

Quarteting in $N = Z$ nuclei

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Outline

- ▶ An exact treatment of the $T = 0$ eigenstates of the isovector pairing Hamiltonian
- ▶ A quartet approximation for proton-neutron pairing Hamiltonians in $N = Z$ systems
- ▶ Spectra of $N = Z$ nuclei in a formalism of α -like quartets

in collaboration with N. Sandulescu (NIPNE, Bucharest)

History of the exact treatments of the isovector pairing Hamiltonian

- ▶ R.W. Richardson, Phys. Rev. **144**, 874 (1966)
H.-T. Chen and R.W. Richardson, Phys. Lett. B **34**, 271 (1971)
H.-T. Chen and R.W. Richardson, Nucl. Phys. A **212**, 317 (1973)
- ▶ Feng Pan and J.P. Draayer, Phys. Rev. C **66**, 044314 (2002)
- ▶ J. Links, H.-Q. Zhou, M.D. Gould, and R.H. McKenzie, J. Phys. A **35**, 6459 (2002)
- ▶ J. Dukelsky, V.G. Gueorguiev, P. Van Isacker, S. Dimitrova, B. Errea, and S. Lerma H., Phys. Rev. Lett. **96**, 072503 (2006)

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The isovector pairing Hamiltonian

$$H^{(iv)} = \sum_{i=1}^{\Omega} \epsilon_i \mathcal{N}_i - g \sum_{i,i'=1}^{\Omega} \sum_{M_T=-1}^1 P_{iM_T}^\dagger P_{i'M_T}$$

$$\mathcal{N}_i = \sum_{\sigma, \tau} a_{i\sigma\tau}^\dagger a_{i\sigma\tau}, \quad P_{iM_T}^\dagger = [a_{i+}^\dagger a_{i-}^\dagger]_{M_T}^{T=1}, \quad (P_{iM_T}^\dagger)^\dagger = P_{iM_T}$$

$$\left(P_{iM_T}^\dagger : \quad M_T = -1 \text{ (pp)}, \quad M_T = 0 \text{ (pn)}, \quad M_T = +1 \text{ (nn)} \right)$$

We confine our analysis to $T = 0$ seniority-zero eigenstates.

The Hilbert space of the model is spanned by the states

$$P_{i_1 M_{T_1}}^\dagger P_{i_2 M_{T_2}}^\dagger \cdots P_{i_N M_{T_N}}^\dagger |0\rangle$$

subject to the condition

$$M_{T_1} + M_{T_2} + \cdots + M_{T_N} = 0$$

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First case: 2 protons and 2 neutrons

Building blocks:

$$B_{\nu\tau}^\dagger = \sum_{k=1}^{\Omega} \frac{1}{2\epsilon_k - E_\nu} P_{k\tau}^\dagger \quad (E_\nu \equiv \text{"pair energy"})$$

Ansatz for the eigenstates:

$$|\Psi\rangle = [B_1^\dagger B_2^\dagger]^{T=0} |0\rangle$$

One finds:

$$\begin{aligned} H^{(iv)} |\Psi\rangle &= (E_1 + E_2) |\Psi\rangle \\ &\quad + \left(1 - g \sum_k \frac{1}{2\epsilon_k - E_1} - \frac{g}{E_2 - E_1}\right) [P^\dagger B_2^\dagger]^{T=0} |0\rangle \\ &\quad + \left(1 - g \sum_k \frac{1}{2\epsilon_k - E_2} - \frac{g}{E_1 - E_2}\right) [P^\dagger B_1^\dagger]^{T=0} |0\rangle \end{aligned}$$

being $P_\tau^\dagger = \sum_k P_{k,\tau}^\dagger$.

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being $P_\tau^\dagger = \sum_k P_{k,\tau}^\dagger$.

Second case: 4 protons and 4 neutrons

Ansatz:

$$|\Psi\rangle = \textcolor{blue}{d_1}[B_1^\dagger B_2^\dagger]^0[B_3^\dagger B_4^\dagger]^0|0\rangle + \textcolor{blue}{d_2}[B_1^\dagger B_3^\dagger]^0[B_2^\dagger B_4^\dagger]^0|0\rangle + \textcolor{blue}{d_3}[B_1^\dagger B_4^\dagger]^0[B_2^\dagger B_3^\dagger]^0|0\rangle$$

