Next-to-leading non-global logarithms in the planar limit

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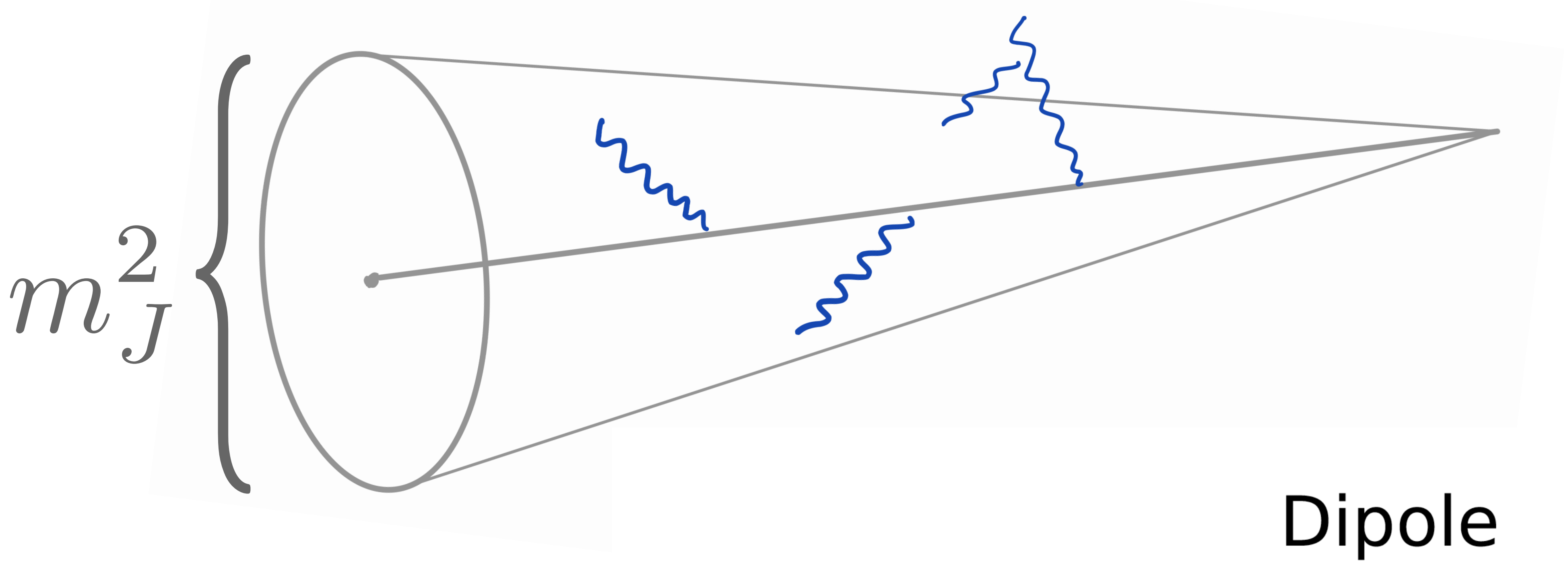
