Radial overlap correction to nuclear superallowed $0^+ \to 0^+$ Fermi β decays using the shell model with Hartree-Fock radial wave functions

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Outline

Low energy tests of the Standard Model

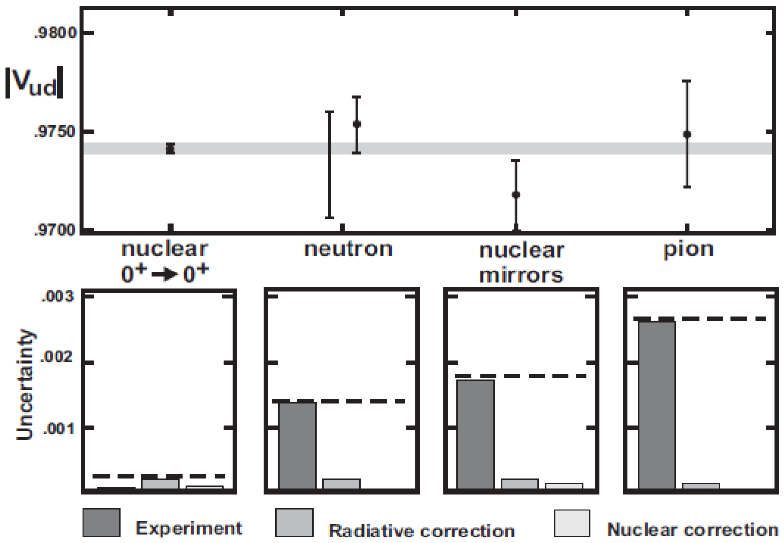
- **CVC**: by analogy with QED, the weak vector current is assumed to be conserved. As a result
  - the Lorentz invariant scalar current is forbidden,
  - and the vector coupling constant, G_V must be a universal constant.
- **CKM unitarity**: the mixing between mass and weak eigenstates is given by the CKM matrix :

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

  - the model itself doesn't give numerical value for the matrix elements, but it requires that $V^\dagger V = \mathbb{1}$.
  - the most dominant element, $|V_{ud}| = G_V/G_\mu$ can be obtained from nuclear physics studies.

Current status of $|V_{ud}|$

- Four semi-leptonic processes have been considered (Hardy&Towner, PRC 91, 025501, 2015)



- Superallowed $0^+ \to 0^+$ (nuclear structure).
- Neutron decay (GT/F ratio).
- Mirror transition (GT/F ratio and nuclear structure).
- Pion decay (very weak BR $\sim 10^{-8}$).

$|V_{ud}|$ from the $0^+ \to 0^+$ process

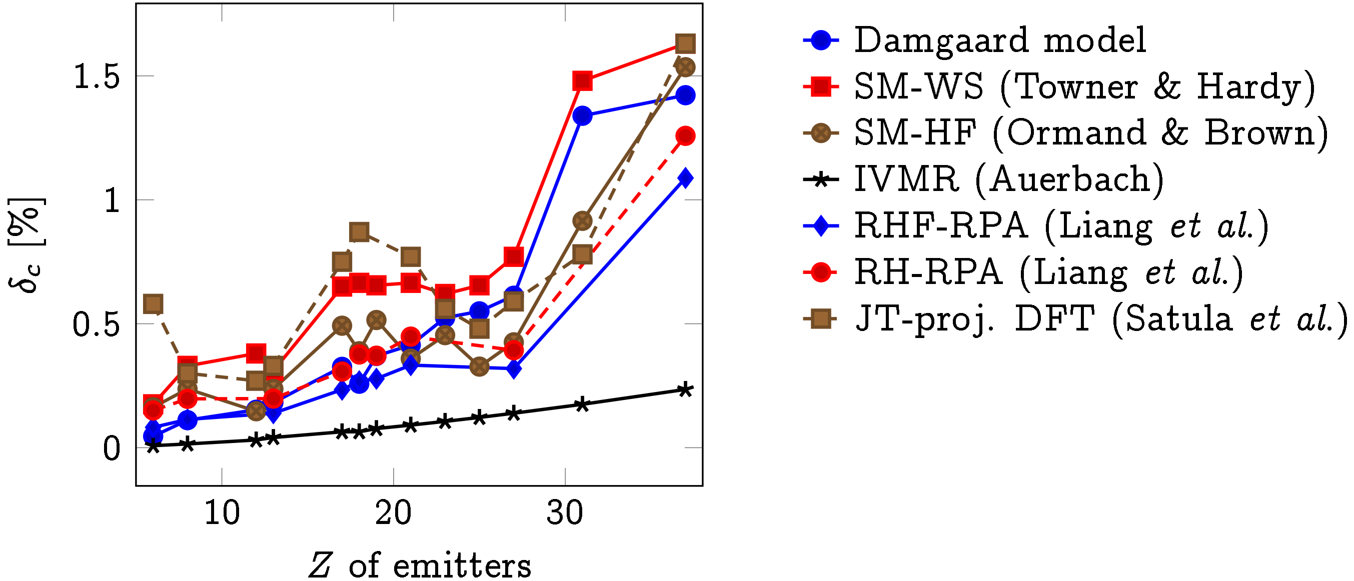
- Occuring between isobaric analogue states ($\Delta J = 0$, $\Delta\pi = NON$, $\Delta T = 0$). Therefore M_F is almost nucleus-independent, except for small ISB effects, $|M_F|^2 = 2(1-\delta_C)$.
- Basic weak-decay equation:

$$\mathcal{F}t = ft(1+\delta'_R)(1-\delta_C+\delta_{NS}) = \frac{K}{2G_V^2(1+\Delta_R^V)}$$

  - $f(Z, Q_{EC})$, statistical rate function,
  - $t = t_{1/2}/BR$, partial half-life,
  - $\delta_C \leq 1\%$, ISB correction,
  - Radiative corrections :
    $\Delta_R^V = (2.361 \pm 0.038)\%$,
    $\delta'_R \sim 1.6\%$ depending on Z and Q_{EC},
    $\delta_{NS} \sim 0.3\%$ depending on nuclear structure.

- For 14 cases (^{10}C to ^{74}Rb), ft has been measured with precision $\leq 0.1\%$, this study is now limited by δ_C.

