

# Particle motion in Hamiltonian Formalism II

## Or how to derive and solve equations of motion

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- 2<sup>nd</sup> order dif. equations of motion from Newton's law (**configuration space**) can be solved by **transforming** them to pairs of 1<sup>st</sup> order ones (in **phase space**)
- Natural appearance of **invariant** of motion ("**energy**")
- Non-linear oscillators have **frequencies** which **depend** on the **invariant** (or "**amplitude**")
- Connected invariant of motion to system's **Hamiltonian** (derived through **Lagrangian**)
- Shown that through the **Hamiltonian**, the **equations of motions** can be **derived**
- **Poisson bracket** operators are helpful for discovering integrals of motion

# Canonical transformations

- ❑ Find a **function** for transforming the Hamiltonian from variable  $(q, p)$  to  $(Q, P)$ , so system becomes **simpler** to study
- ❑ Transformation should be **canonical** (or **symplectic**), so that **Hamiltonian** properties (**phase-space volume**) are preserved

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- These “mixed variable” **generating** functions are derived by

$$F_1(\mathbf{q}, \mathbf{Q}) : p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i} \quad F_3(\mathbf{Q}, \mathbf{p}) : q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$F_2(\mathbf{q}, \mathbf{P}) : p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i} \quad F_4(\mathbf{p}, \mathbf{P}) : q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i}$$

- A general **non-autonomous Hamiltonian** is transformed to

$$H(\mathbf{Q}, \mathbf{P}, t) = H(\mathbf{q}, \mathbf{p}, t) + \frac{\partial F_j}{\partial t}, \quad j = 1, 2, 3, 4$$

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- ❑ One generating function can be constructed by the other through **Legendre transformations**, e.g.

$$F_2(\mathbf{q}, \mathbf{P}) = F_1(\mathbf{q}, \mathbf{Q}) - \mathbf{Q} \cdot \mathbf{P}, \quad F_3(\mathbf{Q}, \mathbf{p}) = F_1(\mathbf{q}, \mathbf{Q}) - \mathbf{q} \cdot \mathbf{p}, \quad \dots$$

with the inner product define as  $\mathbf{q} \cdot \mathbf{p} = \sum_i q_i p_i$

- A fundamental property of canonical transformations is the **preservation of phase space volume**
- This **volume** preservation in phase space can be represented in the **old** and **new variables** as

$$\int \prod_{i=1}^n dp_i dq_i = \int \prod_{i=1}^n dP_i dQ_i$$



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- The volume element in old and new variables are related through the **Jacobian**

$$\prod_{i=1}^n dp_i dq_i = \frac{\partial(P_1, \dots, P_n, Q_1, \dots, Q_n)}{\partial(p_1, \dots, p_n, q_1, \dots, q_n)} \prod_{i=1}^n dP_i dQ_i$$

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- These two relationships imply that the **Jacobian** of a **canonical transformation** should have **determinant** equal to **1**

$$\left| \frac{\partial(P_1, \dots, P_n, Q_1, \dots, Q_n)}{\partial(p_1, \dots, p_n, q_1, \dots, q_n)} \right| = \left| \frac{\partial(p_1, \dots, p_n, q_1, \dots, q_n)}{\partial(P_1, \dots, P_n, Q_1, \dots, Q_n)} \right| = 1$$

- The transformation  $Q = -p$ ,  $P = q$ , which **interchanges conjugate variables** is area preserving, as the Jacobian is

$$\frac{\partial(P,Q)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

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- On the other hand, the transformation from **Cartesian to polar** coordinates  $q = P \cos Q$  ,  $p = P \sin Q$  is not, since

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- There are actually “**polar**” coordinates that are **canonical**, given by  $q = -\sqrt{2P} \cos Q$  ,  $p = \sqrt{2P} \sin Q$  for which

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} \sqrt{2P} \sin Q & \sqrt{2P} \cos Q \\ -\frac{\cos Q}{\sqrt{2P}} & \frac{\sin Q}{\sqrt{2P}} \end{vmatrix} = 1$$

# The Relativistic Hamiltonian for electromagnetic fields

- ❑ Neglecting self fields and radiation, motion can be described by a “single-particle” Hamiltonian

$$H(\mathbf{x}, \mathbf{p}, t) = c\sqrt{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t)\right)^2 + m^2c^2} + e\Phi(\mathbf{x}, t)$$

- ❑  $\mathbf{x} = (x, y, z)$  Cartesian positions
- ❑  $\mathbf{p} = (p_x, p_y, p_z)$  conjugate momenta
- ❑  $\mathbf{A} = (A_x, A_y, A_z)$  magnetic vector potential
- ❑  $\Phi$  electric scalar potential

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- $\Phi$  electric scalar potential

- The ordinary kinetic momentum vector is written

$$\mathbf{P} = \gamma m \mathbf{v} = \mathbf{p} - \frac{e}{c} \mathbf{A}$$

with  $\mathbf{v}$  the velocity vector and  $\gamma = (1 - v^2/c^2)^{-1/2}$  the relativistic factor



