b
UNIVERSITÄT
BERN

AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Two-loop anomalous dimension for the resummation of non-global observables

Xiaofeng Xu

2112.02108 with Thomas Becher and Thomas Rauh

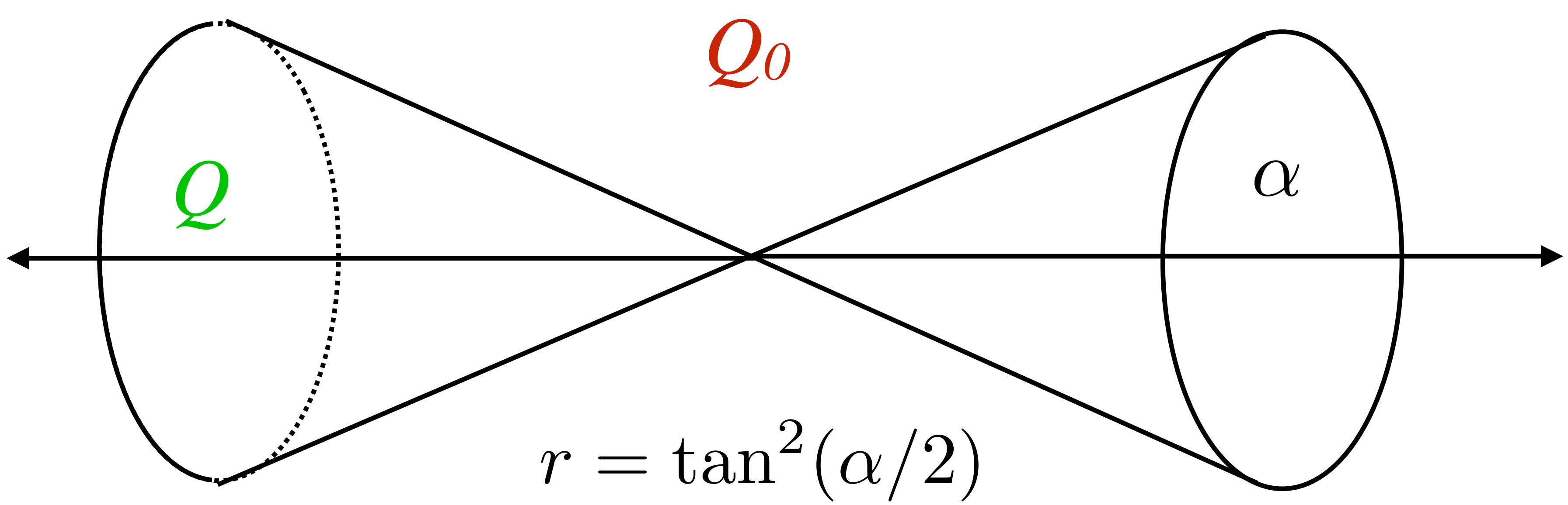
Outline

- Introduction
- Factorization theorem for jet cross sections
- One-loop anomalous dimension $\Gamma^{(1)}$
- Two-loop anomalous dimension $\Gamma^{(2)}$
  - Extraction of $\Gamma^{(2)}$
  - Collinear rearrangement
  - Scheme change
- Conclusion and outlook

Introduction

Non-global logarithms (NGLs) arise when the soft radiation is not distributed evenly. Dasgupta, Salam '02

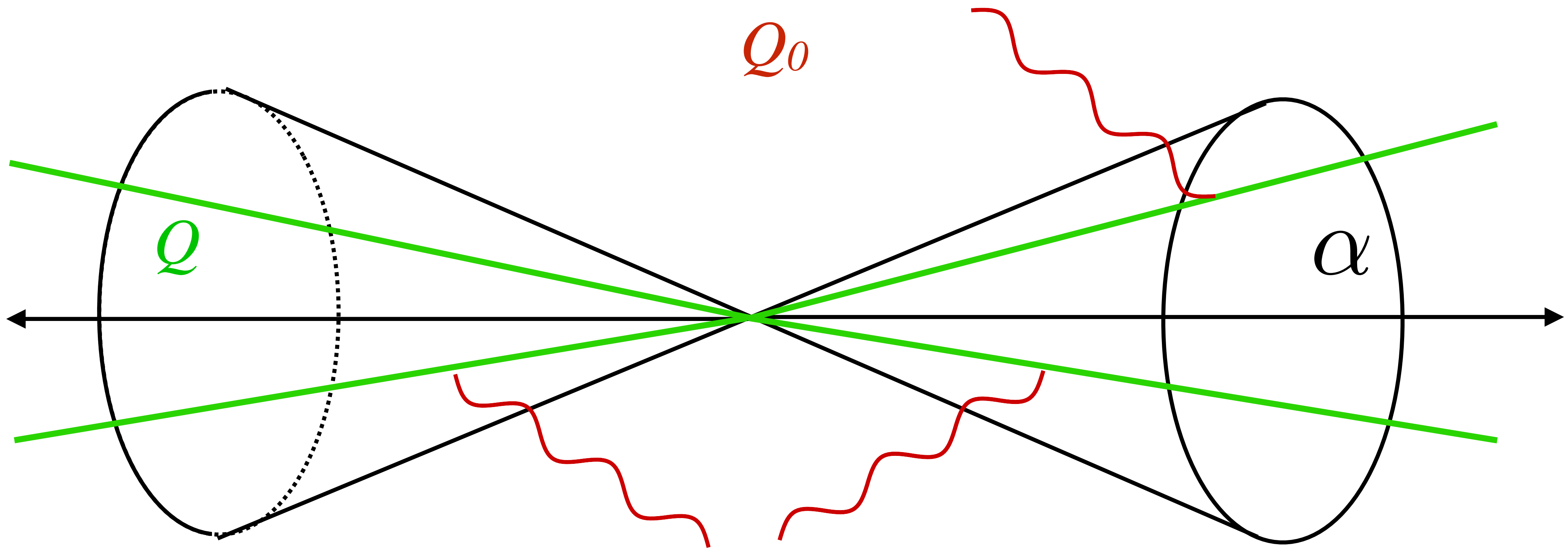
Jet cross sections exhibit such logarithms



Large logarithms: $\sigma = \sigma_0 \left(1 + \frac{\alpha_s C_F}{4\pi}\left(-16 \log r \log(Q_0/Q) - 12 \log r + c_0\right)\right)$

(Sterman Weinberg'1977)

Factorization theorem



$$r = \tan^2(\alpha/2)$$

The amplitude with m hard partons can be factorized as

$$\mathbf{S}_1(n_1)\,\mathbf{S}_2(n_2)\;\ldots\;\mathbf{S}_m(n_m)\,|\mathcal{M}_m(\{\underline{p}\})\rangle$$

Wilson lines: $\boldsymbol{S}_i(n_i) = \exp\left[ig_s \int_0^\infty ds\, n_i \cdot A^a(sn_i)\,\boldsymbol{T}_i^a\right]$

(TB, Neubert, Rothen, Shao '15)

$$\sigma(Q,Q_0) = \sum_{m=m_0}^{\infty} \big\langle \mathcal{H}_m(\{\underline{n}\},Q,\mu) \otimes \mathcal{S}_m(\{\underline{n}\},Q_0,\mu) \big\rangle$$

Color trace

Integration over direction {n}

Hard function with fixed direction $\{\underline{n}\} = \{n_1,\dots,n_m\}$

$$\mathcal{H}_m(\{\underline{n}\},Q,\epsilon) = \frac{1}{2Q^2} \sum_{\text{spins}} \prod_{i=1}^{m} \int \frac{dE_i\, E_i^{d-3}}{\tilde{c}^{\epsilon}\,(2\pi)^2}\, |\mathcal{M}_m(\{\underline{p}\})\rangle\langle\mathcal{M}_m(\{\underline{p}\})|$$
$$\times\, (2\pi)^d\, \delta\Big(Q - \sum_{i=1}^{m} E_i\Big)\, \delta^{(d-1)}(\vec{p}_{\text{tot}})\, \Theta_{\text{in}}(\{\underline{n}\})$$

Soft function along directions $\{\underline{n}\} = \{n_1,\dots,n_m\}$

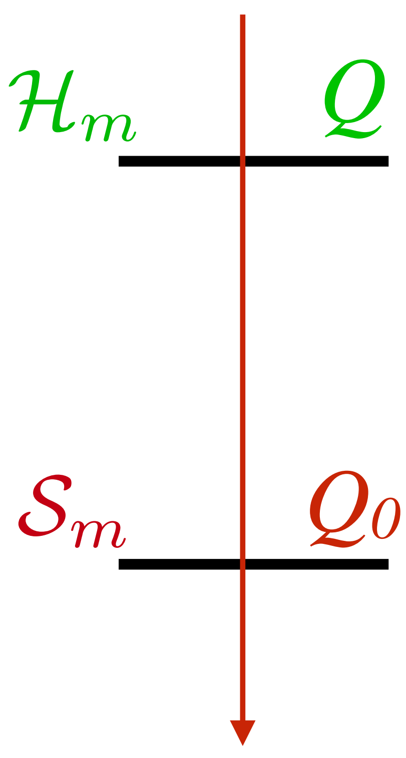
$$\mathcal{S}_m(\{\underline{n}\},Q_0,\epsilon) = \sum\!\!\!\!\!\!\!\int_{X_s} \langle 0|\, \mathbf{S}_1^\dagger(n_1)\, \dots\, \mathbf{S}_m^\dagger(n_m)\, |X_s\rangle\langle X_s|\, \mathbf{S}_1(n_1)\, \dots\, \mathbf{S}_m(n_m)\, |0\rangle\, \theta(Q_0 - 2E_{\text{out}})$$

Renormalization of hard function and renormalization group (RG) evolution

$$\mathcal{H}_m(\{\underline{n}\}, Q, \delta, \mu) = \sum_{l=2}^{m} \mathcal{H}_l^{\text{bare}}(\{\underline{n}\}, Q, \delta, \epsilon)\, \left(\mathbf{Z}^{-1}\right)_{lm}(\{\underline{n}\}, Q, \delta, \epsilon, \mu)$$

$$\frac{d}{d\ln\mu}\mathcal{H}_m(\{\underline{n}\}, Q, \delta, \mu) = -\sum_{l=2}^{m} \mathcal{H}_l(\{\underline{n}\}, Q, \delta, \mu)\, \Gamma_{lm}(\{\underline{n}\}, Q, \delta, \mu)$$

