** a **Taylor map** at some order, and we then lose certain desirable properties of the map
- In particular, a **truncated map** will be usually **non-symplectic**

Symplectic maps

- Consider two sets of canonical variables $\vec{x}_i$, $\vec{x}_f$, which represent the evolution of the system between two points in phase space
- A **map** $\mathcal{M} : \vec{x}_i \mapsto \vec{x}_f$ describes the transformation from one set to the other
- This map is **symplectic,** i.e. it **conserves phase space volumes**, if

$$J^T S J = S$$

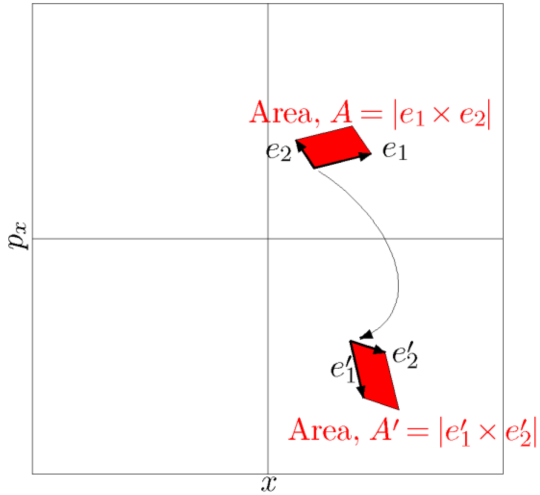
symplecticity condition

$$J_{mn} \equiv \frac{\partial x_{m,f}}{\partial x_{n,i}}$$

Jacobian matrix of the map

$$S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

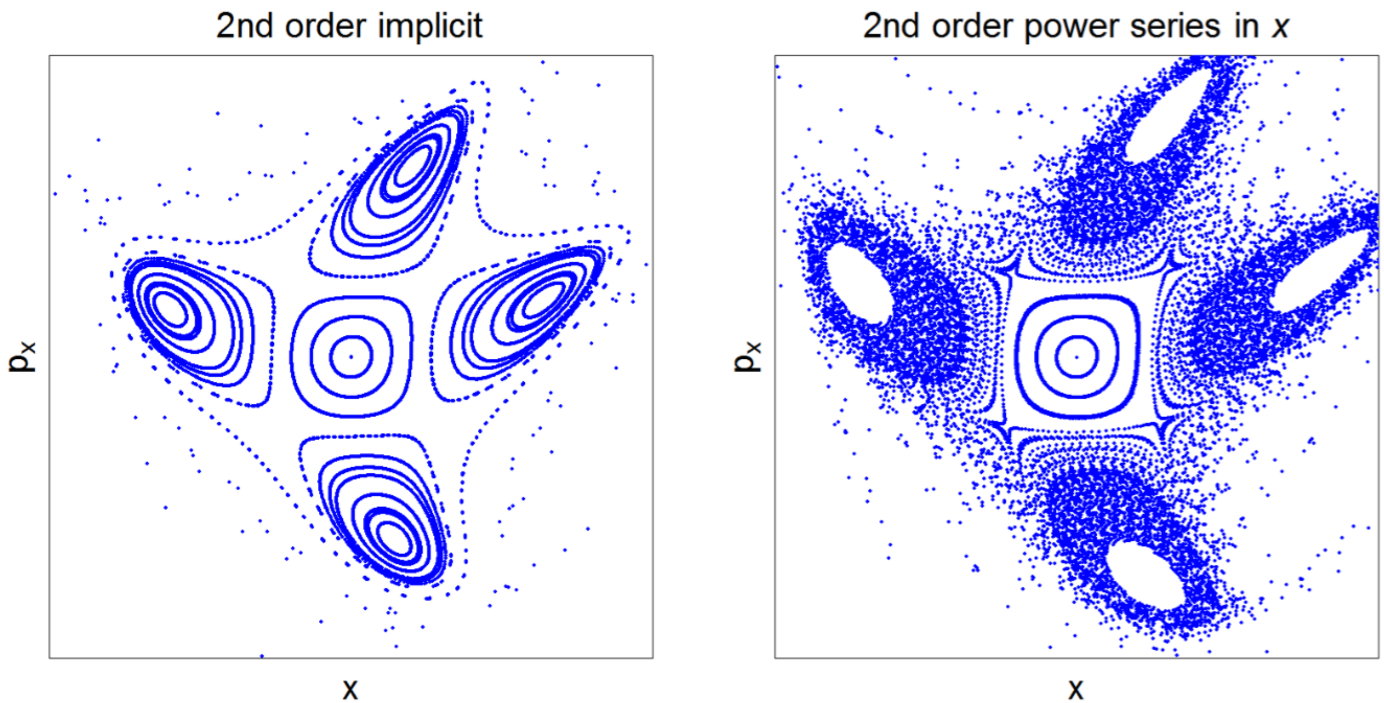
antisymmetric matrix with block diagonals



… this is Liouville's theorem, and is a property of charged particles moving in electromagnetic fields, in the absence of radiation

Symplectic maps

- The effect of **losing symplecticity** becomes apparent if we compare phase space portraits constructed using symplectic (below, left) and non-symplectic (below, right) transfer maps.