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- It is generally a **3 degrees of freedom** one plus time (i.e., **4 degrees of freedom**)
- The Hamiltonian represents the **total energy**

$$H \equiv E = \gamma mc^2 + e\Phi$$

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- The **total kinetic momentum** is

$$P = \left(\frac{H^2}{c^2} - m^2c^2\right)^{1/2}$$

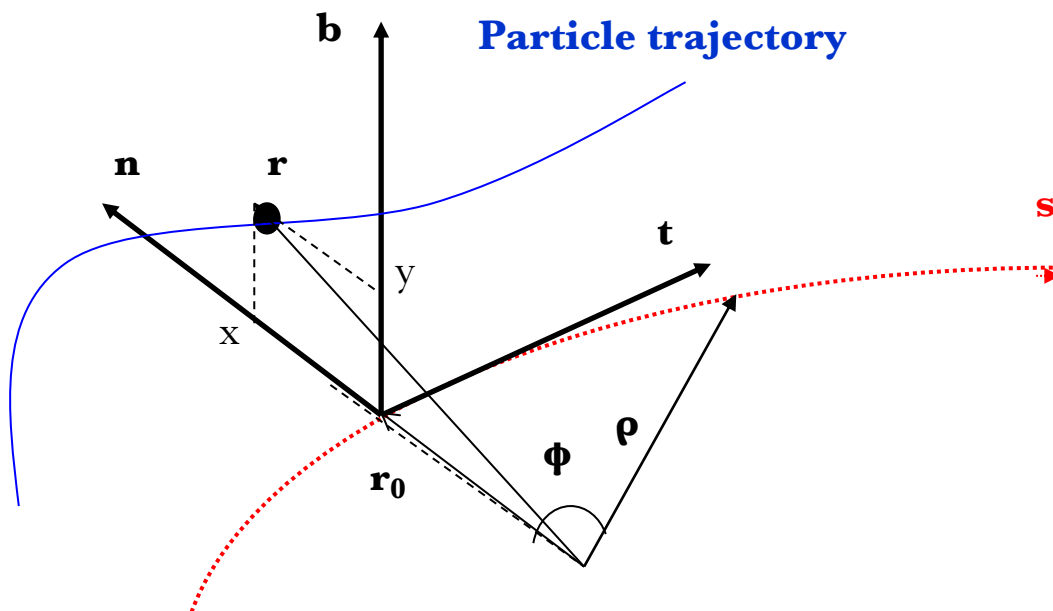
- Using **Hamilton's equations**

$$(\dot{\mathbf{x}}, \dot{\mathbf{p}}) = [(\mathbf{x}, \mathbf{p}), H]$$

it can be shown that motion is governed by **Lorentz equations**

# The Accelerator ring Hamiltonian

- Summary of **canonical transformations** and **approximations** for simplifying Hamiltonian
  - From **Cartesian** to **Frenet-Serret** (rotating) coordinate system (bending in the horizontal plane), useful for **rings**



$$(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{Q}, \mathbf{P}) \text{ or}$$

$$(x, y, z, p_x, p_y, p_z) \mapsto (X, Y, s, P_x, P_y, P_s)$$

- Summary of **canonical transformations** and **approximations** for simplifying Hamiltonian
  - From **Cartesian** to **Frenet-Serret** (rotating) coordinate system (bending in the horizontal plane), useful for **rings**
  - Changing the **independent variable** from time  $t$  to the **path length**  $s$
  - The Hamiltonian can be considered as having **4 degrees of freedom**, where the 4<sup>th</sup> “**position**” is **time** with conjugate momentum  $P_t = -\mathcal{H}$  or  $P_s = -\mathcal{H}$

**Coordinate transformations**

## □ Summary of **canonical transformations** and **approximations** for simplifying Hamiltonian

- From **Cartesian** to **Frenet-Serret** (rotating) coordinate system (bending in the horizontal plane), useful for **rings**
- Changing the **independent variable** from time  $t$  to the **path length**  $s$
- **Electric field** set to **zero**, as **longitudinal** (synchrotron) motion is much **slower** than **transverse** (betatron) one
- Consider **static** and **transverse** magnetic fields

**Coordinate transformations**

**Field approximations**

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□ Consider **static** and **transverse** magnetic fields

□ **Rescale** the momentum with the reference one and move the **origin** to the **periodic orbit**

□ For the **ultra-relativistic limit**  $\beta_0 \rightarrow 1$  ,  $\frac{1}{\beta_0^2 \gamma^2} \rightarrow 0$  the Hamiltonian becomes

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = (1 + \delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(l)}\right) \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2}$$

with  $l = -ct + \frac{s - s_0}{\beta_0}$  and  $\frac{P_t - P_0}{P_0} \equiv \delta$

**Coordinate transformations**

**Field approximations**

- ❑ It is useful for study purposes (especially for finding an “integrable” version of the Hamiltonian) to make an extra **approximation**
- ❑ For this, **transverse momenta** (rescaled to the reference momentum) are considered to be **much smaller than 1**, i.e. the square root can be expanded.



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- ❑ Considering also the large machine approximation  $x \ll \rho$ , (dropping cubic terms), the Hamiltonian is simplified to
 
$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1 + \delta)} - \frac{x(1 + \delta)}{\rho(s)} - e\hat{A}_s$$
- ❑ This expansion may **not** be a **good idea**, especially for **low energy, small size rings**

- Considering the **general expression** of the the **longitudinal component** of the **vector potential** is (see appendix)

- In curvilinear coordinates (curved elements)

$$A_s = \left(1 + \frac{x}{\rho(s)}\right) B_0 \Re e \sum_{n=0}^{\infty} \frac{b_n + ia_n}{n+1} (x + iy)^{n+1}$$

- In Cartesian coordinates  $A_s = B_0 \Re e \sum_{n=0}^{\infty} \frac{b_n + ia_n}{n+1} (x + iy)^{n+1}$

with the **multipole coefficients** being written as

$$a_n = \frac{1}{B_0 n!} \frac{\partial^n B_x}{\partial x^n} \Big|_{x=y=0} \quad \text{and} \quad b_n = \frac{1}{B_0 n!} \frac{\partial^n B_y}{\partial x^n} \Big|_{x=y=0}$$