1. $\mathbf{Z}^{-1}$ and Γ are infinite dimension matrix
2. Compute $\mathcal{H}_m$ at hard scale $\mu_h = Q$ and $\mathcal{S}_m$ at soft scale $\mu_s = Q_0$
3. Solve the RG equation and evolve $\mathcal{H}_m$ from μ_h to μ_s



Relation between renormalization constant and anomalous dimension

Strong coupling renormalization: $Z_\alpha = 1 - \frac{\beta_0}{\epsilon}\frac{\alpha_s}{4\pi}$

$$(\mathbf{Z}^{-1}) = \mathbf{1} + \frac{\alpha_s}{4\pi}\frac{1}{2\epsilon}\mathbf{\Gamma}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{8\epsilon^2}\mathbf{\Gamma}^{(1)}\otimes\mathbf{\Gamma}^{(1)} - \frac{\beta_0}{4\epsilon^2}\mathbf{\Gamma}^{(1)} + \frac{1}{4\epsilon}\mathbf{\Gamma}^{(2)}\right]$$

$\mathbf{\Gamma}^{(1)}$ is (-2) times the one-loop hard function soft divergence

$\mathbf{\Gamma}^{(2)}$ is (-4) times single pole in the two-loop hard function

One-loop anomalous dimension

$$\boldsymbol{\Gamma}^{(1)} = \begin{pmatrix} \mathbf{V}_2 & \mathbf{R}_2 & 0 & 0 & \ldots \\ 0 & \mathbf{V}_3 & \mathbf{R}_3 & 0 & \ldots \\ 0 & 0 & \mathbf{V}_4 & \mathbf{R}_4 & \ldots \\ 0 & 0 & 0 & \mathbf{V}_5 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$\mathbf{R}_m$: real emission

$\mathbf{V}_m$: virtual correction

$$\mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) = \frac{1}{2Q^2} \sum_{\text{spins}} \prod_{i=1}^{m} \int \frac{dE_i\, E_i^{d-3}}{\tilde{c}^\epsilon\, (2\pi)^2} \, |\mathcal{M}_m(\{\underline{p}\})\rangle\langle\mathcal{M}_m(\{\underline{p}\})| \times (2\pi)^d\, \delta\Big(Q - \sum_{i=1}^{m} E_i\Big)\, \delta^{(d-1)}(\vec{p}_{\text{tot}})\, \Theta_{\text{in}}(\{\underline{n}\})$$

Extraction of anomalous dimension from hard function

1. Take the soft limit of $\mathcal{H}_{m+1}(\{n_1, \ldots, n_m, n_q\})$
2. Take the residue of q^0 for loop integrals
3. Put UV cut-off on energy E_q to isolate IR divergence

Real emission

$$\mathcal{H}_{m+1}(\{\underline{n}, n_q\}, Q, \epsilon) = -g_s^2 \int_0^\Lambda \frac{dE_q\, E_q^{d-3}}{\tilde{c}^\epsilon\, (2\pi)^2}\, \theta_{\rm in}(q) \sum_{(ij)} \frac{n_i \cdot n_j}{n_i \cdot q\, n_j \cdot q} \mathbf{T}_i^a\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon)\, \mathbf{T}_j^{\tilde{a}}$$

Virtual corrections

$$\mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) = \frac{g_s^2}{2} \sum_{(ij)} \int \frac{d^d q}{(2\pi)^d} \frac{-i}{q^2 + i0} \frac{n_i \cdot n_j}{[n_i \cdot q + i0]\, [-n_j \cdot q + i0]} \times \mathbf{T}_i \cdot \mathbf{T}_j\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) + \text{h.c.}$$

Real emission

put cut off on E_q

$$\mathcal{H}_{m+1}(\{\underline{n}, n_q\}, Q, \epsilon) = -g_s^2 \int_0^\Lambda \frac{dE_q\, E_q^{d-3}}{\tilde{c}^\epsilon\, (2\pi)^2}\, \theta_{\rm in}(q) \sum_{(ij)} \frac{n_i \cdot n_j}{n_i \cdot q\, n_j \cdot q} \mathbf{T}_i^a\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon)\, \mathbf{T}_j^{\tilde{a}}$$

Virtual corrections

$$\mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) = \frac{g_s^2}{2} \sum_{(ij)} \int \frac{d^d q}{(2\pi)^d} \frac{-i}{q^2 + i0} \frac{n_i \cdot n_j}{[n_i \cdot q + i0]\, [-n_j \cdot q + i0]}$$
$$\times \mathbf{T}_i \cdot \mathbf{T}_j\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) + \text{h.c.}$$

Real emission

put cut off on E_q

$$\mathcal{H}_{m+1}(\{\underline{n}, n_q\}, Q, \epsilon) = -g_s^2 \int_0^\Lambda \frac{dE_q\, E_q^{d-3}}{\tilde{c}^\epsilon\, (2\pi)^2}\, \theta_{\rm in}(q) \sum_{(ij)} \frac{n_i \cdot n_j}{n_i \cdot q\, n_j \cdot q} \mathbf{T}_i^a\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon)\, \mathbf{T}_j^{\tilde{a}}$$

Virtual corrections

1. use identity : $\frac{1}{n_i \cdot q + i0} = \frac{-1}{-n_i \cdot q + i0} - 2\pi i \delta(n_i \cdot q)$

$$\mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) = \frac{g_s^2}{2} \sum_{(ij)} \int \frac{d^d q}{(2\pi)^d} \frac{-i}{q^2 + i0} \frac{n_i \cdot n_j}{[n_i \cdot q + i0]\, [-n_j \cdot q + i0]}$$

$$\times \mathbf{T}_i \cdot \mathbf{T}_j\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) + \text{h.c.}$$

Real emission

put cut off on E_q

$$\mathcal{H}_{m+1}(\{\underline{n}, n_q\}, Q, \epsilon) = -g_s^2 \int_0^\Lambda \frac{dE_q\, E_q^{d-3}}{\tilde{c}^\epsilon\, (2\pi)^2}\, \theta_{\rm in}(q) \sum_{(ij)} \frac{n_i \cdot n_j}{n_i \cdot q\, n_j \cdot q} \mathbf{T}_i^a\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon)\, \mathbf{T}_j^{\tilde{a}}$$

Virtual corrections

1. use identity : $\frac{1}{n_i \cdot q + i0} = \frac{-1}{-n_i \cdot q + i0} - 2\pi i \delta(n_i \cdot q)$

$$\mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) = \frac{g_s^2}{2} \sum_{(ij)} \int \frac{d^d q}{(2\pi)^d} \frac{-i}{q^2 + i0} \frac{n_i \cdot n_j}{[n_i \cdot q + i0]\, [-n_j \cdot q + i0]}$$

$$\times \mathbf{T}_i \cdot \mathbf{T}_j\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) + \text{h.c.}$$

2. Take the residue of q^0

Real emission

put cut off on E_q

$$\mathcal{H}_{m+1}(\{\underline{n}, n_q\}, Q, \epsilon) = -g_s^2 \int_0^{\Lambda} \frac{dE_q\, E_q^{d-3}}{\tilde{c}^{\epsilon}\,(2\pi)^2}\, \theta_{\rm in}(q) \sum_{(ij)} \frac{n_i \cdot n_j}{n_i \cdot q\, n_j \cdot q} \mathbf{T}_i^a\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon)\, \mathbf{T}_j^{\tilde{a}}$$

Virtual corrections

1. use identity : $\frac{1}{n_i \cdot q + i0} = \frac{-1}{-n_i \cdot q + i0} - 2\pi i \delta(n_i \cdot q)$

$$\mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) = \frac{g_s^2}{2} \sum_{(ij)} \int \frac{d^d q}{(2\pi)^d} \frac{-i}{q^2 + i0} \frac{n_i \cdot n_j}{[n_i \cdot q + i0]\,[-n_j \cdot q + i0]} \times \mathbf{T}_i \cdot \mathbf{T}_j\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) + \text{h.c.}$$

3. put cut off on E_q and Integrate out it

2. Take the residue of q^0

Results for one-loop anomalous dimension

$$\mathbf{R}_m = -4 \sum_{(ij)} \mathbf{T}^{a}_{i,L} \mathbf{T}^{\tilde{a}}_{j,R} W^q_{ij} \, \theta_{\rm in}(n_q)$$

$$\mathbf{V}_m = 2 \sum_{(ij)} (\mathbf{T}_{i,L} \cdot \mathbf{T}_{j,L} + \mathbf{T}_{i,R} \cdot \mathbf{T}_{j,R}) \int [d\Omega_q] \, W^q_{ij}$$

$$- 2 \sum_{(ij)} [\mathbf{T}_{i,L} \cdot \mathbf{T}_{j,L} - \mathbf{T}_{i,R} \cdot \mathbf{T}_{j,R}] \times i\pi \, \Pi_{ij}$$

$$W^q_{ij} = \frac{n_i \cdot n_j}{n_i \cdot n_q \, n_j \cdot n_q}$$

1. Collinear divergence when soft emission is collinear to hard partons.
2. Collinear finite by combining real and virtual.