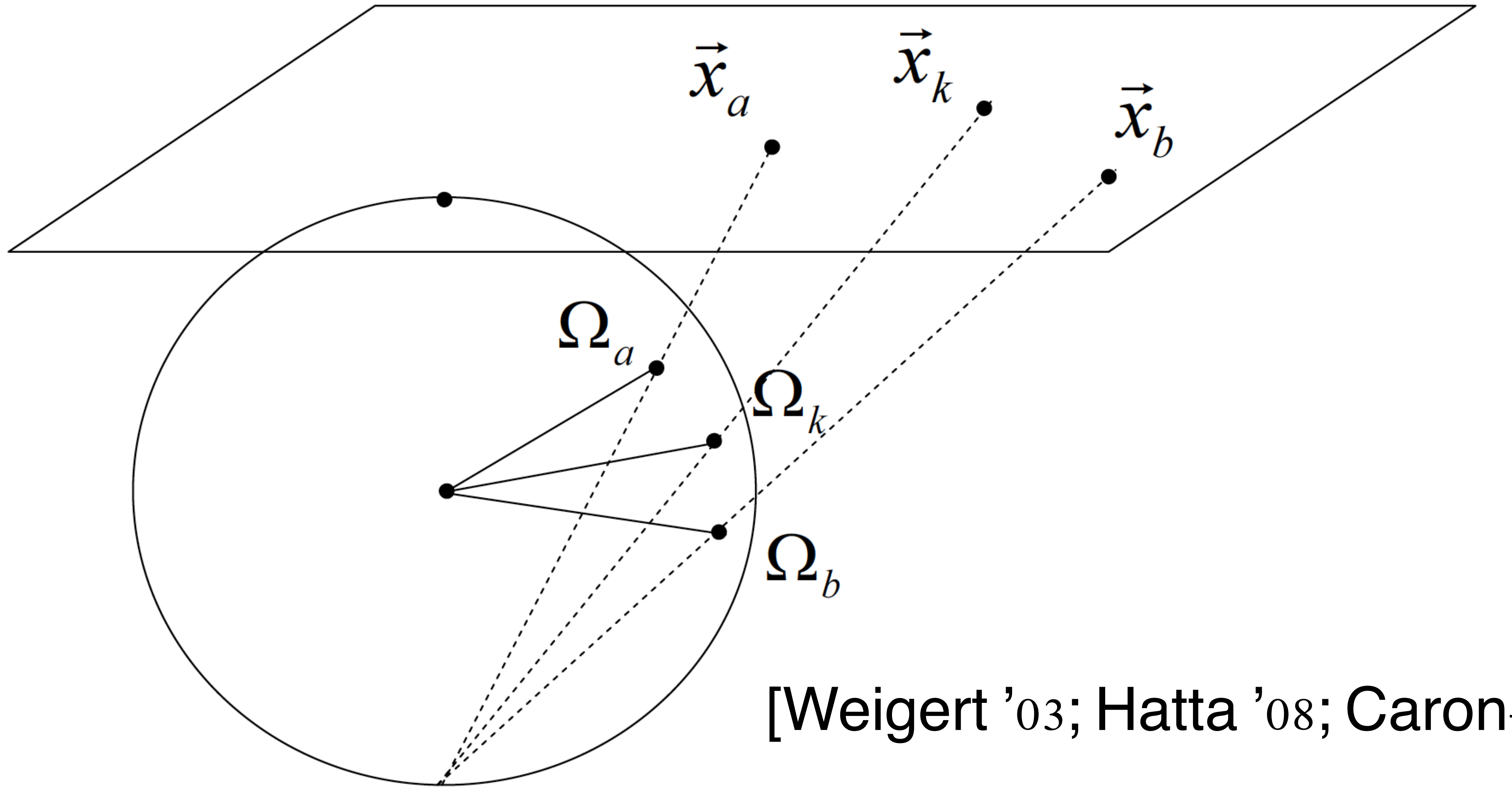
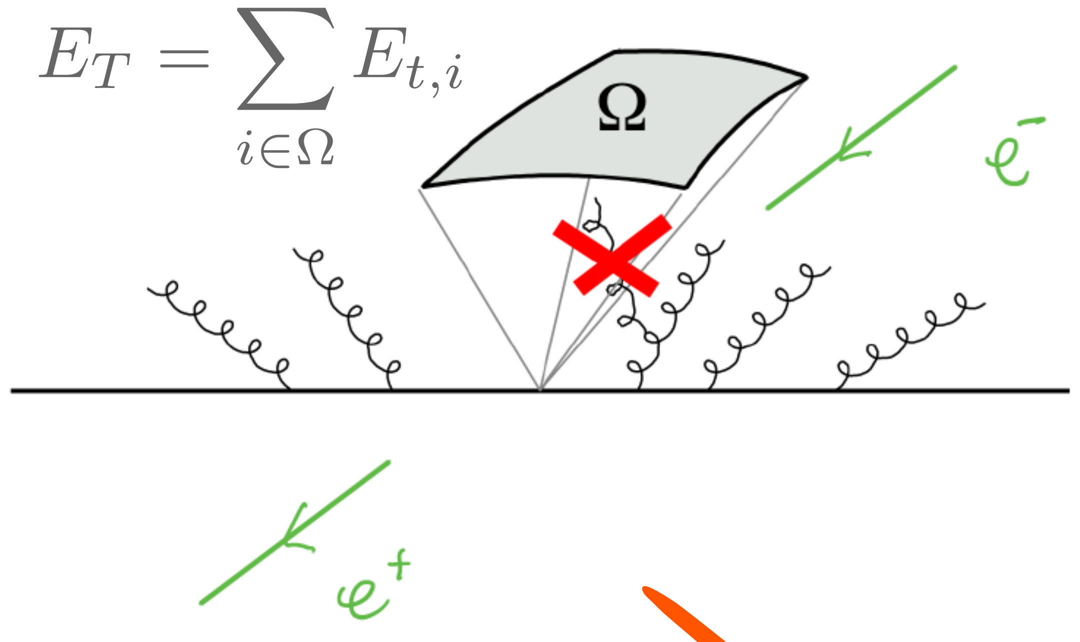
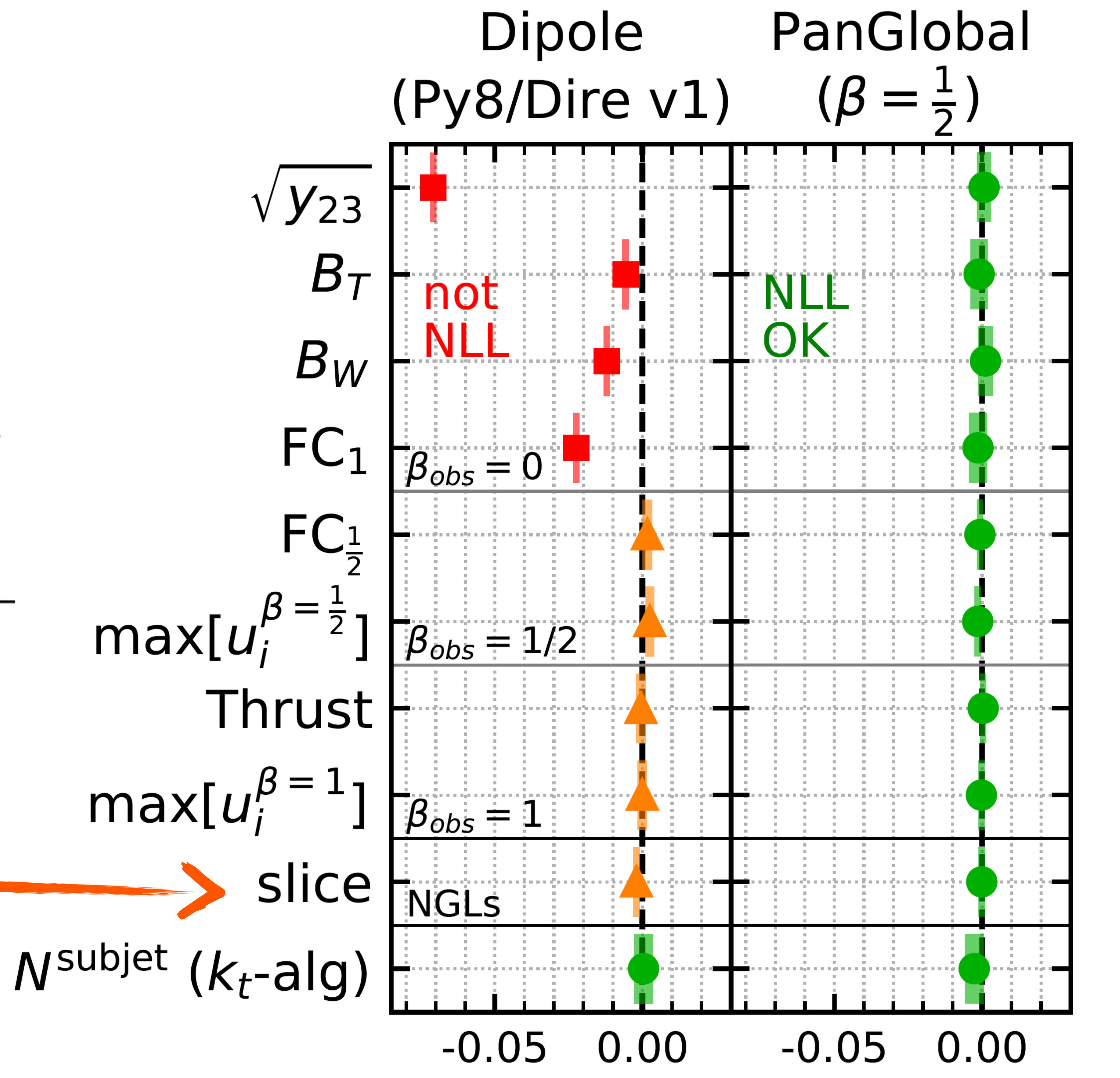
Non-global logarithms are ubiquitous in collider physics