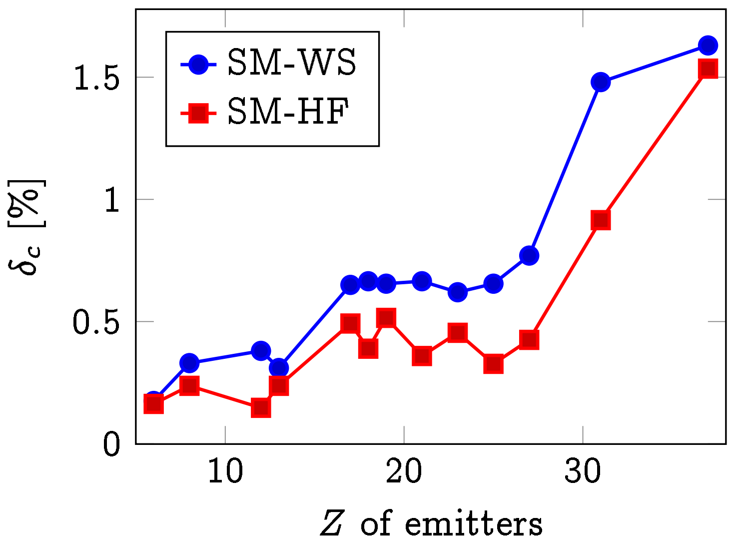
Existing calculations of δ_C



- The calculation of δ_C is strongly model-dependent
- **The shell model (SM-WS & SM-HF) provides the best δ_C values in supporting the Standard Model**

Existing shell-model calculations of δ_C

- While OBTDs can be obtained from the shell model, SPMEs must be evaluated with realistic radial wfs.
  - **Hardy & Towner**: WS potential whose depth and length parameters are adjusted to fix separation energies and charge radii.
  - **Ormand & Brown**: Skyrme-HF potential whose overall strength is scaled to reproduce separation energies.



- With SM-WS, the local variation of δ_C is strongly correlated with $S_n^f - S_p^i$ (PRC 105, 044308, 2022).
- SM-HF yields smaller correction values, and completely different local behavior.
- **The SM-WS agrees better with the Standard model.**
- What's wrong with SM-HF?

Existing shell-model calculations of δ_C

- Possible deficiencies of the SM-HF calculation:
  - Insufficiency of the Slater approximation.
  - Lack of nuclear ISB forces (CIB and CSB forces).
  - The spurious isospin mixing.
  - Exclusion of higher order EM effects such as finite size, Coulomb spin-orbite and vacuum polarization.
  - The approximation for the center-of-mass correction.
  - Presence of correlations and deformation within the data.
  - **All these deficiencies are investigated in the present work.**

Shell-model description of δ_C

- Within the shell model, δ_C can be customarily expressed as,

$$\delta_C = \delta_{C1} + \delta_{C2} + \delta_{C3} + \delta_{C4} + \delta_{C5} + \delta_{C6} \quad \text{with} \quad LO, NLO, N^2LO, N^3LO.$$

  - δ_{C1} accounts for the isospin mixing within the model space,

$$\delta_{C1} = 2 - \frac{2}{\mathcal{M}_F^0} \sum_{k_a k_b} \sqrt{2j_a + 1} X_{ab} \langle f ||| [a^\dagger_{k_b \tau_b} \otimes \tilde{a}_{k_b \tau_b}]^{(0)} || i \rangle ,$$

  - δ_{C2} accounts for the radial mismatch between proton and neutron,

$$\delta_{C2} = \quad -\frac{2}{\mathcal{M}_F^0} \sum_{k_a k_b \pi} X_{ab} \Lambda^{\tau_a \tau_b \pi}_{k_a k_b} \langle f || a^\dagger_{k_a \tau_a} || \pi \rangle^T \langle i || a^\dagger_{k_b \tau_b} || \pi \rangle^T ,$$

where

$$\Lambda^{\tau_a \tau_b \pi}_{k_a k_b} = 1 - \int_0^\infty R^{\tau_a \pi}_{k_a}(r) R^{\tau_a \pi}_{k_b}(r) r^2 dr, \quad X_{ab} = \delta_{J_i J_f} \delta_{l_a l_b} \delta_{j_a j_b} \delta_{\tau_a \tau_b + 1}$$

Shell-model description of δ_C

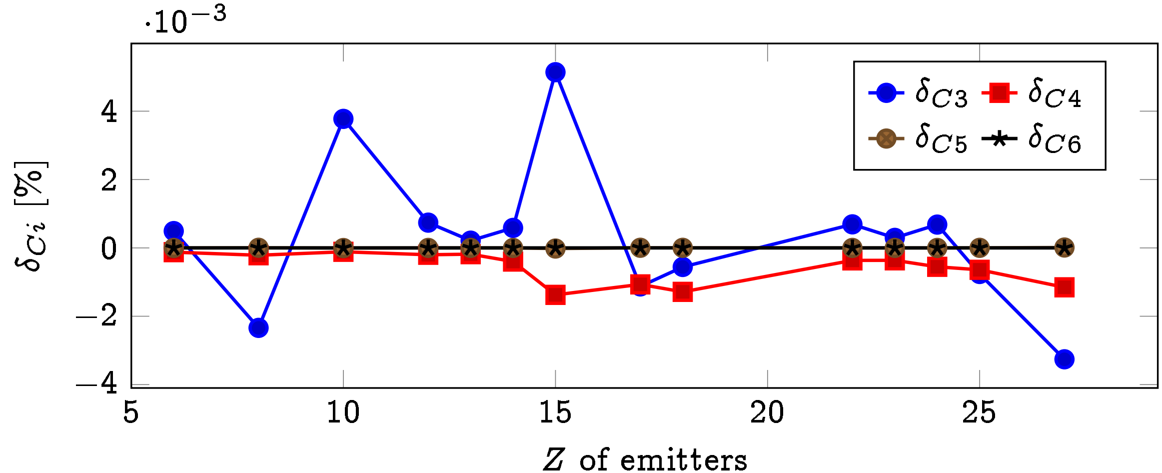
- δ_{C3} is given by

$$\delta_{C3} = -\delta_{C2} - \frac{2}{\mathcal{M}_F^0} \sum_{k_a k_b \pi} X_{ab} \Lambda_{k_a k_b}^{\tau_a \tau_b \pi} \langle f || a_{k_a \tau_a}^{\dagger} || \pi \rangle \langle i || a_{k_b \tau_b}^{\dagger} || \pi \rangle ,$$

- The remaining terms are given by

$$\delta_{C4} = -\frac{(\delta_{C1} + \delta_{C2})^2}{4}, \quad \delta_{C5} = -\delta_{C3}\sqrt{|\delta_{C4}|}, \quad \delta_{C6} = -\frac{(\delta_{C3})^2}{4}$$

- Numerical verification (**Xayavong and Smirnova, arXiv:2201.01035 [nucl-th]**)



All higher order terms (δ_{C3}, δ_{C4}, δ_{C5}, δ_{C6}) are currently negligible

Model spaces and effective interactions

- Core-orbital contribution to δ_{C2} could be greatly amplified due to the dependence of radial wfs. on the excitation energy of the $(A-1)$ system (see PRC 77, 025501, 2008). Therefore the calculation of δ_{C2} generally requires a larger configuration space.
- The following model spaces and effective interactions were employed for our shell-model diagonalizations.