One finds:

$$\begin{aligned} H^{(iv)}|\Psi\rangle &= (E_1 + E_2 + E_3 + E_4)|\Psi\rangle \\ &+ \left(d_1 - \sum_k \frac{g \cdot d_1}{2\epsilon_k - E_1} - \frac{g \cdot d_{123}}{E_2 - E_1} - \frac{g \cdot d_{12}}{E_1 - E_4} - \frac{g \cdot d_{13}}{E_1 - E_3}\right)[P^\dagger B_2^\dagger]^0[B_3^\dagger B_4^\dagger]^0|0\rangle \\ &+ \left(d_1 - \sum_k \frac{g \cdot d_1}{2\epsilon_k - E_2} - \frac{g \cdot d_{123}}{E_1 - E_2} - \frac{g \cdot d_{12}}{E_2 - E_3} - \frac{g \cdot d_{13}}{E_2 - E_4}\right)[P^\dagger B_1^\dagger]^0[B_3^\dagger B_4^\dagger]^0|0\rangle \\ &+ \left(d_1 - \sum_k \frac{g \cdot d_1}{2\epsilon_k - E_3} - \frac{g \cdot d_{123}}{E_4 - E_3} - \frac{g \cdot d_{12}}{E_3 - E_2} - \frac{g \cdot d_{13}}{E_3 - E_1}\right)[P^\dagger B_4^\dagger]^0[B_1^\dagger B_2^\dagger]^0|0\rangle \\ &+ \left(d_1 - \sum_k \frac{g \cdot d_1}{2\epsilon_k - E_4} - \frac{g \cdot d_{123}}{E_3 - E_4} - \frac{g \cdot d_{12}}{E_4 - E_1} - \frac{g \cdot d_{13}}{E_4 - E_2}\right)[P^\dagger B_3^\dagger]^0[B_1^\dagger B_2^\dagger]^0|0\rangle \\ &+ \dots \end{aligned}$$

being: $d_{ij} \equiv d_i + d_j$, $d_{123} \equiv d_1 + d_2 + d_3$

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Ansatz:

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One finds:

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being: $d_{ij} \equiv d_i + d_j$, $d_{123} \equiv d_1 + d_2 + d_3$

The general recipe for an even-even $N = Z$ system

- Adopt the collective pairs

$$B_{\nu\tau}^\dagger = \sum_{k=1}^{\Omega} \frac{1}{2\epsilon_k - E_\nu} P_{k\tau}^\dagger$$

as building blocks.

- Construct the states $|s\rangle$, product of B_ν^\dagger 's arranged into $T = 0$ quartets,

$$|s\rangle = \prod_{q=1}^{N_q} [B_{\nu(1,q,s)}^\dagger B_{\nu(2,q,s)}^\dagger]^{T=0} |0\rangle,$$

such that the space $\{|s\rangle\}$ be invariant under the interchange of any two pairs.

- Expand $|\Psi\rangle$ into this basis: $|\Psi\rangle = \sum_{s=1}^{N_s} d_s |s\rangle$
- Solve the set of equations

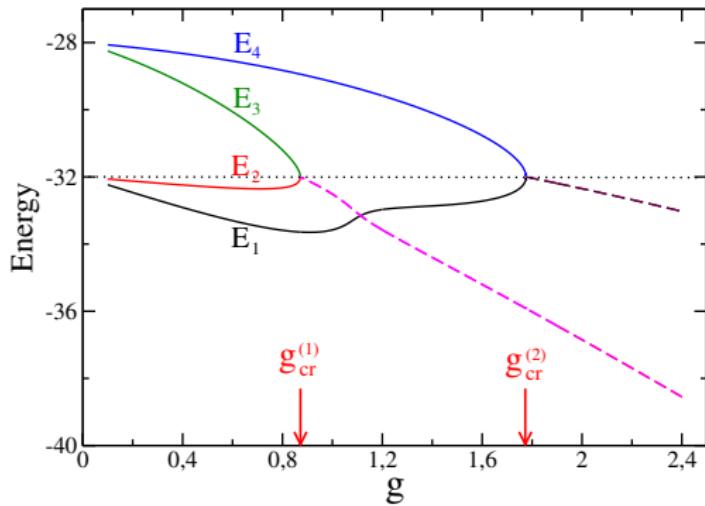
$$\frac{d_s}{g} - \sum_{k=1}^{\Omega} \frac{d_s}{2\epsilon_k - E_\nu} - \sum_{\nu' \neq \nu}^{(1,2N_q)} \frac{S_{\nu'\nu}(s)}{E_{\nu'} - E_\nu} = 0, \quad S_{\nu'\nu}(s) = \sum_t I(t, \nu', \nu, s) d_t$$