- Jets and fiducial cuts, e.g. jet mass, rapidity cuts, isolation, Higgs ggF vs. VBF, …
- Accuracy of parton showers (PS): NLL non-global logarithms ($\alpha_s^n L^{n-1}$) critical for NNLL PS
- Insight into high-energy dynamics (BK/JIMWLK) via stereographic projection of evolution equation









[Dasgupta, Dreyer, Hamilton, PM, Salam, Soyez '20]

[Weigert '03; Hatta '08; Caron-Huot '15]

$$\frac{d^2\Omega_k}{4\pi}\frac{1-\cos\theta_{ab}}{(1-\cos\theta_{ak})(1-\cos\theta_{bk})} = \frac{d^2\vec{x}_k}{2\pi}\frac{(\vec{x}_{ab})^2}{(\vec{x}_{ak})^2(\vec{x}_{bk})^2}$$

Non-global logarithms are ubiquitous in collider physics

- Resummation of LL ($\alpha_s^n L^n$) corrections known for a long time and studied in depth

[Dasgupta, Salam '01–'02; Banfi, Marchesini, Smye '02; Forshaw et al. '06–'09]

Full Nc (FSR) in: [Weigert '03; Hatta, Ueda '13–'20 (+Hagiwara '15)]

(see M. Neubert's talk for issues at hadron colliders)