A	model space	interaction
9-14	$1p_{3/2}1p_{1/2}$	CKP
17-22	$1p_{1/2}1d_{5/2}2s_{1/2}$	REWIL/ZBMI/ZBMII
25-34	$1d_{5/2}2s_{1/2}1d_{3/2}$	USD/USDA/USDB
37-46	$2s_{1/2}1d_{3/2}1f_{7/2}2p_{3/2}$	ZBM2-MOD
49-54	$1f_{7/2}2p_{3/2}2p_{1/2}1f_{5/2}$	GXPF1A/KB3G/FPD6
61-74	$2p_{3/2}2p_{1/2}1f_{5/2}1g_{9/2}$	JUN45/MRG

Evaluation of the overlap integrals in SM-HF

- The radial Skyrme HF equations in a local form

$$\begin{cases} \dfrac{\hbar^2}{2m}\left[-\dfrac{d^2}{dr^2}+\dfrac{l(l+1)}{r^2}\right]u^L_{\alpha_q}(r)+U^L_{\alpha_q}(r,\epsilon_{\alpha_q})u^L_{\alpha_q}(r)=\epsilon_{\alpha_q}u^L_{\alpha_q}(r), \\ u_{\alpha_q}(r)=N_q\left[\dfrac{m^*_q(r)}{m}\right]^{1/2}u^L_{\alpha_q}(r), \end{cases}$$

  - The local energy-dependent potential takes the form

$$\begin{aligned} U^L_{\alpha_q}(r,\epsilon_{\alpha_q}) \quad &= \frac{m^*_q(r)}{m}\left\{ x\cdot U_q(r)+\frac{d^2}{dr^2}\frac{\hbar^2}{4m^*_q(r)}-\frac{m^*_q(r)}{2\hbar^2}\left[\frac{d}{dr}\frac{\hbar^2}{m^*_q(r)}\right]^2 \right. \\ &\left. +\frac{1}{2}W_q(r)\left\langle\boldsymbol{\sigma}\cdot\boldsymbol{l}\right\rangle+\delta_{qp}V_{coul}(r)\right\}+\left[1-\frac{m^*_q(r)}{m}\right]\epsilon_{\alpha_q} \end{aligned}$$

  - x must be adjusted to fix the separation energies.
  - Unlike SM-WS, the charge radii are not fixed in SM-HF.
  - $U_q(r), m^*_q(r), W_q(r), V_{coul}(r)$ and ϵ_{α_q} can be obtained from HF calculation.

Evaluation of the overlap integrals in SM-HF

- Kinetic term

$$\frac{\hbar^2}{m_q^*} = \frac{\hbar^2}{m} + \frac{1}{4}\left[t_1(2+x_1) + t_2(2+x_2)\right]\rho + \frac{1}{4}\left[t_1(1+2x_1) + t_2(1+2x_2)\right]\rho_q$$

- Central term

$$U_q = t_0\left[\left(1+\frac{x_0}{2}\right)\rho - \left(x_0+\frac{1}{2}\right)\rho_q\right] + \frac{t_1}{4}\left\{\left(1+\frac{x_1}{2}\right)\left(\tau - \frac{3}{2}\Delta\rho\right) - \left(x_1+\frac{1}{2}\right)\left(\tau_q - \frac{3}{2}\right.\right.$$

$$+\frac{t_2}{4}\left[\left(1+\frac{x_2}{2}\right)\left(\tau + \frac{1}{2}\Delta\rho\right) + \left(x_2+\frac{1}{2}\right)\left(\tau_q + \frac{1}{2}\Delta\rho_q\right)\right]$$

$$+\frac{t_3}{12}\left[\left(1+\frac{x_3}{2}\right)(2+\gamma)\rho^{\gamma+1} - \left(x_3-\frac{1}{2}\right)\left(2\rho^\gamma\rho_q + \gamma\rho^{\gamma-1}\sum_{q'}\rho_{q'}^2\right)\right]$$

$$-\frac{W_0}{2}\left[\frac{1}{r}(J+J_q) + \frac{1}{2}\frac{d}{dr}(J+J_q)\right],$$

- Spin-orbit term

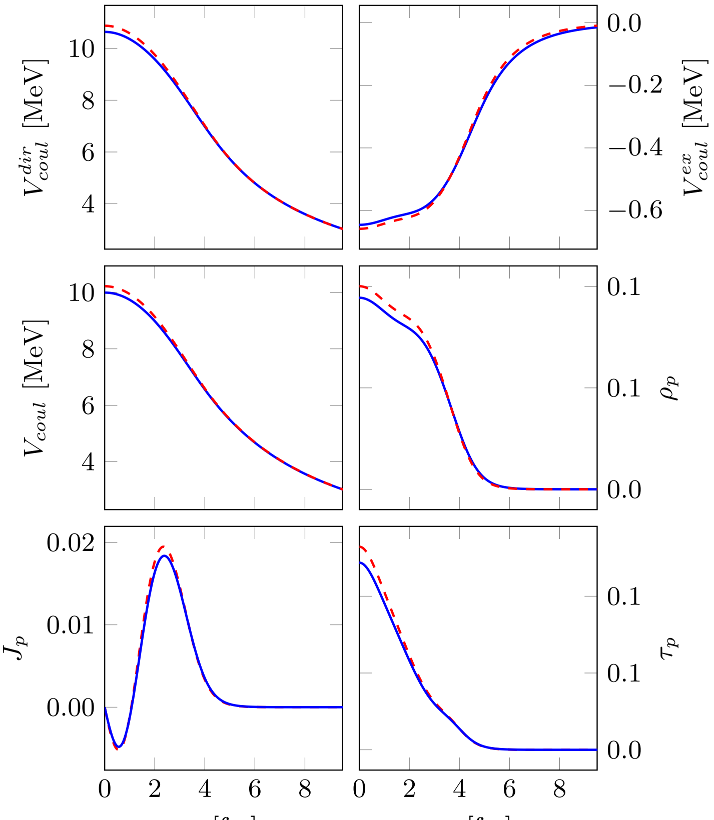
$$W_q = -\frac{1}{8}(t_1x_1 + t_2x_2)J + \frac{1}{8}(t_1 - t_2)J_q + \frac{1}{2}W_0\frac{d}{dr}(\rho+\rho_q)$$

Evidently, isovector component (both physical & spurious) can be induced through the density dependence.

