

Particle motion in Hamiltonian Formalism I

Or how to derive and solve equations of motion

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- The key point is how to derive **equations of motion** and how to **solve (integrate)** them, in order to describe of the **evolution** (dependence with “**time**”) of a **system** (“**particle**”)
- Introduce **formalism of theoretical mechanics** for analysing particle motion in general (linear or non-linear) **dynamical systems**, including **particle accelerators**
- Connect this **formalism** with **concepts** already studied in the introductory school (matrices, synchrotron motion, invariants,...)
- Prepare the **ground** for the approaches followed for **studying non-linear particle motion** in accelerators (in the advanced course)

Equations of motion

- The motion of a “classical” particle in a force field is described by **Newton's law**:

$$m \frac{d^2 u(t)}{dt^2} = \frac{dp_u(t)}{dt} = F(u) = - \frac{\partial V(u)}{\partial u}$$

with u the position

p_u the momentum

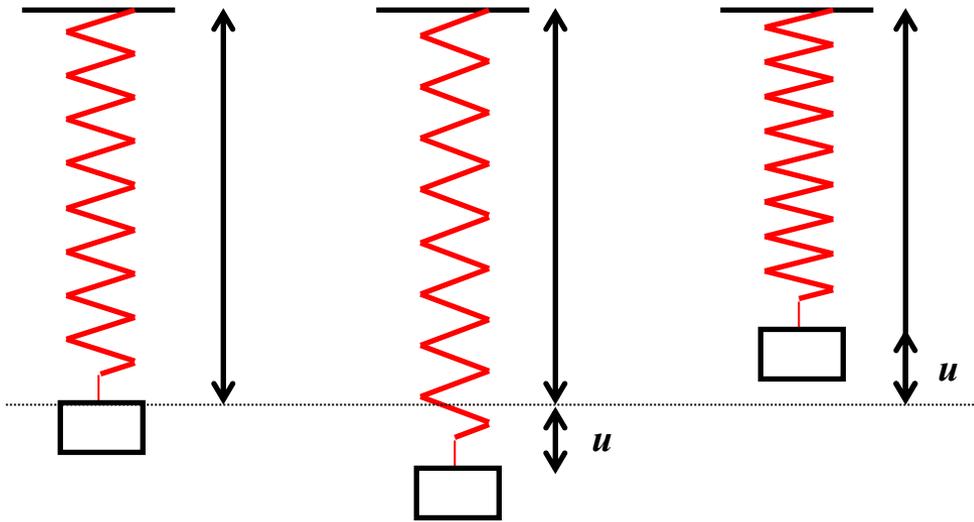
$F(u)$ the force

$V(u)$ the corresponding potential

- It is essential to solve (**integrate**) the differential equation for understanding the evolution of the physical (dynamical) system

- A linear restoring force (**Harmonic oscillator**) is described by

$$\frac{d^2 u(t)}{dt^2} + \omega_0^2 u(t) = 0 \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}}$$



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and the solutions of the **characteristic polynomial** are

$$\lambda^2 + \omega_0^2 = 0 \Rightarrow \lambda_{\pm} = \pm i\omega_0, \text{ which yields the } \mathbf{general\ solution}$$

$$u(t) = ce^{i\omega_0 t} + c^* e^{-i\omega_0 t} = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A \sin(\omega_0 t + \phi)$$

with the "**velocity**"

$$\frac{du(t)}{dt} = -C_1 \omega_0 \sin(\omega_0 t) + C_2 \omega_0 \cos(\omega_0 t) = A \omega_0 \cos(\omega_0 t + \phi)$$

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- Note that a **negative sign** in the differential equation provides a solution described by **hyperbolic sine/cosine** functions
- Note also that for **no restoring force** $\omega_0 = 0$, the motion is **unbounded**

- The **amplitude** and **phase** depend on the **initial conditions**

$$u(0) = u_0 = C_1, \quad \frac{du(0)}{dt} = u'_0 = C_2\omega_0, \quad A = \frac{(u'_0{}^2 + \omega_0^2 u_0^2)^{1/2}}{\omega_0}, \quad \tan(\phi) = \frac{u'_0}{\omega_0 u_0}$$

- The solutions can be re-written thus as

$$u(t) = u_0 \cos(\omega_0 t) + \frac{u'_0}{\omega_0} \sin(\omega_0 t)$$

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or in **matrix form**

$$\begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) \end{pmatrix} \frac{1}{\omega_0} \begin{pmatrix} \sin(\omega_0 t) \\ \cos(\omega_0 t) \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