- Revived interest more recently and new formulations with modern QFT techniques

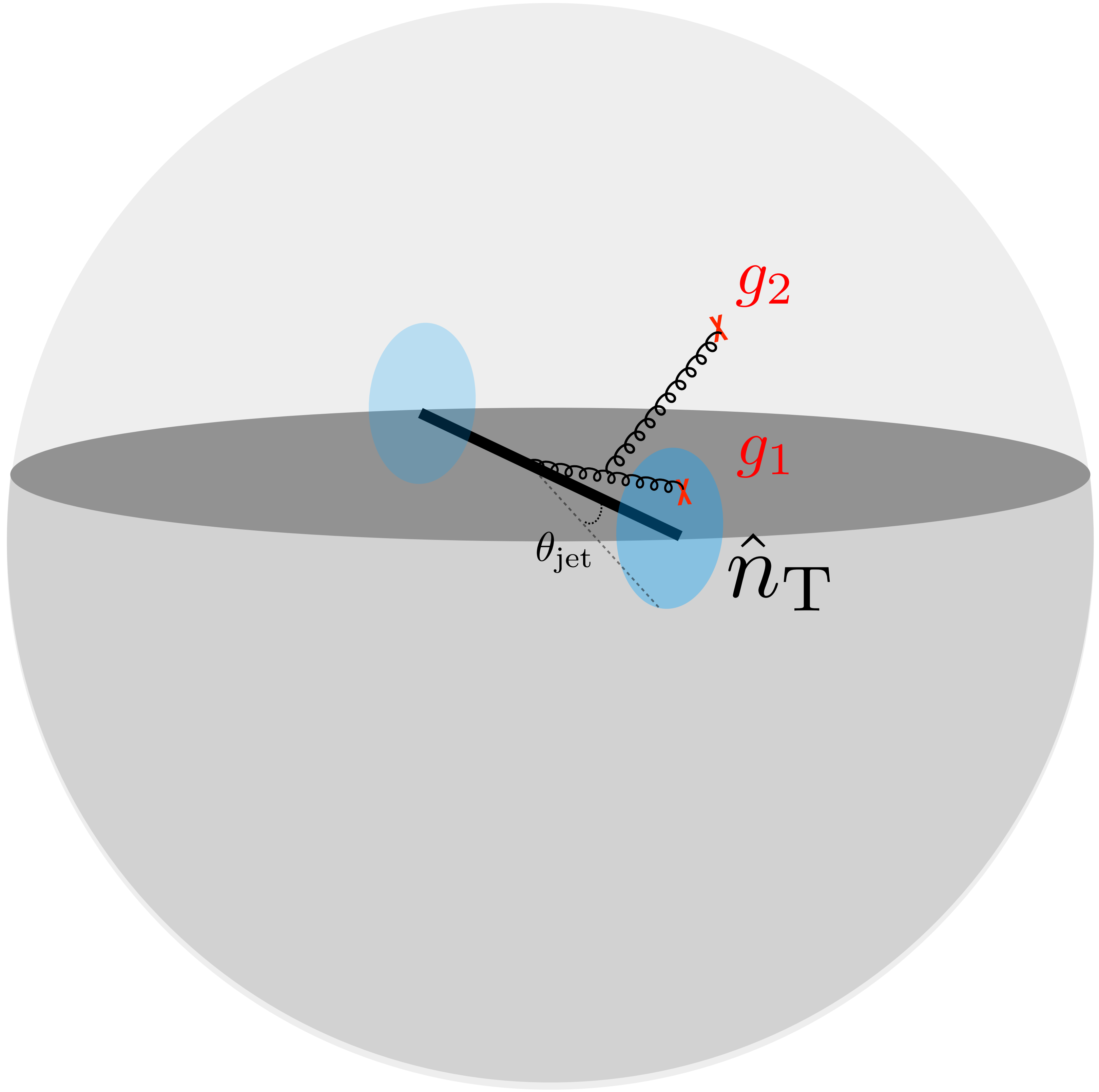
[Becher, Neubert, Rothen, Shao '15–'16 (+ Pecjak '16, Rahn '17, Balsiger '18–'19, Ferroglia '20);