Spurious isospin mixing suppression

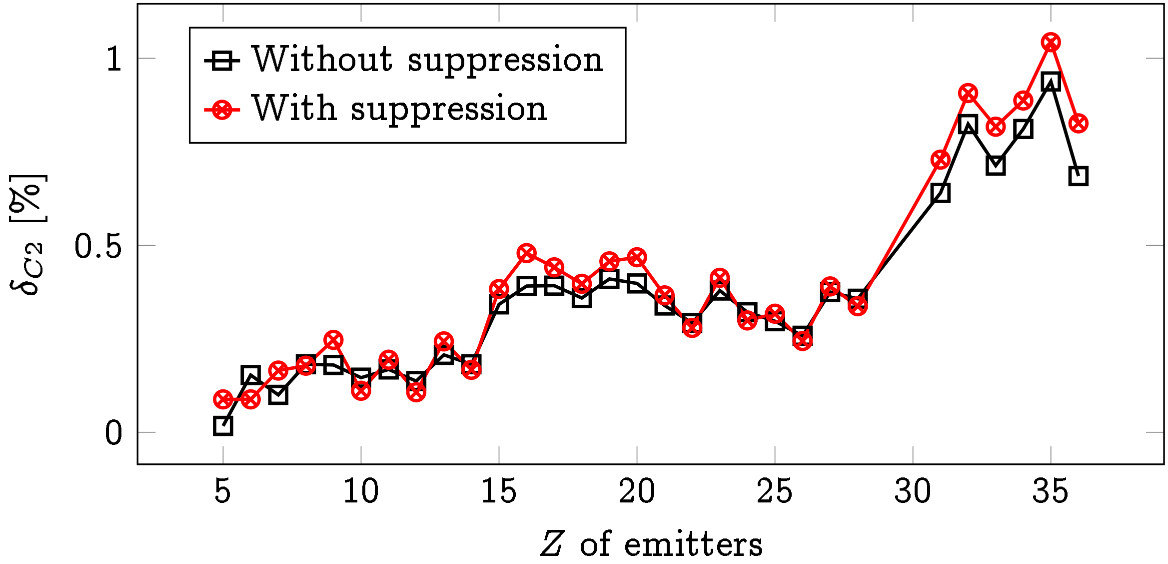
- Our approximation for spurious isospin mixing suppression (PRC 105, 044308, 2022):

- Construct densities and potential from the isospin-invariant HF solution for the $N = Z$ nucleus.
- The suppression leads to a compression of proton densities. As a result, Coulomb potential is increased in the nucleus interior
- The nuclear part which is a functional of the proton densities is also affected. **Therefore the impact of this suppression on δ_{C2} would not be systematic.**



Spurious isospin mixing suppression

- The impact of the spurious isospin suppression on δ_{C2}

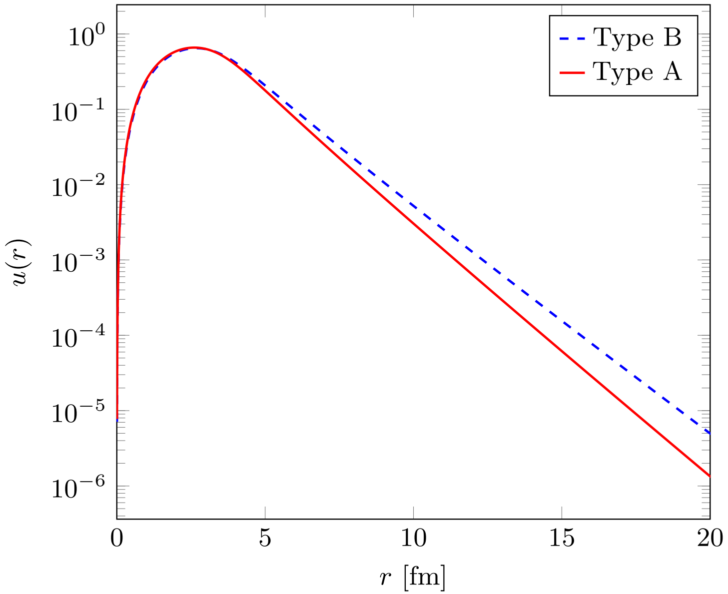


  - It leads to a considerable increase for $30 \leq A \leq 38$ and $A \geq 33$.
  - The emitters with $42 \leq A \leq 54$ are mostly unaffected.
  - Complicated effect in the light-mass region where the nuclear isovector is dominated over the Coulomb.

Center-of-mass correction

- Approximations for the CoM correction

- Exact treatment leads to a nonlocality in coordinate space.
- Type A: $m \to m \times A/(A-1)$ as usually adopted in mean-field calculations using Skyrme interaction
- Type B: $m \to m \times (A-1)/A$ as used with WS potential.