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- By replacing $\omega_0 \rightarrow \sqrt{k_0}$ and $t \rightarrow s$, this becomes the solution of a **quadrupole** (see **Transverse Linear Beam Dynamics** lectures)

- General **transfer matrix** from s_0 to s

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0) \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

- Note that $\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$
which is always true for **conservative systems** ("energy" is constant)

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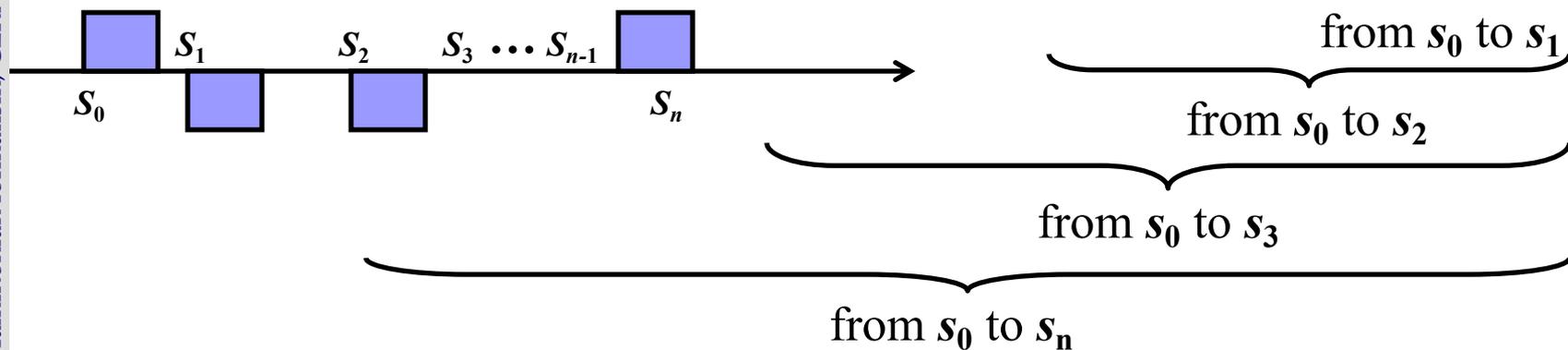
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- The **general solution** can be built by a series of matrix multiplications

$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1}) \dots \mathcal{M}(s_3|s_2) \cdot \mathcal{M}(s_2|s_1) \cdot \underbrace{\mathcal{M}(s_1|s_0)}_{\text{from } s_0 \text{ to } s_1}$$



(see **Transverse Linear Beam Dynamics lectures**)

- Rewrite the differential equation of the harmonic oscillator as a **pair of coupled 1st order equations**

$$\frac{du(t)}{dt} = p_u(t)$$

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- Rewrite the differential equation of the harmonic oscillator as a **pair of coupled 1st order equations**