- The **general non-linear Hamiltonian** can be written as

$$\mathcal{H}(x, y, p_x, p_y, s) = \mathcal{H}_0(x, y, p_x, p_y, s) + \sum_{k_x, k_y} h_{k_x, k_y}(s) x^{k_x} y^{k_y}$$

with the **periodic functions**  $h_{k_x, k_y}(s) = h_{k_x, k_y}(s + C)$

■ Dipole:

$$H = \frac{x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1 + \delta)}$$

■ Quadrupole:

$$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1 + \delta)}$$

■ Sextupole:

$$H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1 + \delta)}$$

■ Octupole:

$$H = \frac{1}{4}k_3(x^4 - 6x^2y^2 + y^4) + \frac{p_x^2 + p_y^2}{2(1 + \delta)}$$

# Linear magnetic fields

■ Assume a simple case of **linear transverse magnetic fields**,

$$B_x = b_1(s)y$$

$$B_y = -b_0(s) + b_1(s)x \quad ,$$

- main bending field
- normalized quadrupole gradient
- magnetic rigidity

$$-B_0 \equiv b_0(s) = \frac{P_0 c}{e \rho(s)} \quad [\text{T}]$$

$$K(s) = b_1(s) \frac{e}{c P_0} = \frac{b_1(s)}{B \rho} \quad [1/\text{m}^2]$$

$$B \rho = \frac{P_0 c}{e} \quad [\text{T} \cdot \text{m}]$$

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$$B \rho = \frac{P_0 c}{e} \quad [\text{T} \cdot \text{m}]$$

- The vector potential has only a longitudinal component which in curvilinear coordinates is

$$B_x = -\frac{1}{1 + \frac{x}{\rho(s)}} \frac{\partial A_s}{\partial y} \quad , \quad B_y = \frac{1}{1 + \frac{x}{\rho(s)}} \frac{\partial A_s}{\partial x}$$

- The previous expressions can be integrated to give

$$A_s(x, y, s) = \frac{P_0 c}{e} \left[ -\frac{x}{\rho(s)} - \left( \frac{1}{\rho(s)^2} + K(s) \right) \frac{x^2}{2} + K(s) \frac{y^2}{2} \right] = P_0 c \hat{A}_s(x, y, s)$$

- The Hamiltonian for linear fields can be finally written as

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1+\delta)} - \frac{x\delta}{\rho(s)} + \frac{x^2}{2\rho(s)^2} + \frac{K(s)}{2} (x^2 - y^2)$$

- Hamilton's equations are
 
$$\frac{dx}{ds} = \frac{p_x}{1+\delta}, \quad \frac{dp_x}{ds} = \frac{\delta}{\rho(s)} - \left( \frac{1}{\rho^2(s)} + K(s) \right) x$$

$$\frac{dy}{ds} = \frac{p_y}{1+\delta}, \quad \frac{dp_y}{ds} = K(s)y$$

and they can be written as two second order uncoupled differential equations, i.e. **Hill's equations** (see **Transverse Dynamics lecture**)

$$x'' + \frac{1}{1+\delta} \left( \overbrace{\frac{1}{\rho(s)^2} + K(s)}^{K_x} \right) x = \frac{\delta}{\rho(s)} \quad \text{with the usual solution for } \delta = 0 \text{ and } u = x, y$$

$$y'' - \frac{1}{1+\delta} \underbrace{K(s)}_{K_y} y = 0$$

$$u(s) = \sqrt{\epsilon\beta(s)} \cos(\psi(s) + \psi_0)$$

$$u'(s) = \sqrt{\frac{\epsilon}{\beta(s)}} (\sin(\psi(s) + \psi_0) + \alpha(s) \cos(\psi(s) + \psi_0))$$

# Action-Angle Variables

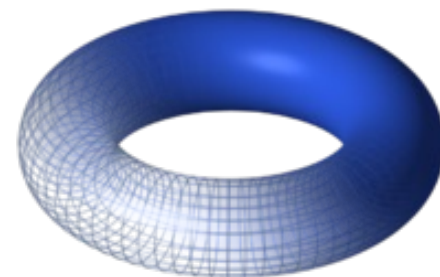


- There is a canonical transformation to some **optimal** set of variables which can simplify the phase-space motion
- This set of variables are the **action-angle** variables
- The action vector is defined as the integral  $\mathbf{J} = \oint \mathbf{p} d\mathbf{q}$  over closed paths in phase space.

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- An **integrable Hamiltonian** is written as a function of only the actions, i.e.  $H_0 = H_0(\mathbf{J})$ . Hamilton's equations give

$$\dot{\phi}_i = \frac{\partial H_0(\mathbf{J})}{\partial J_i} = \omega_i(\mathbf{J}) \Rightarrow \phi_i = \omega_i(\mathbf{J})t + \phi_{i0}$$

$$\dot{J}_i = -\frac{\partial H_0(\mathbf{J})}{\partial \phi_i} = 0 \Rightarrow J_i = \text{const.}$$



i.e. the **actions** are **integrals of motion** and the **angles** are **evolving linearly with time**, with **constant frequencies** which depend on the actions

- The actions define the surface of an **invariant torus**, topologically equivalent to the product of  $n$  circles

- Considering **on-momentum** motion, the Hamiltonian can be written as

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2} + \frac{K_x(s)x^2 - K_y(s)y^2}{2}$$

- The generating function from the original to action angle variables is

$$F_1(x, y, \phi_x, \phi_y; s) = -\frac{x^2}{2\beta_x(s)} [\tan \phi_x(s) + a_x(s)] - \frac{y^2}{2\beta_y(s)} [\tan \phi_y(s) + a_y(s)]$$

