Multi-loop Scattering Amplitudes and Gravitational binary dynamics

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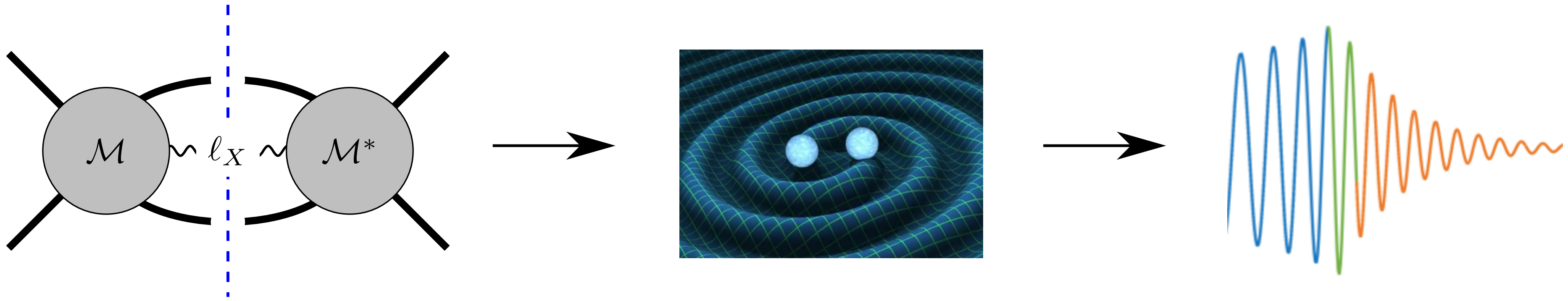
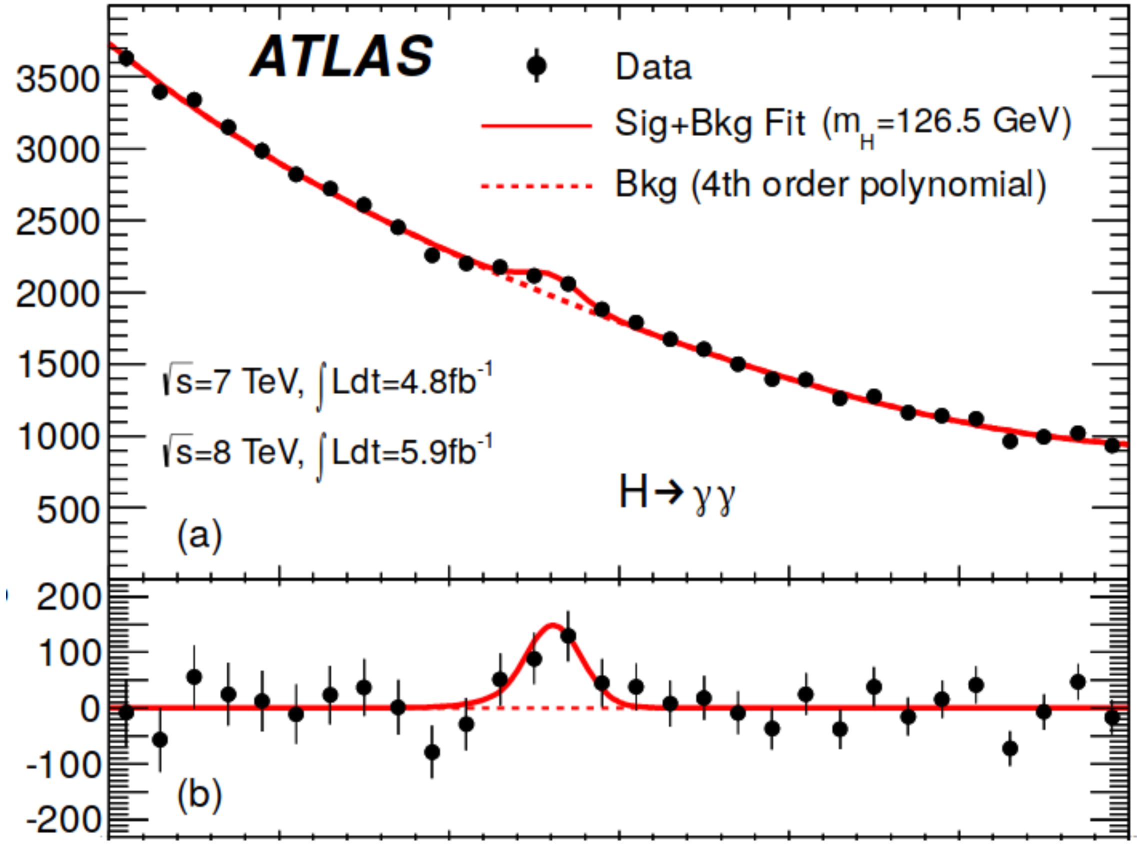


OUTLINE

1. Background - precision gravitational wave physics
2. Classical physics from quantum amplitudes
3. Modern methods for gravity amplitudes
4. Collider-inspired techniques for loop integration
5. Results & comparisons

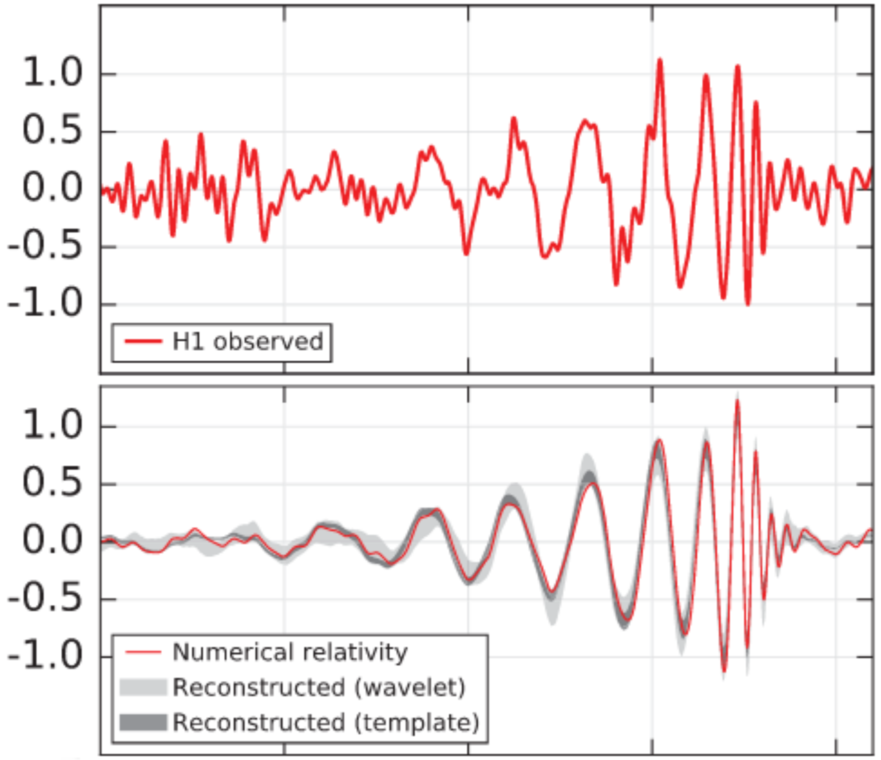
Background

DISCOVERIES OF OUR TIMES



Two fundamental discoveries of our times: **Higgs boson (2012), gravitational waves (2015).**
Spectacular confirmation of SM / GR. Both experiments call for precision theory.

DISCOVERIES OF OUR TIMES

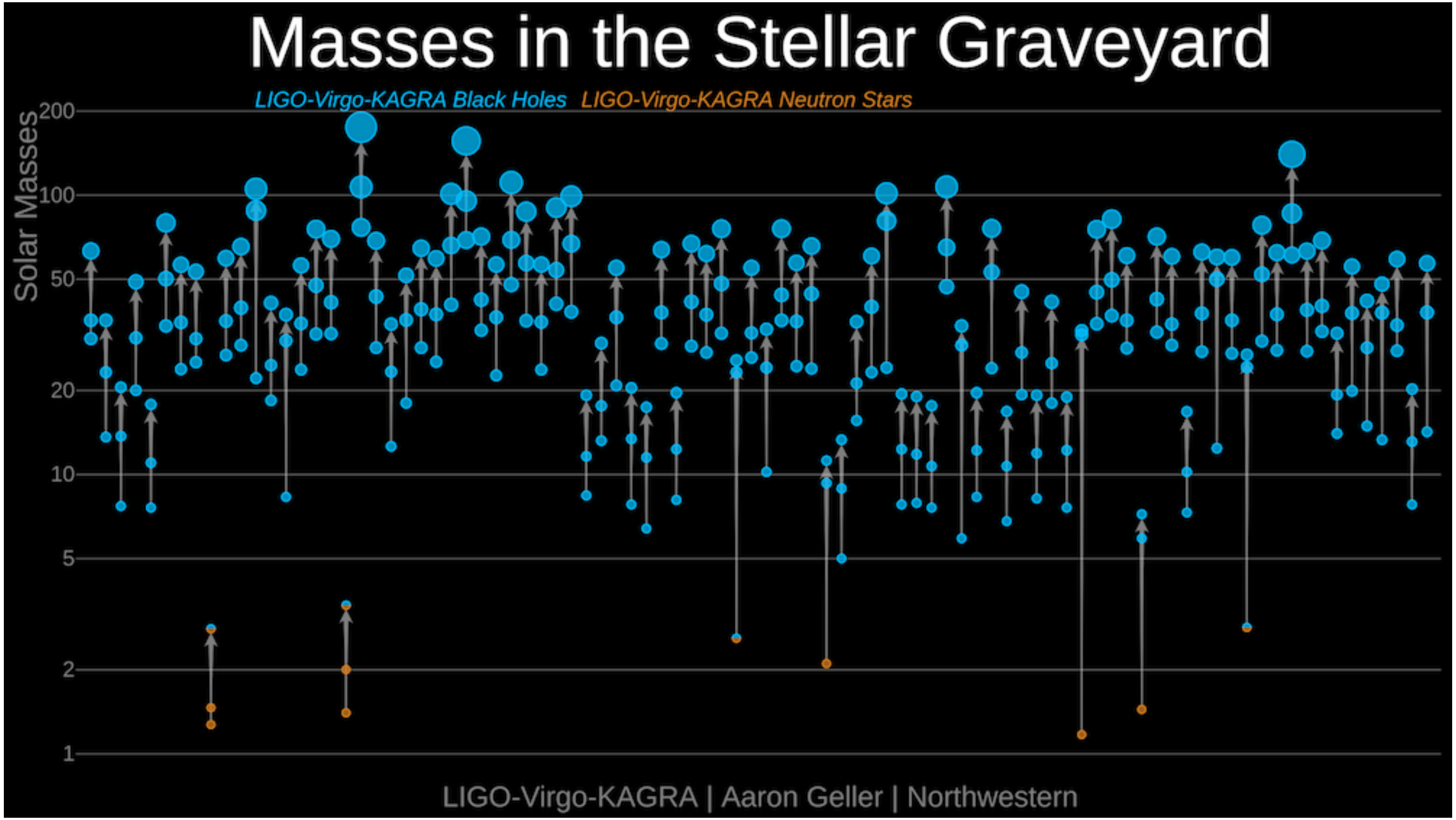


LIGO & VIRGO collaborations, arXiv:1602.03837

Two fundamental discoveries of our times: **Higgs boson (2012), gravitational waves (2015).** Spectacular confirmation of SM / GR. Both experiments call for precision theory.

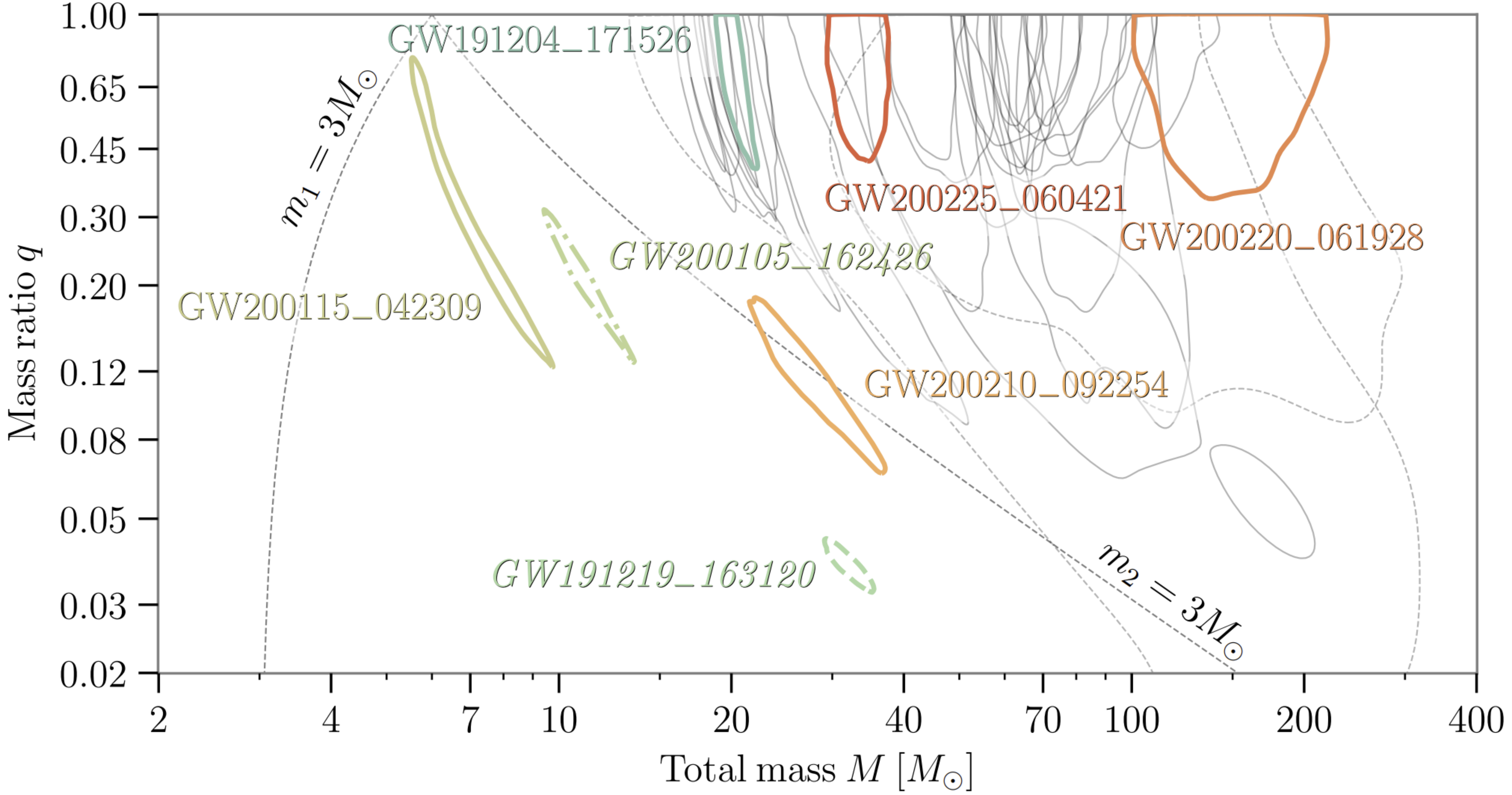
- *Cross-fertilization: scattering amplitudes, loop integrals, effective field theories*

LIGO-VIRGO-KAGRA O3B CATALOG (07 NOV 2021)



LIGO-VIRGO-KAGRA O3B CATALOG (07 NOV 2021)

Credible regions for total mass (x-axis) and mass ratio (y-axis)