$$\frac{du(t)}{dt} = p_u(t) \quad \text{which can be combined to provide}$$

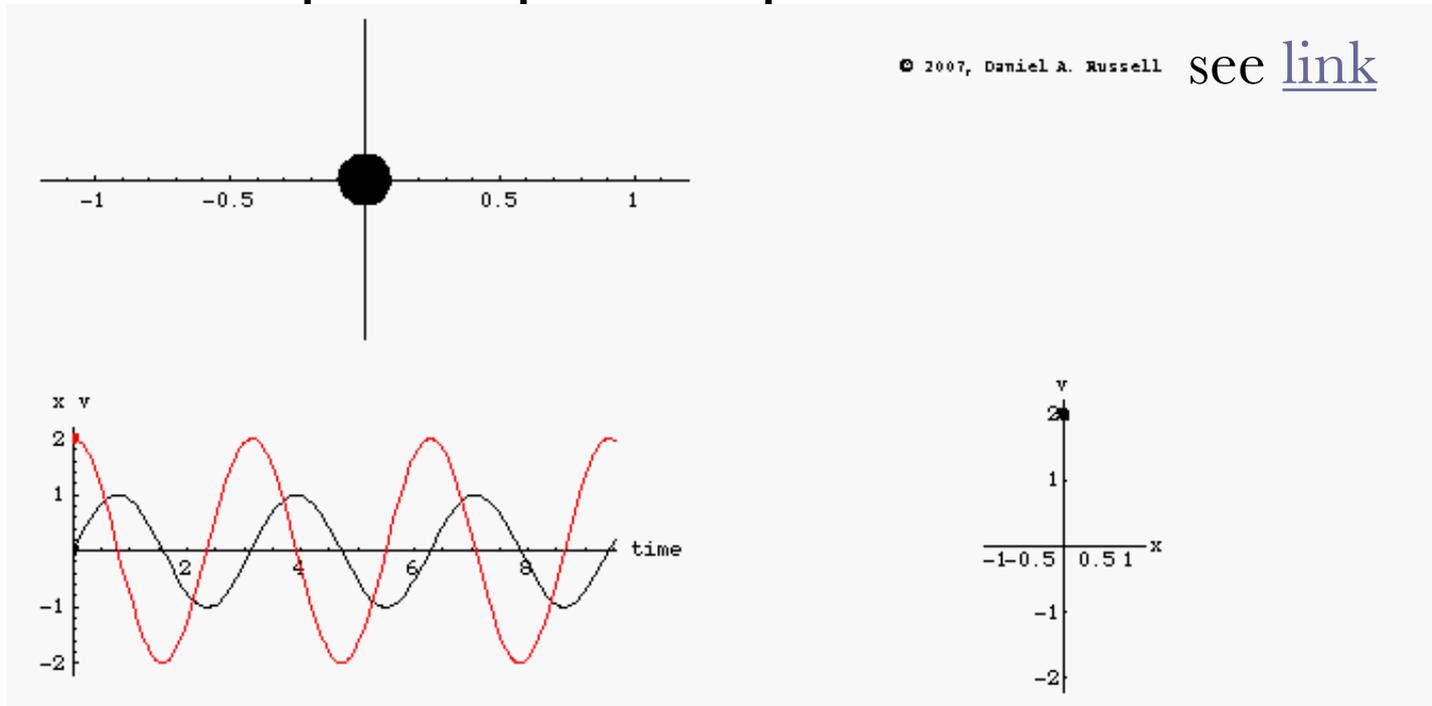
$$\frac{dp_u(t)}{dt} = -\omega_0^2 u(t)$$

$$\frac{dp_u}{dt} p_u + \omega_0^2 u \frac{du}{dt} = \frac{1}{2} \frac{d}{dt} (p_u^2 + \omega_0^2 u^2) = 0 \quad \text{or}$$

$$\frac{1}{2} (p_u^2 + \omega_0^2 u^2) = I_1 \quad \text{with } I_1 \text{ an **integral of motion**}$$

identified as the **mechanical energy** of the system

- The equation $\frac{1}{2} (p_u^2 + \omega_0^2 u^2) = I_1$ describes in general an ellipse in phase space



- Solving the previous equation for p_u , the system can be reduced to a first order equation

$$\frac{du}{dt} = \sqrt{2I_1 - \omega_0^2 u^2}$$

- The last equation can be solved as an explicit integral or “**quadrature**”

$$\int dt = \int \frac{du}{\sqrt{2I_1 - \omega_0^2 u^2}}, \text{ yielding } t + I_2 = \frac{1}{\omega_0} \arcsin \left(\frac{u\omega_0}{\sqrt{2I_1}} \right)$$

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- **Note:** Although the previous route may seem complicated, it becomes more natural when **non-linear** terms appear, where an **ansatz** of the type $u(t) = e^{\lambda t}$ is **not applicable**
- The ability to integrate a differential equation is not just a nice mathematical feature, but deeply characterizes the **dynamical behavior** of the system described by the equation

- The **period** of the harmonic oscillator is calculated through the previous integral after integration between two extrema (when the velocity $\frac{du}{dt} = \sqrt{2I_1 - \omega_0^2 u^2}$ vanishes), i.e. $u_{\text{ext}} = \pm \frac{\sqrt{2I_1}}{\omega_0}$:

$$T = 2 \int_{-\frac{\sqrt{2I_1}}{\omega_0}}^{\frac{\sqrt{2I_1}}{\omega_0}} \frac{du}{\sqrt{2I_1 - \omega_0^2 u^2}} = \frac{2\pi}{\omega_0}$$