- Modelling a storage ring using non-symplectic maps can lead to an inaccurate estimate of the dynamic aperture and the beam lifetime

Symplectic integration

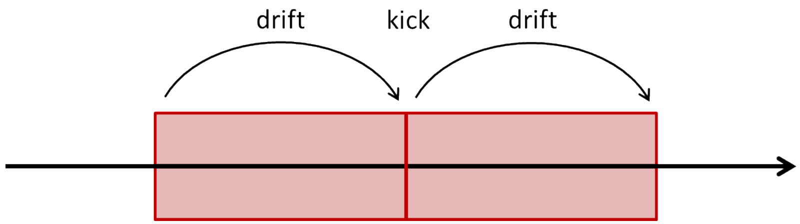
- Consider a sextupole with equations of motion:

$$\frac{dx}{ds} = p_x, \qquad \frac{dp_x}{ds} = -\frac{1}{2}k_2 x^2$$

- Exact solutions using some elementary functions do not exist
- By splitting integration into three steps, it is possible to write an **explicit** and **symplectic** approximate solution

$$\begin{aligned} 0 \le s < L/2: &\qquad x_1 = x_0 + \tfrac{L}{2} p_{x0}, &\qquad p_{x1} &= p_{x0}, \\ s = L/2: &\qquad x_2 = x_1, &\qquad p_{x2} &= p_{x1} - \tfrac{1}{2} k_2 L x_1^2, \\ L/2 < s \le L: &\qquad x_3 = x_2 + \tfrac{L}{2} p_{x2}, &\qquad p_{x3} &= p_{x2} \end{aligned}$$

- This an example of a symplectic integrator known as a **“drift–kick–drift”** approximation