This guarantees that

$$H^{(iv)} |\Psi\rangle = (\sum_\nu E_\nu) |\Psi\rangle$$

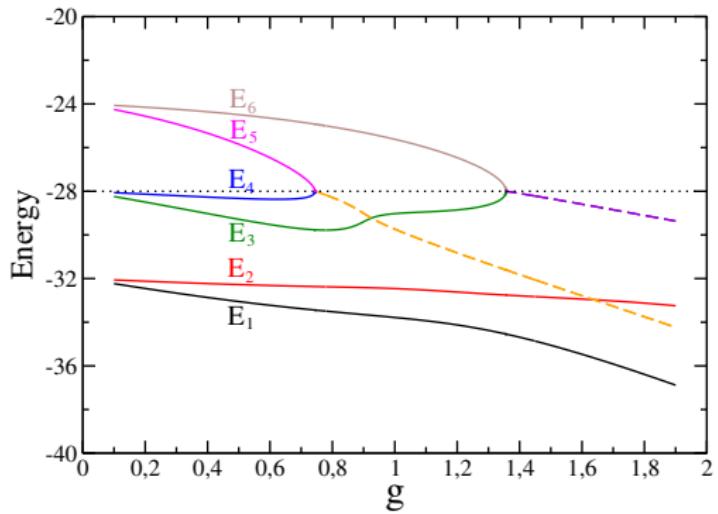
Numerical results: 2 quartets

Pair energies for the ground state of a system of
4 protons and 4 neutrons over 4 equispaced levels



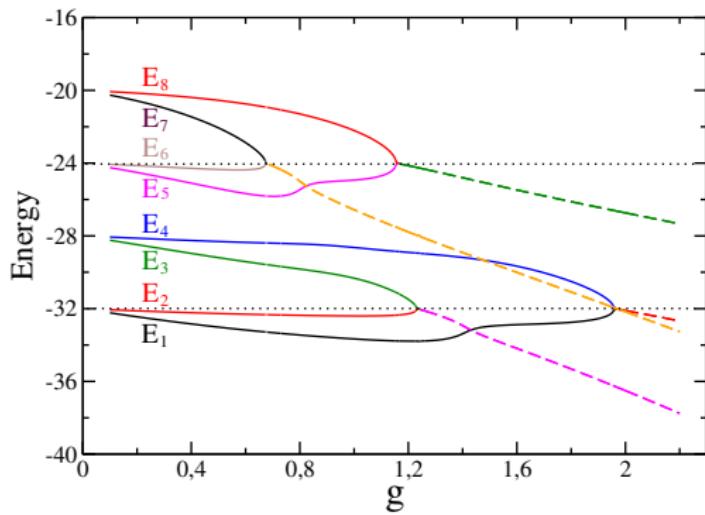
Numerical results: 3 quartets

Pair energies for the ground state of a system of
6 protons and 6 neutrons over 6 equispaced levels



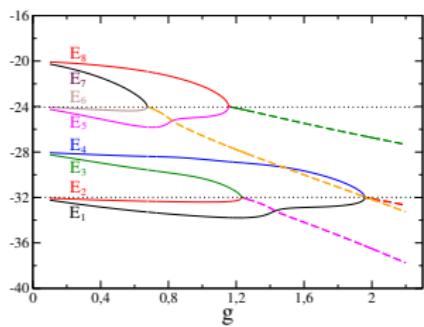
Numerical results: 4 quartets

Pair energies for the ground state of a system of
8 protons and 8 neutrons over 8 equispaced levels

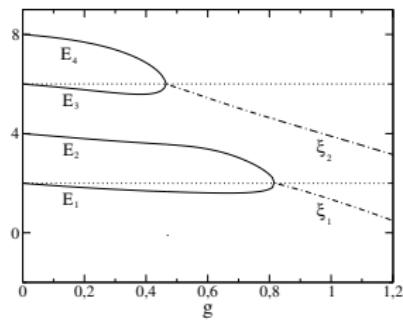


Comparison: isovector vs like-particle pairing

Pair energies for the ground state of a system of **8 protons and 8 neutrons** over 8 equispaced levels



Pair energies for the ground state of a system of **8 like particles** over 8 equispaced levels



R.W. Richardson,
Phys. Lett. **3**, 277 (1963)
R.W. Richardson and N. Sherman,
Nucl. Phys. **52**, 221 (1964)

Conclusions (1st part)

- ▶ The exact $T = 0$ seniority-zero eigenstates of an isovector pairing Hamiltonian are linear superpositions of products of $T = 0$ quartets, each quartet made by two collective $T = 1$ pairs.
- ▶ The isovector pairing Hamiltonian favours the formation of α -like structures in $N = Z$ nuclei.