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- The **period** (or the **frequency**) of linear systems is **independent** of the **integral of motion** (energy)
- Note that this is not true for non-linear systems, e.g. for an oscillator with a **non-linear restoring force** $\frac{d^2u}{dt^2} + k u(t)^3 = 0$

- The integral of motion is $I_1 = \frac{1}{2}p_u^2 + \frac{1}{4}k u^4$ and the

integration yields $T = 2 \int_{-(4I_1/k)^{1/4}}^{(4I_1/k)^{1/4}} \frac{du}{\sqrt{2I_1 - \frac{1}{2}k u^4}} = \sqrt{\frac{1}{2\pi}} \Gamma^2\left(\frac{1}{4}\right) (I_1 k)^{-1/4}$

- This means that the **period** (frequency) **depends** on the **integral of motion** (energy) i.e. the maximum “**amplitude**”

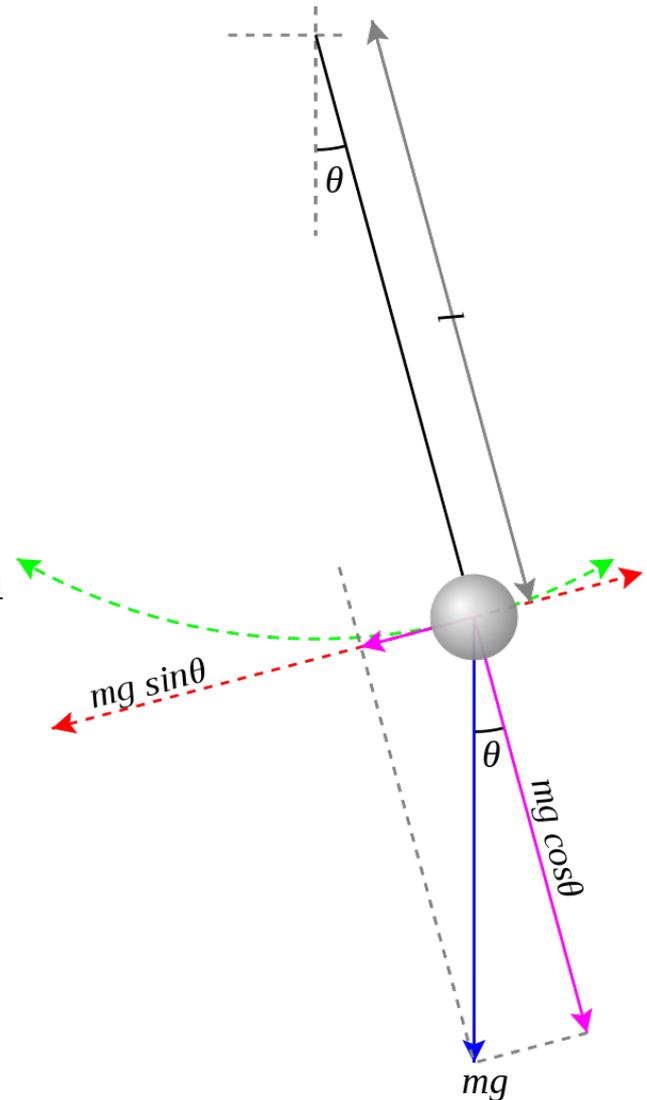
- An important non-linear equation which can be integrated is the one of the **pendulum**, for a string of length L and gravitational constant g

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin\theta = 0$$

- For small displacements it reduces to a **harmonic oscillator** with frequency

$$\omega_0 = \sqrt{\frac{g}{L}}$$

- By appropriate substitutions, this becomes the equation of **synchrotron motion** (see **Longitudinal BD lectures**)



- The **integral of motion** (scaled energy) is

$$\frac{1}{2} \left(\frac{d\theta}{dt} \right)^2 - \frac{g}{L} \cos \theta = I_1 = E'$$

and the quadrature is written as $t = \int_0^\theta \frac{d\theta}{\sqrt{2(I_1 + \frac{g}{L} \cos \theta)}}$
 assuming that for $t = 0$, $\theta_0 = \theta(0) = 0$