Two-loop anomalous dimension

$$\mathbf{\Gamma}^{(2)} = \begin{pmatrix} \mathbf{v}_2 & \mathbf{r}_2 & \mathbf{d}_2 & 0 & \dots \\ 0 & \mathbf{v}_3 & \mathbf{r}_3 & \mathbf{d}_3 & \dots \\ 0 & 0 & \mathbf{v}_4 & \mathbf{r}_4 & \dots \\ 0 & 0 & 0 & \mathbf{v}_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$\mathbf{v}_m$: two-loop virtual

$\mathbf{r}_m$: real-virtual

$\mathbf{d}_m$: double real emission

The extraction of $\Gamma^{(2)}$ is similar to $\Gamma^{(1)}$ but there are some subtleties

1. Which cut-off $\theta(\Lambda - E_q - E_r)$ or $\theta(\Lambda - E_q)\theta(\Lambda - E_r)$
2. Pure collinear singularity $q \parallel r$

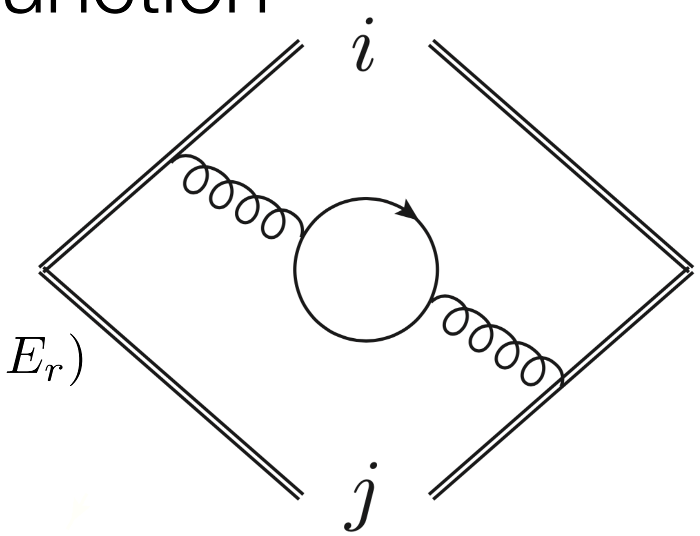
$$\int dE_q dE_r \frac{\theta(\Lambda - E_q)\theta(\Lambda - E_r)}{E_q^{-2\epsilon-1} E_r^{-2\epsilon-1}} = \frac{\Lambda^{-4\epsilon}}{4}\frac{1}{\epsilon^2},$$

$$\int dE_q dE_r \frac{\theta(\Lambda - E_q - E_r)}{E_q^{-2\epsilon-1} E_r^{-2\epsilon-1}} = \frac{\Lambda^{-4\epsilon}}{4}\left(\frac{1}{\epsilon^2} - \frac{2\pi^2}{3}\right) + \mathcal{O}(\epsilon).$$

Double real emissions

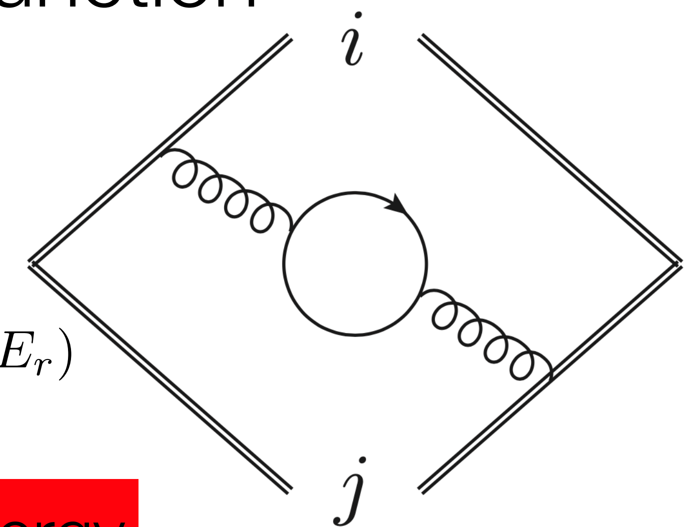
Take soft $q\bar{q}$ as example. The soft limit of hard function

$$\mathcal{H}_{m+2}^{\bar{q}q} = \frac{g_s^4 n_F T_F}{\tilde{c}^{2\epsilon}(2\pi)^4} \sum_{i,j} \mathbf{T}_i^a \, \mathcal{H}_m \mathbf{T}_j^a \, \theta_{\text{in}}(n_q)\theta_{\text{in}}(n_r)$$
$$\times \int \frac{dE_q}{E_q^{2\epsilon}} \frac{dE_r}{E_r^{2\epsilon}} \frac{n_{iq}\, n_{jr} + n_{ir}\, n_{jq} - n_{ij}\, n_{qr}}{(E_q n_{iq} + E_r n_{ir})\,(E_q n_{jq} + E_r n_{jr})\, n_{qr}^2} \, \theta(\Lambda - E_q - E_r)$$



Double real emissions

Take soft $q\bar{q}$ as example. The soft limit of hard function

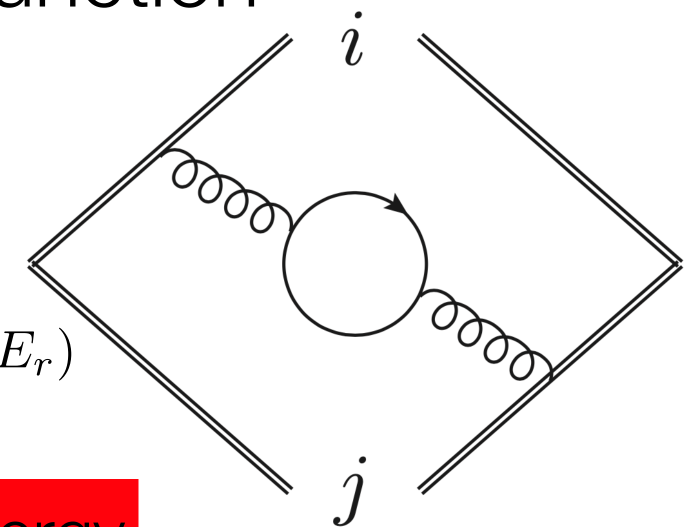


$$\mathcal{H}_{m+2}^{\bar{q}q} = \frac{g_s^4 n_F T_F}{\tilde{c}^{2\epsilon}(2\pi)^4} \sum_{i,j} \mathbf{T}_i^a \, \mathcal{H}_m \mathbf{T}_j^a \, \theta_{\rm in}(n_q)\theta_{\rm in}(n_r)$$
$$\times \int \frac{dE_q}{E_q^{2\epsilon}} \frac{dE_r}{E_r^{2\epsilon}} \frac{n_{iq}\, n_{jr} + n_{ir}\, n_{jq} - n_{ij}\, n_{qr}}{(E_q n_{iq} + E_r n_{ir})\,(E_q n_{jq} + E_r n_{jr})\, n_{qr}^2}\, \theta(\Lambda - E_q - E_r)$$

constraint on total energy

Double real emissions

Take soft $q\bar{q}$ as example. The soft limit of hard function



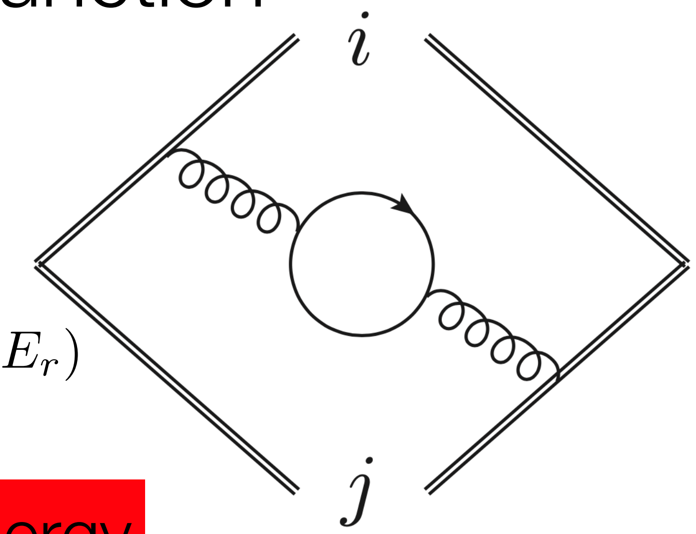
$$\mathcal{H}^{\bar{q}q}_{m+2} = \frac{g_s^4 n_F T_F}{\tilde{c}^{2\epsilon}(2\pi)^4} \sum_{i,j} \mathbf{T}_i^a \, \mathcal{H}_m \mathbf{T}_j^a \, \theta_{\rm in}(n_q)\theta_{\rm in}(n_r)$$
$$\times \int \frac{dE_q}{E_q^{2\epsilon}} \frac{dE_r}{E_r^{2\epsilon}} \frac{n_{iq}\, n_{jr} + n_{ir}\, n_{jq} - n_{ij}\, n_{qr}}{(E_q n_{iq} + E_r n_{ir})\,(E_q n_{jq} + E_r n_{jr})\, n_{qr}^2} \, \theta(\Lambda - E_q - E_r)$$

pure collinear singularity

constraint on total energy

Double real emissions

Take soft $q\bar{q}$ as example. The soft limit of hard function



$$\mathcal{H}^{\bar{q}q}_{m+2} = \frac{g_s^4 n_F T_F}{\tilde{c}^{2\epsilon}(2\pi)^4} \sum_{i,j} \mathbf{T}_i^a \, \mathcal{H}_m \mathbf{T}_j^a \, \theta_{\rm in}(n_q)\theta_{\rm in}(n_r)$$
$$\times \int \frac{dE_q}{E_q^{2\epsilon}} \frac{dE_r}{E_r^{2\epsilon}} \frac{n_{iq}\, n_{jr} + n_{ir}\, n_{jq} - n_{ij}\, n_{qr}}{(E_q n_{iq} + E_r n_{ir})\,(E_q n_{jq} + E_r n_{jr})\, n_{qr}^2} \theta(\Lambda - E_q - E_r)$$

pure collinear singularity

constraint on total energy

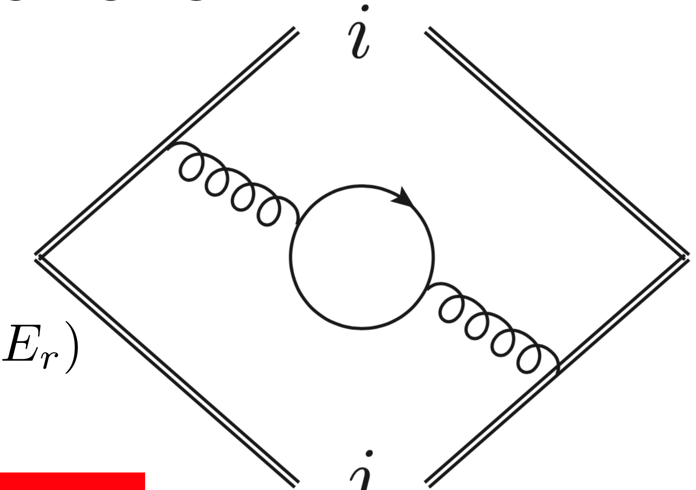
$$\mathcal{H}^{\bar{q}q}_{m+2} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{\mu}{2\Lambda}\right)^{4\epsilon} n_F T_F \sum_{i,j} \mathbf{T}_i^a \, \mathcal{H}_m \mathbf{T}_j^a \, \frac{K^F_{ij;qr} + K^{F,(\epsilon)}_{ij;qr}\epsilon}{\epsilon} \theta_{\rm in}(n_q)\theta_{\rm in}(n_r)$$

$$\mathbf{K}^F_{ij;qr} = \frac{4(n_{iq}n_{jr} + n_{ir}n_{jq} - n_{ij}n_{qr})}{n_{qr}^2(n_{iq}n_{jr} - n_{ir}n_{jq})} \ln\frac{n_{iq}n_{jr}}{n_{ir}n_{jq}} + \frac{8}{n_{qr}^2}$$

$$K^{F,(\epsilon)}_{ij;qr} = -\frac{20}{9\, n_{qr}^2}\left(\frac{n_{ir}}{n_{iq}} - \frac{n_{jr}}{n_{jq}}\right)^2 + \frac{16 W^q_{ij}}{n_{qr}} + \ldots$$

