Averaging and passage through resonances in two-frequency systems near separatrices

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Overview

- We study averaging method for time-periodic perturbations of one-frequency Hamiltonian systems leading to separatrix crossing.
- The time and the angle variable of the unperturbed system are two angle variables, resonances between their frequencies are possible. Resonances accumulate on the separatrices.
- We obtain realistic estimates on the accuracy of averaging method.

Averaging method and obstructions to its use: resonances and separatrix crossings

Perturbations of integrable systems

Unperturbed system

$$\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q},\ \dot{z} = 0$$

- $H = H(p, q, z)$, here z is a scalar or vector parameter, $p, q \in \mathbb{R}^n$ (n-frequency systems).
- This system is completely integrable and can be written as
$$\dot{I} = 0,\ \dot{\varphi} = \omega(I, z),\ \dot{z} = 0, \qquad I, \varphi \in \mathbb{R}^n.$$

Let us add a small perturbation εf:

$$\dot{q} = \frac{\partial H}{\partial p} + \varepsilon f_q(p, q, z),\ \dot{p} = -\frac{\partial H}{\partial q} + \varepsilon f_p(p, q, z),\ \dot{z} = \varepsilon f_z(p, q, z)$$

Perturbations of integrable systems

$$\dot{q} = \frac{\partial H}{\partial p} + \varepsilon f_q(p, q, z),\ \dot{p} = -\frac{\partial H}{\partial q} + \varepsilon f_p(p, q, z),\ \dot{z} = \varepsilon f_z(p, q, z)$$

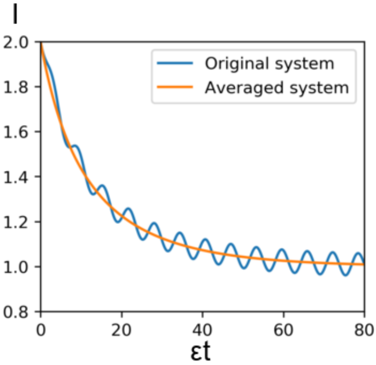
This rewrites in the action-angle variables as

$$\dot{I} = \varepsilon f_I(I, \varphi, z),\ \dot{\varphi} = \omega(I, z) + \varepsilon f_\varphi(I, \varphi, z),\ \dot{z} = \varepsilon f_z(I, \varphi, z),$$

where f_I, f_φ, f_z are the components of f in the action-angle variables:
$f_y = f_q \frac{\partial y}{\partial q} + f_p \frac{\partial y}{\partial p} + f_z \frac{\partial y}{\partial z}, \qquad y = I, \varphi.$

I and z are slow variables of the perturbed system ($\dot{I}, \dot{z} = O(\varepsilon)$), φ is fast variable.

Averaging method

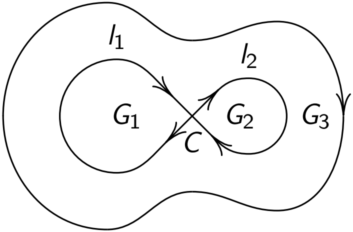


- Evolution of slow variables (I and z) along the solutions of perturbed system can be approximately tracked using the averaged system.

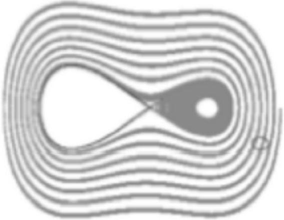
$$\dot{I} = \varepsilon \langle f_I(I, \varphi, z) \rangle_\varphi,$$
$$\dot{z} = \varepsilon \langle f_z(I, \varphi, z) \rangle_\varphi.$$

- For perturbations of one-frequency systems far from separatices this works for all initial data with accuracy $O(\varepsilon)$ for times $\sim \varepsilon^{-1}$ (Fatou, Bogolyubov).