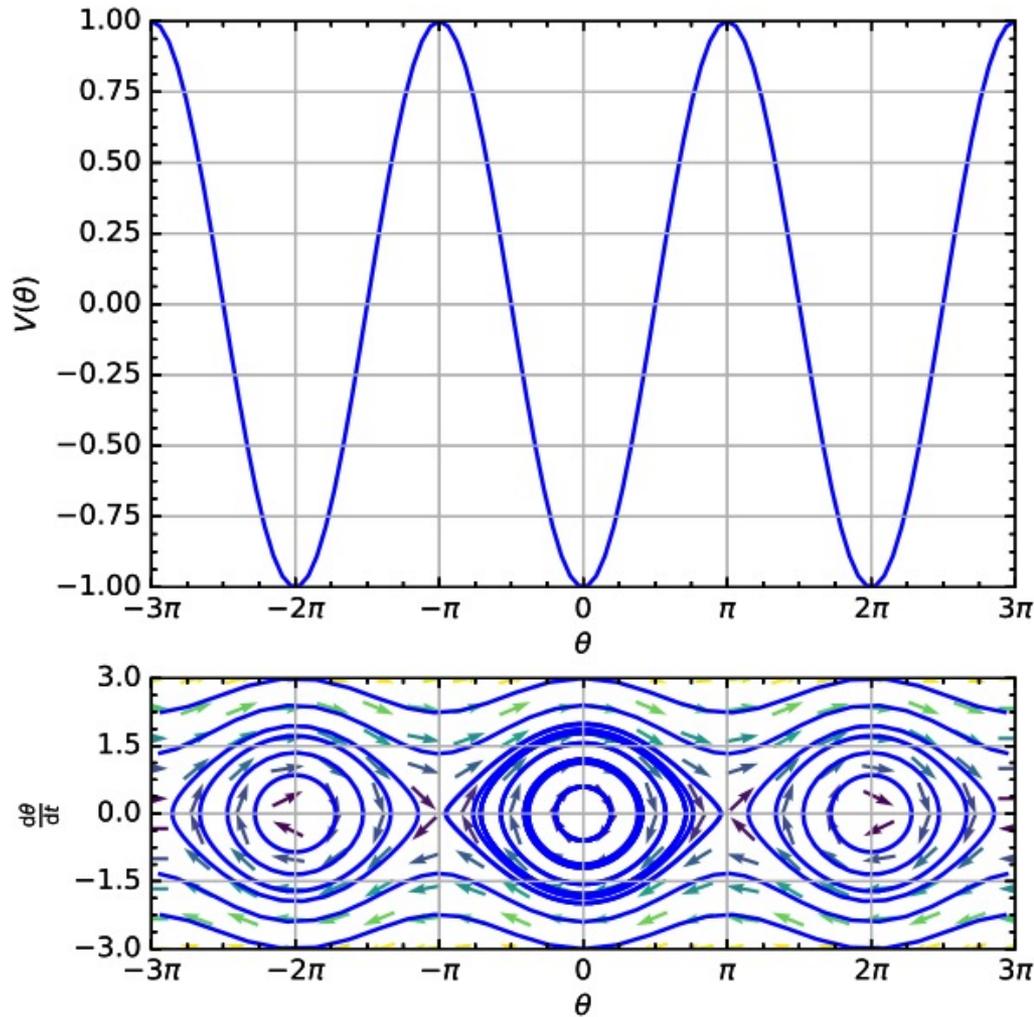
- Using the substitutions $\cos \theta = 1 - 2k^2 \sin^2 \phi$ with

$k = \sqrt{1/2(1 + I_1 L/g)}$, the integral is

$$t = \sqrt{\frac{L}{g}} \int_0^\theta \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad \text{and can be solved using}$$

Jacobi elliptic functions: $\theta(t) = 2 \arcsin \left[k \operatorname{sn} \left(t \sqrt{\frac{g}{L}}, k \right) \right]$

with “sn” representing the **Jacobi elliptic sine**



- Minima and maxima of the potential correspond to stable and unstable fixed points

- For recovering the **period**, the integration is performed between the two extrema, i.e. $\theta = 0$ and $\theta = \arccos(-I_1 L/g)$, corresponding to $\phi = 0$ and $\phi = \pi/2$

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$$T = 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = 4 \sqrt{\frac{L}{g}} \mathcal{K}(k)$$

i.e. the **complete elliptic integral** multiplied by four times the period of the harmonic oscillator