Quartet approximation of the ground state of the IV pairing Hamiltonian (1)

The QCM approximation for the ground state:

► case (A)

$$|QCM\rangle = (Q_{iv}^+)^{n_q} |0\rangle, \quad Q_{iv}^+ = \sum_{ij} x_{ij} [P_i^\dagger P_j^\dagger]^{T=0}$$

► case (B)

$$x_{ij} \rightarrow x_i x_j$$

$$Q_{iv}^+ \rightarrow = [\Gamma^+ \Gamma^+]^{T=0}, \quad \Gamma_t^+ = \sum_i x_i P_{it}^+$$

$$\overline{|QCM\rangle} = (\overline{Q}_{iv}^+)^{n_q} |0\rangle$$

N. Sandulescu et al., PRC 85 (2012) 061303(R)

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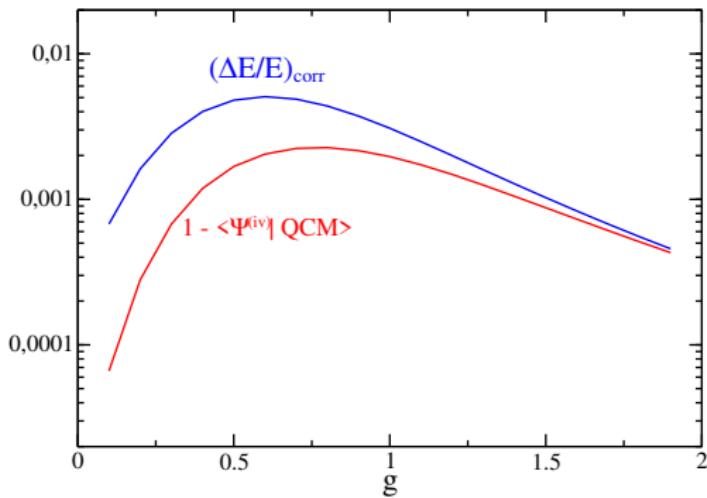
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N. Sandulescu et al., PRC 85 (2012) 061303(R)

Quartet approximation of the ground state of the IV pairing Hamiltonian (2)

6 protons and 6 neutrons over 6 equispaced levels
(case (B))



M.S. and N. Sandulescu, J. Phys. G: Nucl. Phys. 47 (2020) 115101

Quartet approximation of the excited states of the IV pairing Hamiltonian

Excited states built on the QCM ground state:

► case (A)

$$|\Phi_\nu\rangle = \tilde{Q}_\nu^+ (Q_{iv}^+)^{n_q-1} |0\rangle, \quad \tilde{Q}_\nu^+ = \sum_{ij} y_{ij}^{(\nu)} [P_i^+ P_j^+]^{T=0}$$

$$|\Phi_\nu\rangle = \sum_{ij} y_{ij}^{(\nu)} [P_i^+ P_j^+]^{T=0} (Q_{iv}^+)^{n_q-1} |0\rangle$$

► case (B)

$$|\overline{\Phi}_\nu\rangle = \hat{Q}_\nu^+ (\overline{Q}_{iv}^+)^{n_q-1} |0\rangle, \quad \hat{Q}_\nu^+ = [\tilde{\Gamma}_\nu^+ \Gamma^+]^{T=0}, \quad \tilde{\Gamma}_\nu^+ = \sum_i \tilde{x}_i^{(\nu)} P_i^+$$

$$|\overline{\Phi}_\nu\rangle = \sum_i \tilde{x}_i^{(\nu)} [P_i^+ \Gamma^+]^{T=0} (\overline{Q}_{iv}^+)^{n_q-1} |0\rangle$$

Quartet approximation of the excited states of the IV pairing Hamiltonian

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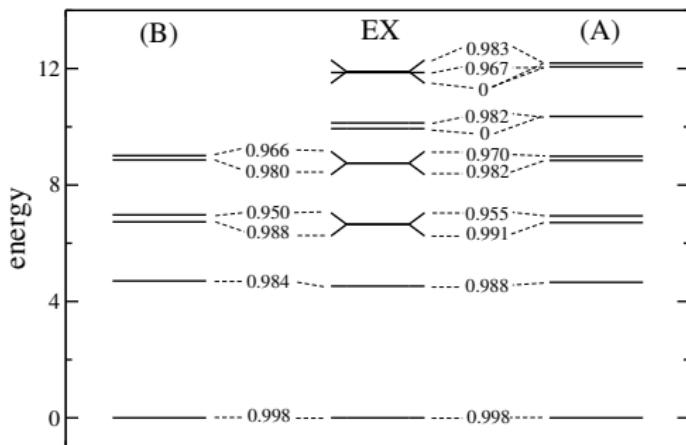
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Quartet approximation of the excited states of the IV pairing Hamiltonian