Separatix crossing for one-frequency systems



Unperturbed system



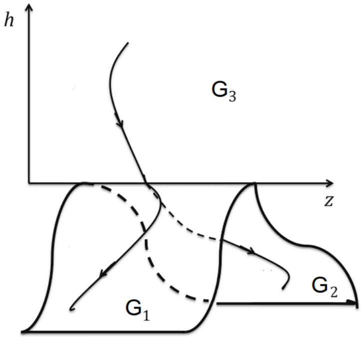
Perturbed system

- Suppose that for all z the unperturbed system has a saddle $C(z)$ with two separatrix loops l_1 and l_2 forming a figure eight.
- Solutions of perturbed system can cross separatrices of the unperturbed system.

Separatix crossing for one-frequency systems

- Set $h(p,q,z) = H(p,q,z) - H(p_C(z), q_C(z), z)$, where $(p_C, q_C) = C(z)$ is the saddle.
- $h = 0$ on separatrices, assume $h > 0$ in G_3 (outside separatices) and $h < 0$ in $G_1 \cap G_2$ (inside)
- We can use energy h instead of action I, then h, z are new slow variables. We can write averaged system in this varables.
- Suppose that h decreases along solutions of averaged system.

Separatix crossing for one-frequency systems



- Gluing together averaged systems in G_3 and G_1 (or G_2) by $h = 0$, we obtain averaged system describing transition from G_3 to G_1 (or G_2).
- Averaging method works[1] for most initial data with measure of exceptional set $O(\varepsilon^r)$, where r can be as large as needed.
- Accuracy $O(\varepsilon)$ before separatrix crossing and $O(\varepsilon|\ln\varepsilon|)$ after (again for times $\sim \varepsilon^{-1}$).
- There are formulas for "probabilities" of transition in G_1 and G_2.

[1] A.I. Neishtadt. "Averaging method for systems with separatrix crossing". *Nonlinearity* 30.7 (2017), p. 2871.

Probabilities of capture

- Initial data for different outcomes (capture in G_1 or G_2) are mixed, so it makes sense to consider captures in G_1 and G_2 as random events with some probabilities.
- One natural definition[2] of the probability of capture is

$$P_i(X_0) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \mu(U_i^\delta(X_0))/\mu(U^\delta),$$

  where $X_0 = (h_0, \varphi_0, z_0)$ is some initial data, U^δ is δ-neighborhood of X_0, $U_i^\delta \subset U^\delta$ is the set of initial conditions captured in G_i, μ is the Lebesgue measure.

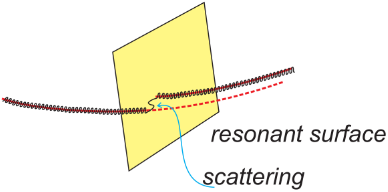
[2] V.I. Arnold. "Small denominators and problems of stability of motion in classical and celestial mechanics". *Russ. Math. Surv* 18.6 (1963), pp. 85–191.

Formula for probabilities

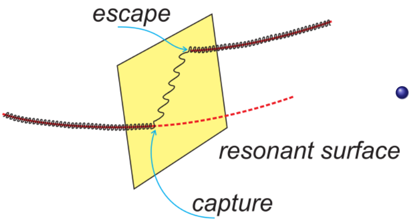
$$P_1 = \frac{\Theta_{1*}}{\Theta_{1*} + \Theta_{2*}}, \qquad P_2 = \frac{\Theta_{2*}}{\Theta_{1*} + \Theta_{2*}}$$

- $\Theta_i(z) = \oint_{l_i} f_h dt$ $(i = 1, 2)$, where $f_h = f_q \frac{\partial h}{\partial q} + f_p \frac{\partial h}{\partial p} + f_z \frac{\partial h}{\partial z}$ and the integral is taken along the separatrix l_i parametrized by the time t of unperturbed system.
- During a pass near l_i the value of h decreases by $\approx \varepsilon\Theta_i$
- $\Theta_{i*} = \Theta_i(z_*)$, where z_* is the value of z when the solution of averaged system crosses separatices