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- The **old variables** with respect to **actions and angles** are

$$u(s) = \sqrt{2\beta_u(s)J_u} \cos \phi_u(s), \quad p_u(s) = -\sqrt{\frac{2J_u}{\beta_u(s)}} (\sin \phi_u(s) + \alpha_u(s) \cos \phi_u(s))$$

and the Hamiltonian takes the form

$$\mathcal{H}_0(J_x, J_y, s) = \frac{J_x}{\beta_x(s)} + \frac{J_y}{\beta_y(s)}$$

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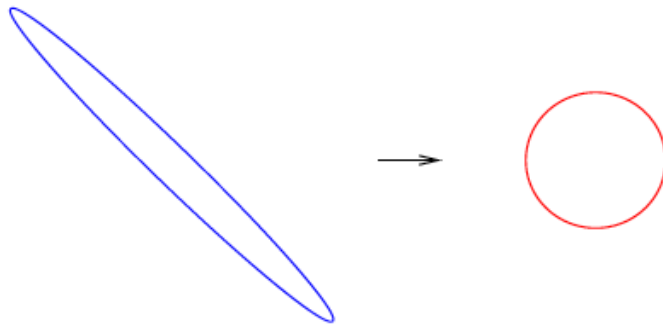
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- The “time” (longitudinal position) dependence can be eliminated by the transformation to **normalized coordinate**

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \cos(\nu\phi) \\ \sin(\nu\phi) \end{pmatrix} \quad \text{with} \quad \nu = \frac{1}{2\pi} \oint \frac{du}{\beta(s)}$$

- Make a coordinate transformation so that we get a simpler form of the matrix, i.e. **ellipses** are transformed to circles (simple rotation)



$$M = \mathcal{A} \circ \mathcal{R} \circ \mathcal{A}^{-1} \quad \text{or :} \quad \mathcal{R} = \mathcal{A}^{-1} \circ M \circ \mathcal{A}$$

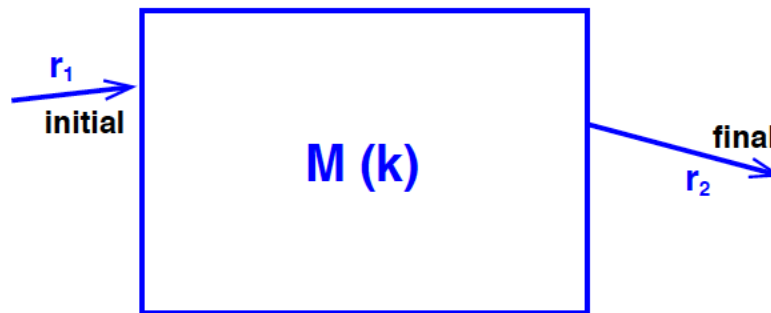
- Using linear algebra, the solution is

$$\mathcal{A} = \begin{pmatrix} \sqrt{\beta(s_0)} & 0 \\ -\frac{\alpha(s_0)}{\sqrt{\beta(s_0)}} & \frac{1}{\sqrt{\beta(s_0)}} \end{pmatrix} \quad \text{and} \quad \mathcal{R} = \begin{pmatrix} \cos(\mu_x) & \sin(\mu_x) \\ -\sin(\mu_x) & \cos(\mu_x) \end{pmatrix}$$

- This transformation can be extended to a **non-linear system** (see **Advanced** course)

# Symplectic maps

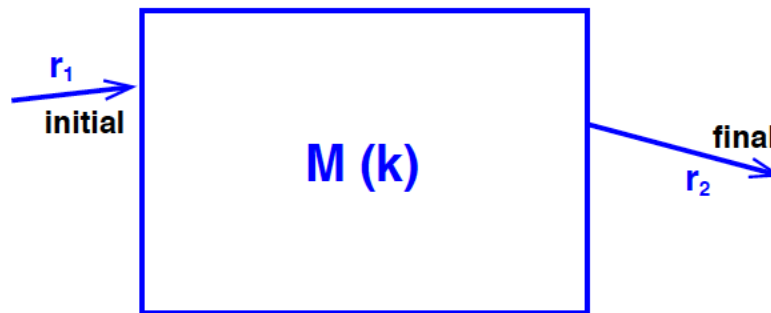
- A generalization of the matrix (which can only describe linear systems), is a **map**, which transforms a system from some initial to some final coordinates



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- Analyzing the map, will give useful information about the behavior of the system
- There are different ways to build the map:
  - Taylor (Power) maps
  - Lie transformations
  - Truncated Power Series Algebra (TPSA), can generate maps from straight-forward tracking
- Preservation of **symplecticity** is important

- Consider two sets of canonical variables  $\mathbf{z}$  ,  $\bar{\mathbf{z}}$  which may be even considered as the evolution of the system between two points in phase space
- A transformation from the one to the other set can be constructed through a **map**  $\mathcal{M} : \mathbf{z} \mapsto \bar{\mathbf{z}}$

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- The **Jacobian matrix** of the map  $M = M(\mathbf{z}, t)$  is composed by the elements  $M_{ij} \equiv \frac{\partial \bar{z}_i}{\partial z_j}$
- The map is **symplectic** if  $M^T J M = J$  where  $J = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$
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- It can be shown that  $\det(M) = 1$
- It can be shown that the variables defined through a symplectic map  $[\bar{z}_i, \bar{z}_j] = [z_i, z_j] = J_{ij}$  which is a known relation satisfied by canonical variables
- In other words, symplectic maps **preserve** Poisson brackets