FUTURE GW DETECTORS

aLigo/ Virgo

— 2020

KAGRA

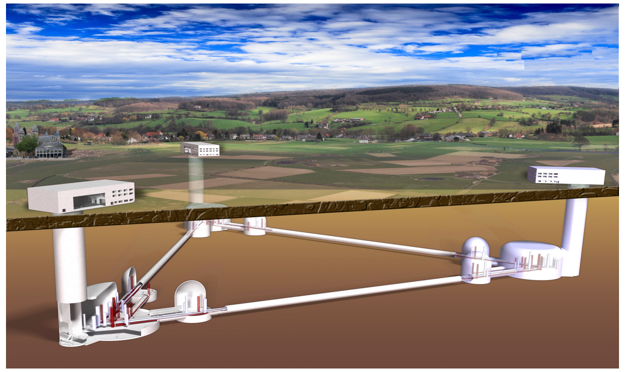
IndIGO

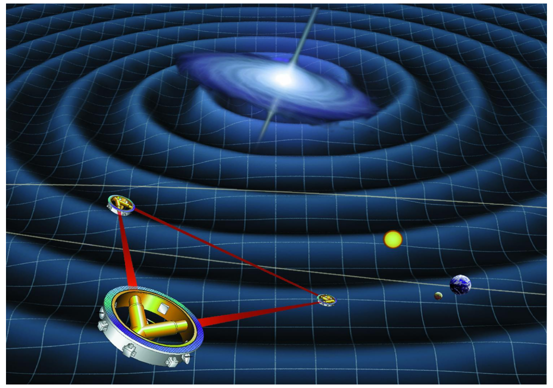
TianQin — 2025

DECIGO

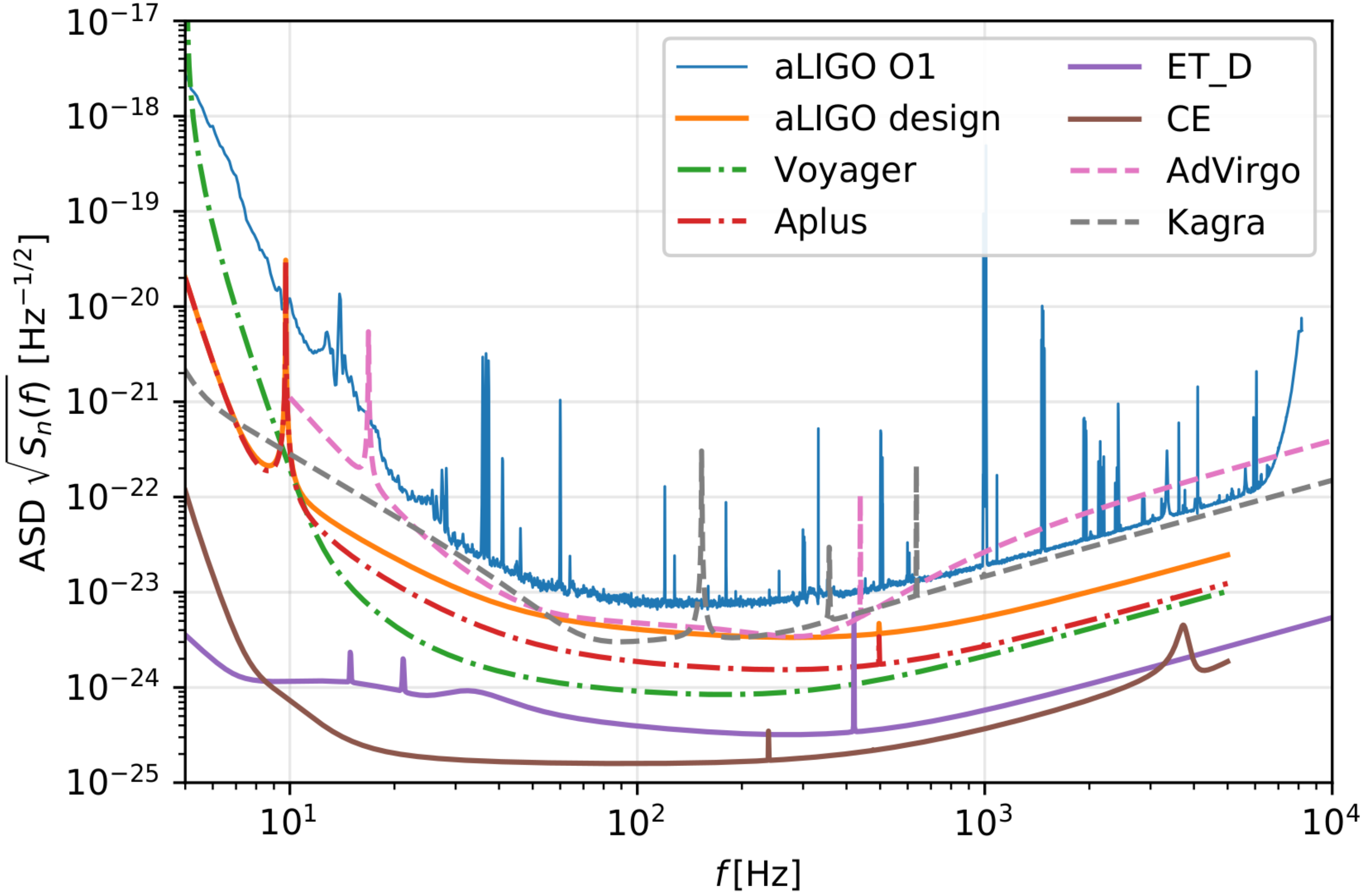
LISA — 2030

Einstein

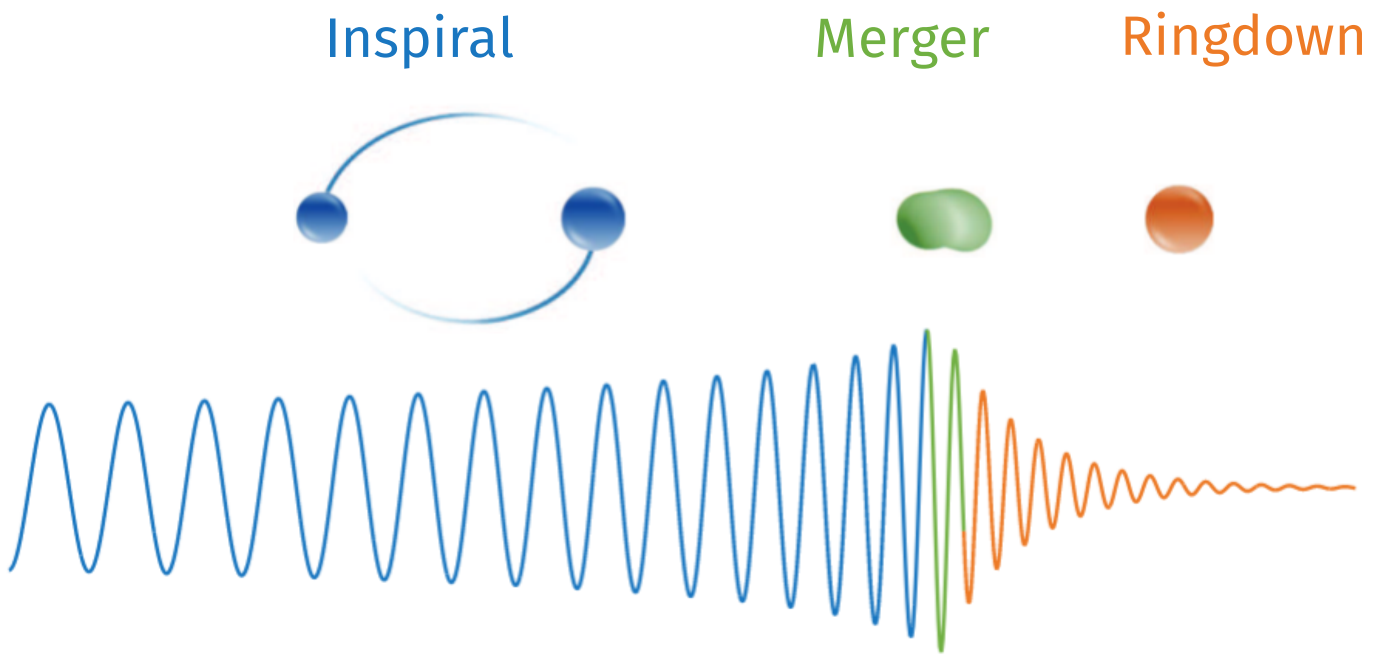




Plot: sensitivity of future GW detectors.



STAGES OF INSPIRAL WAVEFORMS



[Picture: Antelis, Moreno, arXiv:1610.03567]

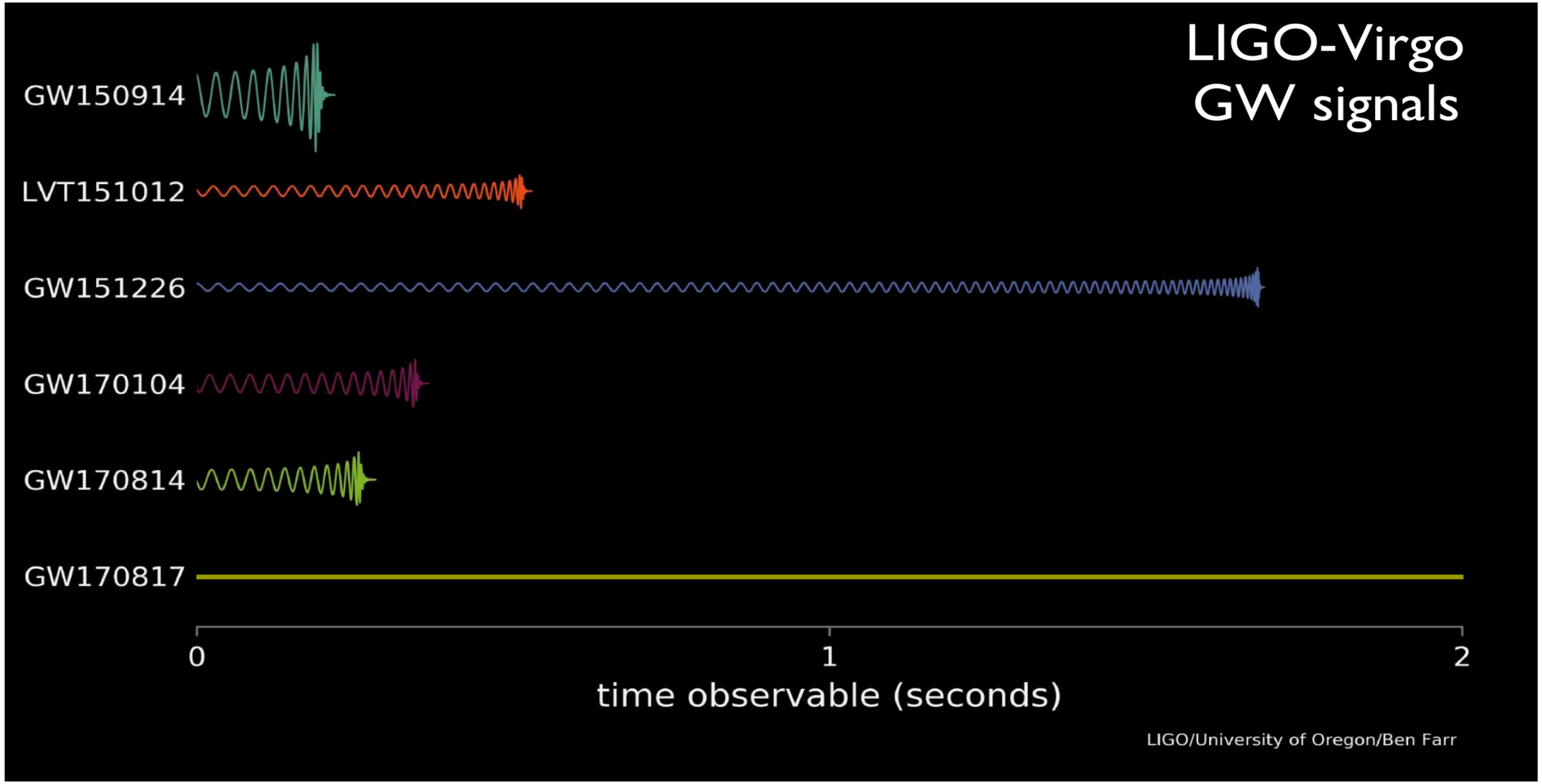
Inspiral Perturbative expansions: post-Newtonian (PN), post-Minkowskian (PM), self-force (SF), semi-analytic models

Merger Numerical relativity - first principles, but too slow to scan over large parameter space.

Ringdown Perturbative Quasi-normal modes.

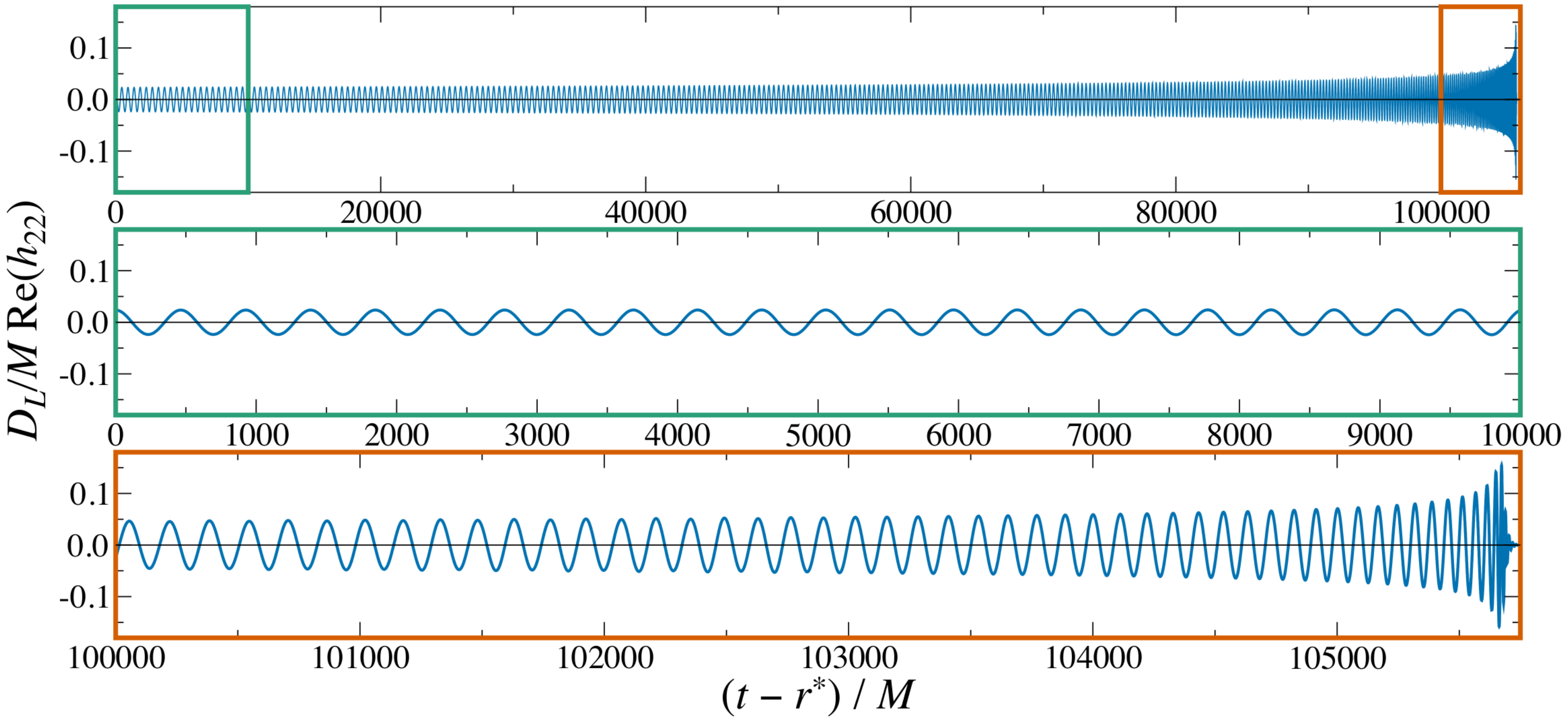
STAGES OF INSPIRAL WAVEFORMS

Inspiral ⟶ Merger ⟶ Ringdown



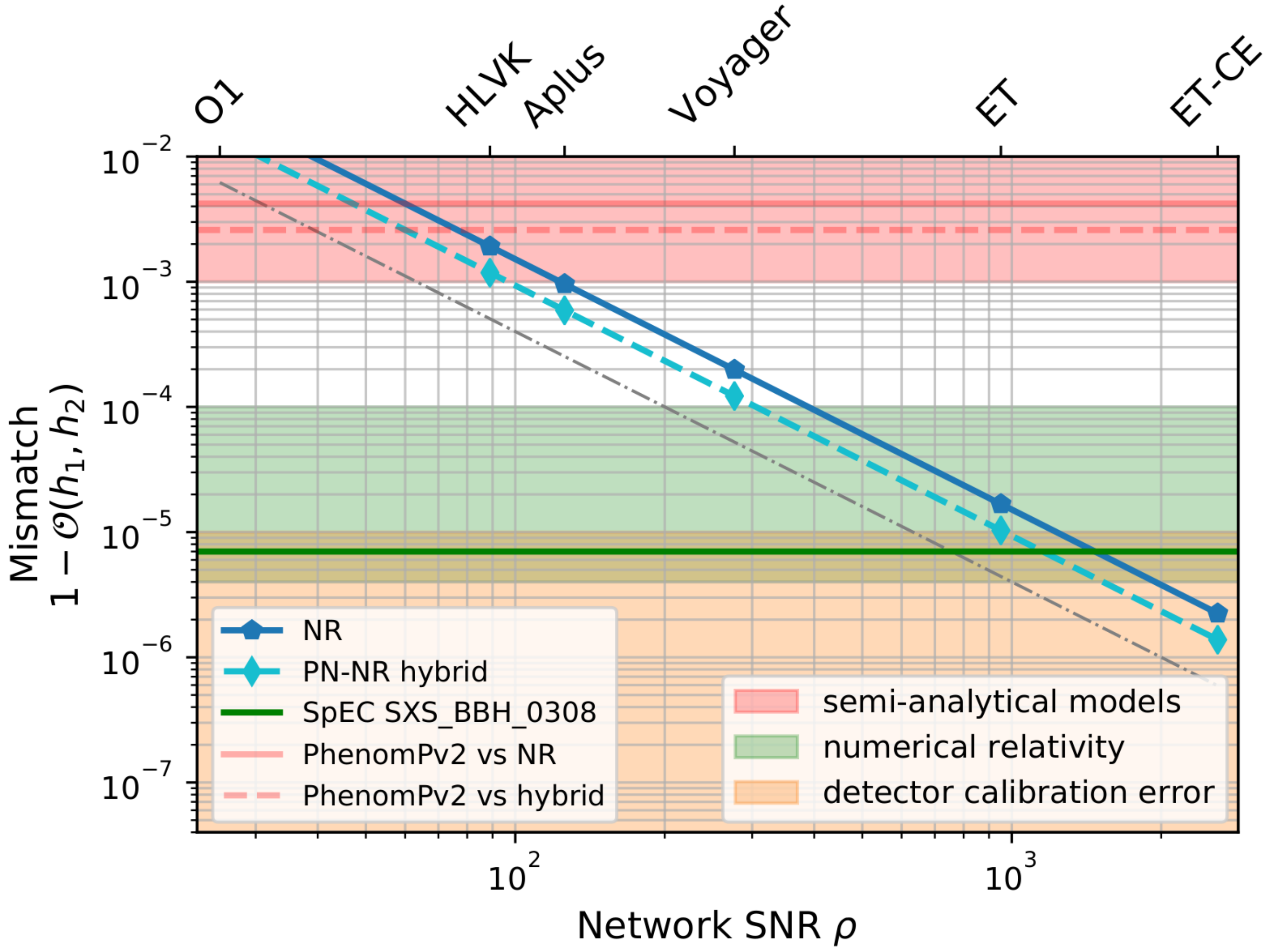
NEED FOR (SEMI-) ANALYTIC CALCULATIONS

- **NR simulations expensive.** Example simuation covering entire signal: 8 months, few million CPU hours. [Szilagyi *et al.* '15]
- 376 GW cycles at ***1 point in parameter space***. Zero spins, mass ratio 7.



[slilde from Alessandra Buonanno]

REQUIREMENTS FOR THEORY PRECISION



Semi-analytic model accuracy needs ~ O(10^3) improvement!

Need ***accurate waveform calculations:*** e.g. at 6th post-Newtonian, 2nd-self force orders.

Test GR, neutron star EOS, exotic objects…

POST-NEWTONIAN (PN) EXPANSION

Joint expansion in $GM/r \sim v^2$, locked by Virial theorem.

Conservative Hamiltonian in c.o.m. frame:

$$m = m_A + m_B, \quad \nu = \mu/m$$
$$\mu = m_A m_B / m$$

$\mathcal{O}(v^2)$ $\mathcal{O}(G)$

$$\frac{H}{\mu} = \boxed{\frac{P^2}{2} - \frac{Gm}{R}} + H_{1\mathrm{PN}} + H_{2\mathrm{PN}} + H_{3\mathrm{PN}} + H_{4\mathrm{PN}} \ldots$$

0PN, Newton

$H_{2\mathrm{PN}}$ (1980), $H_{3\mathrm{PN}}$ (2000), $H_{4\mathrm{PN}}$ (2014)

$$\mathcal{O}(v^4) + \mathcal{O}(Gv^2) + \mathcal{O}(G^2)$$

$$+\frac{1}{c^2}\left\{-\frac{P^4}{8} + \frac{3\nu P^4}{8} + \frac{Gm}{R}\left(-\frac{P_R^2\nu}{2} - \frac{3P^2}{2} - \frac{\nu P^2}{2}\right) + \frac{G^2m^2}{2R^2}\right\}$$

1PN, Einstein, Infeld, Hoffman, 1938

POST-MINKOWSKIAN (PM) EXPANSION

- Expansion in coupling GM/r, **exact velocity dependence** *[Bertotti, Kerr, Plebanski, Portilla, Westpfahl, Gollder, Bel, Damour, Derulle, Ibanez, Martin, Ledvinka, Scaefer, Bicak...]*

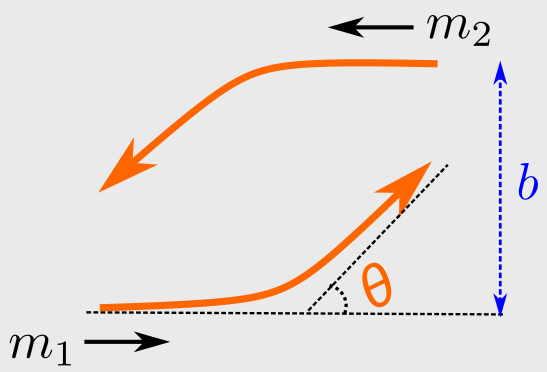
- Most accurate PM binary dynamics until 2019 [Westpfahl, '85]

Scattering angle of two black holes, as function of $m_1, m_2, b, \sigma \equiv v_1 \cdot v_2$,

$$\theta = \frac{4G(m_1+m_2)}{b}\frac{2\sigma^2-1}{2(\sigma^2-1)} + \frac{3\pi}{4}\frac{G^2(m_1+m_2)^2}{b^2}\frac{5\sigma^2-1}{\sigma^2-1} + \mathcal{O}(G^3)$$

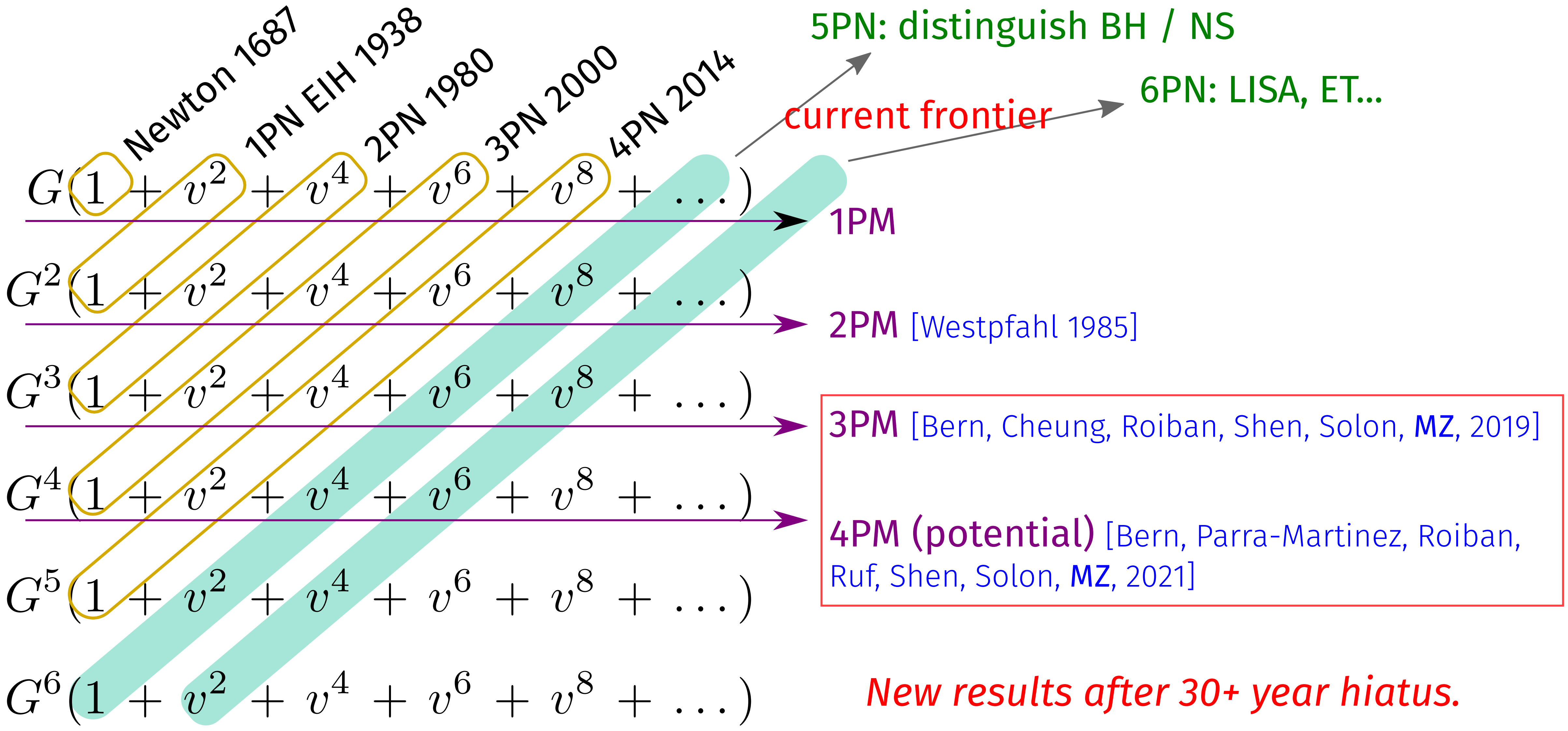
1PM

2PM



- Similar to expansion in **relativistic QFT** - can QFT help push further? What functions appear at higher orders?