Resonances and capture into resonances



Scattering on a resonance



Capture into a resonance

- For two-frequency systems resonances are possible. For evolution of fast variables $\dot{\varphi}_1 = \omega_1(I, z), \dot{\varphi}_2 = \omega_2(I, z)$, resonances are given by

$$\omega_2/\omega_1 = s_2/s_1, \quad (s_1, s_2) \in \mathbb{Z}^2_{>0}.$$

- Most solutions exhibit *scattering on a resonance*: "random" jump of slow variables of magnitude $O(\sqrt{\varepsilon})$
- Some solutions may be *captured into a resonance*: the solution stays near the resonant surface for time $\sim \varepsilon^{-1}$. Measure of initial data that can be captured into resonance is $O(\sqrt{\varepsilon})$.

Two-frequency systems far from separatrices

- Under some genericity condition
  - Accuracy of averaging method $O(\sqrt{\varepsilon}|\ln\varepsilon|)$ holds for most initial data for times $\sim \varepsilon^{-1}$
  - Exceptional set has measure $O(\sqrt{\varepsilon})$.
- Averging for two-frequency systems far from separatrices was studied in papers[3][4], see also review[5]

[3] V.I. Arnold. "Applicability conditions and an error bound for the averaging method for systems in the process of evolution through a resonance". *Doklady Akademii Nauk*. Vol. 161. 1. Russian Academy of Sciences. 1965, pp. 9–12.

[4] A.I. Neishtadt. "Passage through resonances in a two-frequency problem". *Akademiia Nauk SSSR Doklady*. Vol. 221. 1975, pp. 301–304.

[5] A.I. Neishtadt. "Averaging, passage through resonances, and capture into resonance in two-frequency systems". *Russian Mathematical Surveys* 69.5 (2014), p. 771.

Systems with many frequencies

- Under a certain non-degeneracy condition mean error of averaging error over all initial data is $O(\sqrt{\varepsilon})$. [6,7]
- This gives that outside of exceptional set of measure κ accuracy of averaging method is $O(\frac{\sqrt{\varepsilon}}{\kappa})$.
- Taking $\kappa = \varepsilon^{1/4}$ gives accuracy $O(\varepsilon^{1/4})$ and measure of exceptional set $O(\varepsilon^{1/4})$, worse than for two-frequency systems.

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[6] T. Kasuga. "On the adiabatic theorem for the Hamiltonian system of differential equations in the classical mechanics. I, II, III". *Proceedings of the Japan Academy* 37.7 (1961), pp. 366–371, 372–376, 377–382.

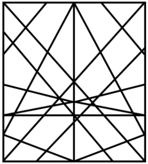
[7] A. I. Neishtadt. "Averaging in multifrequency systems. II". *Doklady Akademii Nauk*. Vol. 226. 6. Russian Academy of Sciences. 1976, pp. 1295–1298.

Systems with many frequencies

- Consider the following unperturbed three-frequency system[8]

$$\dot{\varphi}_1 = I_1,\ \dot{\varphi}_2 = I_2,\ \dot{\varphi}_3 = 1,\ \dot{I}_1 = \dot{I}_2 = 0.$$

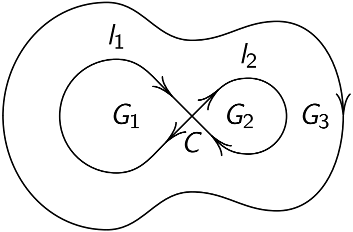
- Resonant surphaces are straight lines with rational equations.
- One has to study *tangencies with resonances* and being near *points of intersection of resonant surphaces*.



[8] V.I. Arnold, V.V. Kozlov, and A.I. Neishtadt. *Mathematical aspects of classical and celestial mechanics*. Vol. 3. Springer Science & Business Media, 2007.