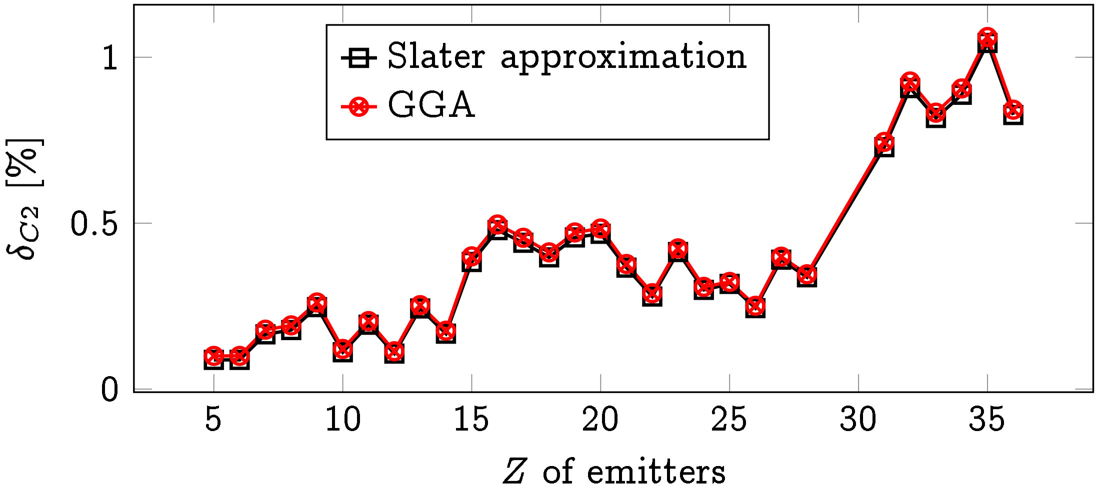


- wfs obtained with type B are more extensive because the actual mass is smaller, then more sensitive to Coulomb.
- **However, the impact of this induced mass uncertainty on δ_{C2} is negligible because they don't break the isospin symmetry** (see PRC 105, 044308, 2022)

Approximations for the Coulomb exchange term

- Exact treatment leads to a nonlocality in coordinate space.
- Slater approximation $V_{sl}^{ex}(r) = -e^2 \left(\frac{3}{\pi}\rho_p\right)^{1/3}$
- Generalized gradient approximation (PRC 105, 044308, 2022)

$$V_{coul}^{ex}(r) = V_{sl}^{ex}(r) \left\{ F(s) - \left[s + \frac{3}{4k_F r} \right] F'(s) + \left[s^2 - \frac{3\rho''_{ch}(r)}{8\rho_{ch}(r)k_F^2} \right] F''(s) \right\},$$



- **δ_{C2} values obtained with GGA are 2-14 % larger**
- **The Slater approximation already works fairy well !**

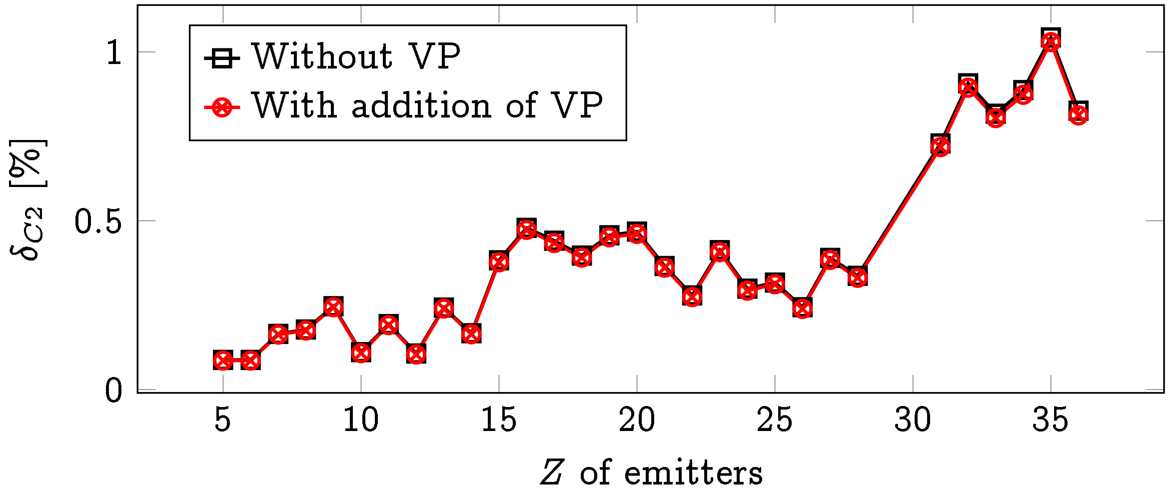
Vacuum polarization

- Vacuum polarization potential can be written as

$$V_{VP}(r) = \frac{2\alpha e^2 \lambda_e}{3r} \int_0^{\infty} dx x \rho_{ch}(r) \left[K_0\left(\frac{2}{\lambda_e}|r-x|\right) - K_0\left(\frac{2}{\lambda_e}|r+x|\right) \right],$$

where

$$K_0(x) = \int_1^{+\infty} dt \left[e^{-xt} \left(\frac{1}{t^2} + \frac{1}{2t^5}\right) \sqrt{t^2-1} \right],$$



- The VP effect is completely negligible.

Coulomb spin-orbit

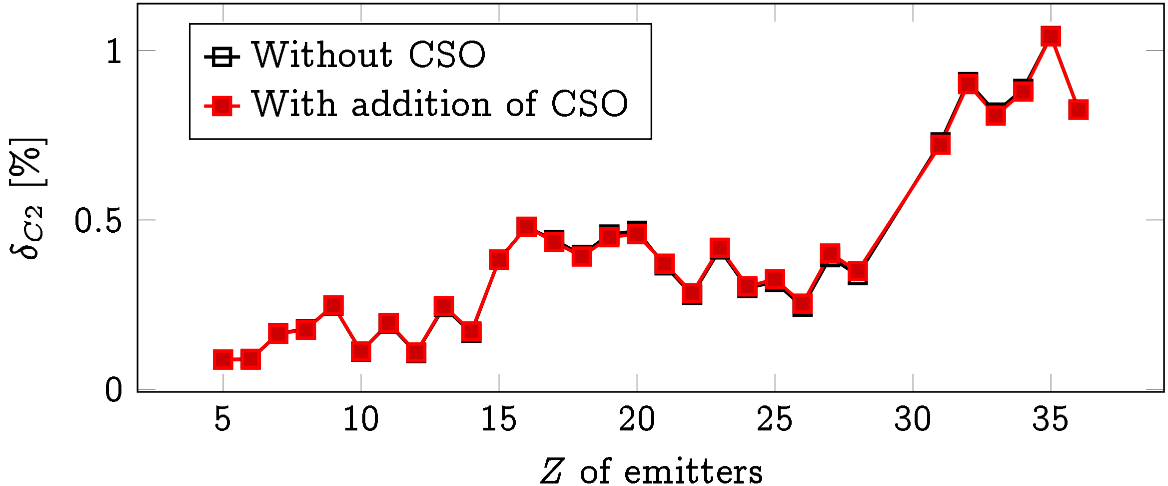
- Coulomb spin-orbit term takes the following (Thomas) form

$$V_{cso}^{q}(r) = \frac{1}{r}\widetilde{W}_q(r)\langle \boldsymbol{\sigma}\cdot\boldsymbol{l}\rangle \quad \text{with} \quad \widetilde{W}_q(r) = \frac{1}{4}\left(\frac{\hbar}{mc}\right)^2 g_q^{\prime}\frac{d}{dr}V_{coul}(r),$$

where

$$g_q^{\prime} = \begin{cases} g_p - 1 & \text{for proton} \\ g_n & \text{for neutron} \end{cases}$$

and g_n and g_p are the neutron and proton g-factors, respectively.