Spectrum of the isovector pairing Hamiltonian for a system
of 6 protons and 6 neutrons over 6 equispaced levels
($g = 1$)



Quartet approximation for a general IV pairing Hamiltonian (1)

The general isovector pairing Hamiltonian in a spherical mean field

$$H = \sum_i \epsilon_i N_i + \sum_{i,j} V_{J=0}^{T=1}(i,j) \sum_{T_z} P_{i,T_z}^+ P_{j,T_z}^+$$
$$N_i = \sum_{\sigma=\pm, \tau=\pm \frac{1}{2}} a_{i\sigma\tau}^\dagger a_{i\sigma\tau}, \quad P_{i,T_z}^+ = \sqrt{\frac{2j_i+1}{2}} [a_i^+ a_i^+]_{T_z}^{T=1, J=0}$$

- The QCM ground state:

$$|QCM\rangle = (Q_{iv}^+)^{n_q} |0\rangle, \quad Q_{iv}^+ = \sum_{ij} x_{ij} [P_i^\dagger P_j^\dagger]^{T=0, J=0}$$

- The excited states built on the QCM ground state

$$|\Phi_{\nu, JJ_z}\rangle = \tilde{Q}_{\nu, JJ_z} (Q_{iv}^+)^{n_q-1} |0\rangle$$
$$\tilde{Q}_{\nu, JJ_z}^+ = \sum_{T'} \sum_{J_1(i_1j_1)} \sum_{J_2(i_2j_2)} Y_{JJ_z}^{(\nu)}(T', J_1(i_1j_1), J_2(i_2j_2)) [P_{J_1, T'}^+(i_1, j_1) P_{J_2, T'}^+(i_2, j_2)]_{J_z}^{J, T=0}$$

Quartet approximation for a general IV pairing Hamiltonian (1)

The general isovector pairing Hamiltonian in a spherical mean field

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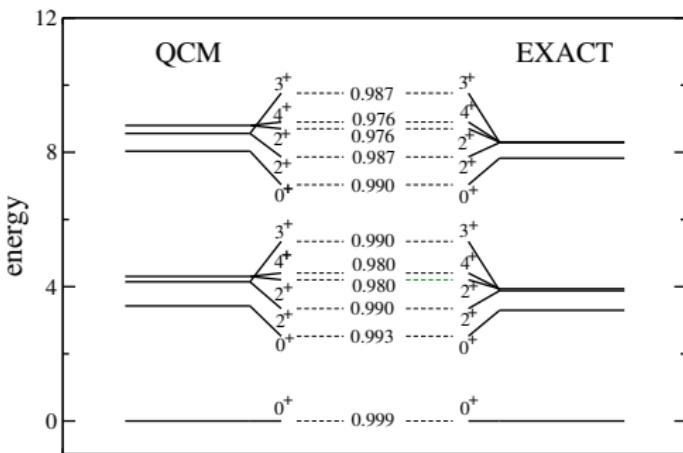
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- The excited states built on the QCM ground state

$$|\Phi_{\nu, JJ_z}\rangle = \tilde{Q}_{\nu, JJ_z} (Q_{iv}^+)^{n_q-1} |0\rangle$$
$$\tilde{Q}_{\nu, JJ_z}^+ = \sum_{T'} \sum_{J_1(i_1j_1)} \sum_{J_2(i_2j_2)} Y_{JJ_z}^{(\nu)}(T', J_1(i_1j_1), J_2(i_2j_2)) [P_{J_1, T'}^+(i_1, j_1) P_{J_2, T'}^+(i_2, j_2)]_{J_z}^{J, T=0}$$

Quartet approximation for a general IV pairing Hamiltonian (2)

Spectrum of the isovector pairing Hamiltonian
for a system of 6 protons and 6 neutrons with
s.p.e.'s and $V_{J=0}^{T=1}(i,j)$ from USDB



The case of the isovector-isoscalar pairing Hamiltonian

$$H = \sum_i \epsilon_i N_i + \sum_{i,j} V_{J=0}^{T=1}(i,j) \sum_{T_z} P_{i,T_z}^+ P_{j,T_z}^- + \sum_{i \leq j, k \leq l} V_{J=1}^{T=0}(ij, kl) \sum_{J_z} D_{ij,J_z}^+ D_{kl,J_z}^-$$

$$D_{j_1 j_2 J_z}^+ = \frac{1}{\sqrt{1 + \delta_{j_1 j_2}}} [a_{j_1}^+ a_{j_2}^+]_{J_z}^{J=1, T=0}$$