NEW RESULTS FOR CONSERVATIVE DYNAMICS



[adapted from Mikhail Solon's slide]

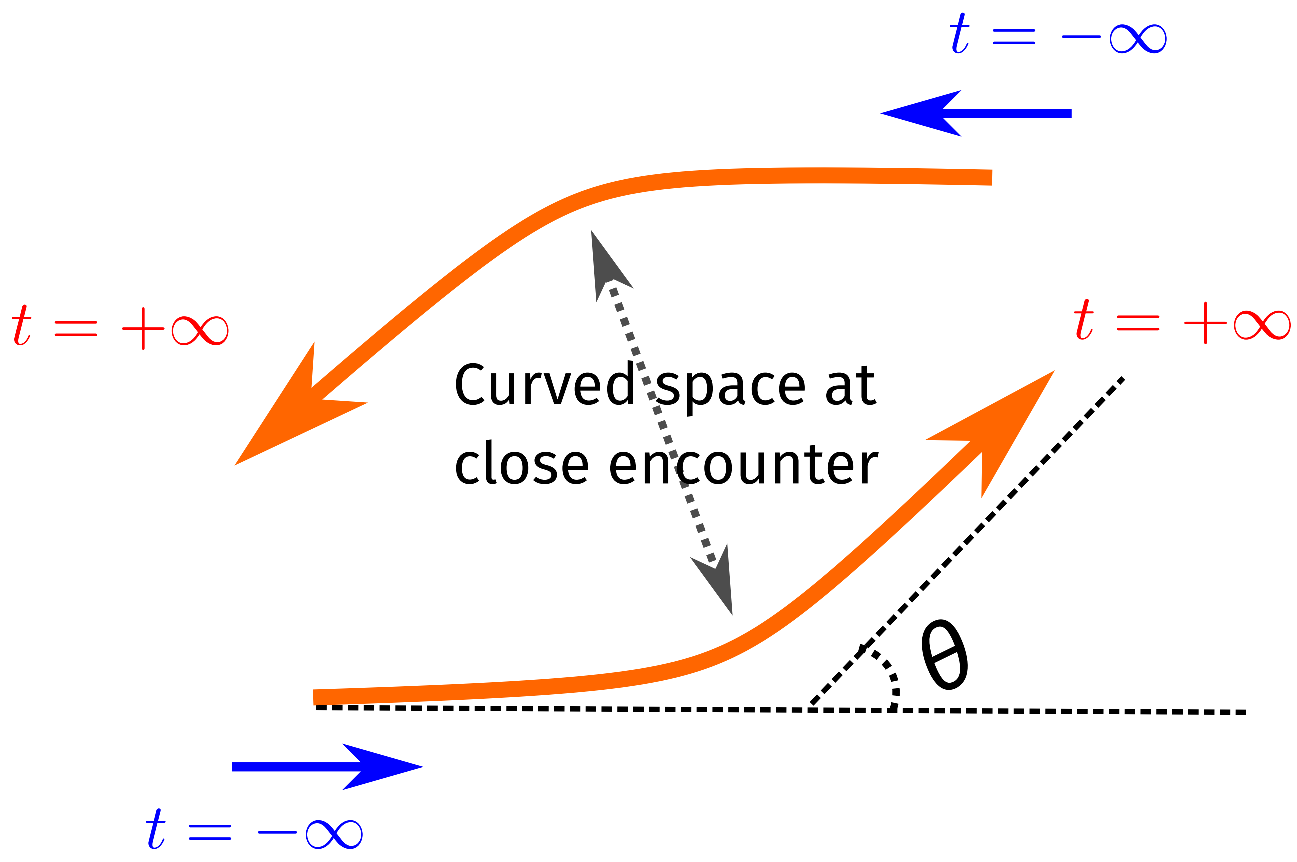
New results after 30+ year hiatus.

Classical from Quantum

HOW QFT HELPS - (1) GAUGE INVARIANCE

GR has ***gauge redundancy***: invariant under general coordinate transformations.

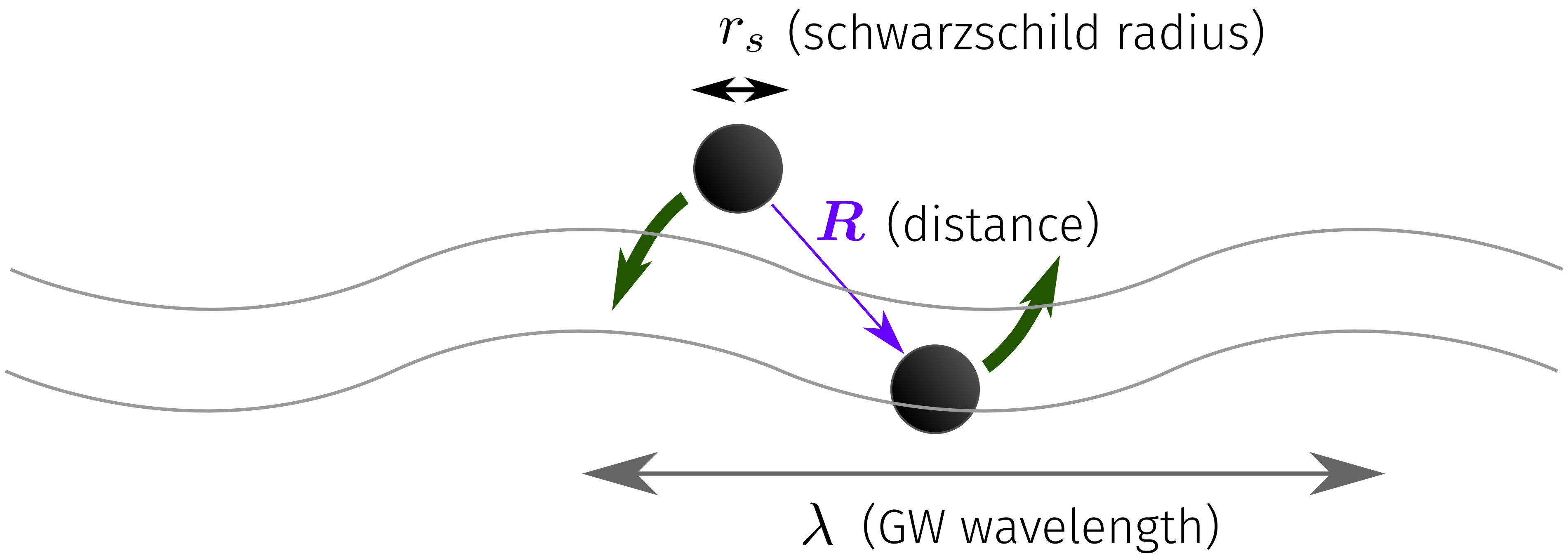
QFT amplitudes defined in asymptotic flat spacetime $t = \pm\infty$
 - gauge invariant, reduces complexity.



HOW QFT HELPS - (2) EFFECTIVE THEORIES

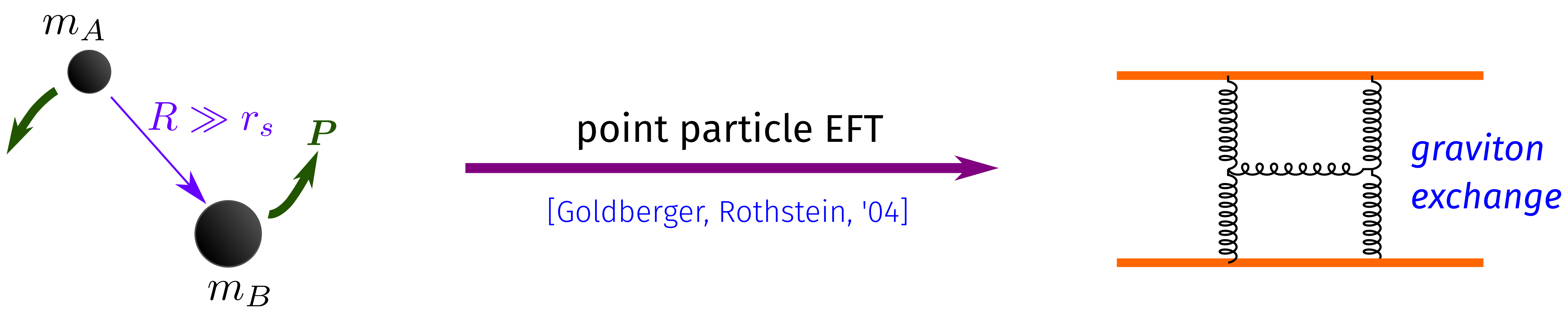
Hierarchy of scales in bound state systems: $R_s \leq r \leq \lambda \implies$

Use **Effective field theory** [Goldberger, Rothstein, '04; Cheung, Rothstein, Solon, '18; Damgaard, Haddad, Helset, '19; Kalin, Porto, '20]



Operator expansion around ***point-particle*** limit.

POINT PARTICLE EFFECTIVE FIELD THEORY



Massive particles (scalar field) coupled to gravity.

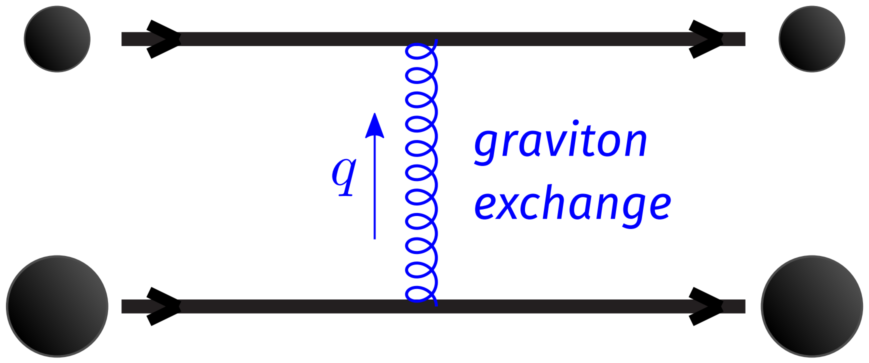
Lagrangian: $S = S_{\text{Einstein-Hilbert}} + S_{\text{point-particle}} + S_{\text{finite-size}}$

Point-particle: scalar field in dynamic metric.

Finite-size (tidal) effect: ***highly suppressed*** for compact objects $\sim \mathcal{O}(G^5)$,

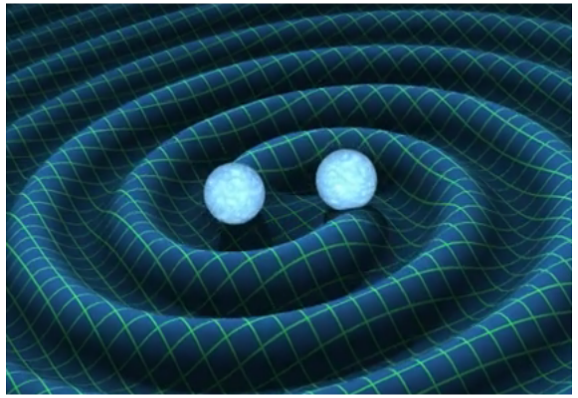
Even more so for black holes $\sim \mathcal{O}(G^6)$.

CLASSICAL FROM QUANTUM: PITFALLS



momentum transfer $q \sim \hbar/R \ll m_1, m_2$

$\hbar \to 0?$



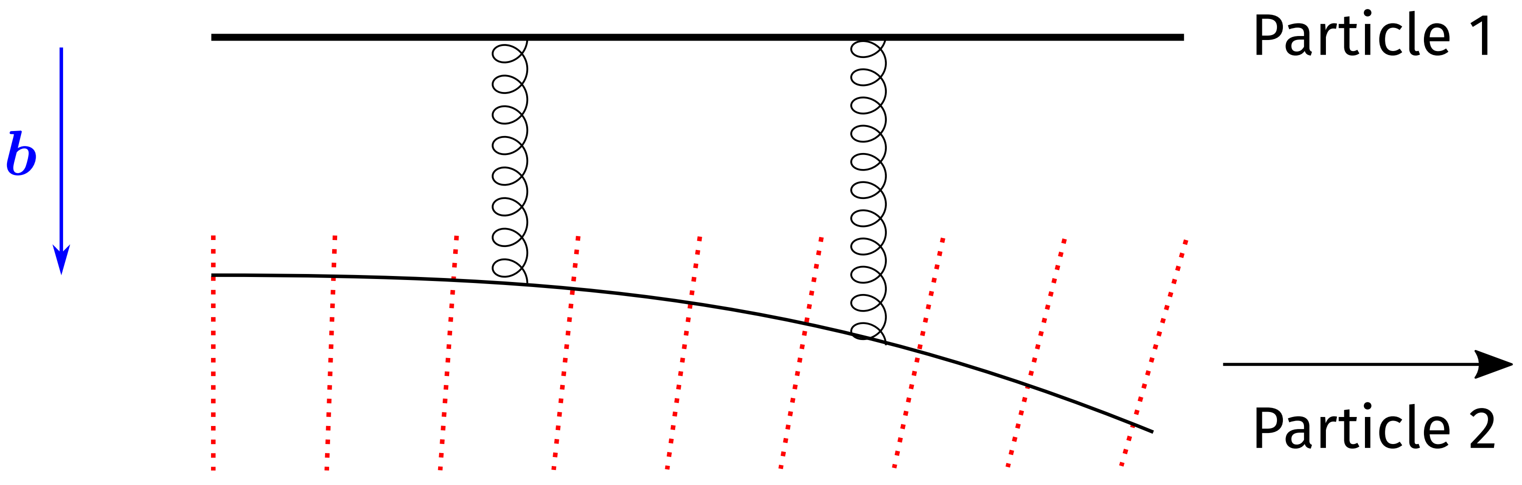
[picture: LIGO]

Naive $\hbar$ expansion won't work. Intuition from JWKB: $\mathcal{M} \sim \exp\left(\frac{i}{\hbar}\int V(x)\,dx\right)$

$\implies \quad 1/\hbar,\ 1/\hbar^2,\ 1/\hbar^3 \ldots$ divergences in perturbative expansion

- referred to as *super-classical* or *classically divergent*

EIKONAL EXPONENTIATION



Conservative amplitude, pure phase: $\tilde{\mathcal{M}}(b) \propto e^{i\delta(b)}$

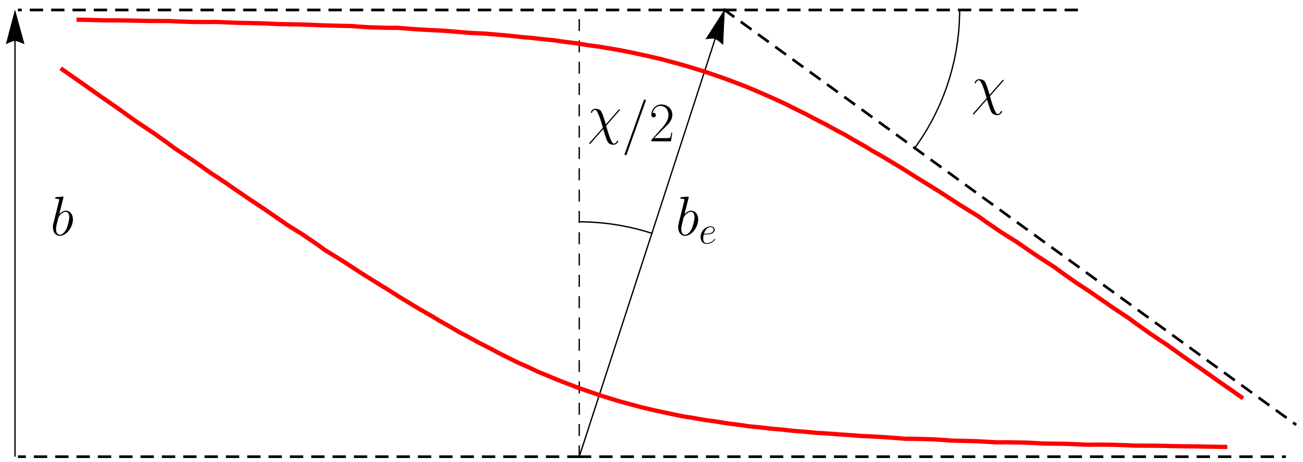
[Glauber, '59; Levy, Sucher, '69; Soldate, '87; 't Hooft, '87; Amati, Ciafoloni, Veneziano, '87, '88, '90, '07; Muzinich, Soldate, '88; Kobat, Ortiz, '92 ...]

Huygens principle: scattering angle ∝ phase gradient $\dfrac{\partial \delta}{\partial b}$

$\tilde{\mathcal{M}}(b)$: Fourier transform of momentum space $\mathcal{M}(q_T)$. Time delay: $\Delta t = \dfrac{\partial \delta}{\partial E}$.