Our result

Setting



- We consider small time-periodic (with period 2π) perturbations of one-frequency systems with separatrix crossing:

$$\dot{q} = \frac{\partial H}{\partial p} + \varepsilon f_q(p, q, z, t), \ \dot{p} = -\frac{\partial H}{\partial q} + \varepsilon f_p(p, q, z, t), \ \dot{z} = \varepsilon f_z(p, q, z, t)$$

- H and f are analytic
- Time t and angle variable φ of the unperturbed system are the two angle variables.
- h decreases along the solutions of averaged system (thus we have transitions from G_3 to G_1 and G_2)
- Some genericity condition

Statement

- Evolution of most initial data in G_3 is described[9] by averaged system describing transition from G_3 to G_1 or to G_2 with accuracy $O(\sqrt{\varepsilon}|\ln \varepsilon|)$ over times $\sim \varepsilon^{-1}$.
- Exceptional set has measure $O(\sqrt{\varepsilon}|\ln^5 \varepsilon|)$.
- Formulas for probabilities of capture in G_1 and G_2 similar to one-frequency case hold $\Theta_i(z) = \frac{1}{2\pi} \int_0^{2\pi} \oint_{l_i} f_h(s, t) ds dt$ $(i = 1, 2)$, the inner integral is taken along the separatrix l_i parametrized by the time s of unperturbed system.

[9] A.I. Neishtadt and A. Okunev. "Averaging and passage through resonances in two-frequency systems near separatrices". *arXiv preprint arXiv:2108.08540* (2021).

Comments

- Estimates on precision $\sqrt{\varepsilon}|\ln \varepsilon|$ is same as far from separatrices and cannot be improved, we do not know if the estimate $\sqrt{\varepsilon}|\ln^5 \varepsilon|$ on the measure of exceptional set can be improved, minimal possible estimate is $\sqrt{\varepsilon}$.
- Slow-fast Hamiltonian systems with two and a half degrees of freedom belong to our class.

$$H(p, q, x, y, t) = H_0(p, q, x, y) + \varepsilon H_1(p, q, x, y, t).$$

Here (q, p) and $(\varepsilon^{-1}x, y)$ are pairs of conjugate variables; variables p, q are fast and x, y are slow. Indeed, we can take $z = (x, y)$.

Outline of the remaining talk

- Resonant zones
- Dynamics near a resonance
- Sketch of proof

Resonant zones

Resonances far from separatrices

- When slow variables are on the resonant surface

$$\omega = s_2/s_1$$

orbits of the unperturbed system $\dot\varphi = \omega, \ \dot t = 1$ span not the whole $\mathbb{T}^2 \ni (\varphi, t)$, but a rational winding of $\mathbb{T}^2$

- Time average becomes space average over these orbits. In terms of the Fourier expansion of the perturbation

$$f(\varphi, t) = \sum_{(m_1,m_2)\in\mathbb{Z}^2} f_{m_1,m_2} e^{i(m_1\varphi + m_2 t)}$$

not only the average $\langle f \rangle_{\varphi,t} = f_{0,0}$ appears in the space average of f, but all coefficients $f_{ks_1,-ks_2}$, $k \in \mathbb{Z}$

- Effect of a resonance $\omega = s_2/s_1$ is determined by these Fourier coefficients $f_{ks_1,-ks_2}$, $k \in \mathbb{Z}$, , $k \neq 0$

Resonant zones far from separatrices

- Effect of a resonance $\omega = s_2/s_1$ is determined by these Fourier coefficients $f_{ks_1, -ks_2}$, $k \in \mathbb{Z}$, , $k \neq 0$
- Fourier coefficients of an analytic functions decrease exponentially

$$\sum_{k \neq 0} \|f_{ks_1, -ks_2}\| < a_s = e^{-K(|s_1| + |s_2|)}$$

Resonant zones far from separatrices

- Effect of a resonance $\omega = s_2/s_1$ is determined by Fourier coefficients that are bounded by
$$a_s = e^{-K(|s_1|+|s_2|)}.$$
- We can consider only $O(\ln^2 \varepsilon)$ resonances with $|s_1|, |s_2| \lesssim |\ln \varepsilon|$, effect of all other resonances is negligebly small
- Resonant zone of the resonance $\hat{\omega} = s_2/s_1$ (i.e., zone where the effect of the resonance is significant) is
$$|\omega - s_2/s_1| \lesssim a_s\sqrt{\varepsilon}.$$
- Total width of all resonant zones is $O(\sqrt{\varepsilon})$.