Larkoski, Moult, Neill '15–'16; Caron-Huot '16; Plaetzer, Ruffa '20; Banfi, Dreyer, PM '21; Becher, Rauh, Xu '21]

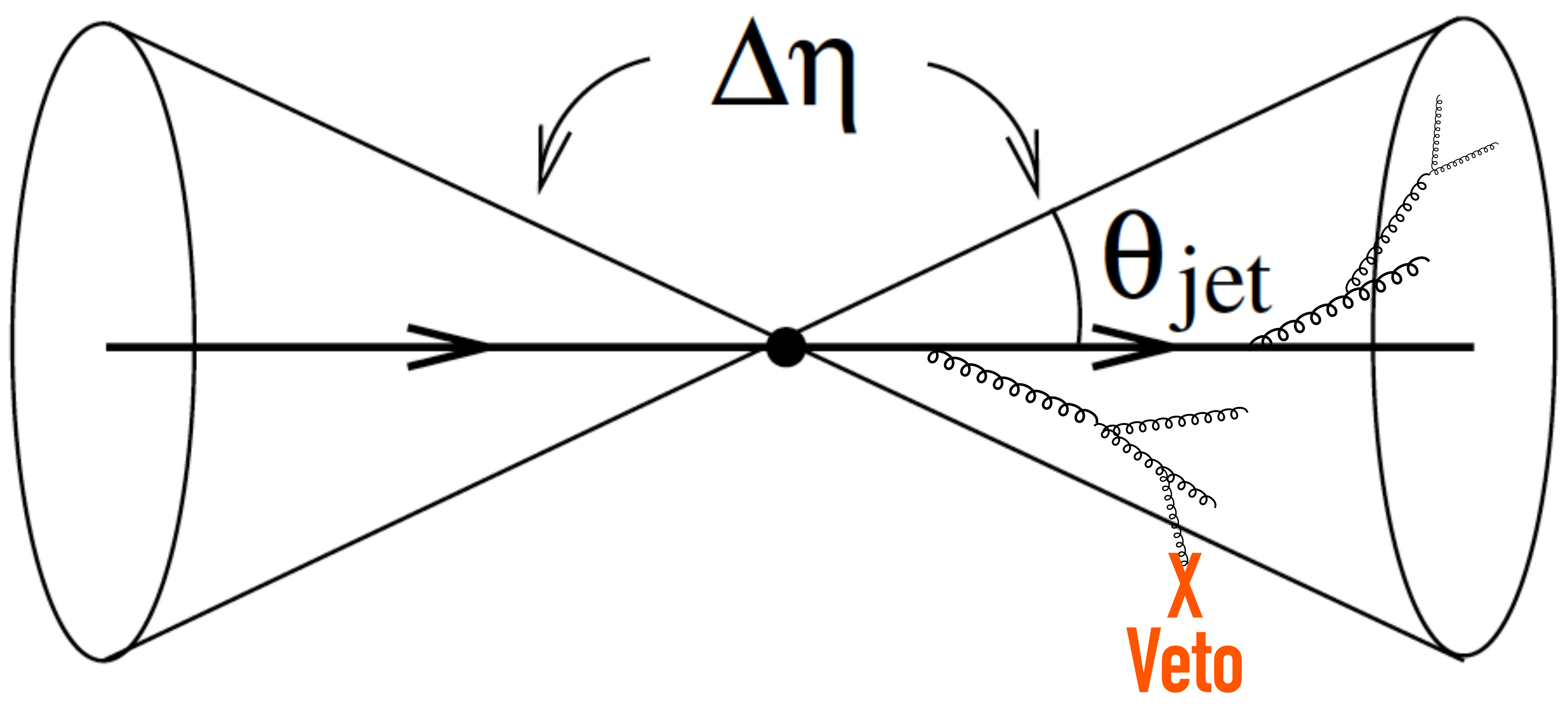
Resummation of NLL corrections remains a great technical challenge due to the complexity of the geometry and colour structure