- The effect of Coulomb spin-orbit is completely negligible.

Finite size effect

- In principle all EM terms should be calculated using $\rho_{ch}(r)$ instead of $\rho_p(r)$.
  - Nuclear charge density can be decomposed as

$$\rho_{ch}(r) = \rho^p_{ch}(r) + \rho^n_{ch}(r) + \rho^{ls}_{ch}(r),$$

where

$$\rho^q_{ch}(\boldsymbol{r}) = \int d\boldsymbol{r}' \rho_q(\boldsymbol{r}') G_q(\boldsymbol{r} - \boldsymbol{r}'),$$

and

$$\rho^{ls}_{ch}(r) = -\left(\frac{\hbar}{mc}\right)^2 \sum_{\alpha,q} \nu^q_\alpha \langle \boldsymbol{\sigma} \cdot \boldsymbol{l} \rangle g'_q \frac{1}{r^2} \frac{d}{dr}\left[r \rho^q_\alpha(r)\right],$$

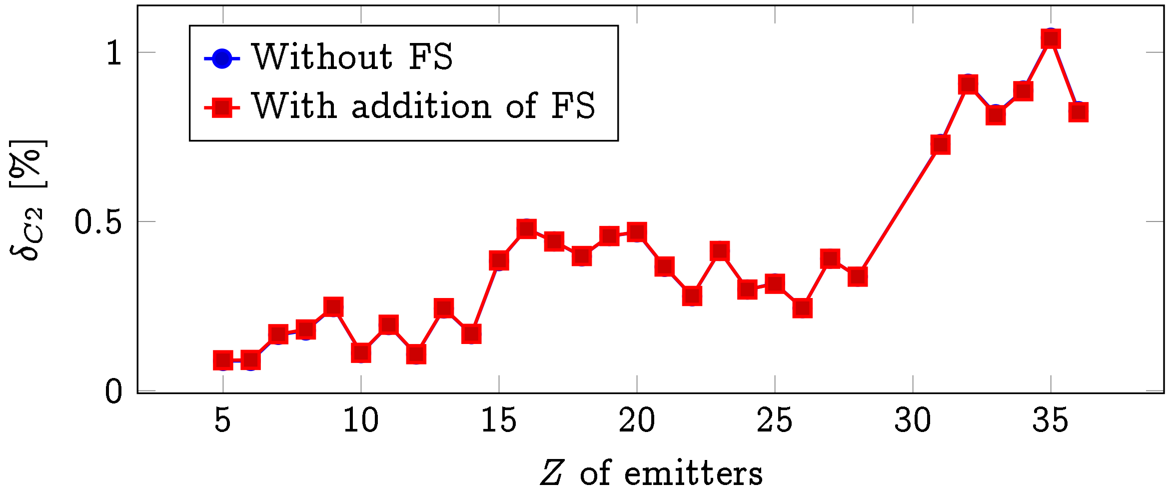
the nucleon charge form factors are given by :

$$G_q(\boldsymbol{r}) = \sum_{i=1}^{n_q} \frac{a^i_q}{(r^i_q \sqrt{\pi})^3} \exp\left[-\frac{\boldsymbol{r}^2}{(r^i_q)^2}\right]$$

with $n_q = 3$ for proton and $n_q = 2$ for neutron.

Finite size effect

- Impact of the FS on δ_{C2}



- The direct impact of the FS on δ_{C2} is completely negligible.
- However the FS has a significant impact on the charge radii which are not fixed in SM-HF. This is one of the reasons behind the discrepancy between SM-WS and SM-HF.

CSB and CIB forces

- We employed the CIB and CSB forces taken from Sagawa, Giai, and Suzuki, PRL 112, 102502 (1995)

$$
\begin{aligned}
v_{CIB} &= \frac{t_{CIB}}{2}\delta\left[P_0^{uz} + \frac{P_1^{uz}}{2}\left(\boldsymbol{k}^2 + \boldsymbol{k}'^2\right) + P_2^{uz}\boldsymbol{k}'\cdot\boldsymbol{k}\right], \\
v_{CSB} &= \frac{t_{CSB}}{2}\delta\left[P_0^{sy} + \frac{P_1^{sy}}{2}\left(\boldsymbol{k}^2 + \boldsymbol{k}'^2\right) + P_2^{sy}\boldsymbol{k}'\cdot\boldsymbol{k}\right],
\end{aligned}
$$

where $t_{CIB} = 4t_{iz}t_{jz}$, $t_{CSB} = 2(t_{iz} + t_{jz})$, $P_i^{sy} = s_i(1 + y_i P_\sigma)$ and $P_i^{uz} = u_i(1 + z_i P_\sigma)$.

- CIB+CSB contribution to the kinetic field

$$
\frac{1}{16}\left[2(u_1 + \tau_{qz}s_1)\rho_q(r) + (u_1 + u_2)\,\rho_{q'}(r)\right],
$$