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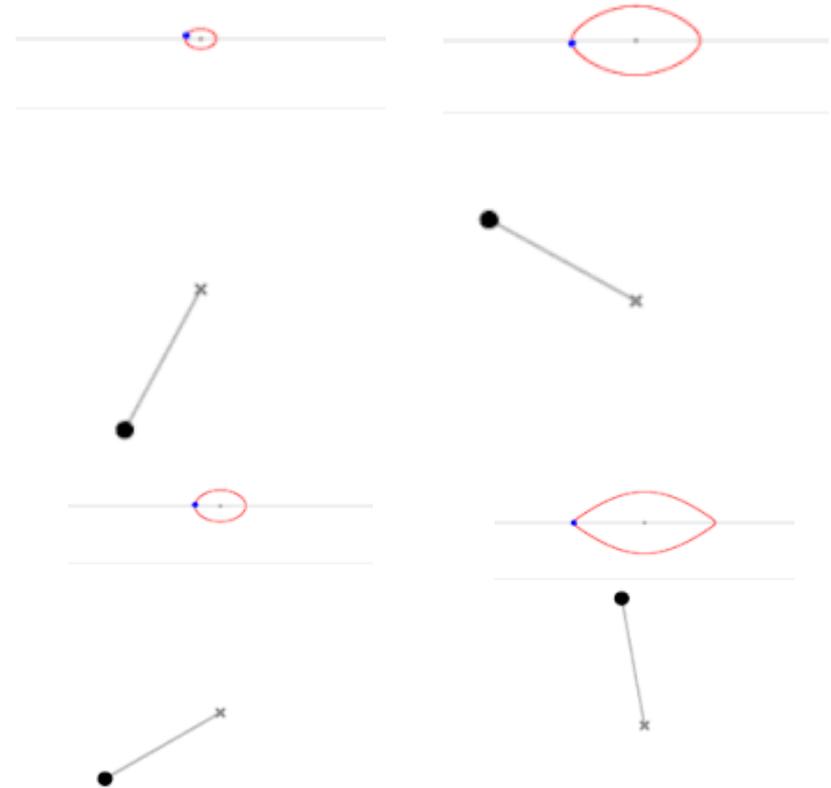
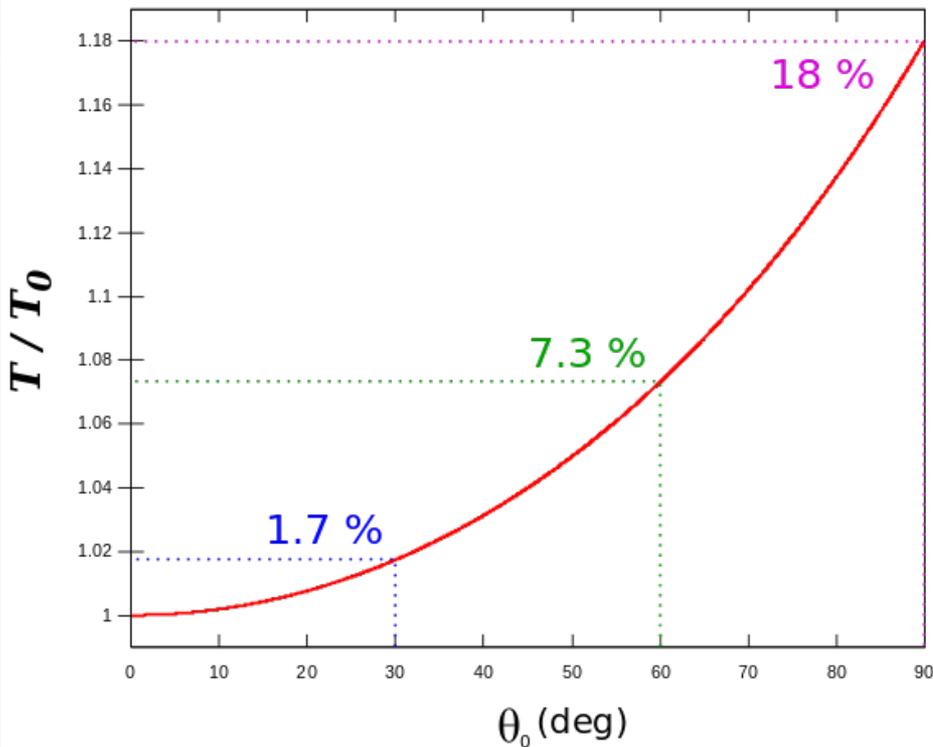
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- By expanding
$$\mathcal{K}(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n} (n!)^2} \right)^2 k^{2n} = \frac{\pi}{2} \left(1 + \frac{1}{4} k^2 + \dots \right)$$