- ▶ The QCM ground state:

$$|\Psi_{gs}\rangle = (Q_{ivs}^+)^{n_q} |0\rangle, \quad Q_{ivs}^+ = Q_{iv}^+ + Q_{is}^+$$

$$Q_{is}^+ = \sum_{j_1 j_2 j_3 j_4} y_{j_1 j_2 j_3 j_4} [D_{j_1 j_2}^+ D_{j_3 j_4}^+]^{J=0}$$

- ▶ The excited states built on the QCM ground state

$$|\Phi_{\nu, JJ_z}\rangle = \tilde{Q}_{\nu, JJ_z} (Q_{ivs}^+)^{n_q - 1} |0\rangle$$

$$\tilde{Q}_{\nu, JJ_z}^+ = \sum_{T'} \sum_{J_1(i_1 j_1)} \sum_{J_2(i_2 j_2)} Y_{JJ_z}^{(\nu)}(T', J_1(i_1 j_1), J_2(i_2 j_2)) [P_{J_1, T'}^+(i_1, j_1) P_{J_2, T'}^+(i_2, j_2)]_{J_z}^{J, T=0}$$

The case of the isovector-isoscalar pairing Hamiltonian

$$H = \sum_i \epsilon_i N_i + \sum_{i,j} V_{J=0}^{T=1}(i,j) \sum_{T_z} P_{i,T_z}^+ P_{j,T_z}^- + \sum_{i \leq j, k \leq l} V_{J=1}^{T=0}(ij,kl) \sum_{J_z} D_{ij,J_z}^+ D_{kl,J_z}^-$$

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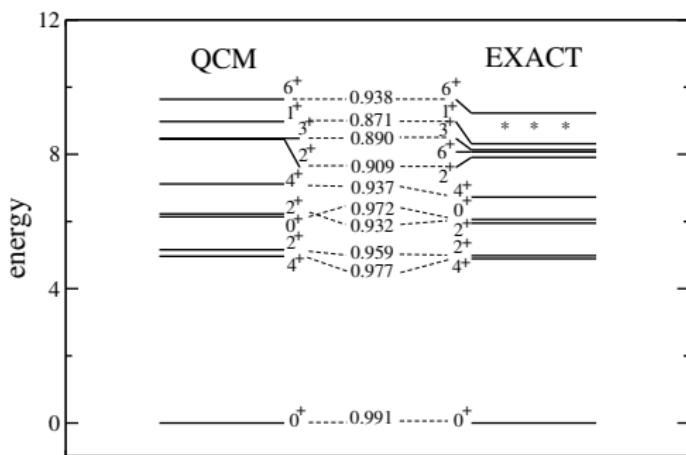
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Quartet approximation for a general IV + IS pairing Hamiltonian

Spectrum of the isovector + isoscalar pairing Hamiltonian
for a system of 6 protons and 6 neutrons with
s.p.e.'s and $V_{J=0}^{T=1}(i,j)$, $V_{J=1}^{T=0}(i,j)$ from USDB



Conclusions (2nd part)

- ▶ The **ground state** of an isovector (plus isoscalar) pairing Hamiltonian in $N = Z$ systems can be well described as a condensate of a $T = 0, J = 0$ quartet built by isovector (plus isoscalar) pairs.
- ▶ The **excited states** of this Hamiltonian can be well described by promoting one of the quartets of the condensate to an excited ($T = 0$) configuration.
- ▶ As a **general conclusion**, $T = 0, J = 0$ quartets play a key role both in the ground and in the excited states of this Hamiltonian.

Spectra of $N = Z$ nuclei in a formalism of α -like quartets

Three basic problems.

1) How to define the quartets:

$$q_{JM}^+ = \sum_{i_1 j_1 J_1} \sum_{i_2 j_2 J_2} \sum_{T'} q_{i_1 j_1 J_1, i_2 j_2 J_2, T'} [[a_{i_1}^+ a_{j_1}^+]^{J_1 T'} [a_{i_2}^+ a_{j_2}^+]^{J_2 T'}]_M^{JT=0}$$

2) How to fix them:

- ▶ from the nearest $T = 0$ one-quartet system ("static" quartets)
(M.S. and N.S., PRL 115 (2015) 112501, PRC 91 (2015) 064318)
- ▶ from intrinsic states ("dynamical" quartets)