- **Symplecticity** guarantees that the **transformations** in phase space are **area preserving**
- To understand what deviation from symplecticity produces consider the simple case of the **quadrupole** with the general matrix written as

$$\mathcal{M}_Q = \begin{pmatrix} \cos(\sqrt{k}L) & \frac{1}{\sqrt{k}} \sin(\sqrt{k}L) \\ -\sqrt{k} \sin(\sqrt{k}L) & \cos(\sqrt{k}L) \end{pmatrix}$$

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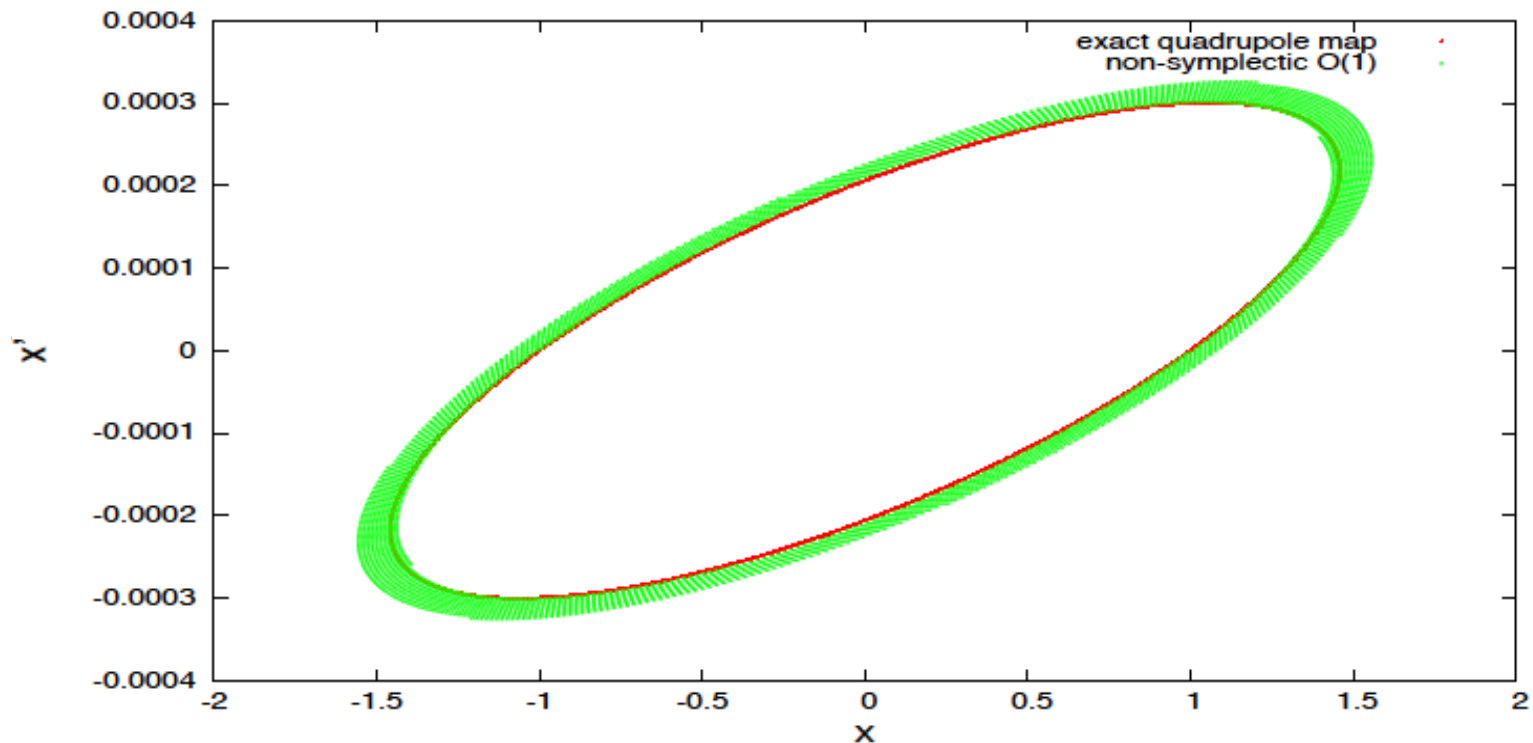
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- Take the Taylor expansion for small lengths, up to first order

$$\mathcal{M}_Q = \begin{pmatrix} 1 & L \\ -kL & 1 \end{pmatrix} + O(L^2)$$

- This is indeed **not symplectic** as the determinant of the matrix is equal to  $1 + kL^2$ , i.e. there is a deviation from symplecticity at 2<sup>nd</sup> order in the quadrupole length

- The iterated non-symplectic matrix does not provide the well-known elliptic trajectory in phase space
- Although the trajectory is very close to the original one, it **spirals outwards towards infinity**



- **Canonical** (or **symplectic**) transformations are necessary for preserving the phase space-volume of
- Starting point relativistic Hamiltonian of particles in E/M fields, and a series of canonical transformations and approximations, the **accelerator ring Hamiltonian** can be derived
- Imposing **linear magnetic fields** in the accelerator Hamiltonian, Hamilton's equations provide the usual **Hill's equation**
- The linear (uncoupled) magnetic field Hamiltonian can be **simplified** through transformation in **action-angle** variables (only function of the actions)
- **Symplectic maps** are essential for preserving the **correct physical time evolution** of linear or non-linear systems





- A fundamental property of Hamiltonian systems is the **preservation of phase space volume** as they evolve
- Let's have a system evolving from  $(p_i q_i) \rightarrow (p'_i q'_i)$  after time  $\delta t$ . By Taylor-expanding and using Hamilton's equations we have:

$$q'_i = q_i(t + \delta t) = q_i(t) + \frac{dq_i}{dt} \delta t + O(\delta t^2) = q_i(t) - \frac{\partial H}{\partial p_i} \delta t + O(\delta t^2)$$