Double real emissions

Take soft $q\bar{q}$ as example. The soft limit of hard function



$$\mathcal{H}^{\bar{q}q}_{m+2} = \frac{g_s^4 n_F T_F}{\tilde{c}^{2\epsilon}(2\pi)^4} \sum_{i,j} \mathbf{T}_i^a \, \mathcal{H}_m \mathbf{T}_j^a \, \theta_{\rm in}(n_q)\theta_{\rm in}(n_r)$$
$$\times \int \frac{dE_q}{E_q^{2\epsilon}} \frac{dE_r}{E_r^{2\epsilon}} \frac{n_{iq}\, n_{jr} + n_{ir}\, n_{jq} - n_{ij}\, n_{qr}}{(E_q n_{iq} + E_r n_{ir})\,(E_q n_{jq} + E_r n_{jr})\, n_{qr}^2} \, \theta(\Lambda - E_q - E_r)$$

pure collinear singularity

constraint on total energy

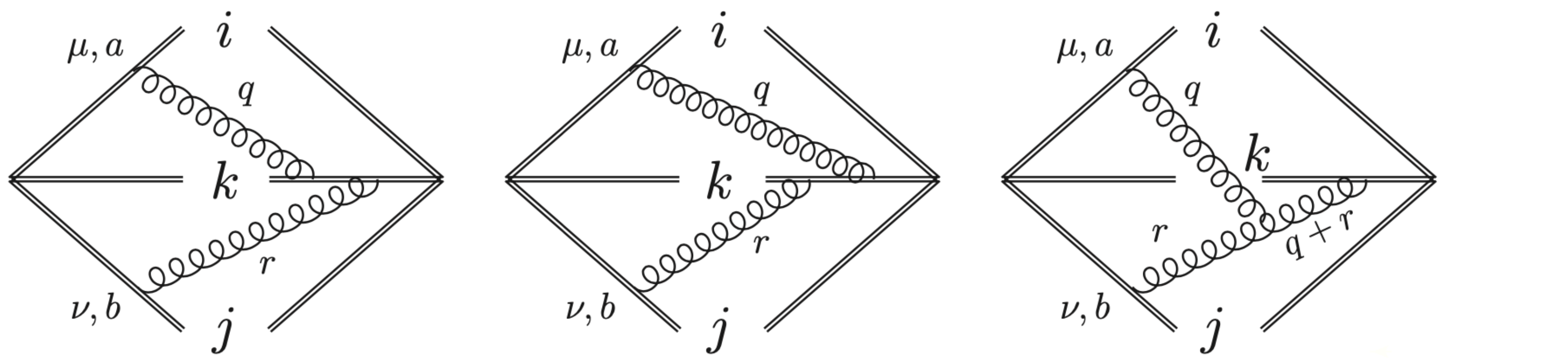
$$\mathcal{H}^{\bar{q}q}_{m+2} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{\mu}{2\Lambda}\right)^{4\epsilon} n_F T_F \sum_{i,j} \mathbf{T}_i^a \, \mathcal{H}_m \mathbf{T}_j^a \, \frac{K^F_{ij;qr} + K^{F,(\epsilon)}_{ij;qr}\epsilon}{\epsilon} \, \theta_{\rm in}(n_q)\theta_{\rm in}(n_r)$$

$$\mathbf{K}^F_{ij;qr} = \frac{4(n_{iq}n_{jr} + n_{ir}n_{jq} - n_{ij}n_{qr})}{n_{qr}^2(n_{iq}n_{jr} - n_{ir}n_{jq})} \ln \frac{n_{iq}n_{jr}}{n_{ir}n_{jq}} + \frac{8}{n_{qr}^2}$$

$$K^{F,(\epsilon)}_{ij;qr} = -\frac{20}{9\, n_{qr}^2} \left(\frac{n_{ir}}{n_{iq}} - \frac{n_{jr}}{n_{jq}}\right)^2 + \frac{16 W^q_{ij}}{n_{qr}} + \ldots$$

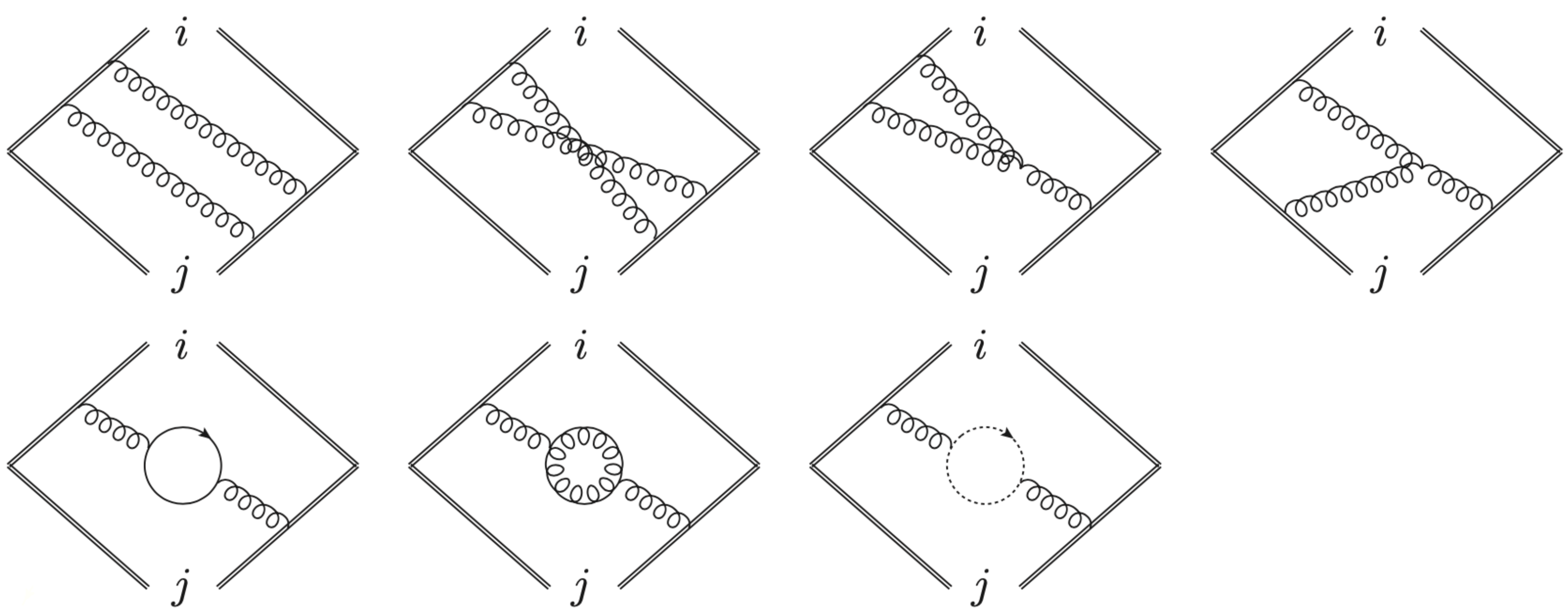
$$\mathbf{d}^F_m = -4 n_F T_F \sum_{(ij)} \mathbf{T}^a_{i,L} \mathbf{T}^a_{j,R} \left[K^F_{ij;qr}\, \theta_{\rm in}(n_r) - \frac{52}{9} W^q_{ij}\, \delta(n_q - n_r) \right] \theta_{\rm in}(n_q)\,.$$

Double-real contribution and $\mathbf{d}_m$



$$\mathbf{d}_m = \sum_{(ij)} \sum_k i f^{abc} \left(\mathbf{T}^a_{i,L} \mathbf{T}^b_{j,L} \mathbf{T}^c_{k,R} - \mathbf{T}^a_{i,R} \mathbf{T}^b_{j,R} \mathbf{T}^c_{k,L} \right) K_{ijk;qr} \theta_{\text{in}}(n_q) \theta_{\text{in}}(n_r)$$

$$- 2 \sum_{(ij)} \mathbf{T}^a_{i,L} \mathbf{T}^a_{j,R} \left(K_{ij;qr} \theta_{\text{in}}(n_q) \theta_{\text{in}}(n_r) + \Gamma_{\text{coll}} W^q_{ij} \, \theta_{\text{in}}(n_q) \delta(n_q - n_r) \right)$$



Angular functions

$$\mathbf{d}_m = \sum_{(ij)} \sum_k i f^{abc} \left(\mathbf{T}^a_{i,L} \mathbf{T}^b_{j,L} \mathbf{T}^c_{k,R} - \mathbf{T}^a_{i,R} \mathbf{T}^b_{j,R} \mathbf{T}^c_{k,L} \right) K_{ijk;qr} \theta_{\text{in}}(n_q) \theta_{\text{in}}(n_r)$$
$$- 2 \sum_{(ij)} \mathbf{T}^a_{i,L} \mathbf{T}^a_{j,R} \left(K_{ij;qr} \theta_{\text{in}}(n_q) \theta_{\text{in}}(n_r) + \Gamma_{\text{coll}} W^q_{ij}\, \theta_{\text{in}}(n_q) \delta(n_q - n_r) \right)$$

Three-leg correlations

$$K_{ijk;qr} = 8 \left(W^q_{ik} W^r_{jk} - W^q_{ik} W^r_{jq} - W^q_{ir} W^r_{jk} + W^q_{ij} W^r_{jq} \right) \ln \left(\frac{n_{kq}}{n_{kr}} \right)$$