3) What to do with them:

Configuration interaction calculations (quartet model (QM))

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Intrinsic states of deformed $N = Z$ nuclei in a formalism of quartets

- “ground” intrinsic state

$$|\Theta_g\rangle = \mathcal{N}_g (Q_g^+)^n |0\rangle, \quad Q_g^+ = \sum_J \alpha_{g,J} (q_g^+)_J{}_0$$

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- “excited” intrinsic states

$$|\Theta_k\rangle = \mathcal{N}_k Q_k^\dagger (Q_g^\dagger)^{(n-1)} |0\rangle, \quad Q_k^\dagger = \sum_J \alpha_{k,J} (q_k^\dagger)_J{}_k$$

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$$k = 0 \rightarrow \text{“}\beta\text{”}, \quad k = 2 \rightarrow \text{“}\gamma\text{”}, \dots$$

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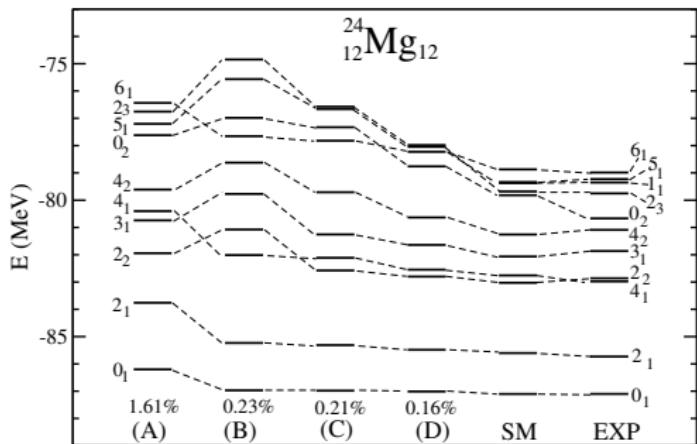
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QM results: ^{24}Mg

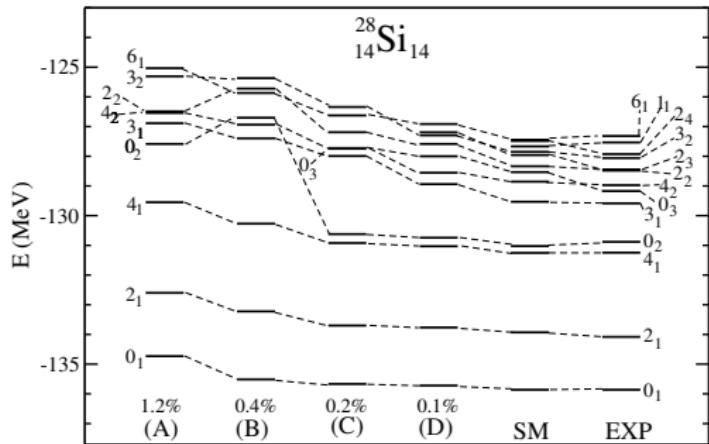


(A) \rightarrow static : $J = 0, 2, 4$
(C) \rightarrow $|\Theta_2\rangle$: $J = 2, 3, 4$

(B) \rightarrow $|\Theta_g\rangle$: $J = 0, 2, 4$
(D) \rightarrow $|\Theta_0\rangle$: $J = 0, 2, 4$

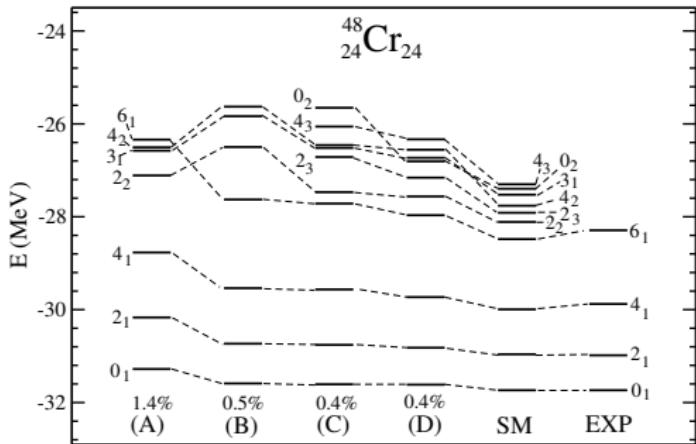
M.S. and N. Sandulescu, PLB 827 (2022) 136987

QM results: ^{28}Si



(A) \rightarrow static : $J = 0, 2, 4$ (B) $\rightarrow |\Theta_g\rangle$: $J = 0, 2, 4$
(C) $\rightarrow |\Theta_0\rangle$: $J = 0, 2, 4$ (D) $\rightarrow |\Theta_3\rangle$: $J = 3, 4$