EIKONAL: KINEMATIC CORRECTIONS

- Eikonal impact parameter defined at closest approach.



$$b_e = \frac{b}{\cos(\chi/2)},$$

$$2\sin\frac{\chi}{2} = \frac{\partial\delta}{\partial b_e}$$

[Figure: Bern, Ita, Parra-Martinez, Ruf, 2002.02459]

- All-order steepest descent argument: *[Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White, '21]*
- Alternative formulation: Radial Action I_r, used in *[Bern, Parra-Martinez, Ruf, Shen, Solon, **MZ**, '21]*

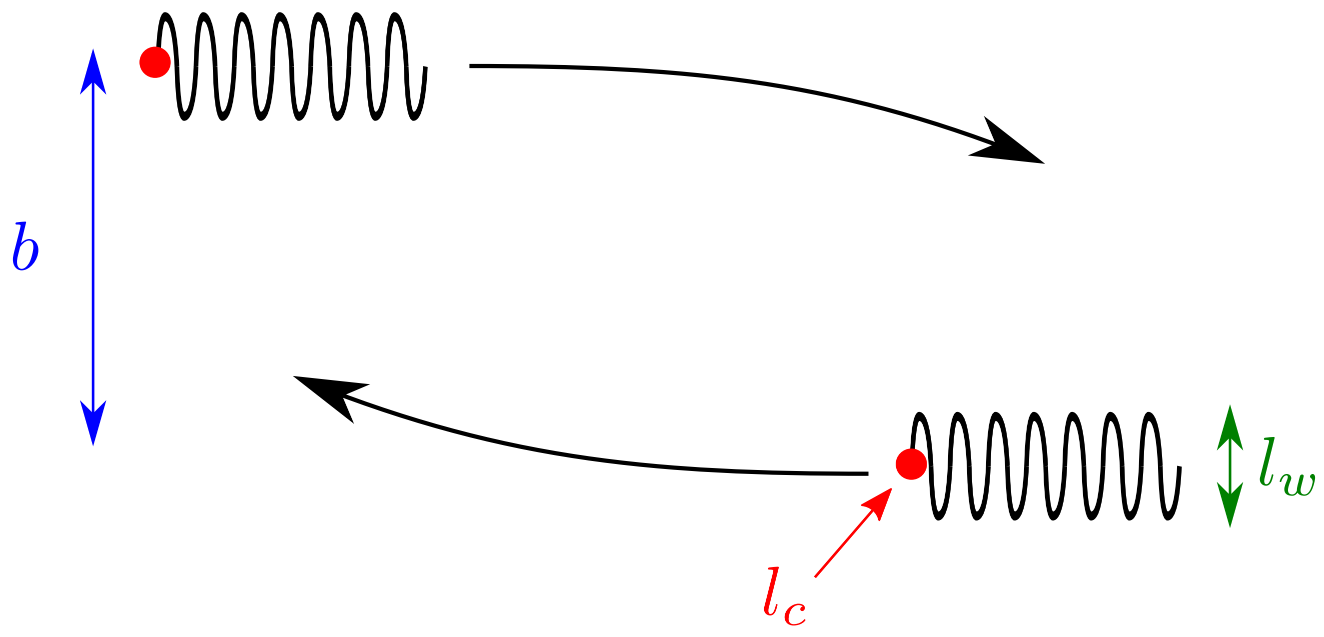
$$\chi = \frac{\partial I_r}{\partial J}, \qquad J \equiv \boldsymbol{p}\cdot\boldsymbol{b}$$

- Also used with HEFT formulation: *[Brandhuber, Chen, Travaglini, Wen, '21]* Phase shift in partial wave scattering: *[Kol, O'Connell, Telem, 2109.12092]* Extensions to spin *[Bern, Roiban, Luna, Shen, **MZ**, '20]* and color charges *[de Cruz, Luna, Scheopner, '21]*

CLASSICAL LIMIT OF QUANTUM OBSERVABLES

[Kosower, Maybe, O'Connel (KMOC), '18]

Compute expectation values of observables from S-matrix,
compoton length l_c $\ll$ wave packet spread l_w $\ll$ impact parameter

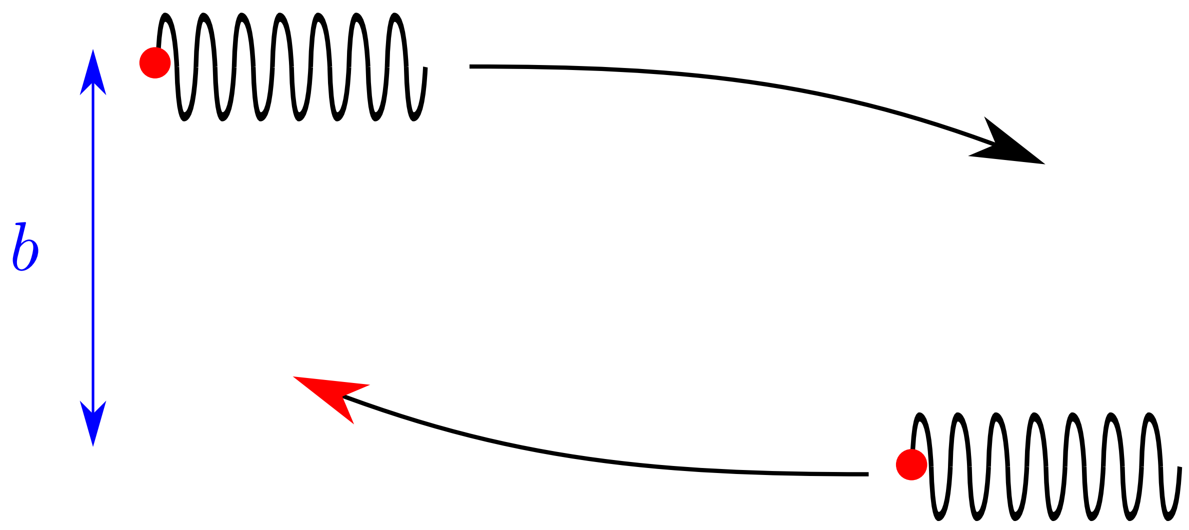


Exact wavepacket shape unimportant for classical limit.

CLASSICAL LIMIT OF QUANTUM OBSERVABLES

[Kosower, Maybe, O'Connel (KMOC), '18]

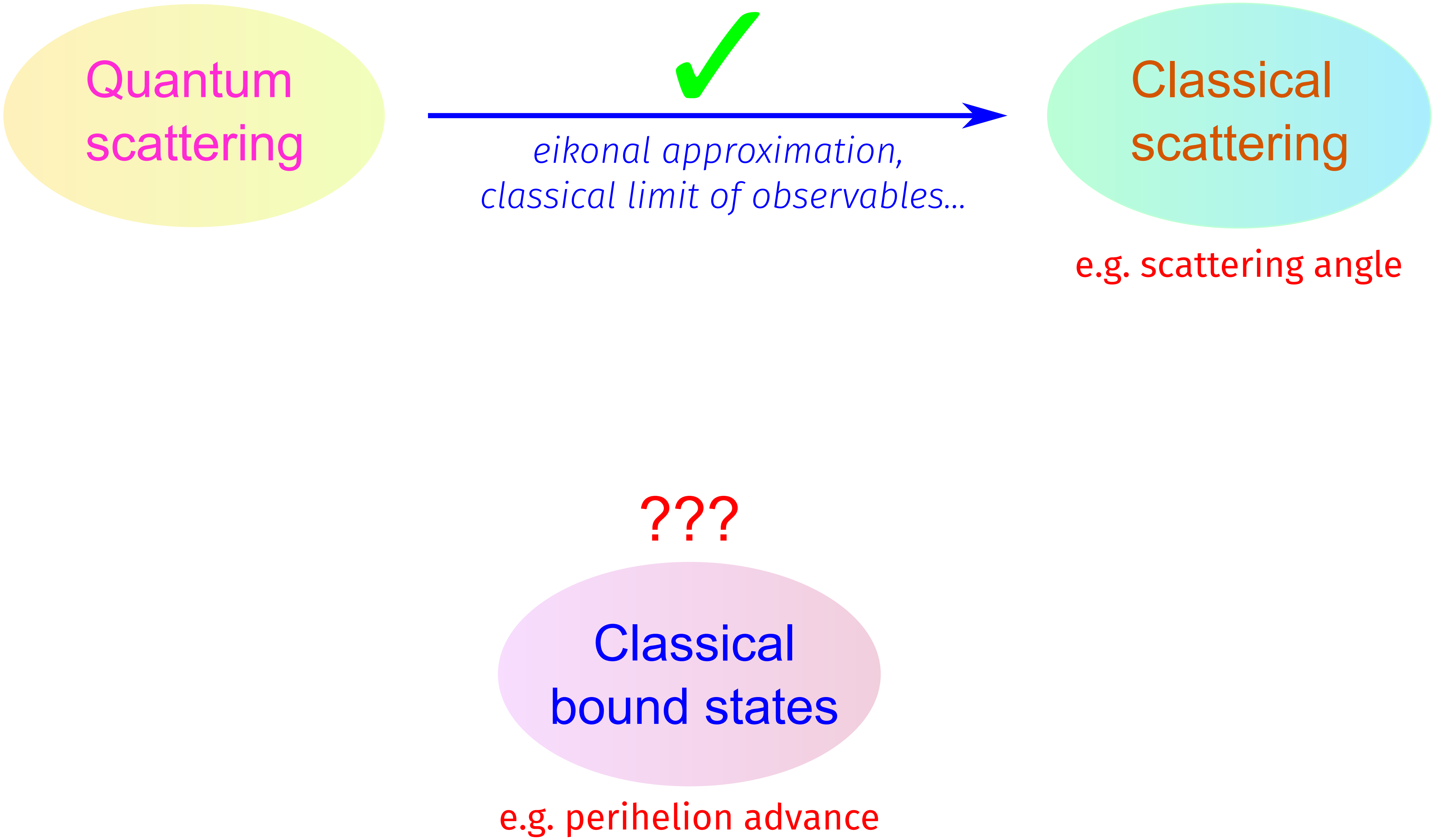
Heuristic: integrate amplitude $\mathcal{M}$ times momentum transfer observable q^μ against wavepacket $e^{i\boldsymbol{b}\cdot\boldsymbol{q}}$.



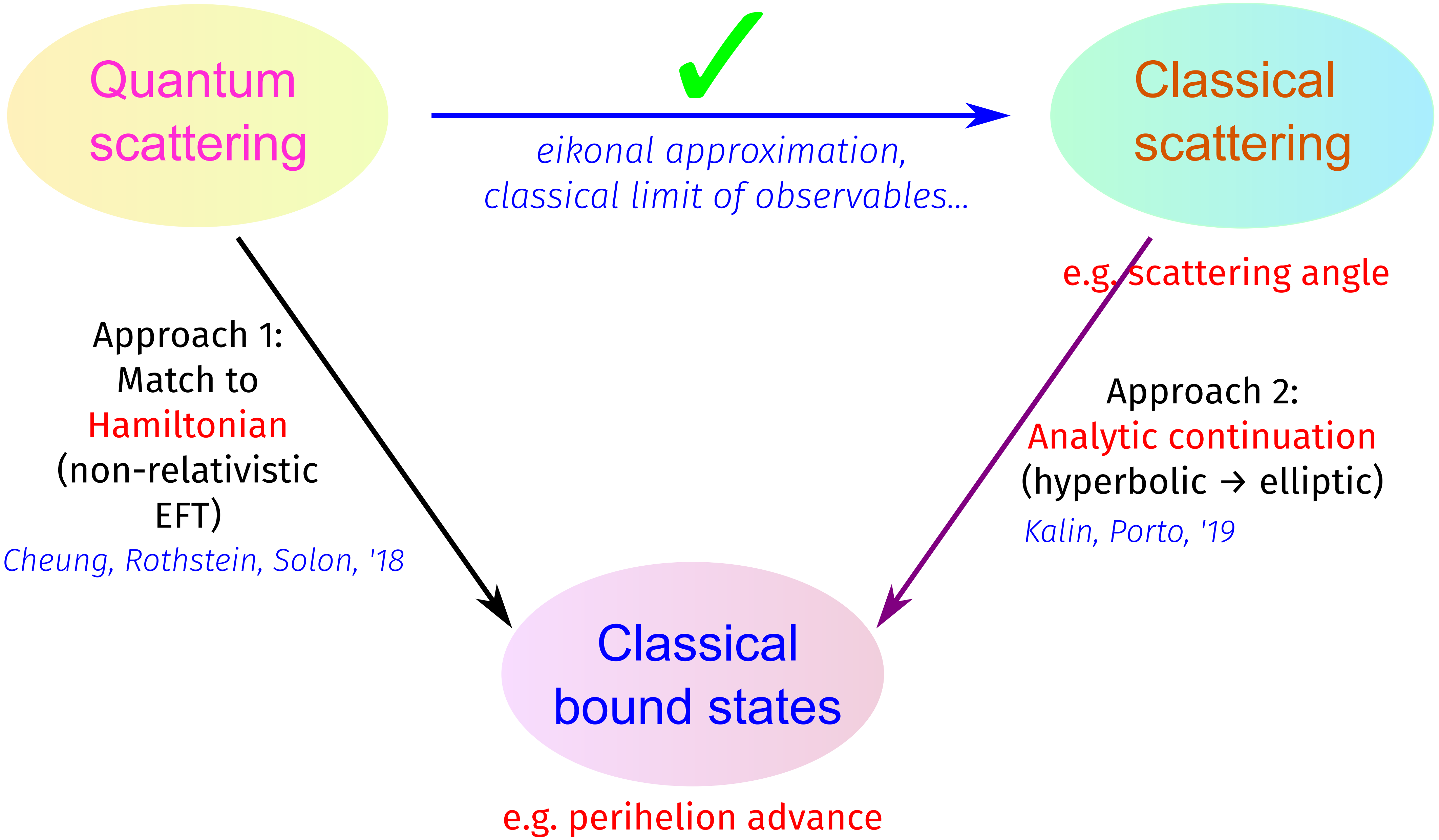
$$\int d^2q_T\,\mathcal{M}(q)\,q^\mu e^{i\boldsymbol{b}\cdot\boldsymbol{q}} \sim \frac{\partial\tilde{\mathcal{M}}(b)}{\partial b}$$

Angle ∝ gradient again!

FROM SCATTERING TO BOUND STATE



FROM SCATTERING TO BOUND STATE



MATCHING TO NON-RELATIVISTIC EFT

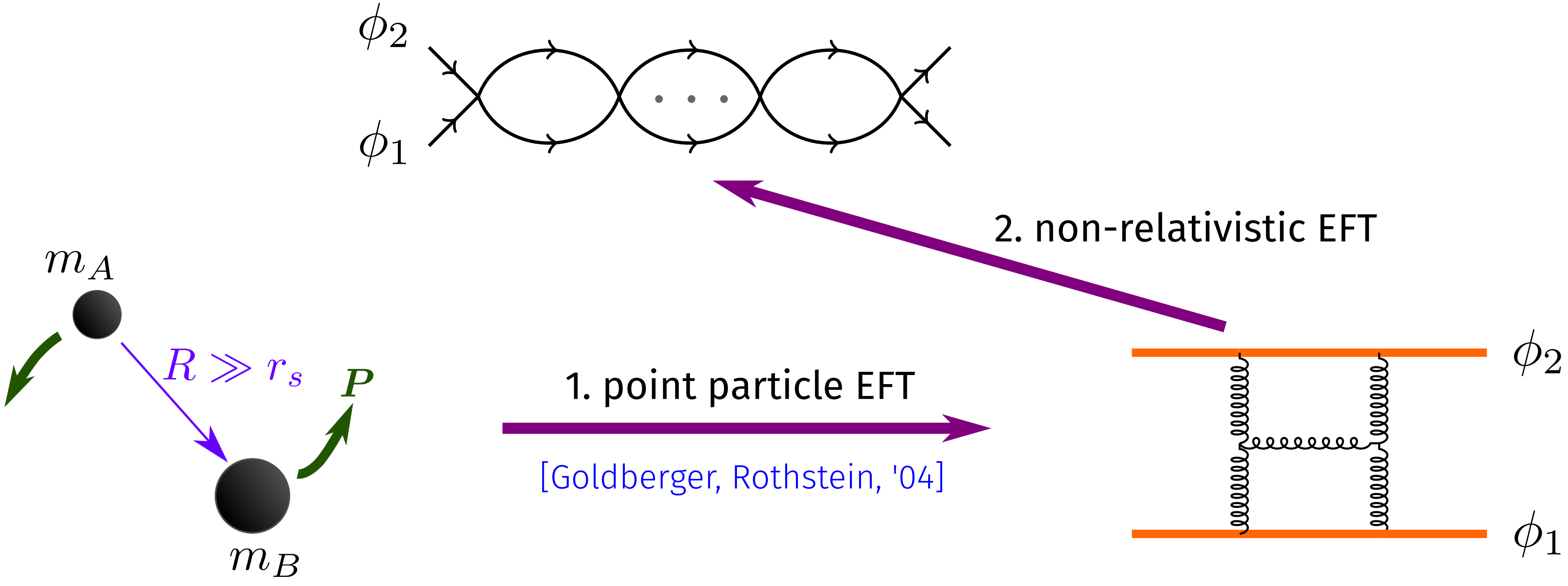
[Cheung, Rothstein, Solon, 1808.02489]

Lagrangian: two scalars, no antiparticles, no particle creation

$$\mathcal{L} = \sum_{i=1,2} \int_{\boldsymbol{k}} \phi_i^\dagger(-\boldsymbol{k}) \left(i\partial_t - \sqrt{\boldsymbol{k}^2 + m_i^2} \right) \phi_i(\boldsymbol{k}) + \int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}') \phi_1^\dagger(-\boldsymbol{k}') \phi_2^\dagger(\boldsymbol{k}') \phi_1(\boldsymbol{k}) \phi_2(-\boldsymbol{k})$$

kinetic term

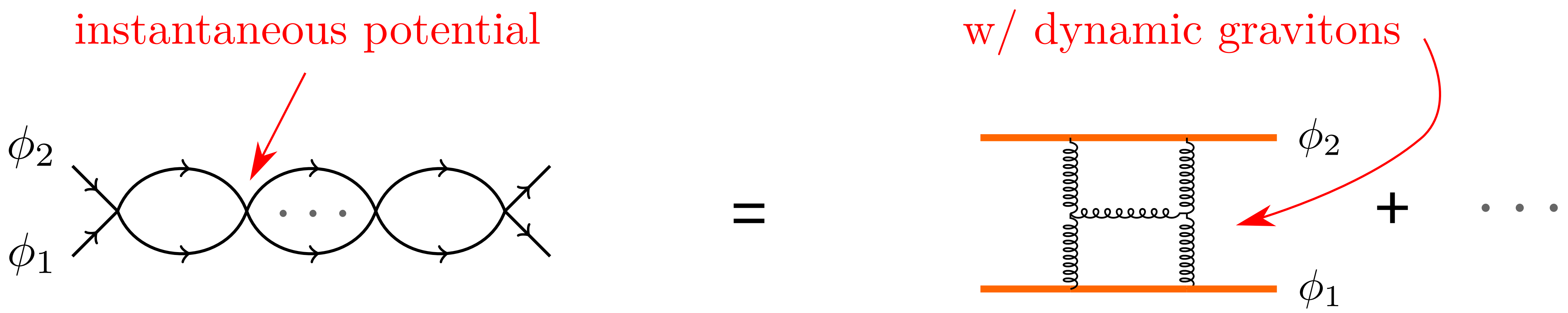
4-scalar contact potential



MATCHING TO NON-RELATIVISTIC EFT

[Cheung, Rothstein, Solon, 1808.02489]

Fix potential from **Matching:** NR EFT amplitude = gravitational scalar amplitude.



EFT amplitude: extremely simple, Feynman diagrams (only bubble diagrams)

Gravity / matter amplitude: need sophisticated modern amplitude methods! See next slides.

$O(G^3)$ / 3PM conservative dynamics: *[Bern, Cheung, Roiban, Shen, Solon, **MZ**, '19 (PRL, JHEP)]* .

$O(G^4)$ / 4PM: *[Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, **MZ**, '21 (PRL)]*

Generalization for spinning BHs: *[Bern, Luna, Roiban, Shen, **MZ**, '20 (PRD)]*

Gravity Amplitudes

PERTURBATIVE GRAVITY

Einstein-Hilbert action

spacetime volume — curvature scalar

$$S_{\text{EH}} = \frac{1}{2\kappa^2} \int d^d x \sqrt{-g}\, R, \qquad \kappa \equiv \sqrt{8\pi G}$$

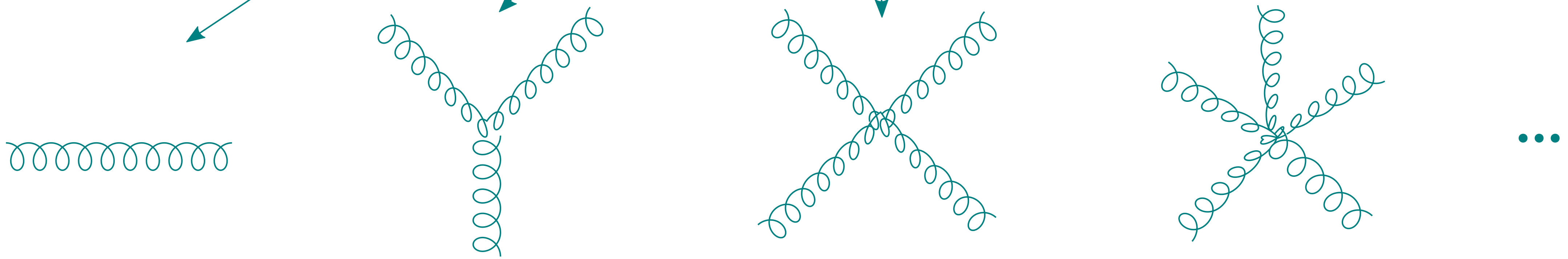
Expand around Minkowskian space: $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$

quadratic kinetic term

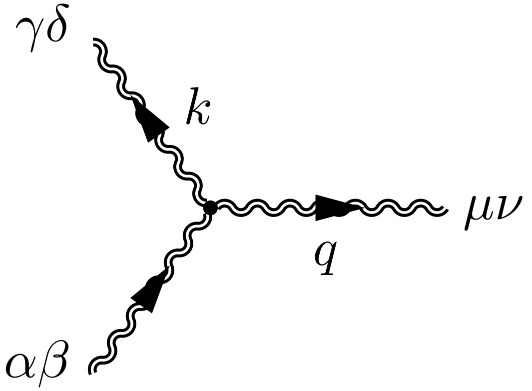
non-linear self couplings...