Effect of resonances far from separatrices

- Effect of a resonance $\omega = s_2/s_1$ is determined by Fourier coefficients that are bounded by
$$a_s = e^{-K(|s_1|+|s_2|)}.$$
- Only finitely many resonances such that capture is possible ($s_1, s_2 \sim 1$, otherwise the effect of $\langle f \rangle_{\varphi,t}$ is larger than the effect of resonance)
- These resonance can capture measure $O(\sqrt{\varepsilon})$, while other trajectroies of perturbed system exhibit scattering with amplitude $O(\sqrt{\varepsilon}|\ln \varepsilon|)$
- For other resonance the amplitude of scattering is proportional to the width of the resonant zone, thus the total scattering caused by other resonances is $O(\sqrt{\varepsilon})$

Resonant zones near separatrices

- Analyticity of the perturbation $f(\varphi, t)$ as function of φ gets worse near separatices (because on the separatices the phase φ is undefined)
- Worse estimate on Fourier coefficients:

$$\sum_{k \neq 0} \|f_{ks_1, -ks_2}\| < b_{s_2} = e^{-c|s_2|}.$$

- We use $h = H(p, q, z) - H(p_C(z), q_C(z), z)$ to measure distance to separatrices.
- It is convenient to measure width of resonant zones in h, not in ω. Widths in h and ω are connected by $\frac{\partial \omega}{\partial h} \sim h^{-1} \ln^{-2} h$.
- Widths of resonant zones in h are

$$O_*(\sqrt{b_{s_2} \varepsilon h}).$$

  Here we say $\psi = O_*(h^a)$ if $\psi = O(h^a |\ln^b h|)$ for some b.
- Total width of all resonant zones is still $O(\sqrt{\varepsilon})$.

Effect of resonant zones near separatrices

- Effect of a resonance $\omega = s_2/s_1$ is determined by Fourier coefficients that are bounded by
$$b_{s_2} = e^{-c|s_2|}.$$
- Resonances such that capture is possible ($s_2 \sim 1$) accumulate on separatrices (e.g., $\omega = 1/s_1$)
- These resonance can capture measure $O_*(\sqrt{\varepsilon h})$, while other trajectroies of perturbed system exhibit scattering with amplitude $O_*(\sqrt{\varepsilon h}|\ln \varepsilon|)$
- For other resonances the amplitude of scattering is proportional to the width of the resonant zone
- Total measure captured in all resonances is $O(\sqrt{\varepsilon})$ and total scattering is $O(\sqrt{\varepsilon}|\ln \varepsilon|)$
- These estimates are valid for $|h| \gtrsim \varepsilon |\ln^5 \varepsilon|$.

Dynamics near a resonance (auxiliary system)

Resonance crossing far from separatrices

- Near the resonance $\omega \approx \hat{\omega} = s_2/s_1$, resonant phase $\gamma = \varphi - \hat{\omega} t$ changes slowly.
- Replace φ by γ, then γ and t will be new phases.
- After a proper coordinate and time change and averaging over the fast phase t, one can get auxiliary system describing perturbed system near this resonance

Auxiliary system (far from separatrices)