QM results: ^{48}Cr



(A) \rightarrow static : $J = 0, 2, 4, 6$ (B) $\rightarrow |\Theta_g\rangle$: $J = 0, 2, 4, 6$
(C) $\rightarrow |\Theta_2\rangle$: $J = 2, 3, 4$ (D) $\rightarrow |\Theta_0\rangle$: $J = 0, 2, 4$

Projection of the intrinsic states

The projection goes through two steps:

- ▶ 1) We project states of good angular momentum from the intrinsic states:

$$\hat{P}_J |\Theta_g\rangle$$

$$\hat{P}_J |\Theta_k\rangle, \quad k = 0, 2, \dots$$

- ▶ 2) We diagonalize the Hamiltonian in the space of the projected states:

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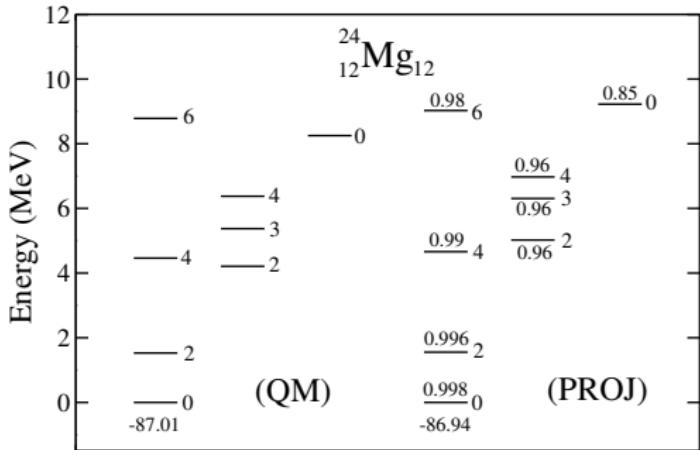
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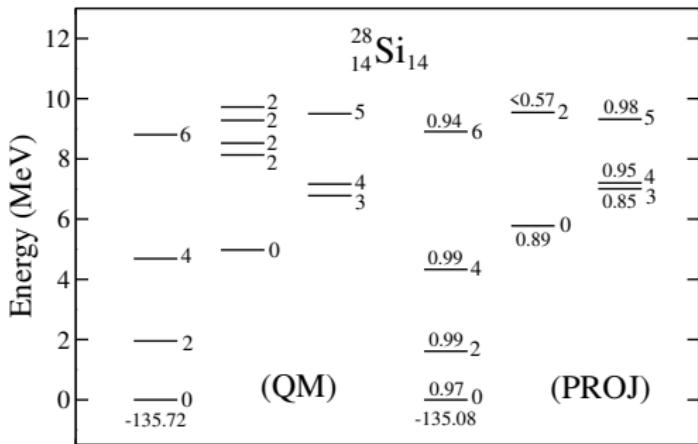
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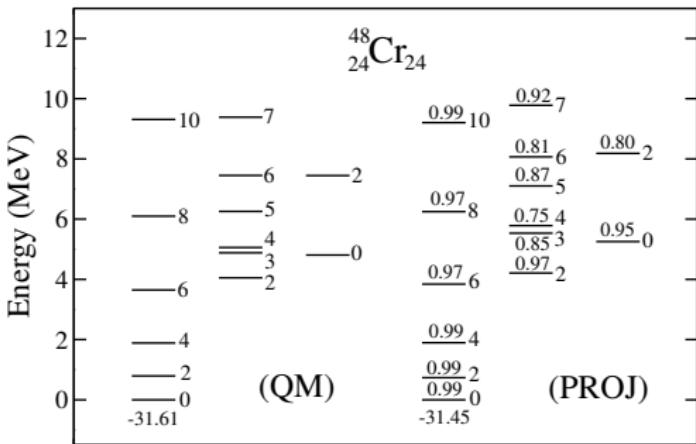
QM vs projection from intrinsic states: ^{24}Mg



QM vs projection from intrinsic states: ^{28}Si



QM vs projection from intrinsic states: ^{48}Cr



Conclusions (3rd part)

- ▶ We have described the spectra of $N = Z$ deformed nuclei in a formalism of α -like quartets.
- ▶ The quartets have been generated by means of proper intrinsic states.
- ▶ Band-like structures have emerged in correspondence with the various sets of quartets employed.
- ▶ The same band-like structures have also been generated by projecting states of good angular momentum from the intrinsic states.