Analytical methods for nonlinear dynamics

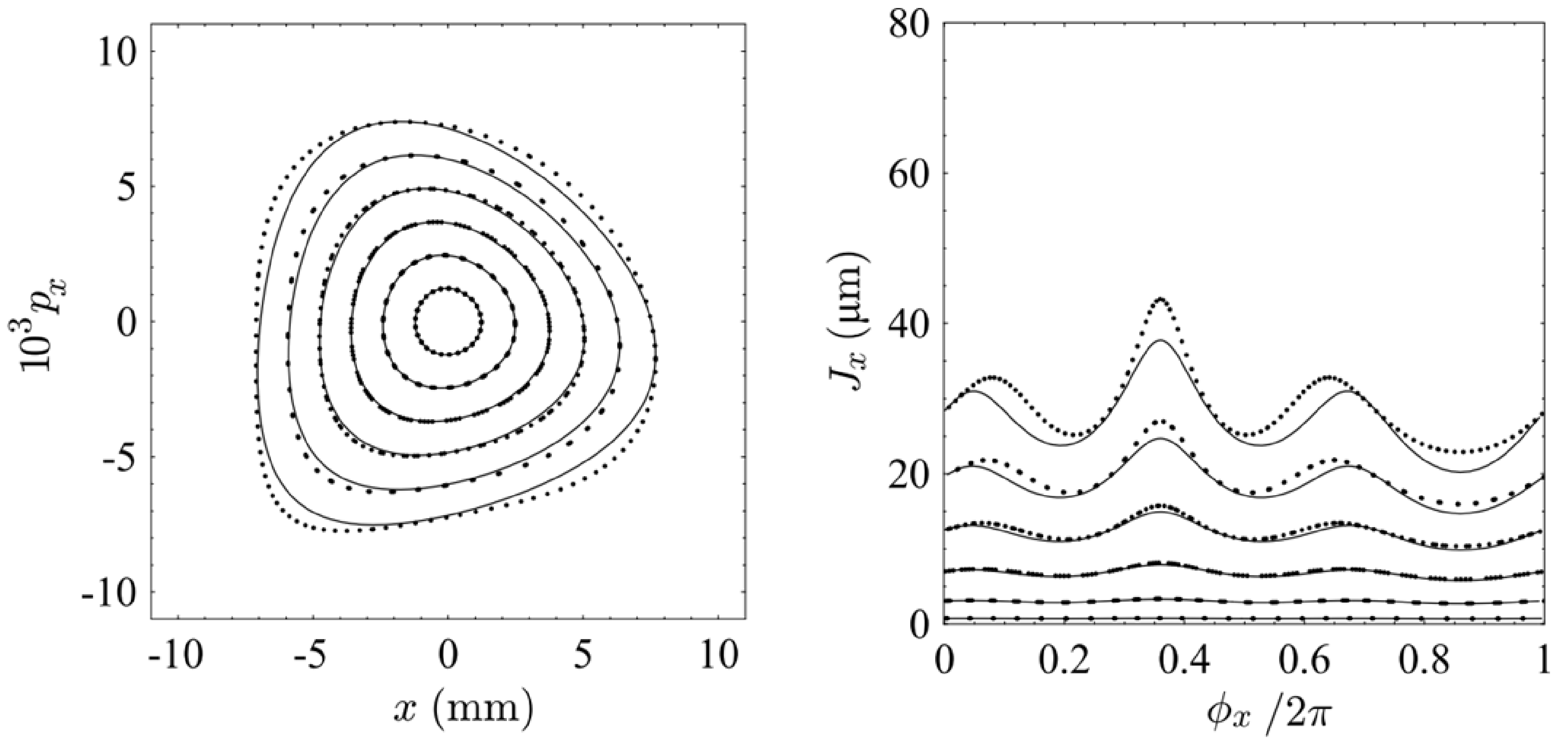
Analytical methods

- There are two approaches widely used in accelerator physics: **perturbation theory** and **normal form** analysis
- In both these techniques, the goal is to construct a **quantity** that is **invariant** under application of the single-turn transfer map. Unfortunately, in both cases the mathematics is complicated and fairly cumbersome
- In the case of a **single sextupole** in a storage ring, we find from **normal form analysis** the following expression for the betatron action as a function of the betatron phase (angle variable):

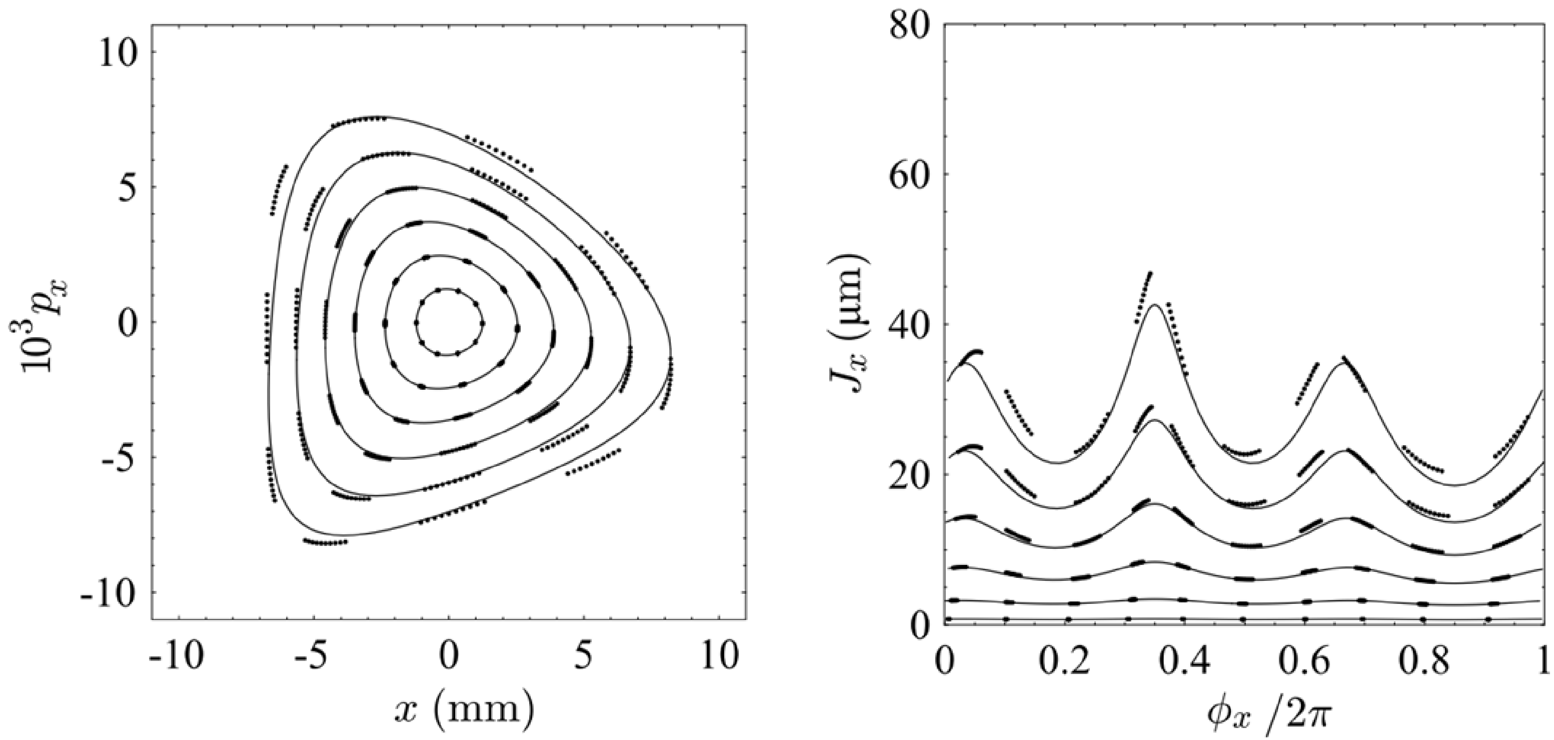
$$J_x \approx I_0 - \frac{k2L}{8}(2\beta_x I_0)^{3/2}\frac{\cos(3\mu_x/2 + 2\phi_x) + \cos(\mu_x/2)}{\sin(3\mu_x/2)} + O(I_0^2)$$