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- Differentiating, we have

$$dq'_i = dq_i(t) - \frac{\partial}{\partial q_i} \left( \frac{\partial H}{\partial p_i} \right) dq_i \delta t + O(\delta t^2)$$

$$dp'_i = dp_i(t) + \frac{\partial}{\partial p_i} \left( \frac{\partial H}{\partial q_i} \right) dp_i \delta t + O(\delta t^2)$$

- Multiplying the two equations

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- Multiplying the two equations

$$dq'_i dp'_i = dq_i dp_i \left[ 1 - \frac{\partial}{\partial q_i} \left( \frac{\partial H}{\partial p_i} \right) + \frac{\partial}{\partial p_i} \left( \frac{\partial H}{\partial q_i} \right) \right] \delta t + O(\delta t^2) \approx dq_i dp_i$$

- From Gauss law of magnetostatics, a vector potential exist

$$\nabla \cdot \mathbf{B} = 0 \rightarrow \exists \mathbf{A} : \mathbf{B} = \nabla \times \mathbf{A}$$

- Assuming transverse 2D field, vector potential has only one component  $A_s$ . The Ampere's law in vacuum (inside the beam pipe)  $\nabla \times \mathbf{B} = 0 \rightarrow \exists V : \mathbf{B} = -\nabla V$

- Using the previous equations, the relations between field components and potentials are

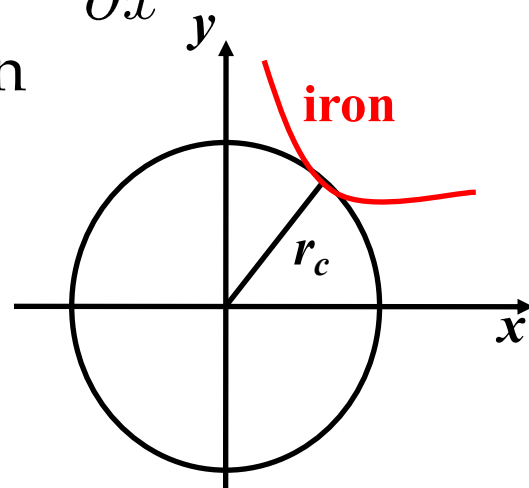
$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y}, \quad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

i.e. Riemann conditions of an analytic function



Exists complex potential of  $z = x + iy$  with power series expansion convergent in a circle with radius  $|z| = r_c$  (distance from iron yoke)

$$\mathcal{A}(x + iy) = A_s(x, y) + iV(x, y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x + iy)^n$$



- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x, y) + iV(x, y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x + iy)^{n-1}$$

- Setting  $b_n = -n\lambda_n$ ,  $a_n = n\mu_n$

$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n - ia_n)(x + iy)^{n-1}$$

- Define normalized coefficients

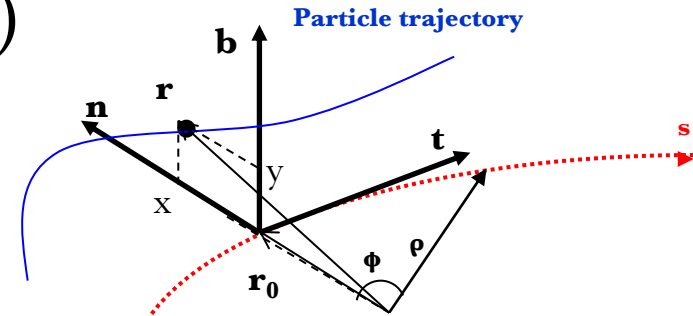
$$b'_n = \frac{b_n}{10^{-4}B_0}r_0^{n-1}, \quad a'_n = \frac{a_n}{10^{-4}B_0}r_0^{n-1}$$

on a reference radius  $r_0$ ,  $10^{-4}$  of the main field to get

$$B_y + iB_x = 10^{-4}B_0 \sum_{n=1}^{\infty} (b'_n - ia'_n)\left(\frac{x + iy}{r_0}\right)^{n-1}$$

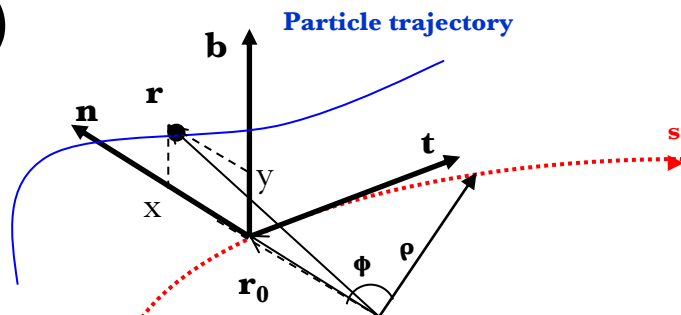
- **Note:**  $n' = n - 1$  is the US convention

- It is useful (especially for **rings**) to transform the Cartesian coordinate system to the **Frenet-Serret system** moving to a closed curve, with path length  $S$



- The **position coordinates** in the two systems are connected by 
$$\mathbf{r} = \mathbf{r}_0(s) + X\mathbf{n}(s) + Y\mathbf{b}(s) = x\mathbf{u}_x + y\mathbf{u}_y + z\mathbf{u}_z$$