with $k = \sqrt{1/2(1 + I_1 L/g)}$, the **“amplitude”**

dependence of the frequency becomes apparent

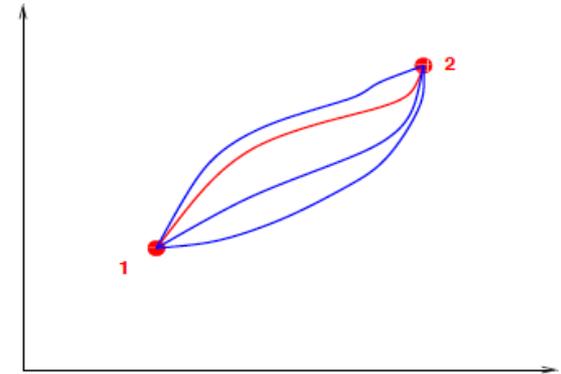
- The deviation from the linear approximation becomes important at **large amplitudes**
- The dependence of frequency with amplitude (**spread**) is useful for **damping instabilities**



Langrangian and Hamiltonian

- Describe motion of particles in q_n coordinates (n degrees of freedom) from time t_1 to time t_2
- It can be achieved by the **Lagrangian function** $L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$ with (q_1, \dots, q_n) the **generalized coordinates** and $(\dot{q}_1, \dots, \dot{q}_n)$ the **generalized velocities**

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- ❑ The Lagrangian is defined as $L = T - V$, i.e. difference between **kinetic** and **potential** energy
- ❑ The integral $W = \int L(q_i, \dot{q}_i, t) dt$ defines the **action**
- ❑ **Hamilton's principle**: system evolves so as the action becomes extremum (principle of **stationary action**)



□ By using **Hamilton's principle**, i.e. $\delta W = 0$, over some time interval t_1 and t_2 for two stationary points $\delta q(t_1) = \delta q(t_2) = 0$ (see appendix), the following differential equations for each degree of freedom are obtained, the **Euler-Lagrange equations**

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

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- In other words, by knowing the form of the Lagrangian, the **equations of motion** can be **derived**

- For a simple **force law** contained in a potential function, governing motion among interacting particles, the Lagrangian is (or as Landau-Lifshitz put it “experience has shown that...”)

$$L = T - V = \sum_{i=1}^n \frac{1}{2} m_i \dot{q}_i^2 - V(q_1, \dots, q_n)$$

- For velocity independent potentials, Lagrange equations become

$$m_i \ddot{q}_i = - \frac{\partial V}{\partial q_i} ,$$

i.e. **Newton's equations.**

- ❑ Some **disadvantages** of the Lagrangian formalism:
 - ❑ **No uniqueness:** different Lagrangians can lead to same equations
 - ❑ **Physical significance** not straightforward (even its basic form given more by “experience” and the fact that it actually works that way!)
 - ❑ **Note:** Lagrangian is very useful in particle physics (invariant under Lorentz transformations)

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 - ❑ **Note:** Lagrangian is very useful in particle physics (invariant under Lorentz transformations)
- ❑ Lagrangian function provides in general n second order differential equations (**coordinate space**)
- ❑ Already observed advantage to move to system of $2n$ first order differential equations, which are more straightforward to solve (**phase space**)
- ❑ Derived by the **Hamiltonian** of the system

- The **Hamiltonian** of the system is defined as the **Legendre transformation** of the Lagrangian

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_i \dot{q}_i p_i - L(\mathbf{q}, \dot{\mathbf{q}}, t)$$

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- **Example:** consider $L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_i m_i \dot{q}_i^2 - V(q_1, \dots, q_n)$

- From this, the momentum can be determined as $p_i = \frac{\partial L}{\partial \dot{q}_i} = m_i \dot{q}_i$

which can be trivially inverted to provide the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \sum_i \frac{p_i^2}{2m_i} + V(q_1, \dots, q_n)$$

□ The **equations of motion** can be derived from the Hamiltonian following the same variational principle as for the Lagrangian (“stationary” action) but also by simply taking the differential of the Hamiltonian (see appendix)

$$\dot{q}_i = \frac{\partial H}{\partial p_i} , \quad \dot{p}_i = -\frac{\partial H}{\partial q} , \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