  where I_0 is a constant (an invariant of the motion), ϕ_x is the angle variable, and μ_x is the phase advance per cell
- The second term becomes **very large** when μ_x is close to **third integer**

$$\mu_x = 0.28 \times 2\pi$$

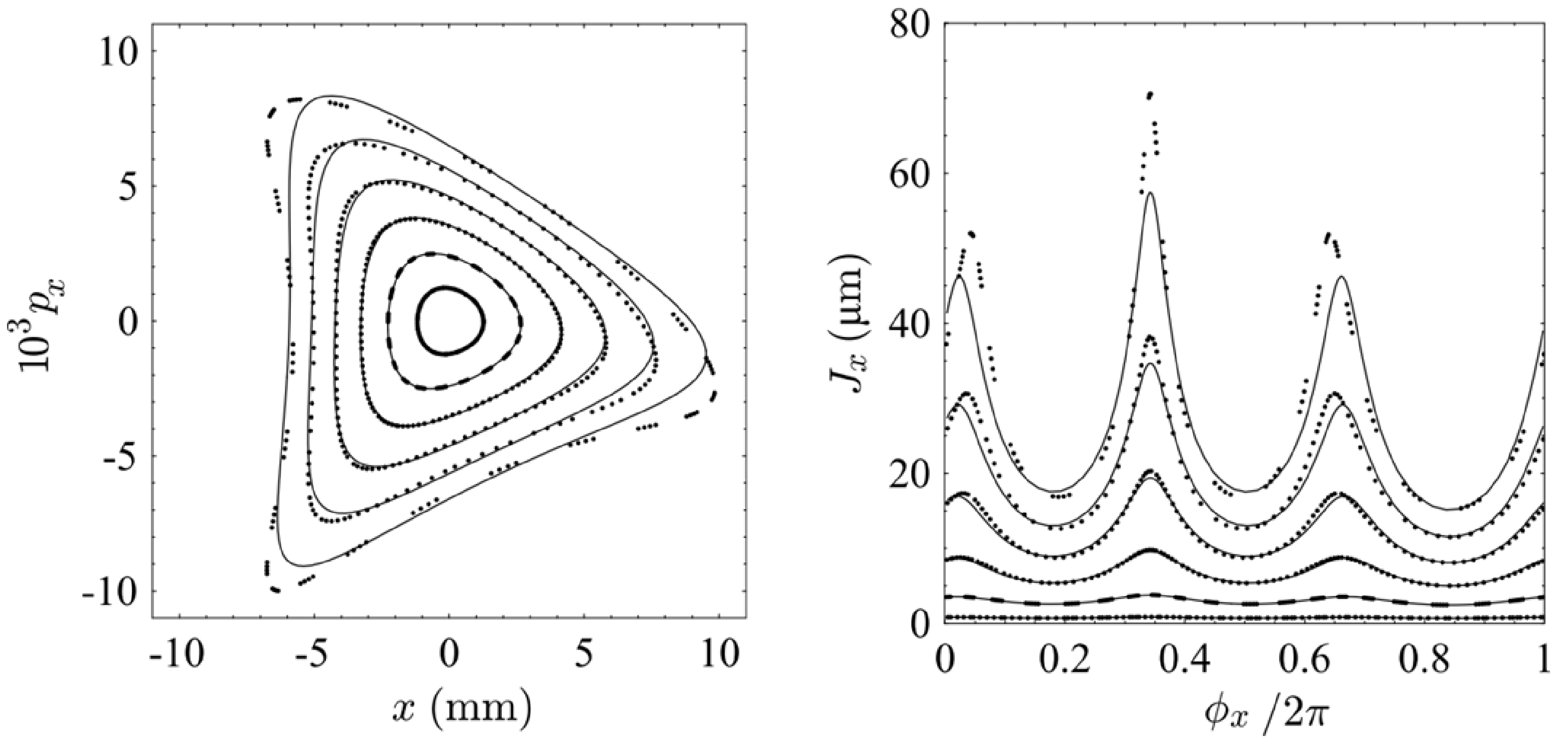


$$\mu_x = 0.30 \times 2\pi$$



Normal form for sextupole

$$\mu_x = 0.315 \times 2\pi$$



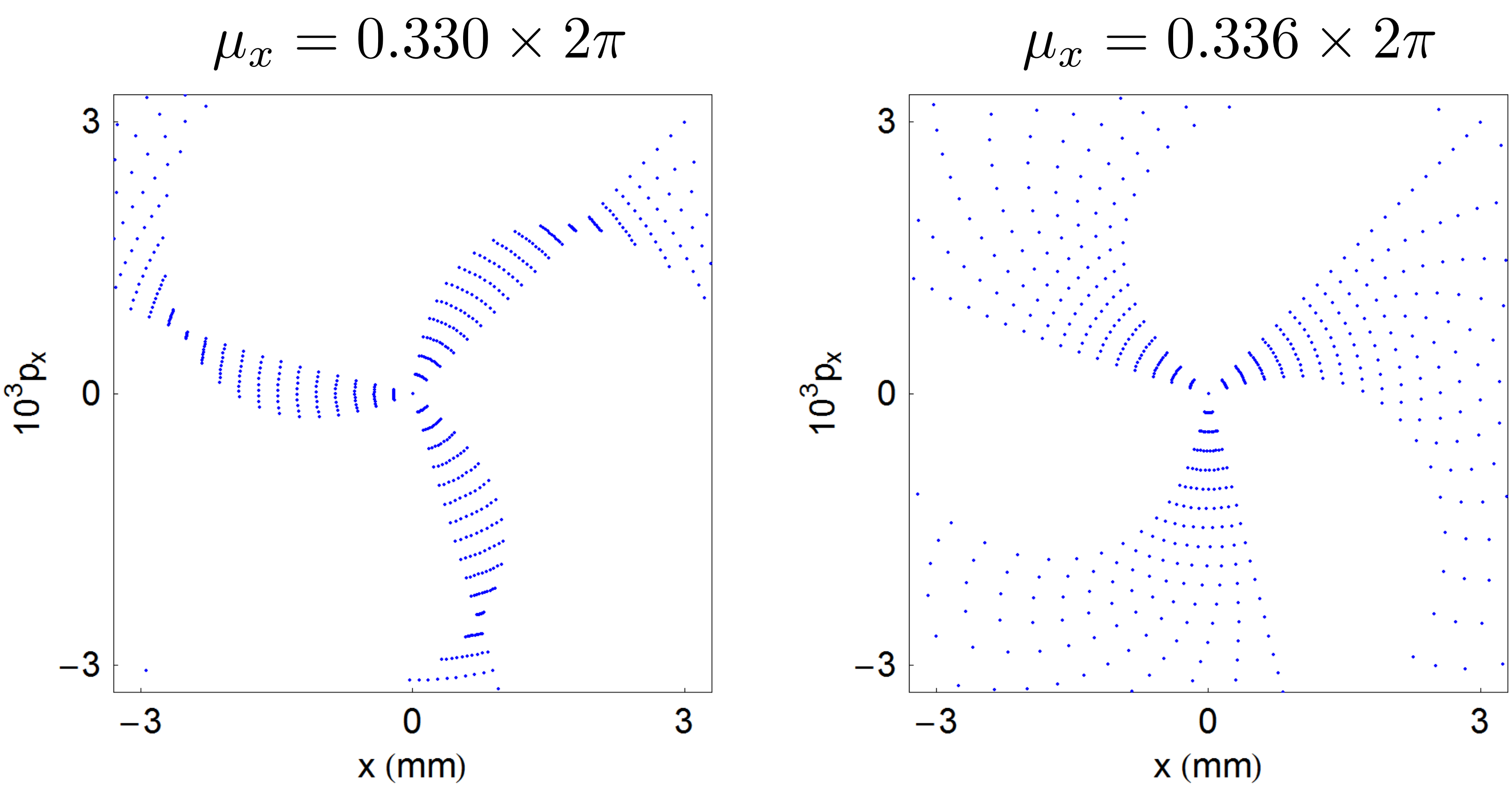
Tune-shift with amplitude

- Close inspection of the plots on the previous slides reveals another effect, in addition to the obvious distortion of the phase space ellipses: the **phase advance** per turn (i.e. the tune) **varies** with increasing **betatron amplitude**
- Normal form analysis (and perturbation theory) can be used to obtain **estimates** for the **tune shift** with amplitude
- In the case of a sextupole, the **tune shift** is **higher-order** in the **sextupole strength**
- An octupole, however, does have a **tune shift** with amplitude in **first-order** of the octupole strength, given by:

$$\nu_x = \nu_{x0} + \frac{k_3 L \beta_x^2}{16\pi} J_x + O(J_x^2)$$

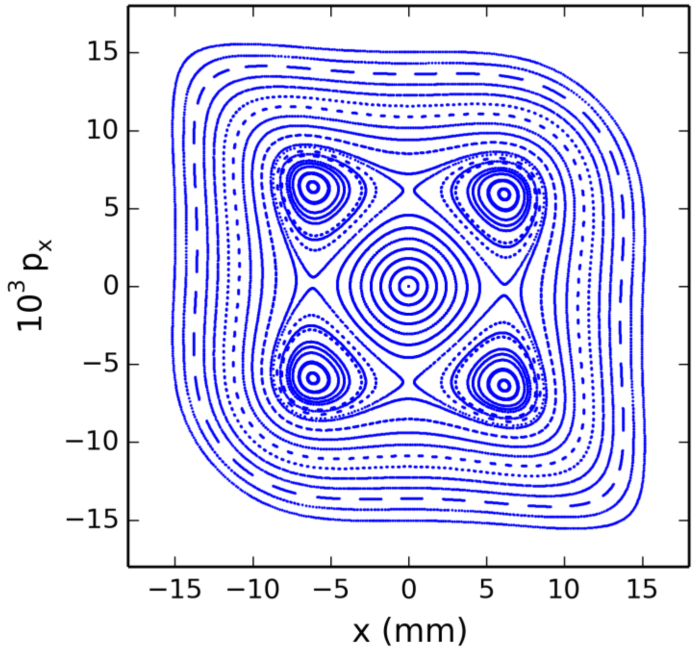
Tune-shift with amplitude

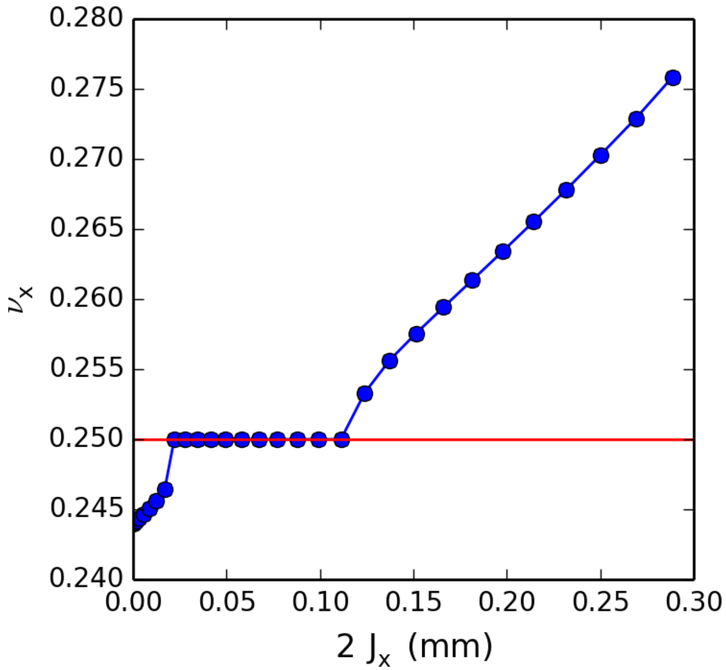
- The tune shift with amplitude becomes obvious if we track a small number of turns (30) in a lattice with a **single octupole**.



Resonant islands of 4th order resonance

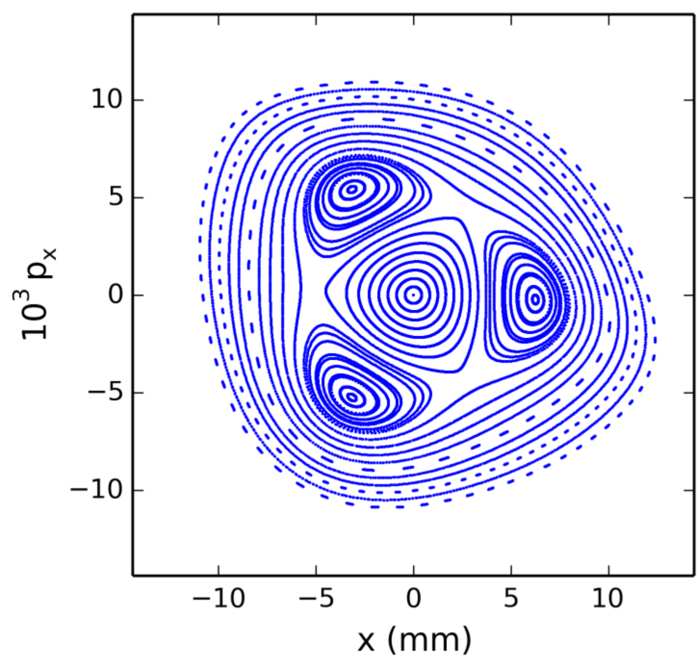
- Simulation of simple storage ring with a **single octupole** close to 4th order resonance
- **Detuning with amplitude** (linear in action)
- Particles in the stable islands have a tune **locked** to the **resonance**