For simplicity, consider the case without parameter z.
Unperturbed auxiliary system:

$$Q' = P, \qquad P' = F(Q).$$

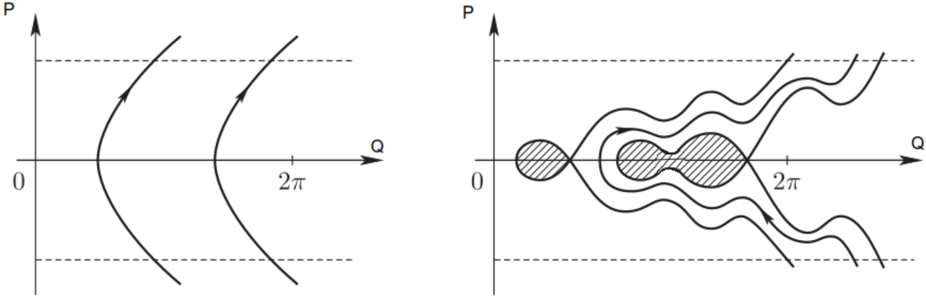
- P measures distance to the resonant surface, $Q \approx \varphi - \hat{\omega} t$ is close to the resonant phase.
- $F(Q)$ is 2π-periodic.

Actual dynamics near resonance is described by a small perturbation of this equation:

$$Q' = P + O(\sqrt{\varepsilon}), \qquad P' = F(Q) + O(\sqrt{\varepsilon}).$$

Weak and strong resonances

$$Q' = P + O(\sqrt{\varepsilon}), \qquad P' = F(Q) + O(\sqrt{\varepsilon}).$$



- Left: weak resonance, capture in resonance is impossible (only scattering)
- Right: strong resonance, capture is possible for initial data near separtrices.

Auxiliary systems near separatrices

- Unperturbed auxiliary system has same form as for resonances far from separatices.
- Perturbation is now splitted into $O_*(\sqrt{\varepsilon/h})$ Hamiltonian perturbation and $O_*(\sqrt{\varepsilon h})$ non-Hamiltonian perturbation.
- This gives **captured** measure $O(\sqrt{\varepsilon h})$ for strong resonances near separatrices.
- **Scattering** is determined by the slow time $\tau = \varepsilon t$ of crossing resonant zone, and this time is proportional to the width $O_*(\sqrt{\varepsilon h})$ of resonant zone in h.

Limit auxiliary systems

Fix $s_2 > 0$ and suppose $s_1 > 0$ is odd and $s_1 \to \infty$.
Then for resonance $\omega = s_2/s_1$ (in G_3, i.e., outside the figure eight) the function F_s describing unperturbed auxiliary system has limit:

$$F_s(Q, z) \to F^*_{s_2, odd}(Q, z).$$

There is another limit for even s_1:

$$F_s(Q, z) \to F^*_{s_2, even}(Q, z).$$

Limit auxiliary systems ($s_2 = 1$)

Denote
$$M_i(Q) = \int_{l_i} f_h(h{=}0, z, \tilde{t}, t{=}\tilde{t}{-}Q)\,d\tilde{t}, \qquad i = 1, 2.$$

Here separatrix l_i is parametrized by the time of unperturbed system $\tilde{t}$. $M_i(Q)$ is 2π-periodic. This is the Melnikov function[10].

Consider the case $s_2 = 1$ first, i.e. resonances $\omega = 1/n$. Then

$$\begin{aligned} -2\pi F^*_{1,even}(Q, z) &= M_1(Q, z) + M_2(Q, z), \\ -2\pi F^*_{1,odd}(Q, z) &= M_1(Q, z) + M_2(Q + \pi, z). \end{aligned}$$

[10] V.K. Melnikov. "On the stability of a center for time-periodic perturbations". *Trudy moskovskogo matematicheskogo obshchestva* 12 (1963), pp. 3–52.

Limit auxiliary systems ($s_2 > 1$)

If $s_2 > 1$, we need to average periodic shifts of $M_i(Q)$.
Given $s_2 \in \mathbb{Z}_{>0}$, set

$$F^*_{s_2,i}(Q,z) = -(2\pi)^{-1}\Big\langle M_i\Big(Q - 2\pi\frac{j}{s_2}\Big)\Big\rangle_{j=0,\dots,s_2-1} \qquad \text{for } i = 1,2,$$

where $\langle\psi(j)\rangle_{j=0,\dots,k-1} = \frac{\psi(0)+\dots+\psi(k-1)}{k}$. Then