- GOAL of this work ⇒ formulate a solution to the problem (in the planar limit) in a way that can be applied to a variety of NG observables and processes

A simple example: cone-jet cross section with a veto

- Consider the production of two cone jets at lepton colliders, with a veto on radiation in the interjet region



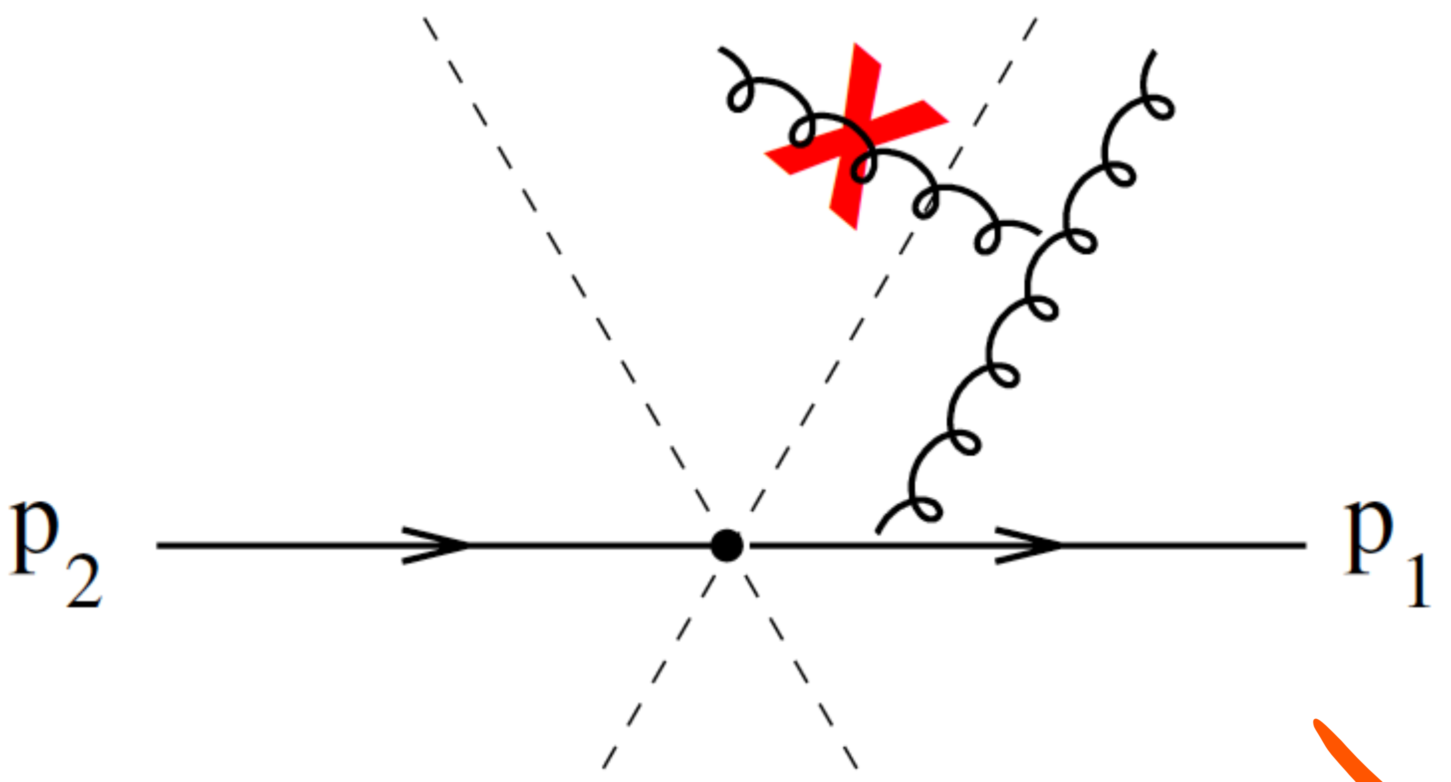


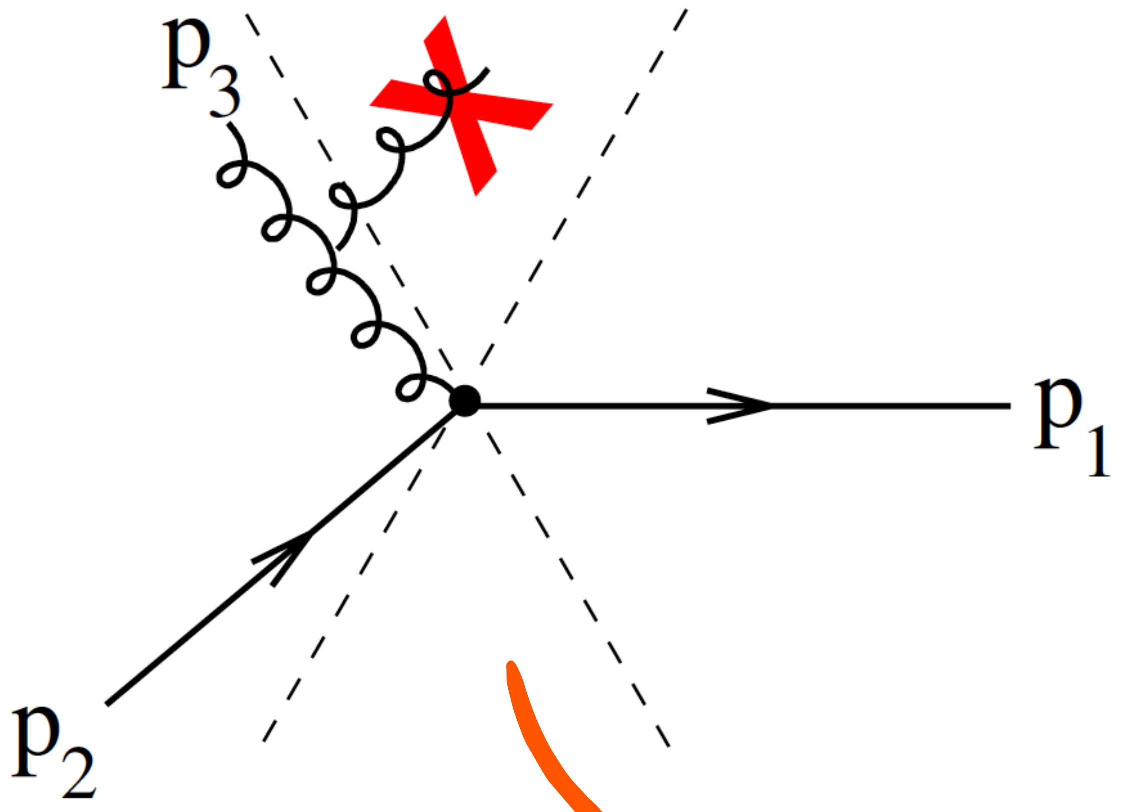
$$\Delta\eta := \ln\frac{1+c}{1-c}, \qquad c = \cos\theta_{\rm jet}$$

Apply a veto e.g. on energy or transverse energy of the radiation in the gap.
Need to calculate distribution of soft gluons on the sphere as a function of the veto scale

Factorisation of the cross section

- Cumulative cross section receives contributions from hard configurations with different multiplicity





$$\Sigma(v) := \sum_{n=2}^{\infty} \mathcal{H}_n \otimes S_n(v) = \mathcal{H}_2 \otimes S_2(v) + \mathcal{H}_3 \otimes S_3(v) + \cdots$$