Two-leg correlations

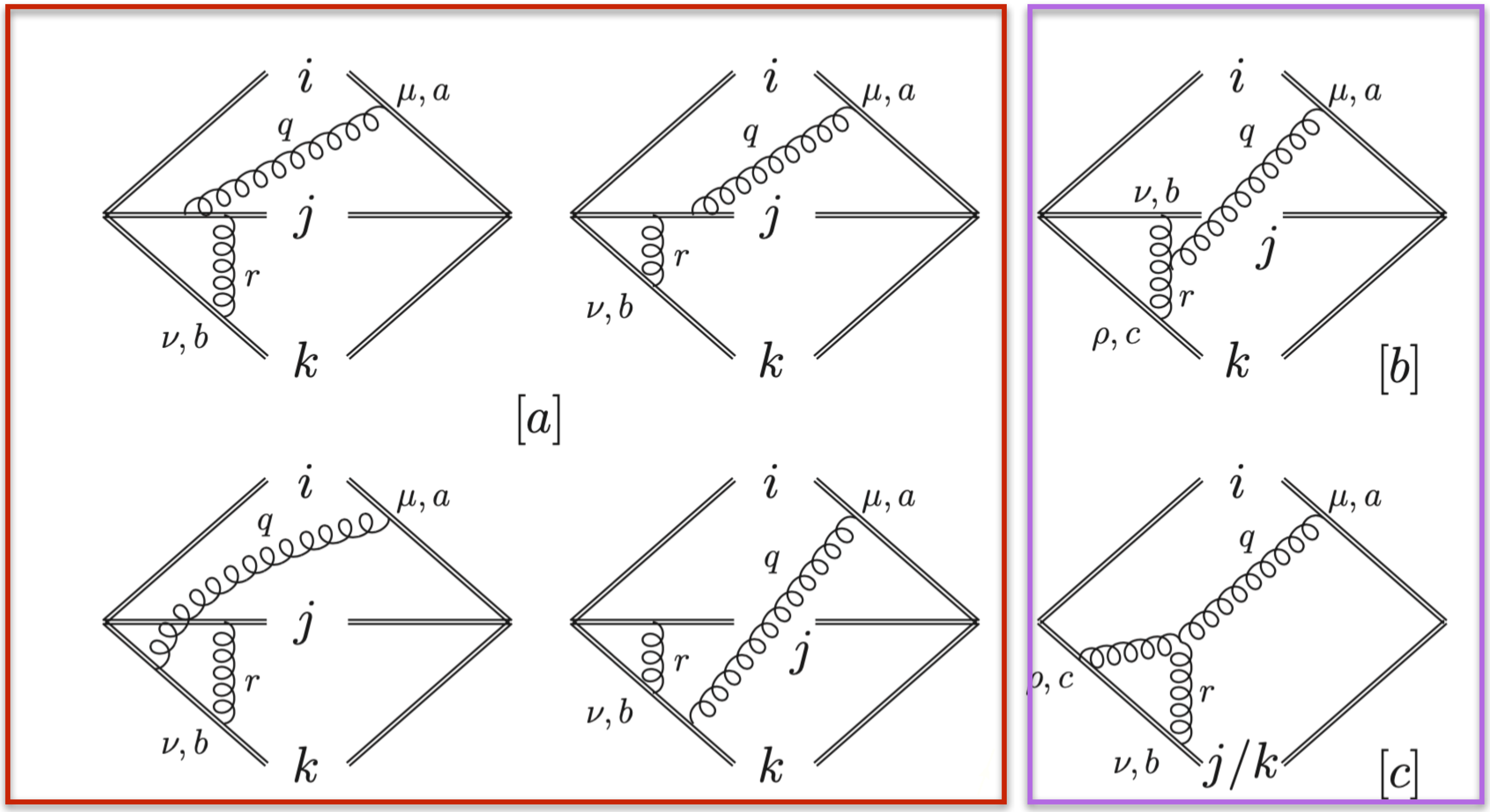
$$K_{ij;qr} = C_A K^{(a)}_{ij;qr} + [n_F T_F - 2C_A] K^{(b)}_{ij;qr} + [C_A - 2n_F T_F + n_S T_S] K^{(c)}_{ij;qr}$$

$$K^{(a)}_{ij;qr} = \frac{4 n_{ij}}{n_{iq} n_{qr} n_{jr}} \left[1 + \frac{n_{ij} n_{qr}}{n_{iq} n_{jr} - n_{ir} n_{jq}} \right] \ln \frac{n_{iq} n_{jr}}{n_{ir} n_{jq}},$$
$$K^{(b)}_{ij;qr} = \frac{8 n_{ij}}{n_{qr} (n_{iq} n_{jr} - n_{ir} n_{jq})} \ln \frac{n_{iq} n_{jr}}{n_{ir} n_{jq}},$$
$$K^{(c)}_{ij;qr} = \frac{4}{n^2_{qr}} \left(\frac{n_{iq} n_{jr} + n_{ir} n_{jq}}{n_{iq} n_{jr} - n_{ir} n_{jq}} \ln \frac{n_{iq} n_{jr}}{n_{ir} n_{jq}} - 2 \right).$$

Caron-Huot ‘15

Real-virtual

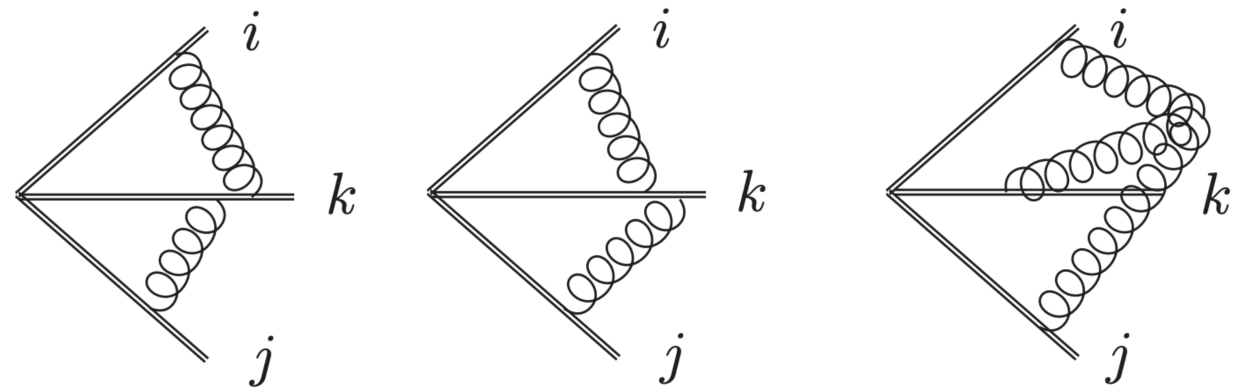
Real-virtual contribution and $\mathbf{r}_m$



$$\mathbf{r}_m = -2\sum_i \sum_{(jk)} i f^{abc} (\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R} - \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L}) \int [d\Omega_r]\, K_{ijk;qr}\theta_{\rm in}(n_q)$$
$$+ 8i\pi \sum_i \sum_{(jk)} i f^{abc} \left(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R} + \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L}\right) W^q_{ij} \ln W^q_{jk}\, \theta_{\rm in}(n_q)$$
$$+ \left(\frac{4\beta_0}{\epsilon} - \frac{8\pi^2 C_A}{3}\right) \sum_{(ij)} \mathbf{T}^a_{i,L}\mathbf{T}^a_{j,R} W^q_{ij}\theta_{\rm in}(n_q)$$

Double virtual corrections

Double-virtual contribution and $\mathbf{v}_m$

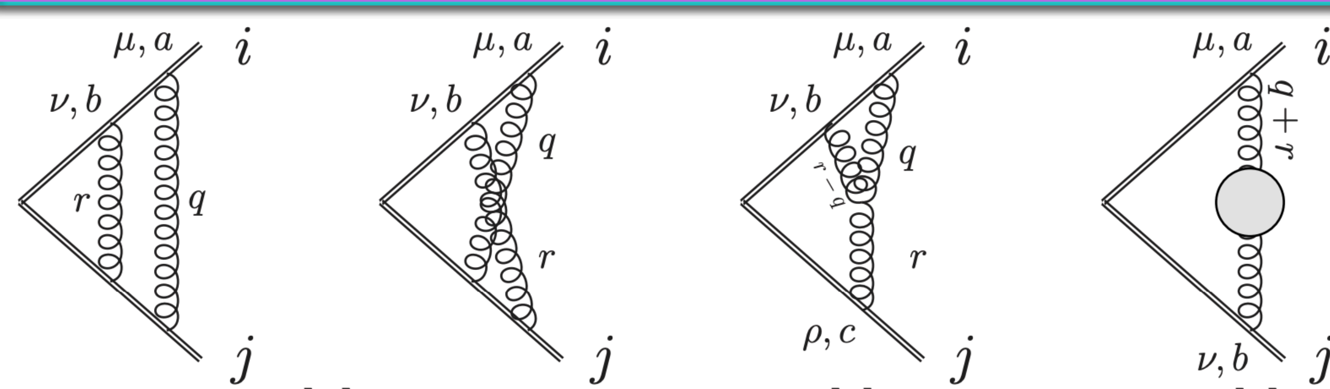


$$\mathbf{v}_m = \sum_{(ijk)} i f^{abc} \left(\mathbf{T}^a_{i,L} \mathbf{T}^b_{j,L} \mathbf{T}^c_{k,L} - \mathbf{T}^a_{i,R} \mathbf{T}^b_{j,R} \mathbf{T}^c_{k,R} \right) \int [d\Omega_q] \int [d\Omega_r] \, K_{ijk;qr}$$

$$+ \sum_{(ij)} \left(\mathbf{T}^a_{i,L} \mathbf{T}^a_{j,L} + \mathbf{T}^a_{i,R} \mathbf{T}^a_{j,R} \right) \Bigg\{ \int [d\Omega_q] \int [d\Omega_r] \, K_{ij;qr}$$

$$+ \left(-\frac{2\beta_0}{\epsilon} + \Gamma_{\mathrm{coll}} + \frac{4\pi^2 C_A}{3} \right) \int [d\Omega_q] \, W^q_{ij} \Bigg\}$$

$$- \sum_{(ij)} \left(\mathbf{T}^a_{i,L} \mathbf{T}^a_{j,L} - \mathbf{T}^a_{i,R} \mathbf{T}^a_{j,R} \right) \gamma_1^{\mathrm{cusp}} \frac{i\pi \Pi_{ij}}{2} \, .$$



[a] [b] [c]