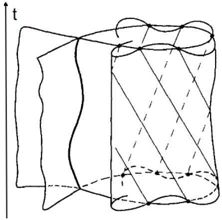
$$\begin{aligned} F^*_{s_2,even}(Q,z) =& F^*_{s,1}(Q,z) + F^*_{s,2}\Big(Q,z\Big), \\ F^*_{s_2,odd}(Q,z) =& F^*_{s,1}(Q,z) + F^*_{s,2}\Big(Q + \frac{\pi}{s_2}, z\Big). \end{aligned}$$

Melnikov function

- Consider the case without z. Melnikov function is used to measure the splitting of separatrices for the perturbed system. In our notation it can be written as (separatrix l_i is parametrized by the time $\tilde{t}$ of unperturbed system; $Q = \tilde{t} - t$)

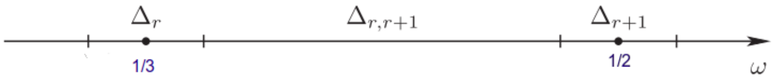
$$M_i(Q) = \int_{l_i} f_h(h{=}0, \tilde{t}, t{=}\tilde{t}{-}Q)d\tilde{t}, \qquad i = 1, 2.$$

- Simple zeroes of Melnikov function correspond to transversal homoclinic intersections, near such intersections there are horseshoes



Proofs: overview

Far from separatices



- We assume ω decreases along solutions of averaged system. For perturbed system $\omega(t)$ passes resonant zones and *non-resonant zones* between them.
- Dynamics between resonant zones is described by standard coordinate change used to justify averaging method.
- Dynamics inside resonant zones is described using auxiliary system.
- This gives estimates on the accuracy of averaging method[11]

[11] A.I. Neishtadt. "Averaging, passage through resonances, and capture into resonance in two-frequency systems". *Russian Mathematical Surveys* 69.5 (2014), p. 771.

Near separatrices

- Dynamics in resonant and non-resonant zones can be studied using same methods, but into account being near separatrices.
  - Phase φ is not defined on separatices and behaves badly near them. Thus many functions used in averaging method are unbounded.
  - E.g., $f_\varphi = O(h^{-1} \ln^{-2} h)$.
  - Also, for a smooth $\psi(p, q, z)$ we have $\frac{\partial \psi}{\partial h} = O(h^{-1} \ln^{-1} h)$ (as $\frac{\partial}{\partial h}$ is taken for fixed φ).
- Immediate neighborhood of separatrices given by $|h| \lesssim \varepsilon |\ln^5 \varepsilon|$ is treated separately, as estimates for resonant zones fail there.

Immediate neighborhood of separatrices

- Immediate neighborhood of separatrices is given by $|h| \lesssim \varepsilon |\ln^5 \varepsilon|$.
- When $h \lesssim \varepsilon \ln^2 \varepsilon$ resonances may overlap (Chirikov criterion: overlap of resonant zones often leads to chaos).

- Divergence of the perturation εf is $O(\varepsilon)$ in action-angle variable.
- Phase volume changes slowly for perturbed system. Argument based on this shows that most solutions leave this zone in time $O(\sqrt{\varepsilon})$, this gives $O(\sqrt{\varepsilon}|\ln^5 \varepsilon|)$ estimate on the measure of exceptional set.

Estimates near separatices

- First part of the problem is obtaining estimates on perturbation in energy-angle variables h, φ, z and other functions related to averaging method.
- We estimate how the perturbation $f(h, \varphi)$ can be continued to complex domain

$$|\operatorname{Im} h| \lesssim h, \qquad |\operatorname{Im} \varphi| \lesssim \omega \sim |\ln^{-1} h|$$

- This gives estimates for Fourier coefficients of the perturbation, required to determine resonant zones.
- This estimates allow to study dynamics in non-resonant zones.