Integral over hard directions

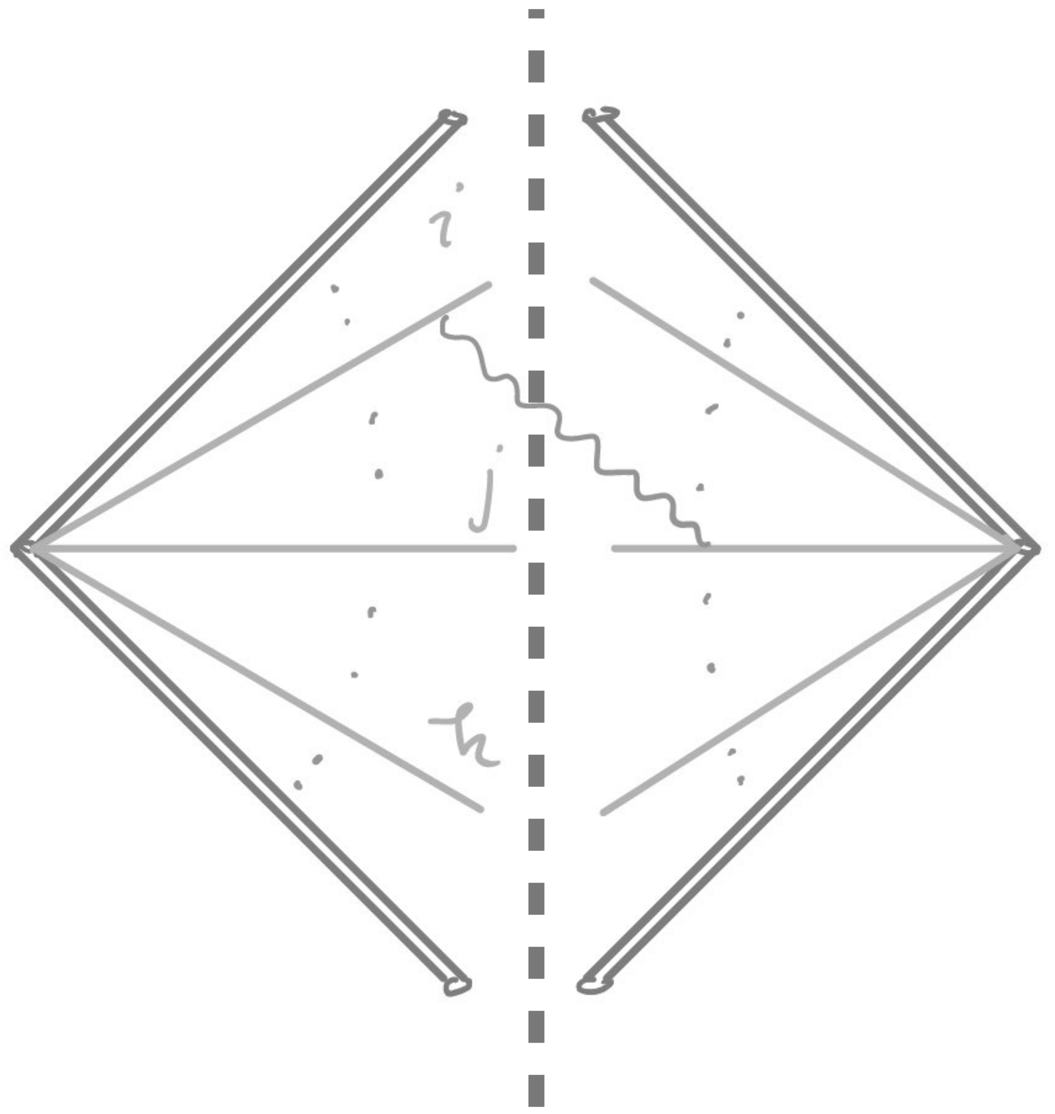
	$\mathcal{H}_2$	$S_2(v)$	$\mathcal{H}_3$	$S_3(v)$
LL	LO	$\mathcal{O}(\alpha_s)$ evolution	-	-
NLL	NLO	$\mathcal{O}(\alpha_s^2)$ evolution	LO	$\mathcal{O}(\alpha_s)$ evolution
	⋮			

The soft factors

Evolution of the soft factors: colour & planar limit

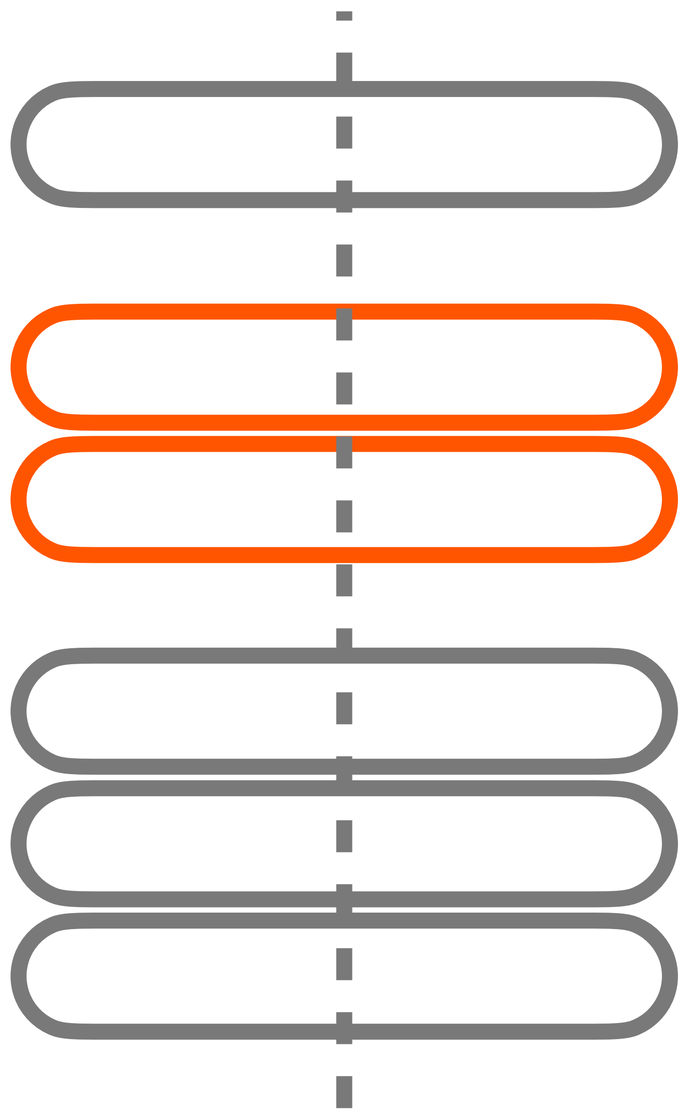
- Complexity growth of colour structure: full squared amplitude can be worked out in large-N_c

e.g. $O(\alpha_s)$ evolution (LL)