(omitting index contractions)

$$S_{\text{EH}} = \int d^d x \left[h\partial^2 h + \kappa h^2 \partial^2 h + \kappa^2 h^3 \partial^2 h + \kappa^3 h^4 \partial^2 h + \ldots \right]$$



PERTURBATIVE GRAVITY



$$\begin{aligned}
\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q) = -\frac{i\kappa}{2}\Big\{ & P_{\alpha\beta,\gamma\delta}\Big[k^\mu k^\nu + (k+q)^\mu (k+q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu} q^2\Big] \\
& + 2q_\lambda q_\sigma \Big[I^{\lambda\sigma,}{}_{\alpha\beta} I^{\mu\nu,}{}_{\gamma\delta} + I^{\lambda\sigma,}{}_{\gamma\delta} I^{\mu\nu,}{}_{\alpha\beta} \\
& \qquad - I^{\lambda\mu,}{}_{\alpha\beta} I^{\sigma\nu,}{}_{\gamma\delta} - I^{\sigma\nu,}{}_{\alpha\beta} I^{\lambda\mu,}{}_{\gamma\delta}\Big] \\
& + \Big[q_\lambda q^\mu (\eta_{\alpha\beta} I^{\lambda\nu,}{}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\nu,}{}_{\alpha\beta}) + q_\lambda q^\nu (\eta_{\alpha\beta} I^{\lambda\mu,}{}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\mu,}{}_{\alpha\beta}) \\
& \quad - q^2 (\eta_{\alpha\beta} I^{\mu\nu,}{}_{\gamma\delta} + \eta_{\gamma\delta} I^{\mu\nu,}{}_{\alpha\beta}) - \eta^{\mu\nu} q^\lambda q^\sigma (\eta_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta} I_{\alpha\beta,\lambda\sigma})\Big] \\
& + \Big[2q^\lambda \Big(I^{\sigma\nu,}{}_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} k^\mu + I^{\sigma\mu,}{}_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} k^\nu \\
& \qquad - I^{\sigma\nu,}{}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} (k+q)^\mu - I^{\sigma\mu,}{}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} (k+q)^\nu \Big) \\
& \quad + q^2 (I^{\sigma\mu,}{}_{\alpha\beta} I_{\gamma\delta,\sigma}{}^\nu + I_{\alpha\beta,\sigma}{}^\nu I^{\sigma\mu,}{}_{\gamma\delta}) \\
& \quad + \eta^{\mu\nu} q^\lambda q_\sigma (I_{\alpha\beta,\lambda\rho} I^{\rho\sigma,}{}_{\gamma\delta} + I_{\gamma\delta,\lambda\rho} I^{\rho\sigma,}{}_{\alpha\beta})\Big] \\
& + \Big[(k^2 + (k+q)^2) \Big(I^{\sigma\mu,}{}_{\alpha\beta} I_{\gamma\delta,\sigma}{}^\nu + I^{\sigma\nu,}{}_{\alpha\beta} I_{\gamma\delta,\sigma}{}^\mu - \frac{1}{2}\eta^{\mu\nu} P_{\alpha\beta,\gamma\delta}\Big) \\
& \quad - ((k+q)^2 \eta_{\alpha\beta} I^{\mu\nu,}{}_{\gamma\delta} + k^2 \eta_{\gamma\delta} I^{\mu\nu,}{}_{\alpha\beta})\Big]\Big\}
\end{aligned}$$

Background field gauge vertex [Holstein,Ross, 0802.0716]

Leading non-linearity: ~100 terms in 3-graviton vertex! Quickly grows out of control.

Double copy & Generalized Unitarity to the rescue

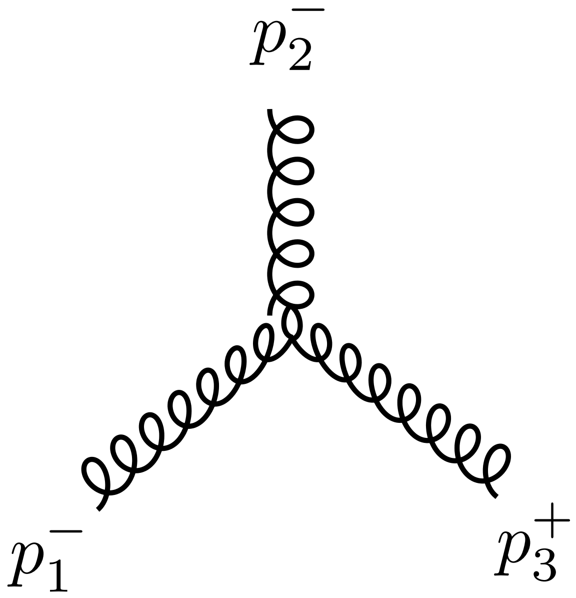
Double copy: gravity from YM^2

Generalized Unitarity: Loops from trees

Alternative based on Feynman diagrams: Tuning nonlinear gauge fixing term: *[Cheung, Remmen, '16, '17; Rafie-Zinedine, '18]*

GRAVITY AMPLITUDES FROM YANG-MILLS

- Gravity = (Yang-Mills)^2. 3-point amplitude example:



Yang-Mills *Gravity*

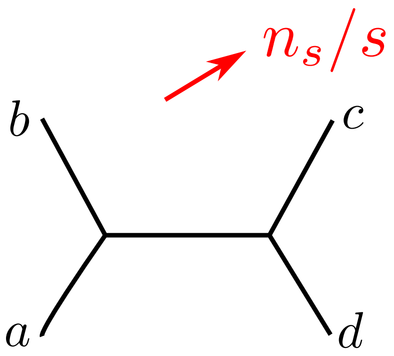
$$\mathcal{A}_3(1^-2^-3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}, \quad \mathcal{M}_3(1^-2^-3^+) = \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}$$

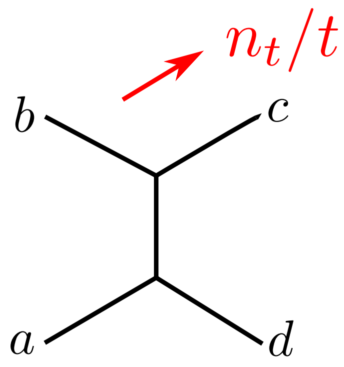
square!

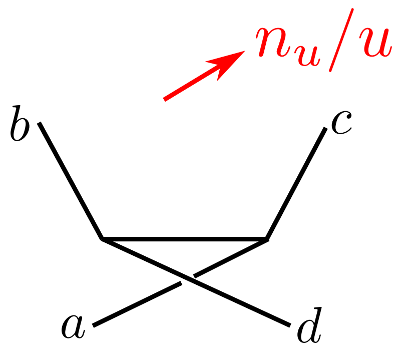
- 4-points and above: generally a ***sum*** of YM × YM expressions.
  - Early example: *Kawai-Lewellen-Tye relations* (string theory)
  - Local Feynman diagram-like version: *BCJ (Bern-Carrasco-Johansson) double copy*.

DOUBLE COPY / COLOR-KINEMATIC DUALITY

- *D* dimensions: easier to get nice analytic integrands from double copy *[Bern, Carrasco, Johansson, '08]*
- Simplest example: ***4-gluon amplitude***, with 4-point vertex "blown up" to 3-pt.

$\mathcal{A}_4 \; = \; C_s$ 

$+ \; C_t$ 

$+ \; C_u$ 

$$C_s = f^{abf} f^{cdf}, \quad C_t = f^{bcf} f^{daf}, \quad C_u = f^{afc} f^{dbf},$$

Jacobi: $C_s + C_t + C_u = 0.$ **Surprise:** $n_s + n_t + n_u = 0.$

color ⟷ kinematics

Gravity: $\mathcal{M}_4 = \mathcal{A}_4\Big|_{C_i \to n_i} = n_s^2/s + n_t^2/t + n_u^2/u$

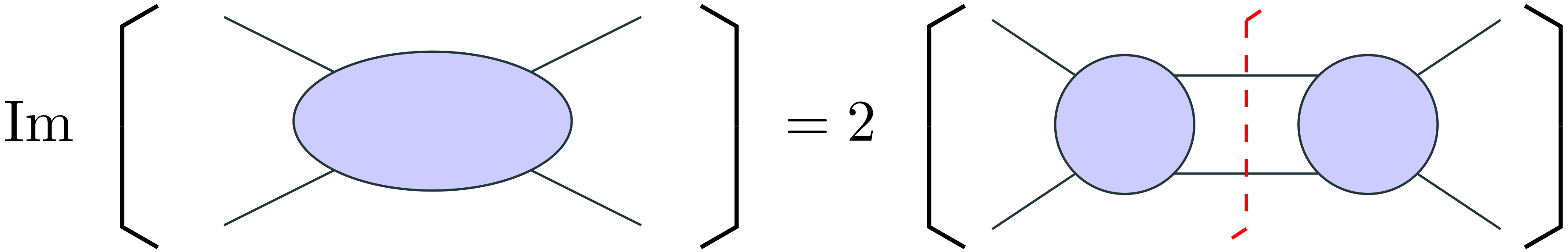
Also applicable with ***massive scalars***, by dimensional reduction.

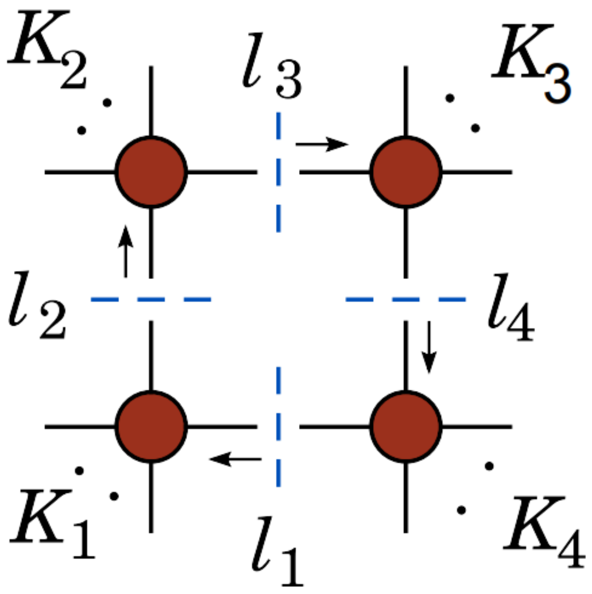
HEFT double copy: *[Brandhuber, Chen, Travaglini, Wen, '21]*

GENERALIZED UNITARITY

[Bern, Dixon, Dunbar, Kosower, '94. Britto, Cachazo, Feng, '04]

Optical theorem: *Imaginary part of forward amplitude = product of two decay amplitudes*





Generalized unitarity: Cutting with ***complex*** l^μ.

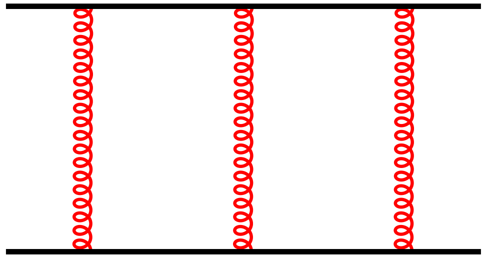
Box diagram example: $l_1^2 = l_2^2 = l_3^2 = l_4^2 = 0$.

Loop integrand factorizes into product of tree amplitudes.

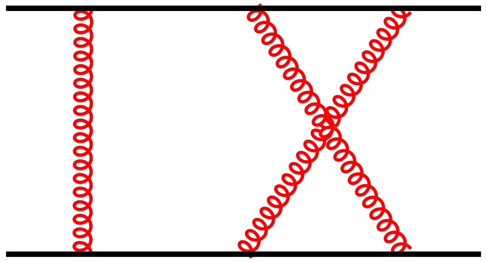
Importing collider methods

CHALLENGES IN LOOP INTEGRATION

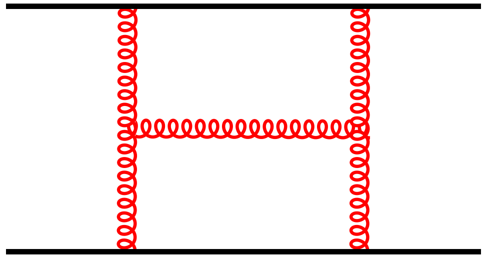
GW physics needs 3 loops and beyond, but exact evalulation is very difficult already at 2 loops: most results are for planar diagrams only, with $m_1 = m_2$.



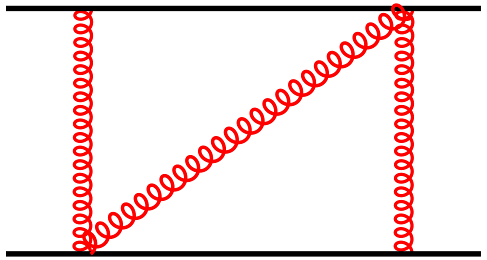
Smirnov, '01;
Henn, Smirnov, '13
Two-mass: Heller '21



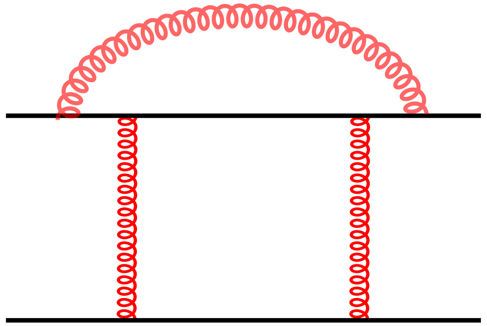
Heinrich, Smirnov, '04
(only the $1/\epsilon^2$ pole)



Leoni, Bianchi, '16;
Kreer, Weinzierl, '21



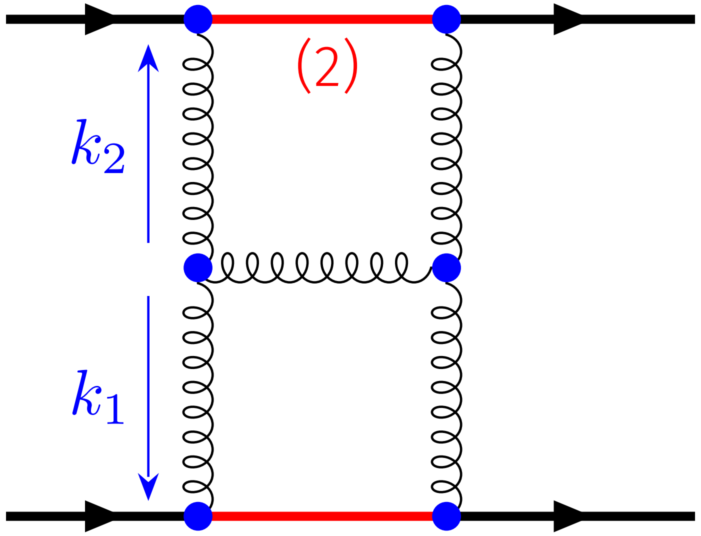
Heller, von Manteuffel, Schabinger, 19;
Broedel, Duhr, Dulat, Penante, Tancredi, '19



Duhr, Smirnov, Tancredi, '21

METHOD OF REGIONS

Feynman integrals expanded at integrand level; sum over "regions" *[Beneke, Smirnov, '98]*



Soft region:

$$|k_1| \sim |k_2| \sim |q| \sim \frac{\hbar}{R} \ll m_1, m_2, \sqrt{s}$$

Kinematics:

$$u_1^2 = u_2^2 = 1,\ u_1 \cdot q = u_2 \cdot q = 0,$$

$$q^2 = t, \quad u_1 \cdot u_2 = \sigma$$

Taylor-expanding matter propagators:

$$(1):\ \ (k_1 + p_1)^2 - m_1^2 \approx 2 m_1 u_1 \cdot k_1 + i0$$

$$(2):\ \ (k_2 + p_2)^2 - m_2^2 \approx 2 m_2 u_2 \cdot k_2 + i0$$

After Expansion in t, remaining nontrivial parameter σ.

DIFFERENTIAL EQUATIONS [Kotikov; Bern, Dixon, Kosower; Remiddi; Gehrmann, Remiddi; Henn]

PN expansion
$q^2 \ll (mv)^2 \ll m^2$

differential equations →
evolution w.r.t. kinematic variable (velocity)
← **consistency conditions:**
cancellation of spurious singularities

PM expansion
$q^2 \ll m^2,\, v \sim \mathcal{O}(1)$

Prolific method for cutting-edge collider amplitude. **Imported into post-Minkowskian gravity:** *[Parra-Martinez, Ruf, **MZ**, '20].* Already widely adopted: *[Kalin, Porto, '20; Di Vecchia, Heissenberg, Russo, Veneziano, '21; Herrmann, Parra-Martinez, Ruf, **MZ**, '21; Bjerrum-Bohr, Damgaard, Plante, Vanhove, '21 ...]*

Amplitude is a sum over "master integrals" by integration by parts: $\mathcal{M} = c_i\, I_i$

Kinematic derivatives also reduced to sum over master integrals themselves: $\dfrac{\partial}{\partial v} I_i = A_{ij}\, I_j$

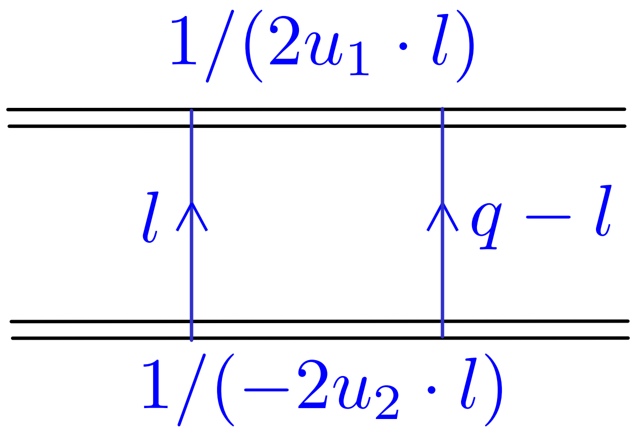
$v = 0$ —— $\partial/\partial v$ ——→ $v = 1$

Near-static boundary conditions

"Post-Minkowskian": exact in velocity

Ultra-relativistic limit

DIFFERENTIAL EQUATIONS



$u_1^2 = u_2^2 = 1,\ u_1 \cdot u_2 = y = \sqrt{1+v^2}$ Rationalization: $y = \frac{1+x^2}{2x}$

Physical region: $0 < x < 1$ Euclidean region: $-1 < x < 0$

symbol letters: $x, 1 \pm x, 1 + x^2$

Similar to letters in massive cusp anomalous dimension
[Bruser, Dlapa, Henn, Yan, '20]

rational matrix

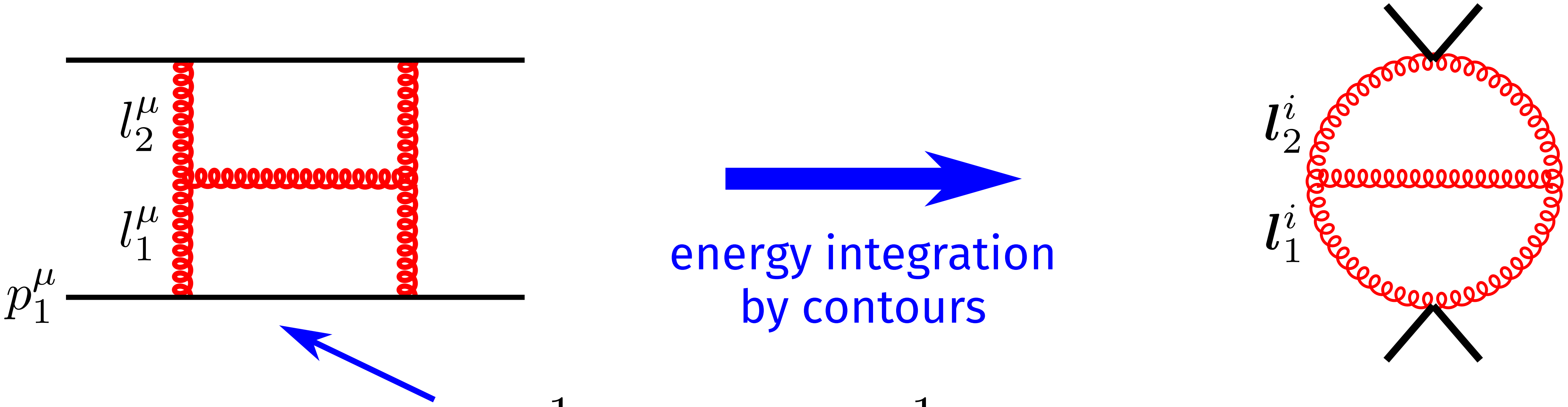
$$\frac{\partial}{\partial x}\vec{I} = \epsilon \sum_r \frac{\partial \log W_r}{\partial x} \mathbb{M}^{(r)} \cdot \vec{I}$$

Canonical form: [Henn, '13]

The last letter only appears at 3 loops. For the potential region, smooth static limit implies that $(1-x)$ is never a first entry.