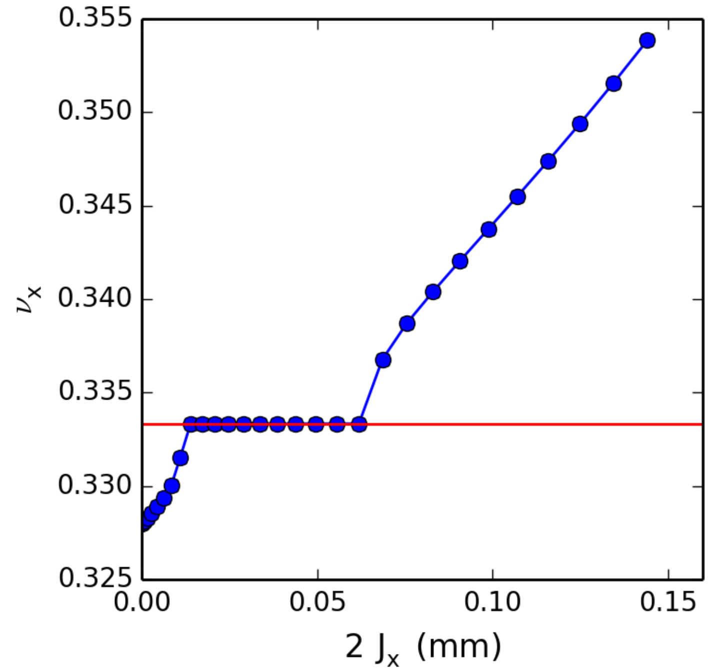




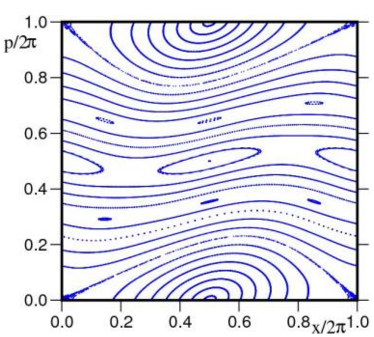
Resonant islands of 3rd order resonance

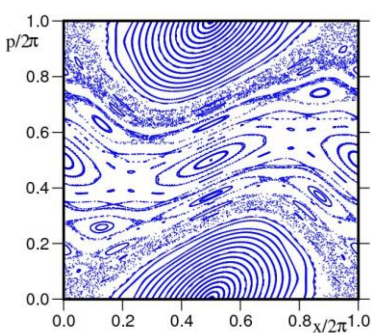
- Simulation of simple storage ring with a **sextupole** and an **octupole** close to **3rd order resonance**
- The **amplitude detuning** induced by the octupole can create **stable islands** even for the **3rd order resonance** (recall the phase-space plot for the case of a single sextupole)

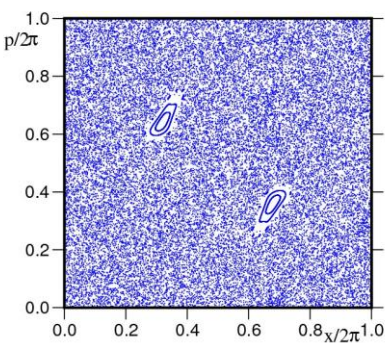


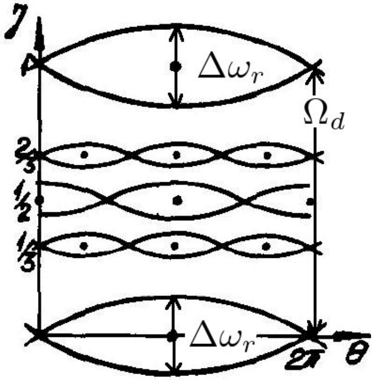


- Perturbation theory and normal form analysis depend on the existence of **constants of motion in the presence of nonlinear perturbations**
  - Constants of motion can exist in the presence of nonlinear perturbations as a consequence of the **Kolmogorov–Arnold–Moser (KAM) theorem**
- **Resonances do not invariably result in loss of stability**
  - Resonances will usually tend to drive the amplitudes of particles with a particular tune to large amplitudes
  - For sufficiently large tune-shift with amplitude, it is possible for there to be a stable region at amplitudes larger than that at which resonance occurs
- The **overlap of two resonances** is associated with a transition from regular to **chaotic motion**: the Chirikov criterion describes the parameter range over which the particle motion becomes chaotic



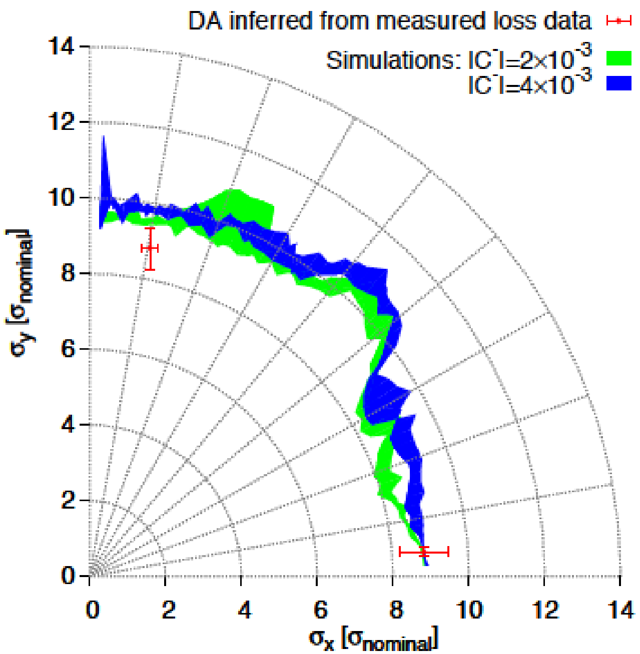






Numerical methods: Dynamic aperture (DA) and Frequency map analysis (FMA)