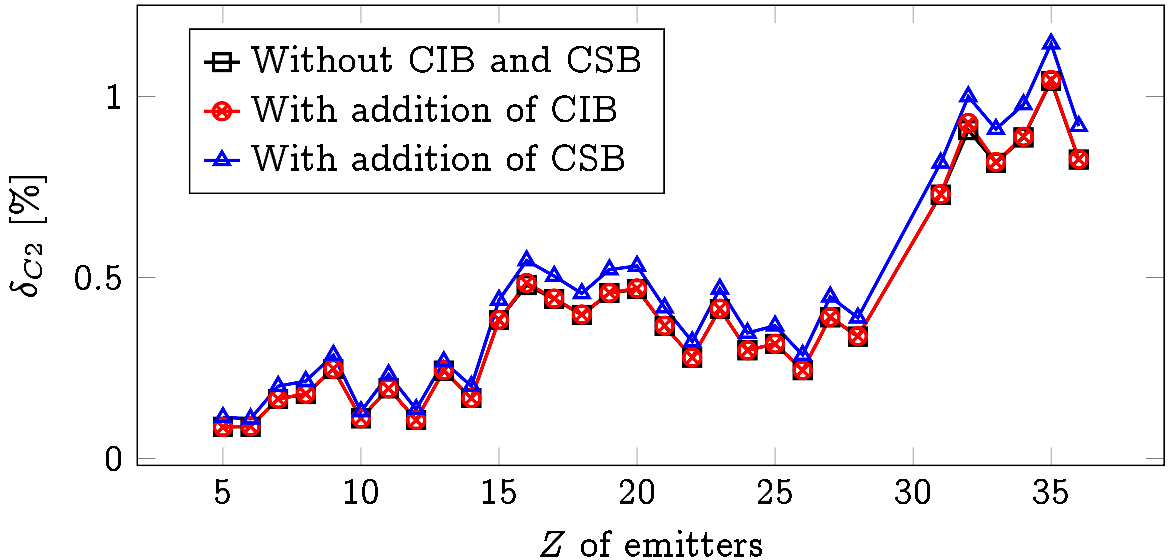
- CIB+CSB contribution to the central field

$$
\begin{aligned}
&\frac{u_0}{4}\left[2\rho_q(r) - \rho_{q'}(r)\right] + \frac{\tau_{qz}s_0}{2}\rho_q(r) - \frac{3u_1}{16}\Delta\rho_q(r) + \frac{1}{32}\left[3u_1 - u_2\right]\Delta\rho_q(r) \\
&+ \frac{u_1}{8}\tau_q(r) - \frac{1}{16}\left[u_1 + u_2\right]\tau_{q'}(r) - \frac{\tau_{qz}}{4}\left[\frac{3s_1}{4}\Delta\rho_q(r) + s_1\tau_q(r)\right],
\end{aligned}
$$

- CIB+CSB contribution to the spin-orbit field

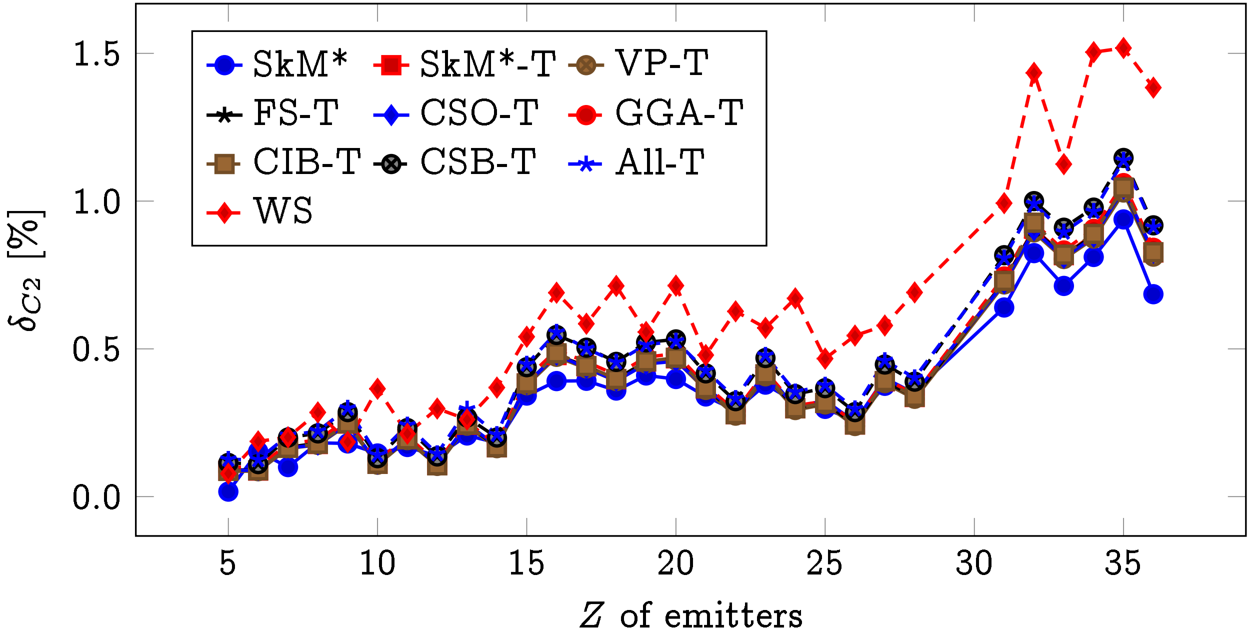
CCSB and CIB forces

- The impact of the CIB and CSB forces on δ_{C2}



  - The CIB contribution is completely negligible
  - The CSB contributes from ranges from 10 to 30 % (in relative %)

Final results



- We have done a remarkable improvement. We observe that SM-HF tends to support SM-WS.
- However, there are still big gaps between the values obtained with SM-WS and SM-HF. Their local variations are also different.

CVC filter

- Both the local variation & the global trends are important in satisfying the CVC hypothesis.
- The CVC requires that $\mathcal{F}t$ mast be nucleus independent.
- Clearly, there is still a problem with the SM-HF.

Calculation	averaged $\mathcal{F}t$ [sec.]	χ^2/ν	scale
SkM*	3077.590 ± 0.921	2.870	1.629
SkM*-T	3076.576 ± 0.958	3.096	1.694
VP-T	3076.730 ± 0.962	3.121	1.701
FS-T	3076.564 ± 0.961	3.113	1.699
CSO-T	3076.557 ± 0.935	2.969	1.654
GGA-T	3076.223 ± 0.968	3.160	1.713
CIB-T	3076.550 ± 0.959	3.106	1.697
CSB-T	3075.099 ± 0.919	2.869	1.625
All-T	3074.807 ± 0.932	2.959	1.649
WS	3073.193 ± 0.707	1.652	1.252