Diagrammatic results and collinear rearrangement

$$
\begin{aligned}
\mathbf{d}_m = & \sum_{(ij)} \sum_{k} i f^{abc} \left(\mathbf{T}^a_{i,L} \mathbf{T}^b_{j,L} \mathbf{T}^c_{k,R} - \mathbf{T}^a_{i,R} \mathbf{T}^b_{j,R} \mathbf{T}^c_{k,L} \right) K_{ijk;qr} \theta_{\text{in}}(n_q) \theta_{\text{in}}(n_r) \\
& - 2 \sum_{(ij)} \mathbf{T}^a_{i,L} \mathbf{T}^a_{j,R} \left(K_{ij;qr} \theta_{\text{in}}(n_q) \theta_{\text{in}}(n_r) + \Gamma_{\text{coll}} W^q_{ij}\, \theta_{\text{in}}(n_q) \delta(n_q - n_r) \right) \\
\mathbf{r}_m = & - 2 \sum_{i} \sum_{(jk)} i f^{abc} (\mathbf{T}^a_{i,L} \mathbf{T}^b_{j,R} \mathbf{T}^c_{k,R} - \mathbf{T}^a_{i,R} \mathbf{T}^b_{j,L} \mathbf{T}^c_{k,L}) \int [d\Omega_r]\, K_{ijk;qr} \theta_{\text{in}}(n_q) \\
& + 8 i \pi \sum_{i} \sum_{(jk)} i f^{abc} \left(\mathbf{T}^a_{i,L} \mathbf{T}^b_{j,R} \mathbf{T}^c_{k,R} + \mathbf{T}^a_{i,R} \mathbf{T}^b_{j,L} \mathbf{T}^c_{k,L} \right) W^q_{ij} \ln W^q_{jk}\, \theta_{\text{in}}(n_q) \\
& + \left(\frac{4\beta_0}{\epsilon} - \frac{8\pi^2 C_A}{3} \right) \sum_{(ij)} \mathbf{T}^a_{i,L} \mathbf{T}^a_{j,R} W^q_{ij} \theta_{\text{in}}(n_q) \\
\mathbf{v}_m = & \sum_{(ijk)} i f^{abc} \left(\mathbf{T}^a_{i,L} \mathbf{T}^b_{j,L} \mathbf{T}^c_{k,L} - \mathbf{T}^a_{i,R} \mathbf{T}^b_{j,R} \mathbf{T}^c_{k,R} \right) \int [d\Omega_q] \int [d\Omega_r]\, K_{ijk;qr} \\
& + \sum_{(ij)} (\mathbf{T}^a_{i,L} \mathbf{T}^a_{j,L} + \mathbf{T}^a_{i,R} \mathbf{T}^a_{j,R}) \left\{ \int [d\Omega_q] \int [d\Omega_r]\, K_{ij;qr} \right. \\
& \left. + \left(-\frac{2\beta_0}{\epsilon} + \Gamma_{\text{coll}} + \frac{4\pi^2 C_A}{3} \right) \int [d\Omega_q]\, W^q_{ij} \right\} \\
& - \sum_{(ij)} (\mathbf{T}^a_{i,L} \mathbf{T}^a_{j,L} - \mathbf{T}^a_{i,R} \mathbf{T}^a_{j,R})\, \gamma_1^{\text{cusp}} \frac{i\pi \Pi_{ij}}{2} \,.
\end{aligned}
$$

Diagrammatic results and collinear rearrangement

$$\mathbf{d}_m = \sum_{(ij)}\sum_k if^{abc}\left(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,R} - \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,L}\right) K_{ijk;qr}\theta_{\rm in}(n_q)\theta_{\rm in}(n_r)$$
$$- 2\sum_{(ij)}\mathbf{T}^a_{i,L}\mathbf{T}^a_{j,R}\left(K_{ij;qr}\theta_{\rm in}(n_q)\theta_{\rm in}(n_r) + \Gamma_{\rm coll}W^q_{ij}\,\theta_{\rm in}(n_q)\delta(n_q - n_r)\right)$$

$$\mathbf{r}_m = -2\sum_i\sum_{(jk)} if^{abc}(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R} - \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L})\int[d\Omega_r]\,K_{ijk;qr}\theta_{\rm in}(n_q)$$
$$+ 8i\pi\sum_i\sum_{(jk)} if^{abc}\left(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R} + \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L}\right) W^q_{ij}\ln W^q_{jk}\,\theta_{\rm in}(n_q)$$
$$+\left(\frac{4\beta_0}{\epsilon} - \frac{8\pi^2 C_A}{3}\right)\sum_{(ij)}\mathbf{T}^a_{i,L}\mathbf{T}^a_{j,R}W^q_{ij}\theta_{\rm in}(n_q)$$

$$\mathbf{v}_m = \sum_{(ijk)} if^{abc}\left(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L} - \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R}\right)\int[d\Omega_q]\int[d\Omega_r]\,K_{ijk;qr}$$
$$+\sum_{(ij)}(\mathbf{T}^a_{i,L}\mathbf{T}^a_{j,L} + \mathbf{T}^a_{i,R}\mathbf{T}^a_{j,R})\left\{\int[d\Omega_q]\int[d\Omega_r]\,K_{ij;qr}\right.$$
$$\left. + \left(-\frac{2\beta_0}{\epsilon} + \Gamma_{\rm coll} + \frac{4\pi^2 C_A}{3}\right)\int[d\Omega_q]\,W^q_{ij}\right\}$$
$$-\sum_{(ij)}(\mathbf{T}^a_{i,L}\mathbf{T}^a_{j,L} - \mathbf{T}^a_{i,R}\mathbf{T}^a_{j,R})\,\gamma_1^{\rm cusp}\frac{i\pi\Pi_{ij}}{2}\,.$$

$$\int[d\Omega_r]\,K_{ij;qr} = 2W^q_{ij}\left[\beta_0\left(\frac{1}{\epsilon} + \ln(2W^q_{ij})\right) + \frac{1}{4}\gamma_1^{\rm cusp} - \frac{\Gamma_{\rm coll}}{2} - \frac{2\pi^2 C_A}{3} + \frac{c_R - 1}{3}C_A\right]$$

Result for $\Gamma^{(2)}$

$$
\begin{aligned}
\boldsymbol{d}_m = & \sum_{(ij)} \sum_{k} i f^{abc} \left(\boldsymbol{T}^a_{i,L} \boldsymbol{T}^b_{j,L} \boldsymbol{T}^c_{k,R} - \boldsymbol{T}^a_{i,R} \boldsymbol{T}^b_{j,R} \boldsymbol{T}^c_{k,L} \right) K_{ijk;qr} \theta_{\text{in}}(n_q) \theta_{\text{in}}(n_r) \\
& - 2 \sum_{(ij)} \boldsymbol{T}^c_{i,L} \boldsymbol{T}^c_{j,R} K_{ij;qr} \theta_{\text{in}}(n_q) \theta_{\text{in}}(n_r) \, , \\
\boldsymbol{r}_m = & - 2 \sum_{i} \sum_{(jk)} i f^{abc} (\boldsymbol{T}^a_{i,L} \boldsymbol{T}^b_{j,R} \boldsymbol{T}^c_{k,R} - \boldsymbol{T}^a_{i,R} \boldsymbol{T}^b_{j,L} \boldsymbol{T}^c_{k,L}) \int \left[d^2 \Omega_r \right] K_{ijk;qr} \theta_{\text{in}}(n_q) \\
& - \sum_{(ij)} \boldsymbol{T}^a_{i,L} \boldsymbol{T}^a_{j,R} \left\{ W^q_{ij} \left[4 \beta_0 \ln(2 W^q_{ij}) + \gamma_1^{\text{cusp}} \right] - 2 \int \left[d^2 \Omega_r \right] K_{ij;qr} \right\} \theta_{\text{in}}(n_q) \\
& + 8 i \pi \sum_{i} \sum_{(jk)} i f^{abc} \left(\boldsymbol{T}^a_{i,L} \boldsymbol{T}^b_{j,R} \boldsymbol{T}^c_{k,R} + \boldsymbol{T}^a_{i,R} \boldsymbol{T}^b_{j,L} \boldsymbol{T}^c_{k,L} \right) W^q_{ij} \ln W^q_{jk} \, \theta_{\text{in}}(n_q) \, , \\
\boldsymbol{v}_m = & \sum_{(ijk)} i f^{abc} \left(\boldsymbol{T}^a_{i,L} \boldsymbol{T}^b_{j,L} \boldsymbol{T}^c_{k,L} - \boldsymbol{T}^a_{i,R} \boldsymbol{T}^b_{j,R} \boldsymbol{T}^c_{k,R} \right) \int \left[d^2 \Omega_q \right] \int \left[d^2 \Omega_r \right] K_{ijk;qr} \\
& + \sum_{(ij)} \frac{1}{2} (\boldsymbol{T}^a_{i,L} \boldsymbol{T}^a_{j,L} + \boldsymbol{T}^a_{i,R} \boldsymbol{T}^a_{j,R}) \int \left[d^2 \Omega_q \right] W^q_{ij} \left[4 \beta_0 \ln(2 W^q_{ij}) + \gamma_1^{\text{cusp}} \right] \\
& - i \pi \sum_{(ij)} \frac{1}{2} (\boldsymbol{T}^a_{i,L} \boldsymbol{T}^a_{j,L} - \boldsymbol{T}^a_{i,R} \boldsymbol{T}^a_{j,R}) \, \Pi_{ij} \, \gamma_1^{\text{cusp}} \, .
\end{aligned}
$$

Scheme change

For the renormalization conditions for soft function

$$\mathcal{S}^{\mathrm{ren}(1)} = \mathcal{S}^{(1)} - \frac{1}{2\epsilon}\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{1}\,,$$

$$\mathcal{S}^{\mathrm{ren}(2)} = \mathcal{S}^{(2)} - \frac{1}{8\epsilon^2}\left[\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{1} + 2\beta_0\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{1}\right] - \frac{1}{4\epsilon}\left[\mathbf{\Gamma}^{(2)}\hat{\otimes}\mathbf{1} + 2\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathcal{S}^{\mathrm{ren}(1)} + 4\beta_0\mathcal{S}^{\mathrm{ren}(1)}\right]$$

Angular integration in d dimension

Scheme change

For the renormalization conditions for soft function

$$\mathcal{S}^{\mathrm{ren}(1)} = \mathcal{S}^{(1)} - \frac{1}{2\epsilon}\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{1}\,,$$

$$\mathcal{S}^{\mathrm{ren}(2)} = \mathcal{S}^{(2)} - \frac{1}{8\epsilon^2}\left[\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{1} + 2\beta_0\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{1}\right] - \frac{1}{4\epsilon}\left[\mathbf{\Gamma}^{(2)}\hat{\otimes}\mathbf{1} + 2\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathcal{S}^{\mathrm{ren}(1)} + 4\beta_0\mathcal{S}^{\mathrm{ren}(1)}\right]$$