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- These are indeed $2n + 2$ equations describing the motion in the “**extended**” phase space $(q_1, \dots, q_n, p_1, \dots, p_n, t, -H)$

- ❑ The variables $(q_1, \dots, q_n, p_1, \dots, p_n, t, -H)$ are called **canonically conjugate** (or canonical) and define the evolution of the system in **phase space**
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- ❑ These variables have the special property that they preserve volume in phase space, i.e. satisfy the well-known **Liouville's theorem**
- ❑ The variables used in the **Lagrangian do not necessarily have this property**
- ❑ Hamilton's equations can be written in **vector form**
 $\dot{\mathbf{z}} = \mathbf{J} \cdot \nabla H(\mathbf{z})$ with $\mathbf{z} = (q_1, \dots, q_n, p_1, \dots, p_n)$
and $\nabla = (\partial q_1, \dots, \partial q_n, \partial p_1, \dots, \partial p_n)$
- ❑ The $2n \times 2n$ matrix $\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$ is called the **symplectic matrix**

- ❑ Crucial step in study of Hamiltonian systems is identification of **integrals of motion**
- ❑ Consider a **time dependent function** of phase space. Its time evolution is given by

$$\begin{aligned} \frac{d}{dt} f(\mathbf{p}, \mathbf{q}, t) &= \sum_{i=1}^n \left(\frac{dq_i}{dt} \frac{\partial f}{\partial q_i} + \frac{dp_i}{dt} \frac{\partial f}{\partial p_i} \right) + \frac{\partial f}{\partial t} \\ &= \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} \right) + \frac{\partial f}{\partial t} = [H, f] + \frac{\partial f}{\partial t} \end{aligned}$$

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- ❑ If a quantity is explicitly **time-independent** and its Poisson bracket with the Hamiltonian vanishes (i.e. **commutes** with the H), it is a **constant** (or **integral**) of motion (as an **autonomous** Hamiltonian itself)

- 2nd order dif. equations of motion from Newton's law (**configuration space**) can be solved by **transforming** them to pairs of 1st order ones (in **phase space**)
- Natural appearance of **invariant** of motion (“**energy**”)
- Non-linear oscillators have **frequencies** which **depend** on the **invariant** (or “**amplitude**”)
- Connected invariant of motion to system's **Hamiltonian** (derived through **Lagrangian**)
- Shown that through the **Hamiltonian**, the **equations of motions** can be **derived**
- **Poisson bracket** operators are helpful for discovering integrals of motion

- The variation of the action can be written as

$$\delta W = \int_{t_1}^{t_2} (L(q + \delta q, \dot{q} + \delta \dot{q}, t) - L(q, \dot{q}, t)) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt$$

- Taking into account that $\delta \dot{q} = \frac{d\delta q}{dt}$, the 2nd part of the integral can be integrated by parts giving

$$\delta W = \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q dt = 0$$

- The first term is zero because $\delta q(t_1) = \delta q(t_2) = 0$ so the second integrand should also vanish, providing the following differential equations for each degree of freedom, the **Lagrange equations**

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

- The **equations of motion** can be derived from the Hamiltonian following the same variational principle as for the Lagrangian (“least” action) but also by simply taking the differential of the Hamiltonian

$$dH = \sum_i p_i d\dot{q}_i + \dot{q}_i dp_i - \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} d\dot{q}_i - \underbrace{\frac{\partial L}{\partial q_i}}_{\dot{p}_i} dq_i - \frac{\partial L}{\partial t} dt$$

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or

$$dH(q, p, t) = \sum_i \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt$$

- By equating terms, **Hamilton's equations** are derived

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

- These are indeed $2n + 2$ equations describing the motion in the “**extended**” phase space $(q_1, \dots, q_n, p_1, \dots, p_n, t, -H)$

□ The Poisson brackets between two functions of a set of canonical variables can be defined by the differential **operator**

$$[f, g] = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} \right)$$

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- From this definition, and for any three given functions, the following **properties** can be shown

$$[af + bg, h] = a[f, h] + b[g, h], \quad a, b \in \mathbb{R} \quad \text{bilinearity}$$

$$[f, g] = -[g, f] \quad \text{anticommutativity}$$

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0 \quad \text{Jacobi's identity}$$

$$[f, gh] = [f, g]h + g[f, h] \quad \text{Leibniz's rule}$$

- Poisson brackets operation satisfies a **Lie algebra**