Test of the local variation of δ_{C2}

- Consider the following decay chain

$$a \to b \to c$$

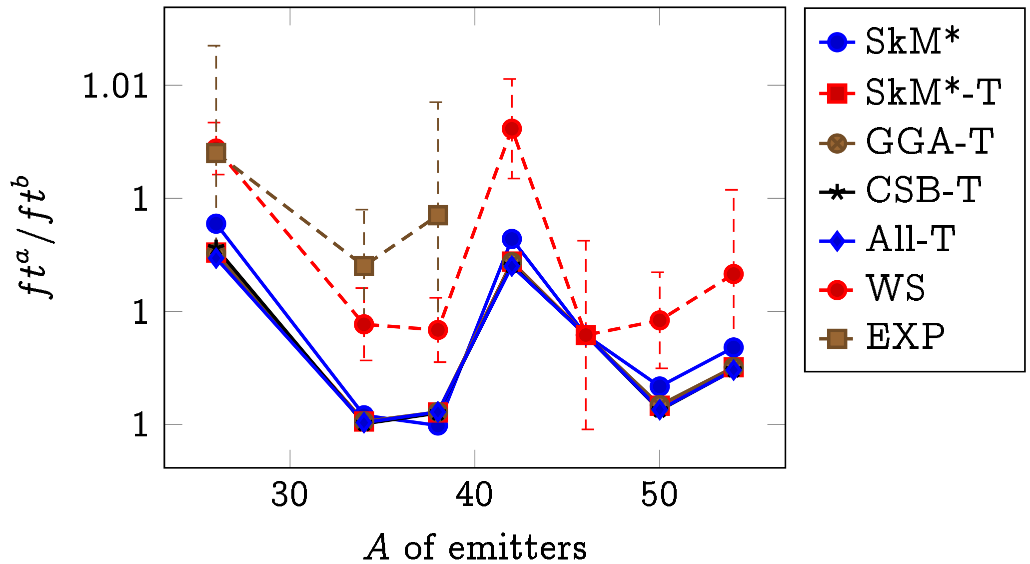
  where a denotes the even-even emitter and b the odd-odd emitter.
- With CVC validated, the so-called mirror ft ratio is given by

$$\frac{ft^a}{ft^b} = 1 + (\delta_R^b - \delta_R^a) + (\delta_{NS}^b - \delta_{NS}^b) - (\delta_C^b - \delta_C^a)$$

  - The theoretical uncertainty on $(\delta_R^b - \delta_R^a)$, $(\delta_{NS}^b - \delta_{NS}^b)$ and $(\delta_C^b - \delta_C^a)$ is much smaller than that on the individual corrections. Therefore the data of mirror ft ratio can serve as an accurate test of our theoretical model, In particular ft^a/ft^b is very sensitive to the local property of δ_{C2}.

Test of the local variation of δ_{C2}

- The mirror ft ratio



- Unluckyly, only the data for $A = 26, 34, 38$ are precise enough for this test to be meaningful.
- For most cases, the SM-HF fails to reproduce the experimental/WS data.

Post-HF effects

- In principle, HF theory can only describe spherical close shell nuclei, any post-HF effects should be subtracted from data before fitting.
- The energy of an open shell nuclei can be decomposed as

$$E = E_{HF} + E_{PHF},$$

where E_{PHF} is the post-HF contribution.

- Suppose that E_{PHF} is dominated by the Wigner energy, namely

$$E_{PHF} = W|N - Z| + d\pi_{pn}\delta_{NZ},$$

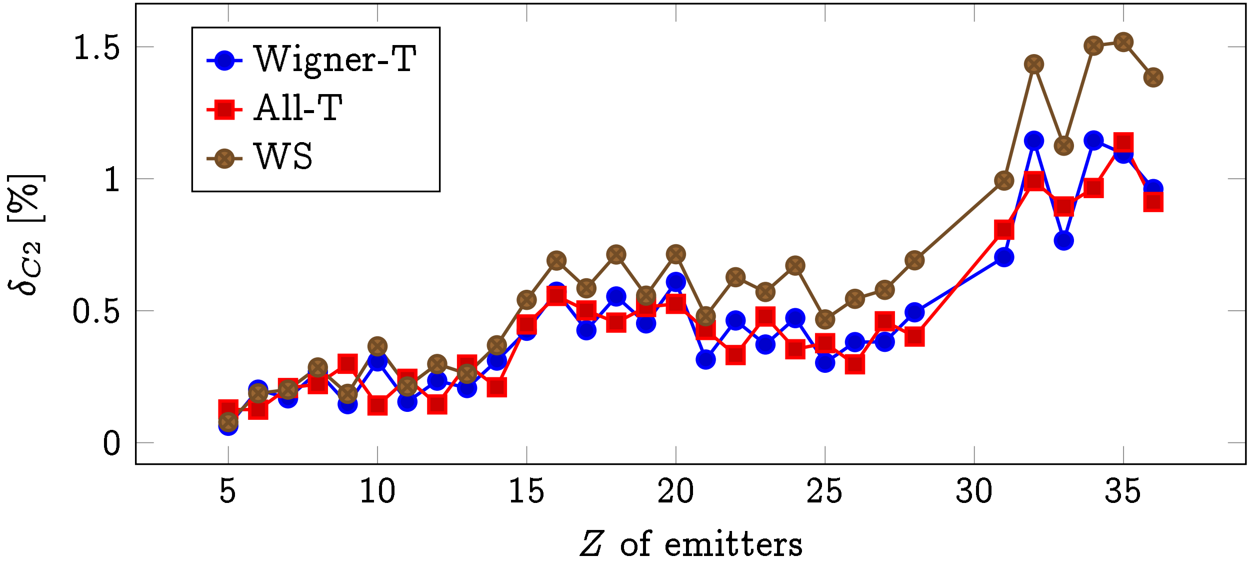
where $W \approx 3d/2$ and $\pi_{pn} = 1(0)$ for odd-odd(other) nuclei. Subsequently the Coulomb displacement energy is given by

$$CDE = CDE_{HF} \pm 2d = S_n^f - S_p^i,$$

where $+(-)$ corresponds to the odd-odd (even-even) emitters.

- **As a sensitivity study, we adjust d to reproduce the experimental or WS values of ft^a/ft^b.**

Post-HF effects



- By subtracting the Wigner energy contribution, the local property of δ_{C2} is remarkably improved.
- This result indicates that δ_{C2} is very sensitive to post-HF effects.

Summary

- Our calculation using SM-HF leads to a considerable improvement over the existing calculations. However, there is still a problem with this model.
- The calculations using SM-HF doesn't pass the CVC filter. It fails to reproduce the right local variation, unless the post HF contribution is subtracted from data. Moreover, the δ_{C2} values obtained with SM-HF are globally smaller.
- We found that δ_{C2} is sensitive to the Wigner energy.
- Perhaps better agreement between SM-WS and SM-HF will be obtained if post-HF effects are properly treated.
- A better result may be obtained if one is able to fix the charge radii.
- More reliable spurious isospin suppression is needed.