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- The **Frenet-Serret unit vectors** and their derivatives are defined as  $(\mathbf{t}, \mathbf{n}, \mathbf{b}) = (\frac{d}{ds}\mathbf{r}_0(s), -\rho(s)\frac{d^2}{ds^2}\mathbf{r}_0(s), \mathbf{t} \times \mathbf{n})$

$$\frac{d}{ds} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\rho(s)} & 0 \\ \frac{1}{\rho(s)} & 0 & -\tau(s) \\ 0 & 0 & \tau(s) \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

with  $\rho(s)$  the **radius of curvature** and  $\tau(s)$  the **torsion** which vanishes in case of planar motion

□ We are seeking a canonical transformation between

$$\begin{aligned} (\mathbf{q}, \mathbf{p}) &\mapsto (\mathbf{Q}, \mathbf{P}) \text{ or} \\ (x, y, z, p_x, p_y, p_z) &\mapsto (X, Y, s, P_x, P_y, P_s) \end{aligned}$$

□ The **generating** function is

$$(\mathbf{q}, \mathbf{P}) = -\left( \frac{\partial F_3(\mathbf{p}, \mathbf{Q})}{\partial \mathbf{p}}, \frac{\partial F_3(\mathbf{p}, \mathbf{Q})}{\partial \mathbf{Q}} \right)$$

□ By using the **relationship** for the **positions**,

$$\mathbf{r} = \mathbf{r}_0(s) + X\mathbf{n}(s) + Y\mathbf{b}(s) = x\mathbf{u}_x + y\mathbf{u}_y + z\mathbf{u}_z$$

the generating function is

$$F_3(\mathbf{p}, \mathbf{Q}) = -\mathbf{p} \cdot \mathbf{r}$$



□ For planar motion, the momenta are

$$\mathbf{P} = (P_X, P_Y, P_s) = \mathbf{p} \cdot \left( \frac{\partial F_3}{\partial X}, \frac{\partial F_3}{\partial Y}, \frac{\partial F_3}{\partial s} \right) = \mathbf{p} \cdot (\mathbf{n}, \mathbf{b}, (1 + \frac{X}{\rho})\mathbf{t})$$

□ Taking into account that the **vector potential** is also transformed in the same way

$$(A_X, A_Y, A_s) = \mathbf{A} \cdot (\mathbf{n}, \mathbf{b}, (1 + \frac{X}{\rho})\mathbf{t})$$

the **new Hamiltonian** is given by

$$\mathcal{H}(\mathbf{Q}, \mathbf{P}, t) = c \sqrt{(P_X - \frac{e}{c}A_X)^2 + (P_Y - \frac{e}{c}A_Y)^2 + \frac{(P_s - \frac{e}{c}A_s)^2}{(1 + \frac{X}{\rho(s)})^2}} + m^2c^2 + e\Phi$$

- ❑ It is more convenient to use the **path length**  $s$ , instead of the **time** as **independent variable**
- ❑ The Hamiltonian can be considered as having **4 degrees of freedom**, where the 4<sup>th</sup> “**position**” is **time** and its conjugate momentum is  $P_t = -\mathcal{H}$

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- ❑ The Hamiltonian can be considered as having **4 degrees of freedom**, where the 4<sup>th</sup> “**position**” is **time** and its conjugate momentum is  $P_t = -\mathcal{H}$
- ❑ In the same way, the new Hamiltonian with the path length as the independent variable is just  $P_s = -\tilde{\mathcal{H}}(X, Y, t, P_X, P_Y, P_t, s)$  with

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right) \sqrt{\left(\frac{P_t + e\Phi}{c}\right)^2 - m^2c^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- ❑ It can be proved that this is indeed a **canonical transformation**
- ❑ Note the existence of the **reference orbit** for **zero vector potential**, for which  $(X, Y, P_X, P_Y, P_s) = (0, 0, 0, 0, P_0)$

- Due to the fact that **longitudinal** (synchrotron) motion is **much slower** than the **transverse** (betatron) one, the electric field can be set to **zero** and the Hamiltonian is written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right) \sqrt{\underbrace{\left(\frac{\mathcal{H}}{c}\right)^2 - m^2c^2}_{P^2} - \left(P_x - \frac{e}{c}A_X\right)^2 - \left(P_Y - \frac{e}{c}A_Y\right)^2}$$

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- The Hamiltonian is then written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right) \sqrt{(P^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2)}$$

- If **static** magnetic fields are considered, the time dependence is also dropped, and the system is having **2 degrees of freedom + “time”** (path length)

- Due to the fact that **total momentum** is **much larger** than the transverse ones, another transformation may be considered, where the transverse momenta are rescaled

$$(\mathbf{Q}, \mathbf{P}) \mapsto (\bar{\mathbf{q}}, \bar{\mathbf{p}}) \text{ or}$$

$$(X, Y, t, P_X, P_Y, P_t) \mapsto (\bar{x}, \bar{y}, \bar{t}, \bar{p}_x, \bar{p}_y, \bar{p}_t) = (X, Y, -c t, \frac{P_X}{P_0}, \frac{P_Y}{P_0}, -\frac{P_t}{P_0 c})$$

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- The new variables are indeed canonical if the Hamiltonian is also rescaled and written as

$$\bar{\mathcal{H}}(\bar{x}, \bar{y}, \bar{t}, \bar{p}_x, \bar{p}_y, \bar{p}_t) = \frac{\tilde{\mathcal{H}}}{P_0} = -e\bar{A}_s - \left(1 + \frac{\bar{x}}{\rho(s)}\right) \sqrt{\bar{p}_t^2 - \frac{m^2 c^2}{P_0} - (\bar{p}_x - e\bar{A}_x)^2 - (\bar{p}_y - e\bar{A}_y)^2}$$

$$\text{with } (\bar{A}_x, \bar{A}_y, \bar{A}_s) = \frac{1}{P_0 c} (A_x, A_y, A_s)$$

$$\text{and } \frac{m^2 c^2}{P_0} = \frac{1}{\beta_0^2 \gamma_0^2}$$

□ Along the reference trajectory  $\bar{p}_{t0} = \frac{1}{\beta_0}$  and

$$\left. \frac{d\bar{t}}{ds} \right|_{P=P_0} = \left. \frac{\partial \bar{H}}{\partial \bar{p}_t} \right|_{P=P_0} = -\bar{p}_{t0} = -\frac{1}{\beta_0}$$