BOUNDARY CONDITIONS - POTENTIAL REGION

- In the potential region, static boundary values of integrals are ***3D spatial integrals***
  - instantaneous post-Newtonian potential.



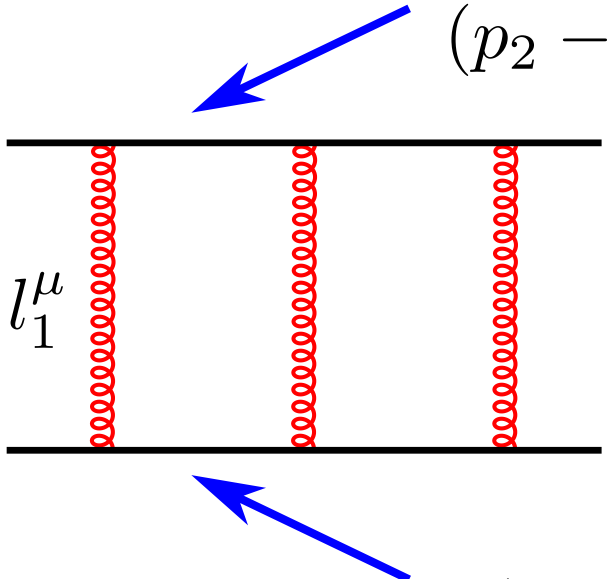
$$\frac{1}{(p_1+l_1)^2-m_1^2} \approx \frac{1}{2m_1 u_1 \cdot l_1}$$

- 3D propagator integrals known to very high orders. ***Differential equations*** give velocity dependence (PM expansion) "for free".

BOUNDARY CONDITIONS - DIVERGENCES

- Iterated graviton exchange creates divergent terms near the static limit.

$$\frac{1}{(p_2 - l_1)^2 - m_2^2} \approx \frac{1}{-2m_2 u_2 \cdot l_1}$$

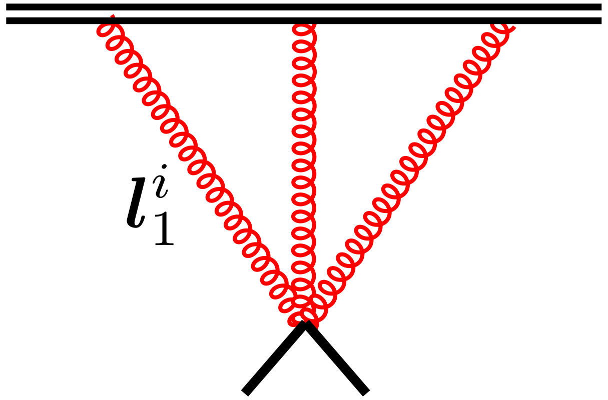


$$\frac{1}{(p_1 + l_1)^2 - m_1^2} \approx \frac{1}{2m_1 u_1 \cdot l_1}$$

energy integration
by contours

$$\frac{1}{v^2}$$

$$\frac{1}{\boldsymbol{n}_z \cdot \boldsymbol{l}_1}$$

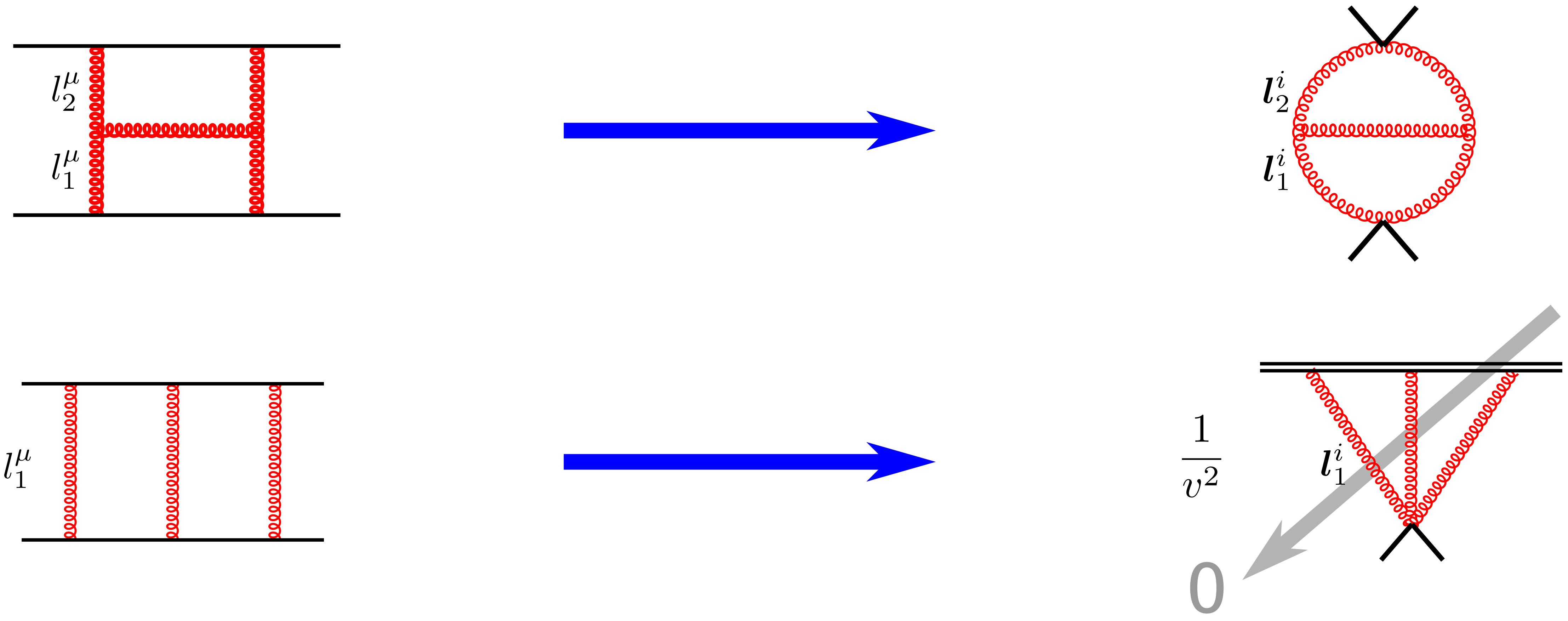


- Not "genuine" higher-loop correction - can we skip evaluating these integrals?

RADIAL ACTION MIRACLE

Bern, Parra-Martinez, Ruf, Shen, Solon, ***MZ****, '21.* *HEFT version: Brandhuber, Chen, Travaglini, Wen, '21*

- Remove divergences from boundary conditions, then solve DEs.

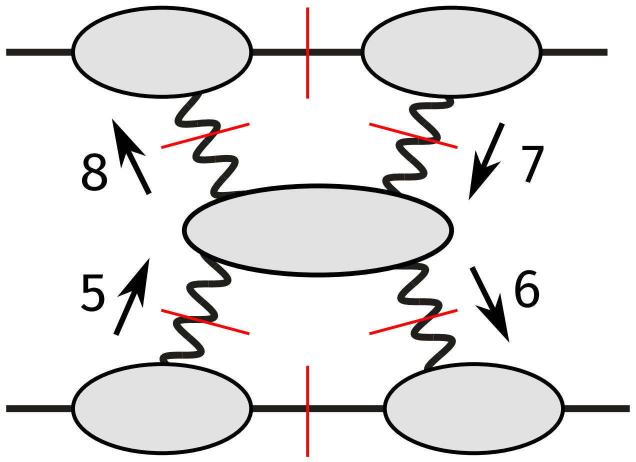


- Fourier transformed amplitude turns into a finite quantity, the ***radial action*** $I_r, \ \chi = \partial I_r / \partial J.$

Results and Comparisons

3PM / 2-LOOP AMPLITUDE - EXAMPLE

[Bern, Cheung, Roiban, Shen, Solon, **MZ**, '19]



Traditional Feynman diagram:

expect more than 10^5 terms.

- 100 terms per 3-graviton vertex
- 3 terms per graviton-scalar vertex
- 3 terms per graviton propagator

All multiplied together!

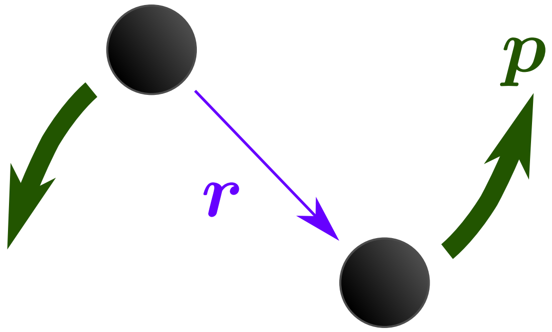
Gravity tree = (Yang-Mills tree)^2 by KLT. **Cut loop amplitude = product of trees** by generalized unitarity.

$$\text{Cut} = -i\left\{2t^2 m_1^2 m_2^4 + \frac{1}{t^6}\left[\text{Tr}[\not{7}\not{2}\not{8}\not{6}\not{1}\not{5})^4] + (7 \leftrightarrow 8)\right]\right\}\left(\frac{1}{(k_5 - k_8)^2} + \frac{1}{(k_6 + k_8)^2}\right)$$

Stay in 4D if you can - spinor helicity amplitudes are simple, though with spurious singularities.

RESULT: 3PM CONSERVATIVE POTENTIAL

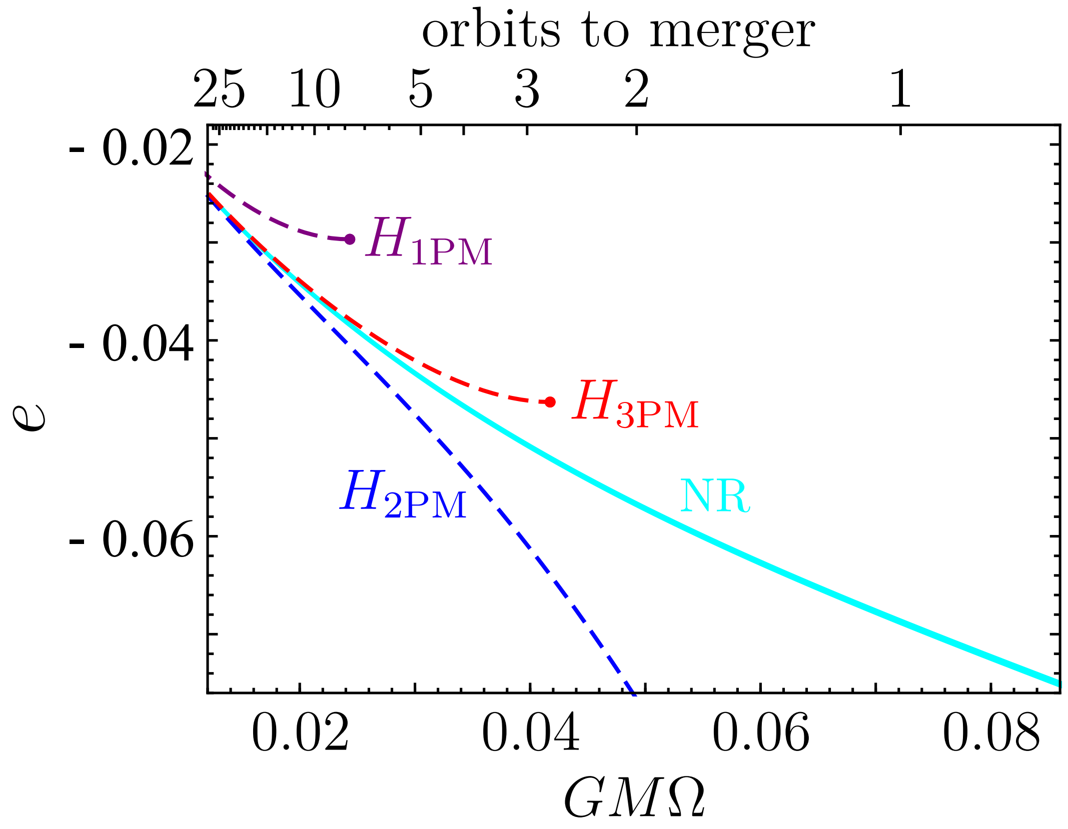
[Bern, Cheung, Roiban, Shen, Solon, **MZ** '19 (PRL)]



$$H^{3\mathrm{PM}}(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V^{3\mathrm{PM}}(\boldsymbol{p},\boldsymbol{r})$$

$$V^{3\mathrm{PM}}(\boldsymbol{p},\boldsymbol{r}) = c_1(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right) + c_2(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^2 + c_3(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^3$$

Westpfahl, '85 Our new result



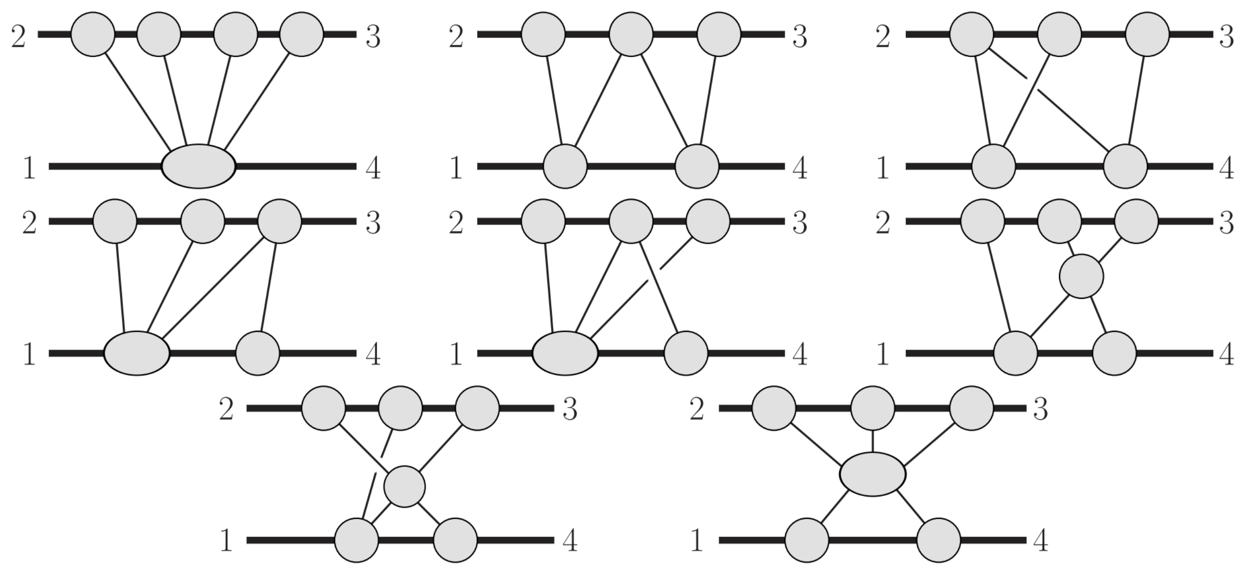
Binding energy: ***improved agreement with numerical relativity*** over lower PM orders.
[Antonelli, Buonanno, Steinhoff, Vines, '19]

Higher orders needed to compete with PN theory.
Next slides: our new result for **4PM!**

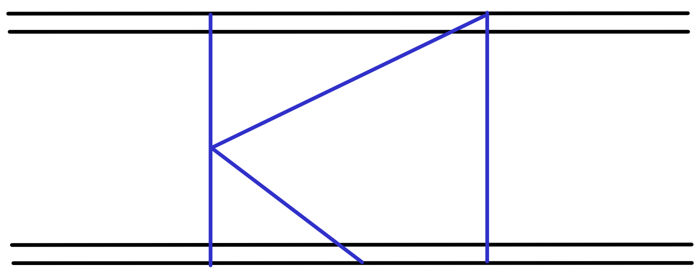
4PM / 3-LOOP - POTENTIAL REGION

*[Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, **MZ**, 2101.07254 (PRL)]*

- ***Loop integrand*** from 8 generalized unitarity cuts.



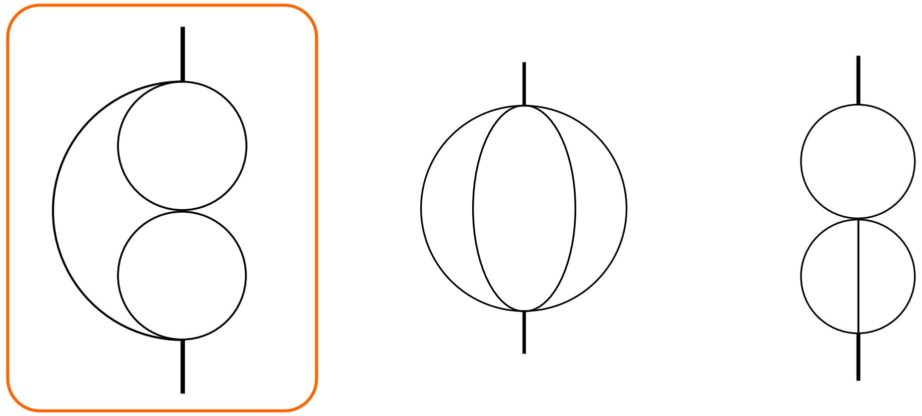
- ***Integration:***
  - IBP reduction with FIRE6 *[Smirnov]*.
  - Used `epsilon` *[Prauso, '17]* to find canonical form for DEs + one elliptic sector



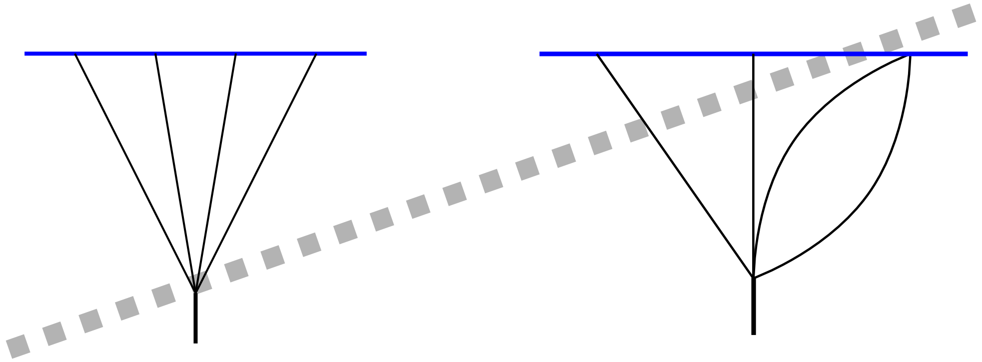
(3 master integrals)

- ***Boundary conditions*** in terms of 3D integrals

Contribute to elliptic integrals after solving DEs

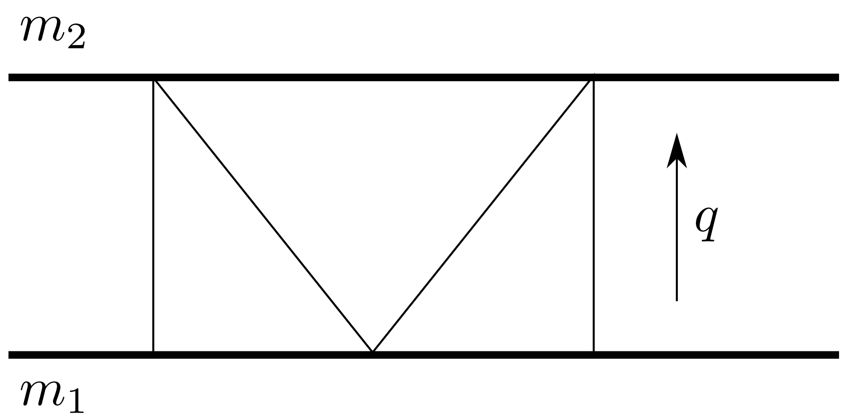


Divergent 3D integrals set to zero



3-LOOP AMPLITUDE

[Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, **MZ**, 2101.07254 (PRL)]



$+ \ldots$

An infrared divergence!

$$\mathcal{M}_4(\boldsymbol{q}) = G^4 M^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{1/3}\tilde{\mu}^2}\right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^p + \nu\left(\frac{\mathcal{M}_4^t}{\epsilon} + \mathcal{M}_4^f\right)\right] + \text{Iterations}$$