Derivation of auxiliary system

Derivation far from separatrices: coordinate change

$$\dot{I} = \varepsilon f_I(I, \varphi, t), \qquad \dot{\varphi} = \omega(I) + \varepsilon f_\varphi(I, \varphi, t).$$

- Fix rational $\hat{\omega}$, consider resonance $\omega \approx \hat{\omega}$, define $\hat{I}$ by $\omega(\hat{I}) = \hat{\omega}$.
- New variables $\gamma = \varphi - \hat{\omega} t$, $J = I - \hat{I}$. Near resonance $J, \dot{\gamma} \approx 0$.
- $\dot{\gamma} = \omega - \hat{\omega} + O(\varepsilon) = \widehat{\frac{\partial \omega}{\partial I}} J + O(J^2) + O(\varepsilon)$
- $\dot{J} = \varepsilon f_I(\hat{I}, \gamma, t) + O(\varepsilon J)$
- Set $\alpha = \sqrt{\varepsilon / \widehat{\frac{\partial \omega}{\partial I}}}$, $\beta = \sqrt{\varepsilon \widehat{\frac{\partial \omega}{\partial I}}} \sim \sqrt{\varepsilon}$
- Set $P = J/\alpha$, $Q = \gamma$. Assume $P \lesssim 1$. We get

$$\dot{Q} = \beta P + O(\varepsilon), \qquad \dot{P} = \beta f_I(\hat{I}, Q + \hat{\omega} t, t) + O(\varepsilon).$$

Derivation far from separatrices: averaging and time change

$$\dot I = \varepsilon f_I(I, \varphi, t), \qquad \dot\varphi = \omega(I) + \varepsilon f_\varphi(I, \varphi, t).$$

Recall $\alpha = \sqrt{\varepsilon / \frac{\hat{\partial \omega}}{\partial I}}$, $\beta = \sqrt{\varepsilon \frac{\hat{\partial \omega}}{\partial I}} \sim \sqrt{\varepsilon}$,

$$P = (I - \hat I)/\alpha, \qquad Q = \varphi - \hat\omega t.$$

$$\dot Q = \beta P + O(\varepsilon), \qquad \dot P = \beta f_I(\hat I, Q + \hat\omega t, t) + O(\varepsilon).$$

- We see that t is fast compared with P, Q. Let us apply averaging over t and replace the true system by averaged system
- Denote $F(Q) = \langle f_I(\hat I, Q + \hat\omega t, t) \rangle_{t \in [0, 2\pi]}$, then we get

$$\dot Q = \beta P + O(\varepsilon), \qquad \dot P = \beta F(Q) + O(\varepsilon).$$

- Taking new time $\tau = \beta t$ and denoting $a' = \frac{\mathrm{d}a}{\mathrm{d}\tau}$, we get

$$Q' = P + O(\sqrt{\varepsilon}), \qquad P' = F(Q) + O(\sqrt{\varepsilon}).$$

Derivation near separatrices

- Perturbation εf grows near separatrices in variables I, φ, but the divergence remains $O(\varepsilon)$
- We separate Hamiltonian part of perturbation in action-angle variables so that non-Hamiltonian part does not grow near separatrices
- Small parameter when using averaging for the transition to auxiliary system is $\beta = \sqrt{\varepsilon \frac{\partial \hat{\omega}}{\partial I}} = O_*(\sqrt{\varepsilon / h})$. Single step of averaging method gives accuracy $O_*(\sqrt{\varepsilon / h}) \ll \sqrt{\varepsilon}$ and is not enough, we use many steps for better accuracy[12].
- To apply multistep averaging, estimates on complex continuation of perturbation εf in action-angle variables are used
- Resulting unperturbed auxiliary system is as far from separatices; perturbation (for auxiliary system) consists of $O_*(\sqrt{\varepsilon / h})$ Hamiltonian part and $O_*(\sqrt{\varepsilon h})$ non-Hamiltonian part.

[12] A.I. Neishtadt. "The separation of motions in systems with rapidly rotating phase". *Journal of Applied Mathematics and Mechanics* 48.2 (1984), pp. 133–139.

Thank you!