Angular integration in d dimension

Change to $\overline{\mathrm{MS}}$ scheme

$$\mathcal{S}^{\mathrm{ren}(2)} = \mathcal{S}^{(2)} - \frac{1}{8\epsilon^2}\left[\mathbf{\Gamma}^{(1)} \otimes_2 \mathbf{\Gamma}^{(1)} \otimes_2 \mathbf{1} + 2\beta_0\mathbf{\Gamma}^{(1)} \otimes_2 \mathbf{1}\right] - \frac{1}{4\epsilon}\left[\bar{\mathbf{\Gamma}}^{(2)} \otimes_2 \mathbf{1} + 2\mathbf{\Gamma}^{(1)} \otimes_2 \bar{\mathcal{S}}^{\mathrm{ren}(1)} + 4\beta_0\bar{\mathcal{S}}^{\mathrm{ren}(1)}\right]$$

Angular integration in 4 dimension

Scheme change

For the renormalization conditions for soft function

$$\mathcal{S}^{\mathrm{ren}(1)} = \mathcal{S}^{(1)} - \frac{1}{2\epsilon}\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{1}\,,$$

$$\mathcal{S}^{\mathrm{ren}(2)} = \mathcal{S}^{(2)} - \frac{1}{8\epsilon^2}\left[\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{1} + 2\beta_0\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathbf{1}\right] - \frac{1}{4\epsilon}\left[\mathbf{\Gamma}^{(2)}\hat{\otimes}\mathbf{1} + 2\mathbf{\Gamma}^{(1)}\hat{\otimes}\mathcal{S}^{\mathrm{ren}(1)} + 4\beta_0\mathcal{S}^{\mathrm{ren}(1)}\right]$$

Angular integration in d dimension

Change to $\overline{\mathrm{MS}}$ scheme

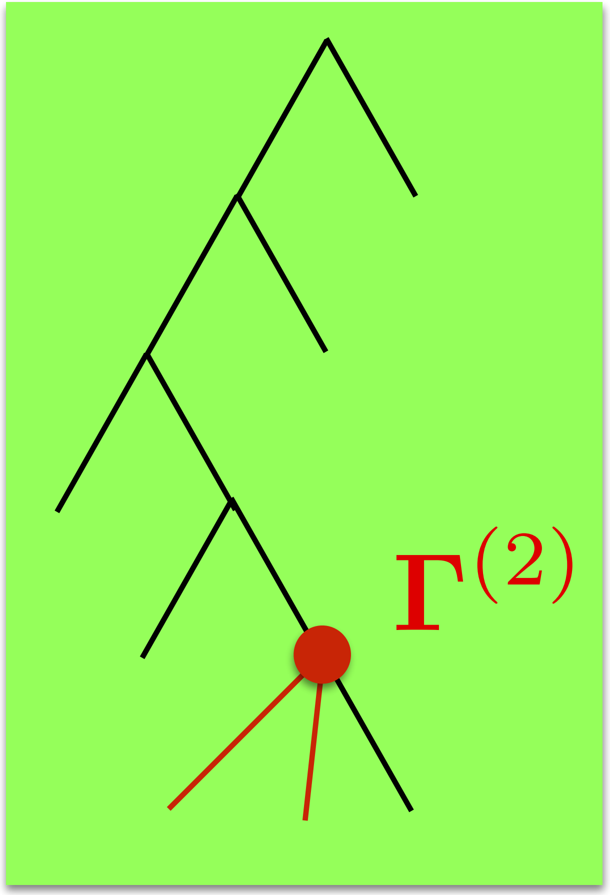
Angular integration in 4 dimension

$$\mathcal{S}^{\mathrm{ren}(2)} = \mathcal{S}^{(2)} - \frac{1}{8\epsilon^2}\left[\mathbf{\Gamma}^{(1)}\otimes_2\mathbf{\Gamma}^{(1)}\otimes_2\mathbf{1} + 2\beta_0\mathbf{\Gamma}^{(1)}\otimes_2\mathbf{1}\right] - \frac{1}{4\epsilon}\left[\bar{\mathbf{\Gamma}}^{(2)}\otimes_2\mathbf{1} + 2\mathbf{\Gamma}^{(1)}\otimes_2\bar{\mathcal{S}}^{\mathrm{ren}(1)} + 4\beta_0\bar{\mathcal{S}}^{\mathrm{ren}(1)}\right]$$

$$\bar{\mathbf{\Gamma}}^{(2)}\otimes_2\mathbf{1} = \mathbf{\Gamma}^{(2)}\otimes_2\mathbf{1} - 2\beta_0\,\mathbf{\Gamma}^{(1)}\otimes_\epsilon\mathbf{1} - \left(\mathbf{\Gamma}^{(1)}\otimes_2\mathbf{\Gamma}^{(1)}\otimes_\epsilon\mathbf{1} - \mathbf{\Gamma}^{(1)}\otimes_\epsilon\mathbf{\Gamma}^{(1)}\otimes_2\mathbf{1}\right)$$

Outlook and conclusion

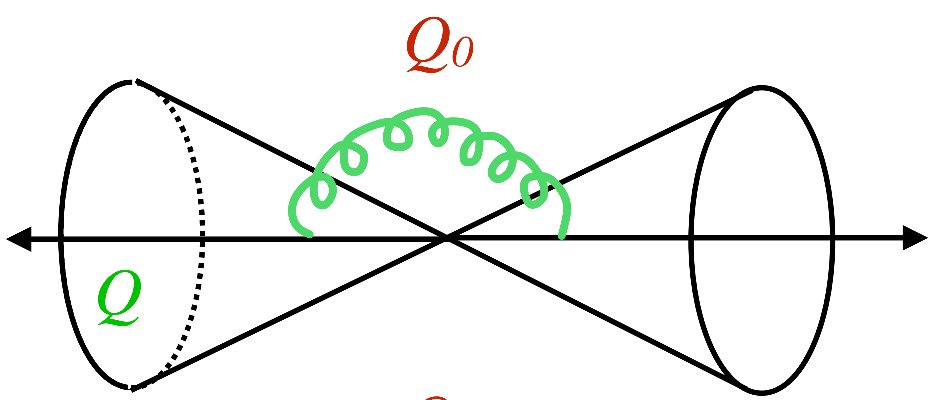
1. We present two-loop anomalous dimension.
2. Reproduce divergence of two-loop soft functions.
3. The anomalous dimension is finite combining $\mathbf{d}_m$, $\mathbf{r}_m$ and $\mathbf{v}_m$.
4. The angular integrations are in 4 dimension, which is suitable for parton shower.



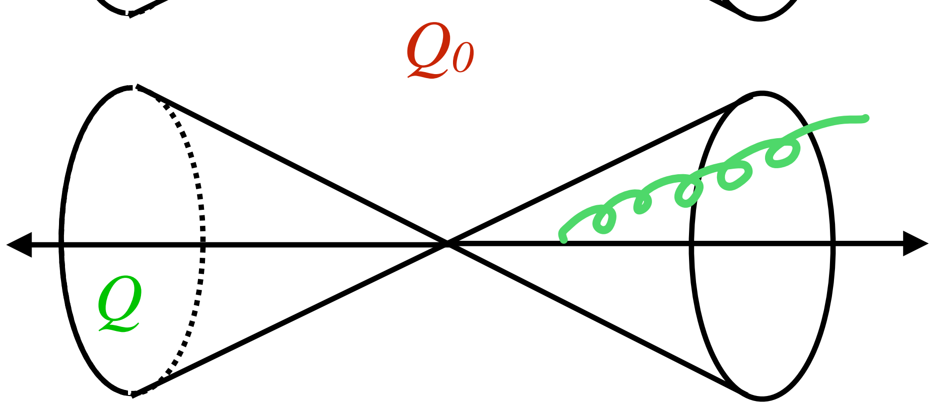
Back up

NLO cross sections with region expansion

Hard mode: $p_h \sim Q(1,1,1)$

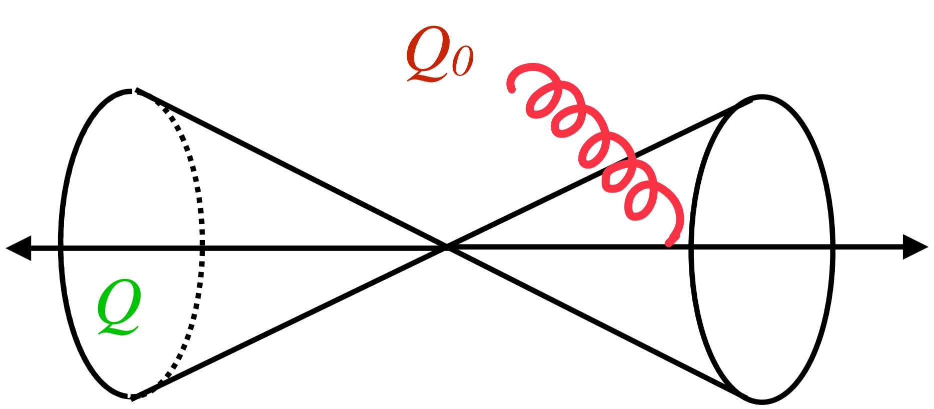


$$\sigma_0 \frac{\alpha_s C_F}{4\pi} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left[-\frac{4}{\epsilon^2} - \frac{6}{\epsilon} + \ldots\right]$$



$$\sigma_0 \frac{\alpha_s C_F}{4\pi} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left[+\frac{4}{\epsilon^2} + \frac{6}{\epsilon} - \frac{4\ln(r)}{\epsilon} + \ldots\right]$$

Soft mode: $p_s \sim Q_0(1,1,1)$



$$\sigma_0 \frac{\alpha_s C_F}{4\pi} \left(\frac{\mu^2}{Q_0^2}\right)^{\epsilon} \left[\frac{4\ln(r)}{\epsilon} + \ldots\right]$$