$M = m_1 + m_2$

$\nu = \frac{m_1 m_2}{M^2}$, **1SF** term vanishes in probe limit

$\frac{1}{r^4}$ potential after 3D Fourier transform

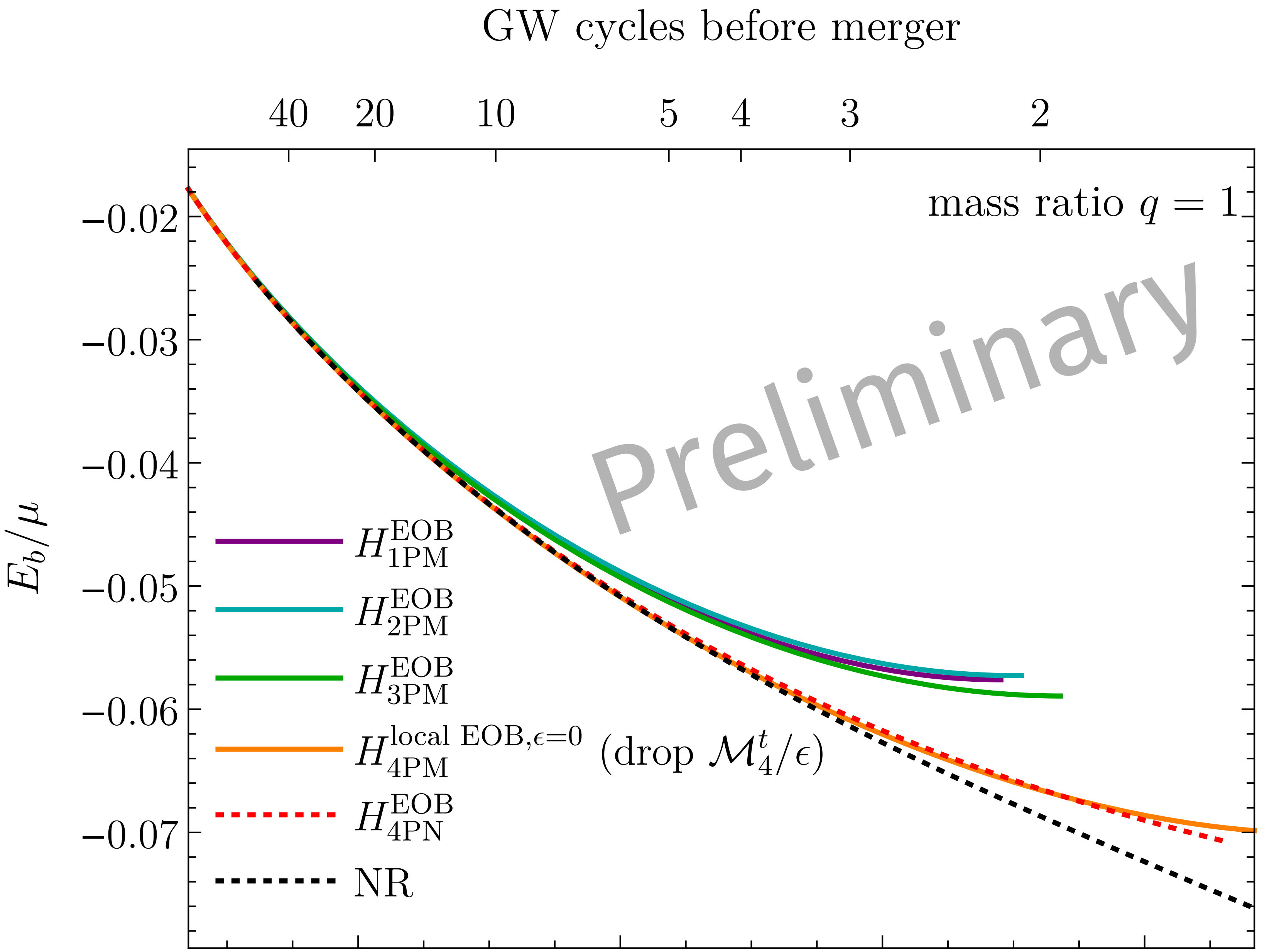
$$\mathcal{M}_4^{\rm p} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}, \quad \mathcal{M}_4^{\rm t} = h_1 + h_2 \log\left(\tfrac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}}.$$

$$\sigma \equiv \frac{p_1 \cdot p_2}{m_1 m_2}$$

h_i are rational functions, see next slide

4PM BINDING ENERGY VS. NUMERICAL RELATIVITY

[Khalil, Buonanno, Steinhoff, Vines, preliminary]



4PM (potential region) $\sim G^4$

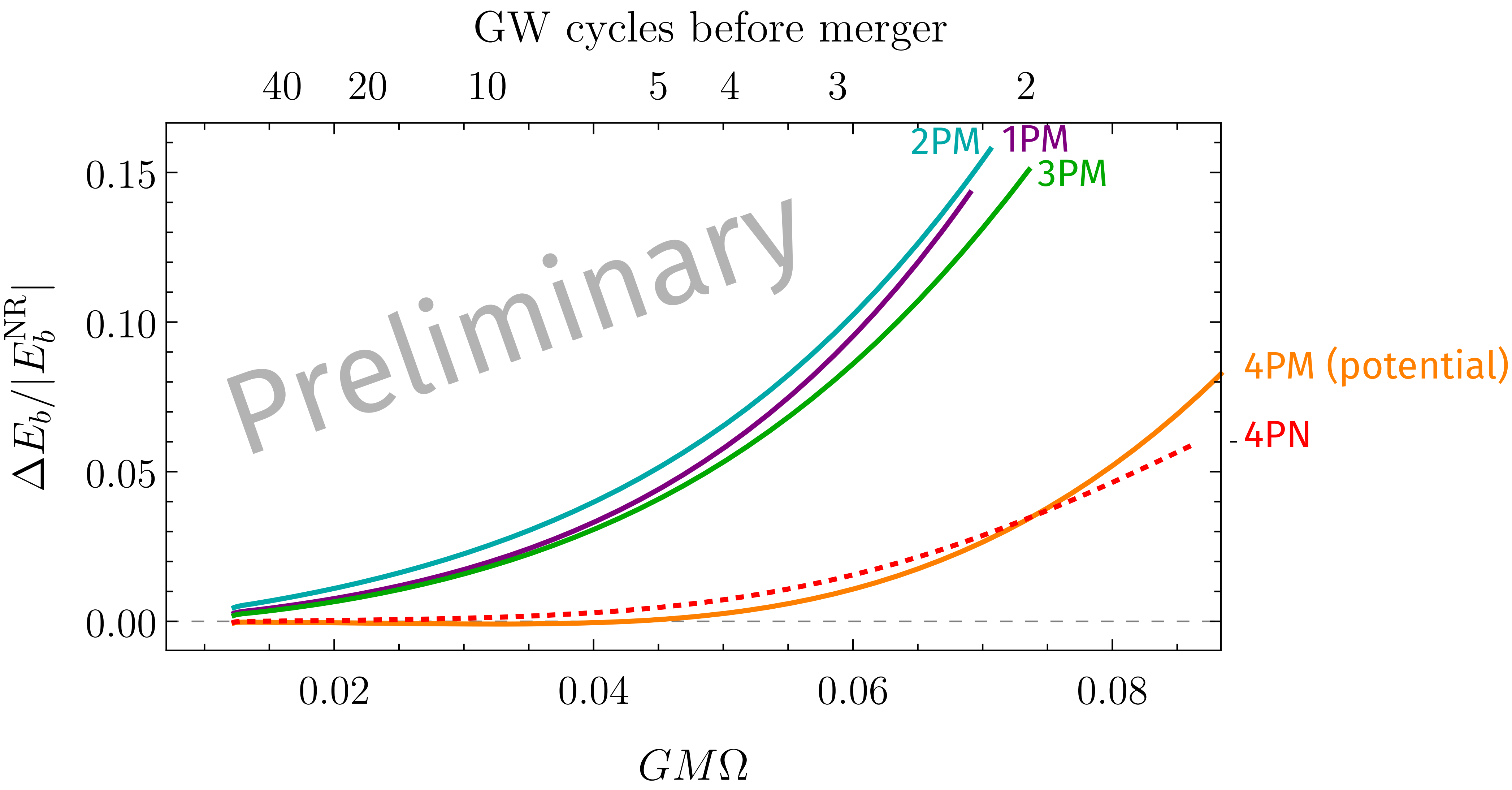
4PN $\sim G^n (v^2)^{5-n},\ 0 \leq n \leq 5$

NR

4PM BINDING ENERGY VS. NUMERICAL RELATIVITY

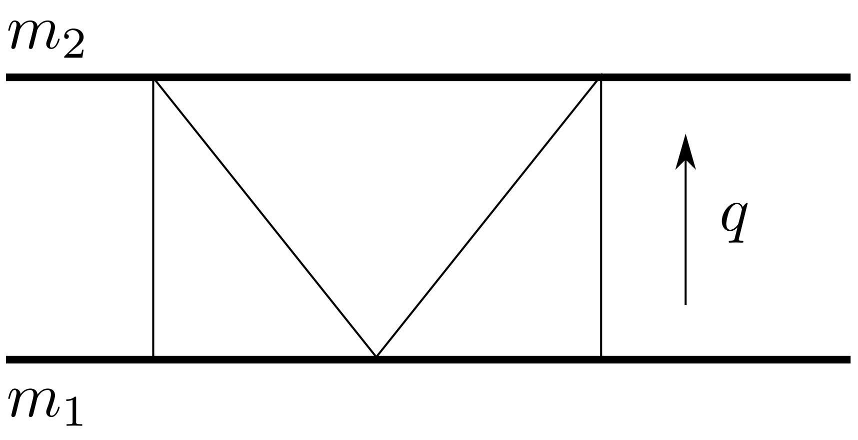
[Khalil, Buonanno, Steinhoff, Vines, preliminary]

- Same plot shown as relative deviation from NR.



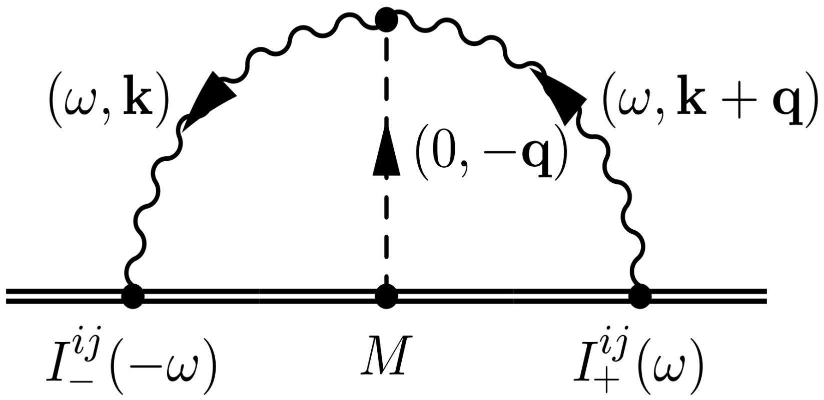
- Post-Minksowskian prediction starts to become competitive - more to come!

IR DIVERGENCE IN 2-BODY POTENTIAL



potential region / near-zone

$+$



radiation region / far-zone

[Gally, Leibovich, Porto, Ross, '15]

Potential region: spatial momentum exchange between two bodies

$$\mathcal{M}_4^{\rm pot}(\boldsymbol{q}) \propto \left[\frac{\mathcal{M}_4^t}{\epsilon} + {\rm finite}\right] + {\rm Iterations}$$

Radiation region: couples to multipole moments of binary system

$$\mathcal{M}_4^{\rm rad}(\boldsymbol{q}) \propto \left[-\frac{\mathcal{M}_4^t}{\epsilon} + 2\log(v^2) + {\rm finite}\right]$$

We just computed this for unbound orbits - see next slide

Divergence cancels in the sum, leaving $\log(v)$ term analogous to Lamb shift in QED with $\log(\alpha)$ term.

FINITE 3-LOOP AMPLITUDE: POTENTIAL + TAIL

[Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, **MZ**, 2112.10750 (PRL)]

$$\mathcal{M}_4^{\text{cons}} = G^4 M^7 \nu^2 |\boldsymbol{q}| \pi^2 \left[\mathcal{M}_4^{\text{probe}} + \nu \left(4\mathcal{M}_4^{\text{tail}} \log\left(\frac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}} \right) \right] + \text{Iterations}$$

$p_\infty \equiv \sqrt{(u_1 \cdot u_2)^2 - 1}$

$$\mathcal{M}_4^{\text{probe}} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}, \quad \mathcal{M}_4^{\text{tail}} = r_1 + r_2 \log\left(\frac{\sigma+1}{2}\right) + r_3 \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}},$$

$$\mathcal{M}_4^{\pi^2} = r_4 \pi^2 + r_5\, K\left(\frac{\sigma - 1}{\sigma + 1}\right) E\left(\frac{\sigma-1}{\sigma+1}\right) + r_6\, K^2\left(\frac{\sigma - 1}{\sigma + 1}\right) + r_7\, E^2\left(\frac{\sigma - 1}{\sigma + 1}\right),$$

$$\mathcal{M}_4^{\text{rem}} = r_8 + r_9 \log\left(\frac{\sigma + 1}{2}\right) + r_{10} \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} + r_{11} \log(\sigma) + r_{12} \log^2\left(\frac{\sigma + 1}{2}\right) + r_{13} \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} \log\left(\frac{\sigma + 1}{2}\right)$$
$$+ r_{14} \frac{\text{arccosh}^2(\sigma)}{\sigma^2 - 1} + r_{15}\, \text{Li}_2\left(\frac{1-\sigma}{2}\right) + r_{16}\, \text{Li}_2\left(\frac{1 - \sigma}{1 + \sigma}\right) + r_{17} \frac{1}{\sqrt{\sigma^2 - 1}} \left[\text{Li}_2\left(-\sqrt{\frac{\sigma - 1}{\sigma + 1}}\right) - \text{Li}_2\left(\sqrt{\frac{\sigma - 1}{\sigma + 1}}\right) \right]$$

complete elliptic integrals of the 1st & 2nd kind

polylogarithms up to transcendental weight 2

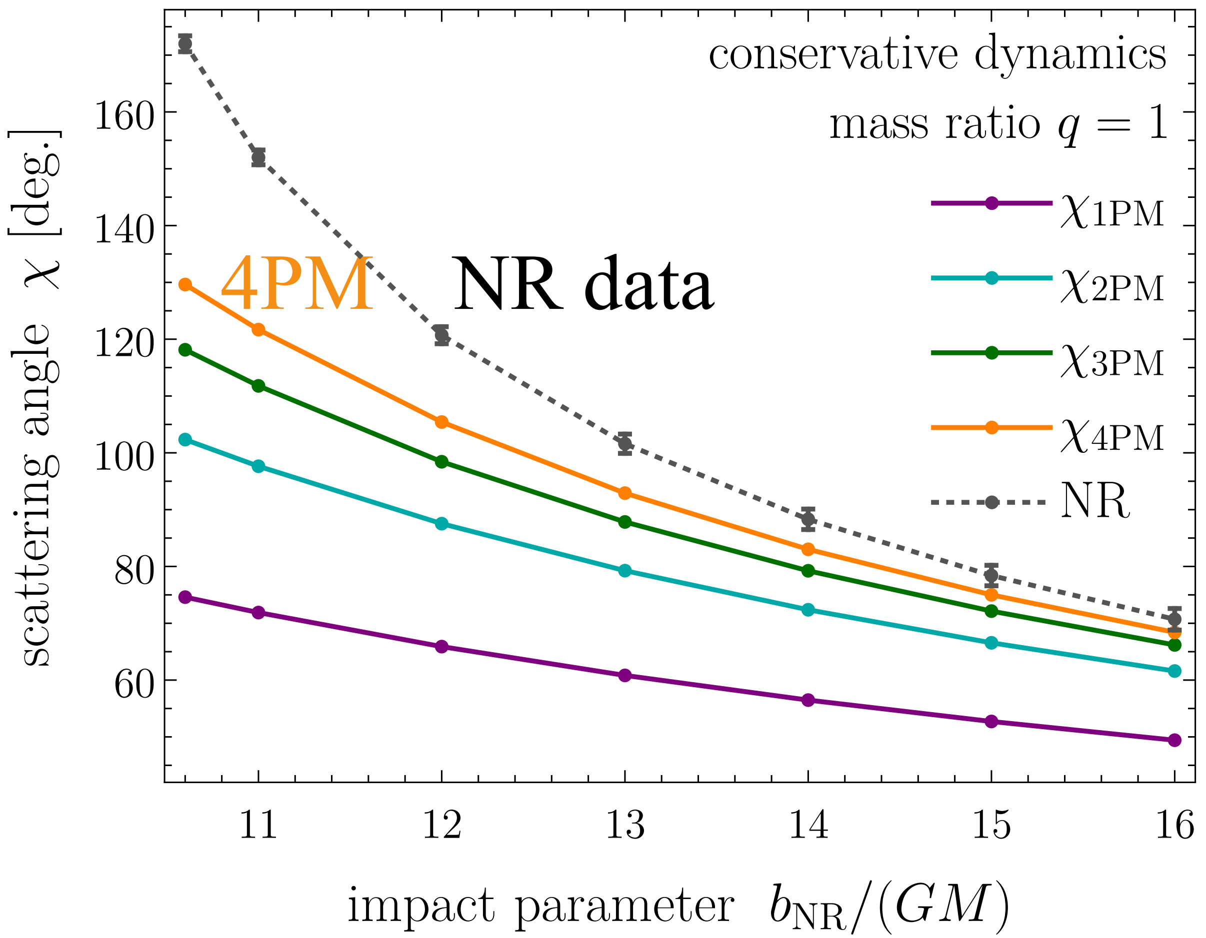
Rational functions:

$$r_1 = \frac{1151 - 3336\sigma + 3148\sigma^2 - 912\sigma^3 + 339\sigma^4 - 552\sigma^5 + 210\sigma^6}{12(\sigma^2 - 1)}, \quad r_2 = \frac{1}{2}\left(5 - 76\sigma + 150\sigma^2 - 60\sigma^3 - 35\sigma^4\right), \quad \ldots$$

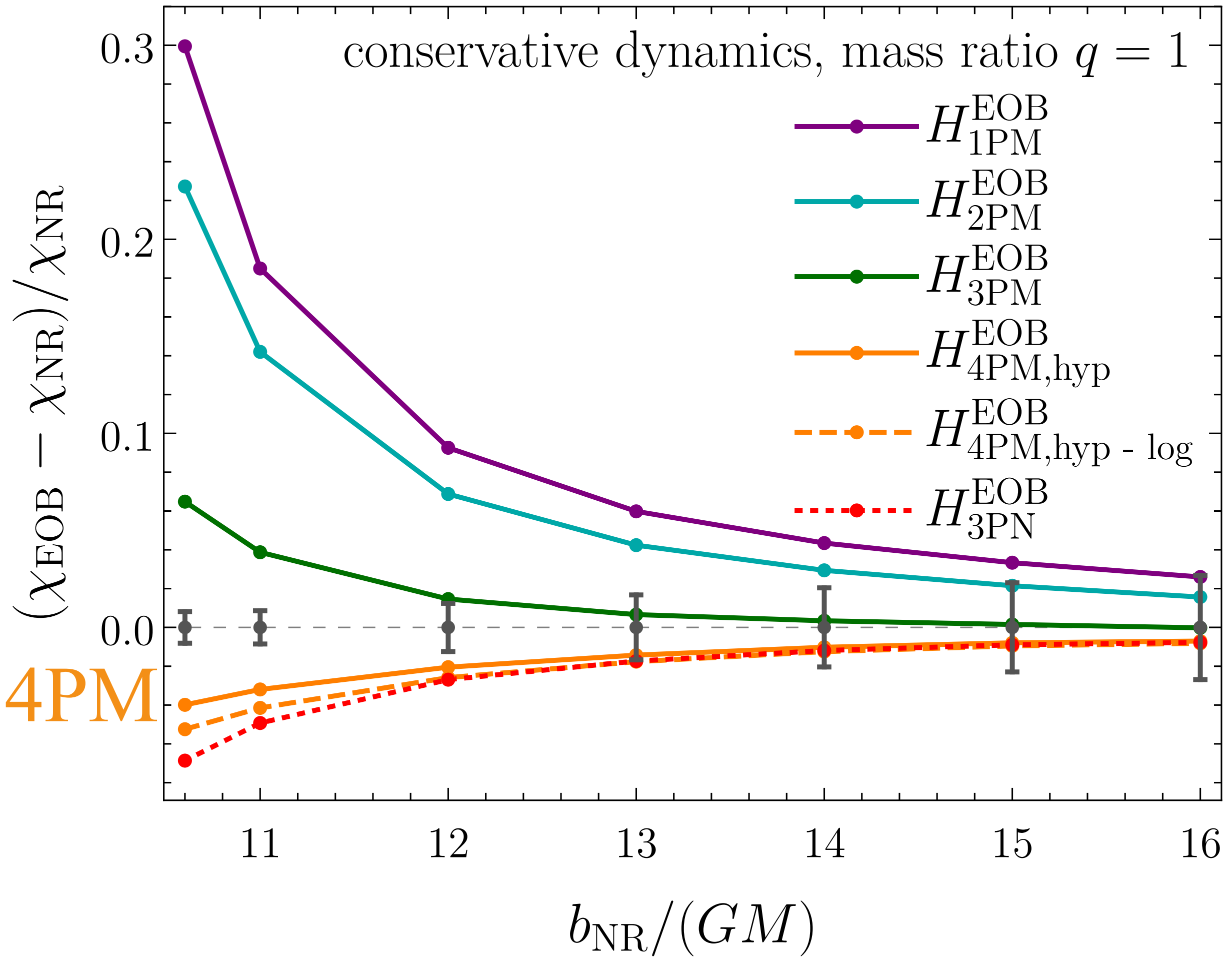
4PM SCATTERING ANGLE V.S. NUMERICAL RELATIVITY

(slide by Chia-Hsien Shen)

Original angle in PM perturbation



EOB-improved angle

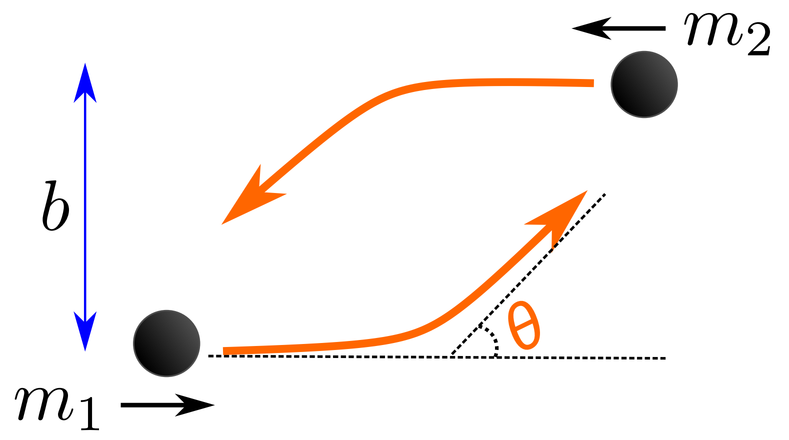


[Khalil, Buonanno, Steinhoff, Vines, forthcoming]
with numerical data from [Damour, Guercilena, Hinder, Hopper, Nagar, Rezzolla]

NEW FRONTIER: RADIATIVE DYNAMICS

What's the energy loss in black hole scattering, at lowest order in G?