□ It is thus useful to **move the reference frame** to the **reference trajectory** for which another canonical transformation is performed

$$(\bar{\mathbf{q}}, \bar{\mathbf{p}}) \mapsto (\hat{\mathbf{q}}, \hat{\mathbf{p}}) \text{ or}$$

$$(\bar{x}, \bar{y}, \bar{t}, \bar{p}_x, \bar{p}_y, \bar{p}_t) \mapsto (\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = (\bar{x}, \bar{y}, \bar{t} + \frac{s - s_0}{\beta_0}, \bar{p}_x, \bar{p}_y, \bar{p}_t - \frac{1}{\beta_0})$$



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□ The mixed variable generating function is

$$(\hat{\mathbf{q}}, \bar{\mathbf{p}}) = \left( \frac{\partial F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \hat{\mathbf{p}}}, \frac{\partial F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \bar{\mathbf{q}}} \right) \text{ providing}$$

$$F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}}) = \bar{x}\hat{p}_x + \bar{y}\hat{p}_y + \left( \bar{t} + \frac{s - s_0}{\beta_0} \right) \left( \hat{p}_t + \frac{1}{\beta_0} \right)$$

□ The Hamiltonian is then

$$\hat{\mathcal{H}}(\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = \frac{1}{\beta_0} \left( \frac{1}{\beta_0} + \hat{p}_t \right) - e\hat{A}_s - \left( 1 + \frac{\hat{x}}{\rho(s)} \right) \sqrt{\left( \hat{p}_t + \frac{1}{\beta_0} \right)^2 - \frac{1}{\beta_0^2 \gamma_0^2} - (\hat{p}_x - e\hat{A}_x)^2 - (\hat{p}_y - e\bar{A}_y)^2}$$

□ First note that  $\hat{p}_t = \bar{p}_t - \frac{1}{\beta_0} = \bar{p}_t - \bar{p}_{t0} = \frac{P_t - P_0}{P_0} \equiv \delta$   
and  $l = \hat{t}$

□ In the **ultra-relativistic limit**  $\beta_0 \rightarrow 1$  ,  $\frac{1}{\beta_0^2 \gamma^2} \rightarrow 0$   
and the Hamiltonian is written as

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = (1+\delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right) \sqrt{(1+\delta)^2 - (p_x - e\hat{A}_x)^2 - (p_y - e\hat{A}_y)^2}$$

where the “hats” are dropped for simplicity

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where the “hats” are dropped for simplicity

□ If we consider **only transverse field** components,  
the **vector potential** has **only a longitudinal**  
component and the Hamiltonian is written as

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = (1 + \delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right) \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2}$$

□ Note that the Hamiltonian is **non-linear** even in the  
absence of any field component (i.e. for a drift)!

- The Poisson bracket properties satisfy what is mathematically called a **Lie** algebra
- They can be represented by (Lie) operators of the form  
 $: f : g = [f, g]$  and  $: f :^2 g = [f, [f, g]]$  etc.

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- For a Hamiltonian system  $H(\mathbf{z}, t)$  there is a **formal solution** of the equations of motion  $\frac{d\mathbf{z}}{dt} = [H, \mathbf{z}] =: H : \mathbf{z}$  written as  $\mathbf{z}(t) = \sum_{k=0}^{\infty} \frac{t^k :H: {}^k}{k!} \mathbf{z}_0 = e^{t:H:} \mathbf{z}_0$  with a symplectic map  $\mathcal{M} = e^{t:H:}$

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- The 1-turn accelerator map can be represented by the composition of the maps of each element  
 $\mathcal{M} = e^{:f_2:} e^{:f_3:} e^{:f_4:} \dots$  where  $f_i$  (called the generator) is the Hamiltonian for each element, a polynomial of degree  $m$  in the variables  $z_1, \dots, z_n$

- Consider the 1D quadrupole Hamiltonian

$$H = \frac{1}{2} (k_1 x^2 + p^2)$$

- For a quadrupole of length  $L$ , the map is written as

$$e^{\frac{L}{2} : (k_1 x^2 + p^2) :}$$

- Consider the 1D quadrupole Hamiltonian

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- For a quadrupole of length  $L$ , the map is written as

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- Its application to the transverse variables is

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2)}:x = \sum_{n=0}^{\infty} \left( \frac{(-k_1 L^2)^n}{(2n)!} x + L \frac{(-k_1 L^2)^n}{(2n+1)!} p \right)$$

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2)}:p = \sum_{n=0}^{\infty} \left( \frac{(-k_1 L^2)^n}{(2n)!} p - \sqrt{k_1} \frac{(-k_1 L^2)^n}{(2n+1)!} p \right)$$



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- This finally provides the usual quadrupole matrix

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2)}:x = \cos(\sqrt{k_1} L)x + \frac{1}{\sqrt{k_1}} \sin(\sqrt{k_1} L)p$$

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2)}:p = -\sqrt{k_1} \sin(\sqrt{k_1} L)x + \cos(\sqrt{k_1} L)p$$