- The most direct way to evaluate the non-linear dynamics performance of a ring is the computation of **Dynamic Aperture** (short: DA), which is the **boundary of the stable region in co-ordinate space**
- Need a **symplectic tracking code** to follow particle trajectories (a lot of initial conditions) for a number of turns until the particles start getting lost → this boundary defines the **Dynamic aperture**



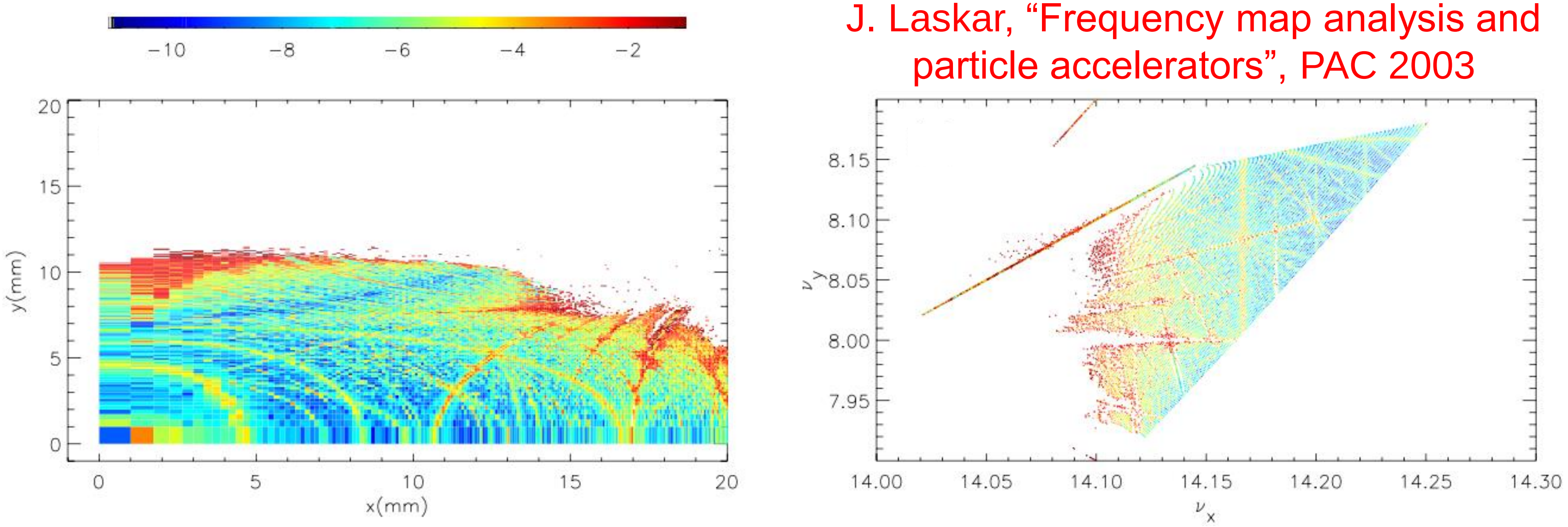
DA of the LHC

… very good agreement between tracking simulations and measurements in the machine

E.Mclean, PhD thesis, 2014

Frequency Map (example for ALS)

J. Laskar, "Frequency map analysis and particle accelerators", PAC 2003



- Numerically integrate the phase space trajectories through the lattice for sufficient number of turns (i.e. perform particle tracking)
- Compute through advanced Fourier methods ν_x and ν_y after sufficient number of turns, and plot the tune variation by color code in the configuration space x-y (left) and the tune space ν_x-ν_y (right)
  - **regular motion** corresponds to small tune diffusion)
  - **Chaotic motion** associated with large tune diffusion, occurs at the dynamic aperture of the machine
  - **Resonances** appear as curves in initial condition space

Conclusions and Summary

Summary

- **Nonlinear dynamics** appear in a wide variety of accelerator systems, including **single-pass** systems (such as bunch compressors) and **multi-turn** systems (such as storage rings)
- It is possible to model nonlinear dynamics in a given component or section of beamline by representing the **transfer map** as a **power series**
- **Conservation** of **phase space volumes** is an important feature of the beam dynamics in many systems. To conserve phase space volumes, transfer maps must be **symplectic**
- In general, (**truncated**) **power series** maps are **not symplectic**
- To construct a **symplectic transfer map**, the equations of motion in a given accelerator component must be solved using a **symplectic integrator** (e.g. the "drift–kick–drift" approximation for a multipole magnet)

Summary

- **Common features** of nonlinear dynamics in accelerators include **phase space distortion**, **tune shifts with amplitude**, **resonances**, and **chaotic** particle trajectories at large amplitudes (**dynamic aperture** limits)
- Analytical methods such as **perturbation theory** and **normal form** analysis can be used to estimate the impact of nonlinear perturbations in terms of quantities such as **resonance strengths** and **tune shifts with amplitude**
- **Frequency map analysis** provides a useful numerical tool for characterising tune shifts and resonance strengths from tracking data
- This can give some **insight** into **limitations** on the **dynamic aperture**