[Herrmann, Parra-Martinez, Ruf, MZ, arXiv:2101.07255 (PRL)]

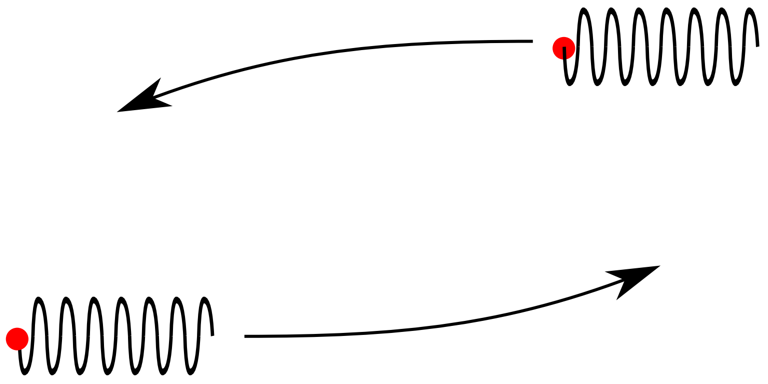


No exact analytic result until our paper, despite studies dating back to 1970s

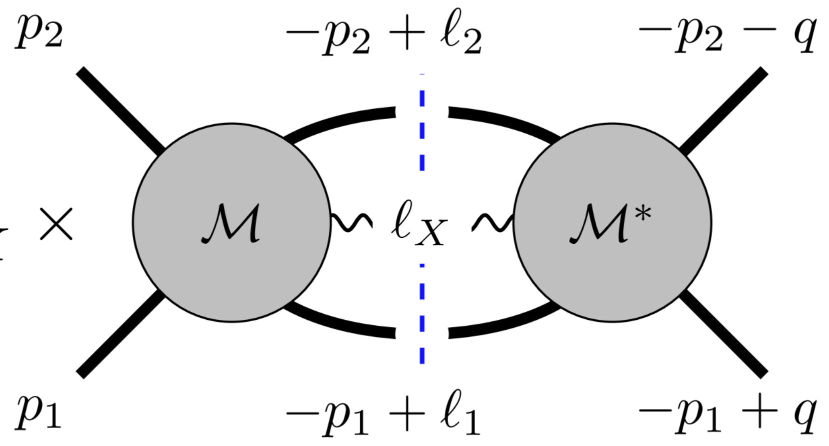
[Ruffini, Wheeler, '72; Kovac, Throne, '78; Peters, 70]

Consider quantum scattering of wavepackets in the classical limit

[Kosower, Maybee, O'Connell, '18]

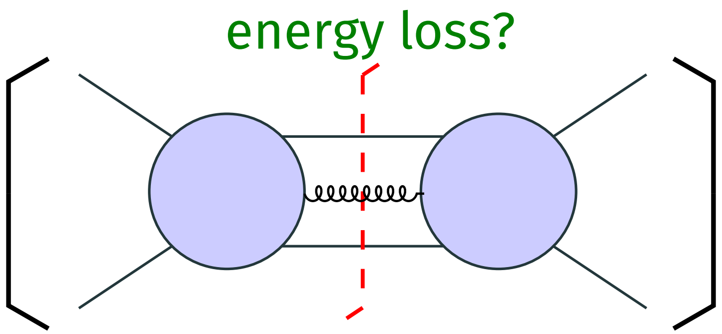


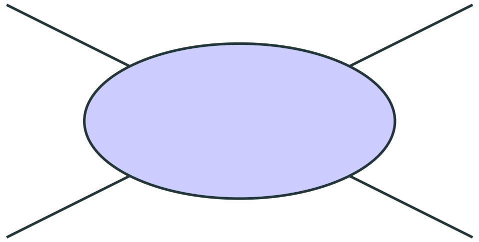
$$\sum_X \int \mathrm{d}\Phi_{2+X}\ \ell_X^\mu \times$$



COLLIDER METHODS MEET GRAVITY

[Herrmann, Parra-Martinez, Ruf, MZ, 2101.07255 (PRL), 2104.03957 (PRD)]





Reverse unitarity *[Anastasiou, Melnikov, '02; Anastasiou, Dixon, Melnikov, '03; Anastasiou, Duhr, Dulat, Furlan, Herzog, Mistlberger, '15]*

$$2\pi i\,\delta(k^2 - m^2) = \frac{1}{(k^2 - m^2 - i\epsilon)} - \frac{1}{(k^2 - m^2 + i\epsilon)}$$

relativistic mass-shell condition for phase-space

propagator for virtual particles

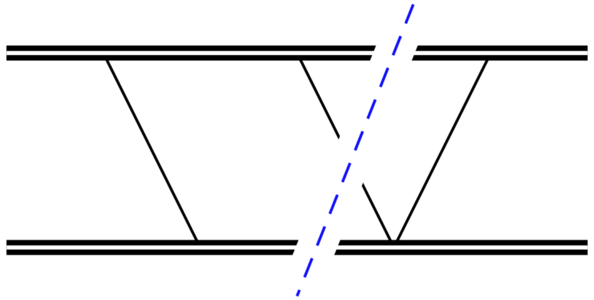
Phase space integrals treated like ***loop integrals.***

- Technique instrumental for LHC Higgs cross section at NNLO, N^3LO

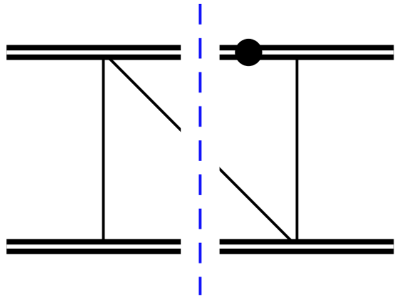
EXAMPLE PHASE SPACE INTEGRAL FROM DIFF. EQS.

[Herrmann, Parra-Martinez, Ruf, MZ, 2101.07255 (PRL), 2104.03957 (PRD)]

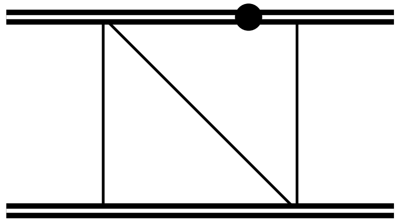
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1-x^2}{2x}\right.$$



$$\left.\vphantom{\frac{1}{x}}\right] = \frac{1}{x}$$



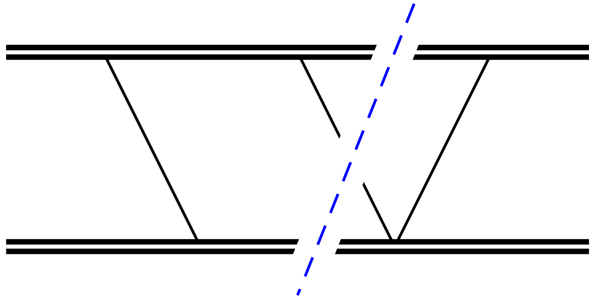
$$= \frac{1}{x}\, 2\,\mathrm{Im}$$



$$x = \sigma - \sqrt{\sigma^2 - 1}, \quad \sigma \equiv \frac{p_1 \cdot p_2}{m_1 m_2}, \quad 0 < x < 1$$

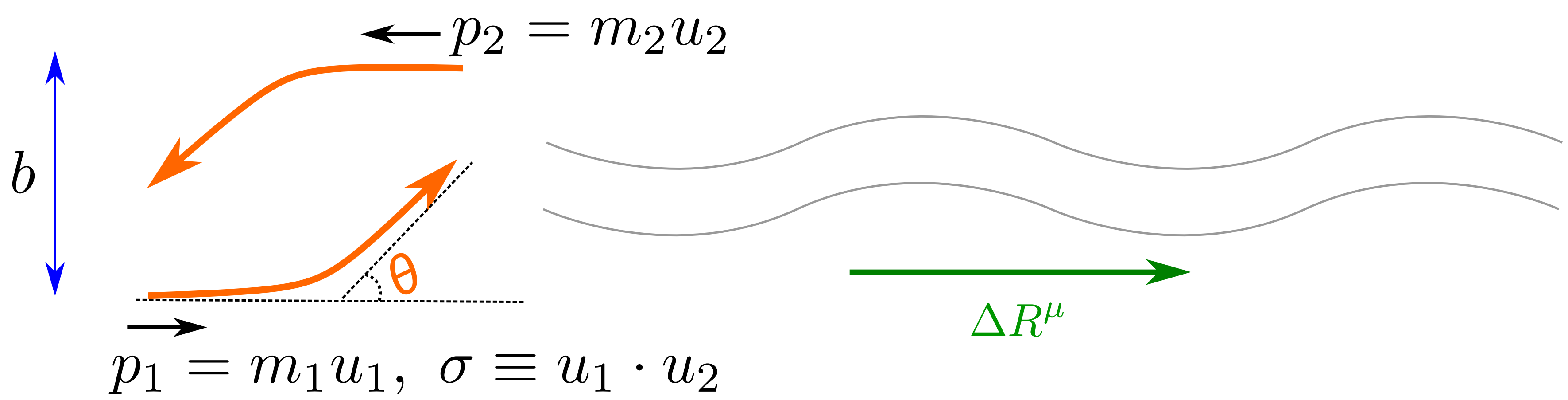
ultra-relativistic (↑ 0) static (↑ 1)

$$\Longrightarrow$$



$$= \frac{2x}{1-x^2}\pi^3 \log(x) = -\frac{2x}{1-x^2}\pi^3 \,\mathrm{arccosh}(\sigma)$$

RESULT: ENERGY LOSS IN SCATTERING



$$\Delta R^\mu = \frac{G^3 m_1^2 m_2^2}{|b|^3} \frac{u_1^\mu + u_2^\mu}{\sigma + 1} \mathcal{E}(\sigma) + \mathcal{O}(G^4)\,.$$

normalized velocity 4-vectors of the incoming BHs

"Superclassical" terms cancel after summing over diagrams.

$$\mathcal{E}(\sigma) = f_1 + f_2 \log\left(\frac{\sigma+1}{2}\right) + f_3 \frac{\sigma \operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\,,$$

$$f_1 = \frac{210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151}{48\left(\sigma^2-1\right)^{3/2}}\,,$$

$$f_2 = -\,\frac{35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5}{8\sqrt{\sigma^2-1}}\,, \qquad f_3 = \frac{\left(2\sigma^2-3\right)\left(35\sigma^4 - 30\sigma^2 + 11\right)}{8(\sigma^2-1)^{3/2}}$$

RADIATED MOMENTUM: COMPARISONS

$$\Delta E^{\text{hyperbolic}} = \frac{G^3 m_1^2 m_2^2}{|b|^3 \sqrt{1 + \frac{2(\sigma-1)m_1 m_2}{(m_1+m_2)^2}}} \mathcal{E}(\sigma)\,, \qquad \sigma \equiv \frac{p_1 \cdot p_2}{m_1 m_2}$$

Ultra-relativistic limit:

$$\mathcal{E}(\sigma) \sim \frac{35}{8}\pi(1 + 2\log(2))\sigma^3 \approx 32.7983\sigma^3$$

Agrees with numerical result of *[Bini, Damour, Geralico, '21]*, 32.7985 ± 0.0016. Disagree with *[Peters, '70; Kovac, Throrne, '78]*

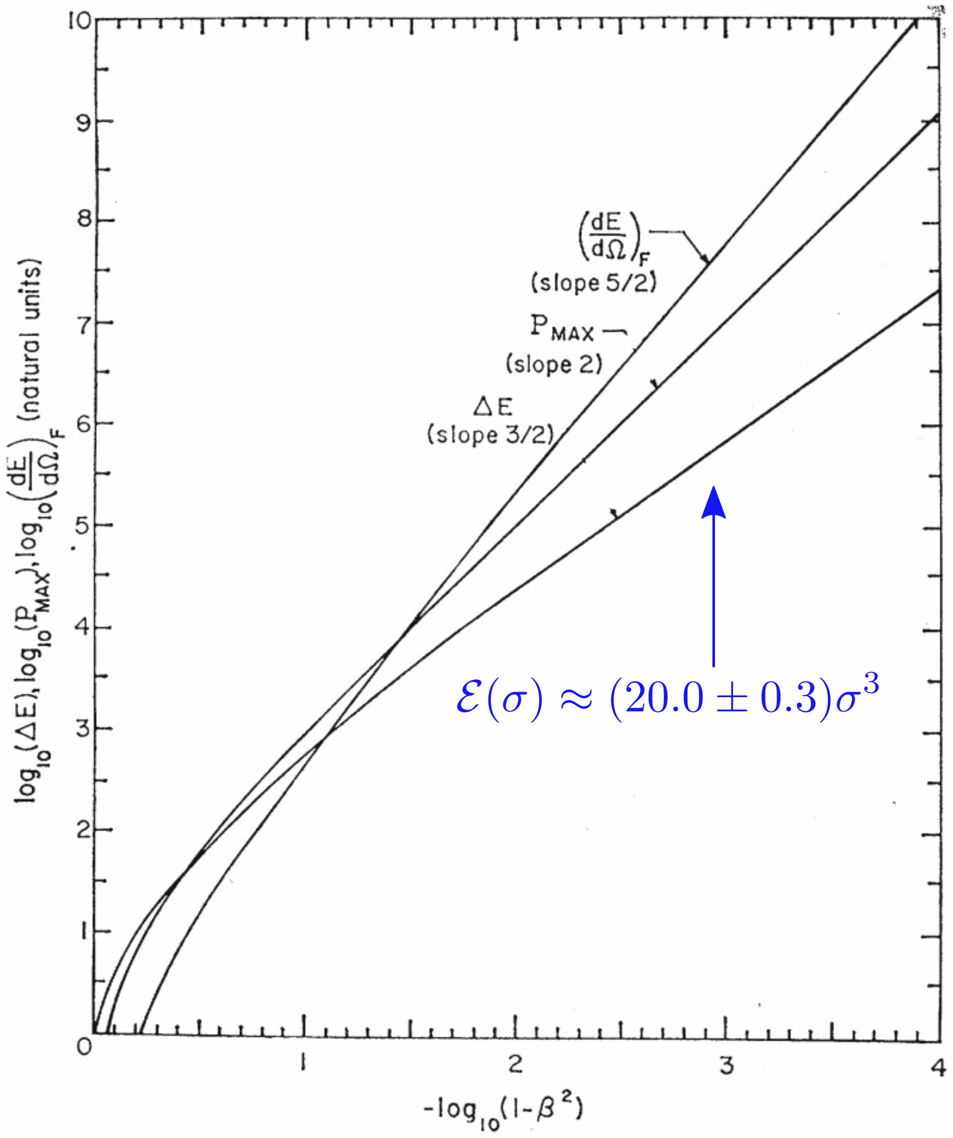


Figure 4 of [Peters, '70]

Small-velocity limit:

$$\mathcal{E}(\sigma) = \frac{37v}{15} + \frac{2393v^3}{840} + \frac{61703v^5}{10080} + \frac{12755740946147v^{15}}{762814660608} + \dots$$

Leading term agrees with *[Ruffini, Wheeler, '72]*

Up to v^{15}: agrees with *[Bini, Damour, Geralico, '21]*,

RADIATED MOMENTUM: COMPARISONS

$$\Delta E^{\text{hyperbolic}} = \frac{G^3 m_1^5 m_2^5 (\sigma^2-1)^{3/2}}{J^3 \left(1+\frac{2(\sigma-1)m_1 m_2}{(m_1+m_2)^2}\right)^2 M^3}\mathcal{E}(\sigma), \quad \sigma \equiv \frac{p_1\cdot p_2}{m_1 m_2}, \quad M \equiv m_1+m_2$$

By ***analytic continuation*** into $\sigma < 1$, [Kalin, Porto, '19; Bini, Damour, Geralico, '20]

$$\Delta E^{\text{elliptic}}(\sigma, J) \equiv \Delta E^{\text{hyperbolic}}(\sigma, J) - \Delta E^{\text{hyperbolic}}(\sigma, -J)\Big|_{\sqrt{\sigma^2-1}\to-\sqrt{1-\sigma^2}}$$

Energy loss per orbit consistent with ***3PN results*** $\sim \mathcal{O}(G^3 v \cdot G^n v^{2m}),\ n+m \leq 3$

[Blanchet, Schaefer, '89; Peters, Mathews, '63; Peters, '64; Wagoner, Will, '76; Junker, Schaefer, '92; Gopakumar, Iyer, '97, '01; Arun, Blanchet, Iyer, Qusailah, '08]

Consistent with 4PM "tail" in [Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, **MZ**, '21]

$$\mathcal{M}_4^{\text{pot}}(\boldsymbol{q}) \propto \left[\mathcal{M}_4^p + \nu\left(\frac{\mathcal{M}_4^t}{\epsilon} + \mathcal{M}_4^f\right)\right] + \text{Iterations}, \quad \Delta E \propto \mathcal{M}_4^t$$

Proportinality predicted by [Bini, Damour, '17; Bini, Damour, Geralico, '20; Blanchet, Foffa, Larroutorou, Sturani, '19]

DISCUSSIONS & OUTLOOK

- Obtained new results for *post-Minkowskian binary dynamics*, in some cases beyond best classical calculations.
- Start to compete with post-Newtonian theory, and offers ***new analytic insights.***
- Relies on modern methods for *scattering amplitudes* (double copy, generalized unitarity), *EFT* (inspired by NRQED/QCD), *advanced integration methods* (IBP, DE, reverse unitarity) …
- Exciting new frontier of ***radiative dynamics.*** Need vast improvements to become mature.
- Rich physics to be explored - spin, tidal effects, radiation reaction, tail effects, angular momentum loss… Preparing for *coming decades of GW physics!*