Collinear rearrangement

Collinear singularity in $\mathbf{v}_m$

$$
\begin{aligned}
\mathbf{v}_m = &\sum_{(ijk)} i f^{abc} \left(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L} - \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R}\right) \int [d\Omega_q] \int [d\Omega_r]\, K_{ijk;qr} \\
&+ \sum_{(ij)} \left(\mathbf{T}^a_{i,L}\mathbf{T}^a_{j,L} + \mathbf{T}^a_{i,R}\mathbf{T}^a_{j,R}\right) \Bigg\{ \int [d\Omega_q] \int [d\Omega_r]\, K_{ij;qr} \\
&+ \left(-\frac{2\beta_0}{\epsilon} + \Gamma_{\rm coll} + \frac{4\pi^2 C_A}{3}\right) \int [d\Omega_q]\, W^q_{ij} \Bigg\} \\
&- \sum_{(ij)} \left(\mathbf{T}^a_{i,L}\mathbf{T}^a_{j,L} - \mathbf{T}^a_{i,R}\mathbf{T}^a_{j,R}\right) \gamma_1^{\rm cusp} \frac{i\pi\Pi_{ij}}{2}\,.
\end{aligned}
$$

Collinear rearrangement

Collinear singularity in $\mathbf{v}_m$

$$\mathbf{v}_m = \sum_{(ijk)} i f^{abc} \left(\mathbf{T}^a_{i,L} \mathbf{T}^b_{j,L} \mathbf{T}^c_{k,L} - \mathbf{T}^a_{i,R} \mathbf{T}^b_{j,R} \mathbf{T}^c_{k,R} \right) \int [d\Omega_q] \int [d\Omega_r]\, K_{ijk;qr}$$

$$+ \sum_{(ij)} \left(\mathbf{T}^a_{i,L} \mathbf{T}^a_{j,L} + \mathbf{T}^a_{i,R} \mathbf{T}^a_{j,R} \right) \left\{ \int [d\Omega_q] \int [d\Omega_r]\, K_{ij;qr} + \left(-\frac{2\beta_0}{\epsilon} + \Gamma_{\rm coll} + \frac{4\pi^2 C_A}{3} \right) \int [d\Omega_q]\, W^q_{ij} \right\}$$

$$- \sum_{(ij)} \left(\mathbf{T}^a_{i,L} \mathbf{T}^a_{j,L} - \mathbf{T}^a_{i,R} \mathbf{T}^a_{j,R} \right) \gamma_1^{\rm cusp} \frac{i\pi \Pi_{ij}}{2} \, .$$

Collinear rearrangement

Collinear singularity in $\mathbf{v}_m$

$$
\begin{aligned}
\mathbf{v}_m = & \sum_{(ijk)} i f^{abc} \left(\mathbf{T}^a_{i,L} \mathbf{T}^b_{j,L} \mathbf{T}^c_{k,L} - \mathbf{T}^a_{i,R} \mathbf{T}^b_{j,R} \mathbf{T}^c_{k,R} \right) \int [d\Omega_q] \int [d\Omega_r] \, K_{ijk;qr} \\
& + \sum_{(ij)} \left(\mathbf{T}^a_{i,L} \mathbf{T}^a_{j,L} + \mathbf{T}^a_{i,R} \mathbf{T}^a_{j,R} \right) \left\{ \int [d\Omega_q] \int [d\Omega_r] \, K_{ij;qr} \right. \\
& \left. + \left(- \frac{2\beta_0}{\epsilon} + \Gamma_{\rm coll} + \frac{4\pi^2 C_A}{3} \right) \int [d\Omega_q] \, W^q_{ij} \right\} \\
& - \sum_{(ij)} \left(\mathbf{T}^a_{i,L} \mathbf{T}^a_{j,L} - \mathbf{T}^a_{i,R} \mathbf{T}^a_{j,R} \right) \gamma_1^{\rm cusp} \frac{i\pi \Pi_{ij}}{2} \, .
\end{aligned}
$$

$$
\begin{aligned}
\int [d\Omega_r] \, K_{ij;qr} = 2 W^q_{ij} \Bigg[& \beta_0 \left(\frac{1}{\epsilon} + \ln(2W^q_{ij}) \right) + \frac{1}{4} \gamma_1^{\rm cusp} \\
& - \frac{\Gamma_{\rm coll}}{2} - \frac{2\pi^2 C_A}{3} + \frac{c_R - 1}{3} C_A \Bigg]
\end{aligned}
$$

Collinear rearrangement

Collinear singularity in $\mathbf{v}_m$

$$\mathbf{v}_m = \sum_{(ijk)} i f^{abc} \left(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L} - \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R} \right) \int [d\Omega_q] \int [d\Omega_r]\, K_{ijk;qr}$$

$$+ \sum_{(ij)} \left(\mathbf{T}^a_{i,L}\mathbf{T}^a_{j,L} + \mathbf{T}^a_{i,R}\mathbf{T}^a_{j,R} \right) \left\{ \int [d\Omega_q] \int [d\Omega_r]\, K_{ij;qr} + \left(-\frac{2\beta_0}{\epsilon} + \Gamma_{\rm coll} + \frac{4\pi^2 C_A}{3} \right) \int [d\Omega_q]\, W^q_{ij} \right\}$$

$$- \sum_{(ij)} \left(\mathbf{T}^a_{i,L}\mathbf{T}^a_{j,L} - \mathbf{T}^a_{i,R}\mathbf{T}^a_{j,R} \right) \gamma_1^{\rm cusp} \frac{i\pi \Pi_{ij}}{2} \,.$$

$$\int [d\Omega_r]\, K_{ij;qr} = 2W^q_{ij} \left[\beta_0 \left(\frac{1}{\epsilon} + \ln(2W^q_{ij}) \right) + \frac{1}{4}\gamma_1^{\rm cusp} - \frac{\Gamma_{\rm coll}}{2} - \frac{2\pi^2 C_A}{3} + \frac{c_R - 1}{3} C_A \right]$$

$$\mathbf{v}_m = \sum_{(ijk)} i f^{abc} \left(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L} - \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R} \right) \int [d\Omega_q] \int [d\Omega_r]\, K_{ijk;qr}$$

$$+ \sum_{(ij)} \frac{\mathbf{T}^a_{i,L}\mathbf{T}^a_{j,L} + \mathbf{T}^a_{i,R}\mathbf{T}^a_{j,R}}{2} \int [d\Omega_q]\, W^q_{ij} \left[4\beta_0 \ln(2W^q_{ij}) + \gamma_1^{\rm cusp} + \frac{4C_A(c_R - 1)}{3} \right]$$

$$- \sum_{(ij)} \left(\mathbf{T}^a_{i,L}\mathbf{T}^a_{j,L} - \mathbf{T}^a_{i,R}\mathbf{T}^a_{j,R} \right) \gamma_1^{\rm cusp} \frac{i\pi \Pi_{ij}}{2} \,.$$

Collinear rearrangement in $\mathbf{d}_m$ and $\mathbf{r}_m$

$$\mathbf{r}_m = -2\sum_i \sum_{(jk)} i f^{abc}(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R} - \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L}) \int [d\Omega_r]\, K_{ijk;qr}\theta_{\rm in}(n_q)$$
$$+ 8i\pi \sum_i \sum_{(jk)} i f^{abc}\left(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R} + \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L}\right) W^q_{ij} \ln W^q_{jk}\, \theta_{\rm in}(n_q)$$
$$+ \left(\frac{4\beta_0}{\epsilon} - \frac{8\pi^2 C_A}{3}\right) \sum_{(ij)} \mathbf{T}^a_{i,L}\mathbf{T}^a_{j,R} W^q_{ij}\theta_{\rm in}(n_q)$$

Replace the β_0 term by $K_{ij;qr}$
And combine with $\Gamma_{\rm coll}$ term in $\mathbf{d}_m$

$$\mathbf{d}_m = \sum_{(ij)} \sum_k i f^{abc}\left(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,R} - \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,L}\right) K_{ijk;qr}\theta_{\rm in}(n_q)\theta_{\rm in}(n_r)$$
$$-2\sum_{(ij)} \mathbf{T}^a_{i,L}\mathbf{T}^a_{j,R}\left(K_{ij;qr}\theta_{\rm in}(n_q)\theta_{\rm in}(n_r) + \Gamma_{\rm coll} W^q_{ij}\, \theta_{\rm in}(n_q)\delta(n_q - n_r)\right)$$

$$\int [d\Omega_r]\, K_{ij;qr} = 2W_{ij}^q \Big[\beta_0 \left(\frac{1}{\epsilon} + \ln(2W_{ij}^q) \right) + \frac{1}{4}\,\gamma_1^{\mathrm{cusp}} - \frac{\Gamma_{\mathrm{coll}}}{2} - \frac{2\pi^2 C_A}{3} + \frac{c_R - 1}{3} C_A \Big]$$

Collinear rearrangement in $\mathbf{d}_m$ and $\mathbf{r}_m$

$$\begin{aligned}\mathbf{r}_m = &-2 \sum_i \sum_{(jk)} i f^{abc} (\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R} - \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L}) \int [d\Omega_r]\, K_{ijk;qr}\theta_{\mathrm{in}}(n_q) \\ &+ 8i\pi \sum_i \sum_{(jk)} i f^{abc} \left(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,R} + \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,L}\right) W_{ij}^q \ln W_{jk}^q\, \theta_{\mathrm{in}}(n_q) \\ &+ \left(\frac{4\beta_0}{\epsilon} - \frac{8\pi^2 C_A}{3} \right) \sum_{(ij)} \mathbf{T}^a_{i,L}\mathbf{T}^a_{j,R} W_{ij}^q \theta_{\mathrm{in}}(n_q)\end{aligned}$$

Replace the β_0 term by $K_{ij;qr}$

And combine with Γ_{coll} term in $\mathbf{d}_m$

$$\begin{aligned}\mathbf{d}_m = &\sum_{(ij)} \sum_k i f^{abc} \left(\mathbf{T}^a_{i,L}\mathbf{T}^b_{j,L}\mathbf{T}^c_{k,R} - \mathbf{T}^a_{i,R}\mathbf{T}^b_{j,R}\mathbf{T}^c_{k,L}\right) K_{ijk;qr}\theta_{\mathrm{in}}(n_q)\theta_{\mathrm{in}}(n_r) \\ &-2 \sum_{(ij)} \mathbf{T}^a_{i,L}\mathbf{T}^a_{j,R} \left(K_{ij;qr}\theta_{\mathrm{in}}(n_q)\theta_{\mathrm{in}}(n_r) + \Gamma_{\mathrm{coll}} W_{ij}^q\, \theta_{\mathrm{in}}(n_q)\delta(n_q - n_r) \right)\end{aligned}$$