$\Rightarrow$ ['t Hooft '73]

$\mathbf{T}_i \cdot \mathbf{T}_j \sim N_c\, \delta_{j,i\pm 1}$



$$\mathcal{A}_{12}^2 = \bar{\alpha}^n(\mu)(2\pi)^{2n}(\mu^{2\epsilon})^n \sum_{\pi_n} \frac{(p_1 \cdot p_2)}{(p_1 \cdot k_{i_1})(k_{i_1} \cdot k_{i_2}) \ldots (k_{i_n} \cdot p_2)}$$

[Bassetto, Ciafaloni, Marchesini '83; Fiorani, Marchesini, Reina '88]

- Define Laplacian soft factors and their evolution equations with energy scale (e.g. dipole k_t)

$$S_n(v) = \int \frac{d\nu}{2\pi i} e^{\nu v} Z_{12\ldots n}[Q;u]\,; \quad Q\partial_Q Z_{12}[Q;u] = \mathbb{K}[Z[Q,u],u]\,; \quad u = e^{-\nu v(k)} = \text{source}$$

$\downarrow$

$Z_{123} = Z_{12}Z_{23}$ in planar limit

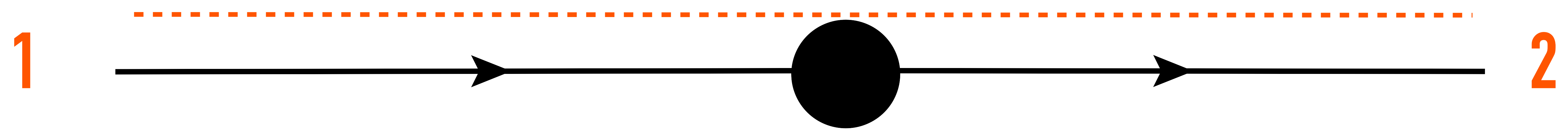
Evolution of the soft factors: integral equations and geometry

- Evolution equations can be expressed conveniently in integral form
  - At LL (one loop kernel) they reduce to the BMS equation (**definition of Z_{12}**)

$$Z_{12}[Q;\{u\}] = \Delta_{12}(Q) + \int [dk_a]\bar{\alpha}(k_{ta})w_{12}^{(0)}(k_a)\,\frac{\Delta_{12}(Q)}{\Delta_{12}(k_{ta})} \times Z_{1a}[k_{ta};\{u\}]Z_{a2}[k_{ta};\{u\}]u(k_a)\Theta(Q-k_{ta})$$

Sudakov: no-emission probability
(defined by unitarity $Z_{12}[Q;\{u=1\}] = 1$,
standard evolution of soft virtual squared amplitudes)

[Dasgupta, Salam '01; Banfi, Marchesini, Smye '01]



Symmetries of squared amplitude allow for an iterative reconstruction (strongly) ordered in dipole k_t

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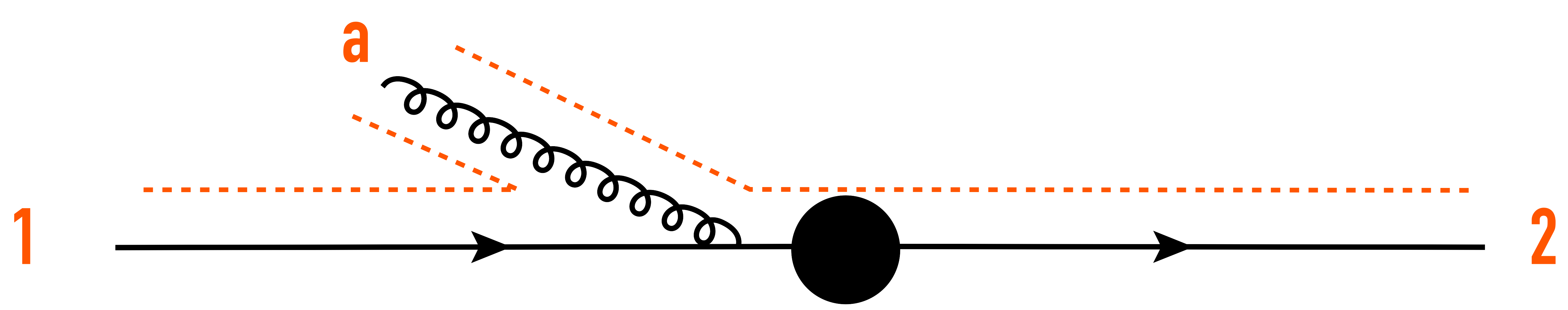

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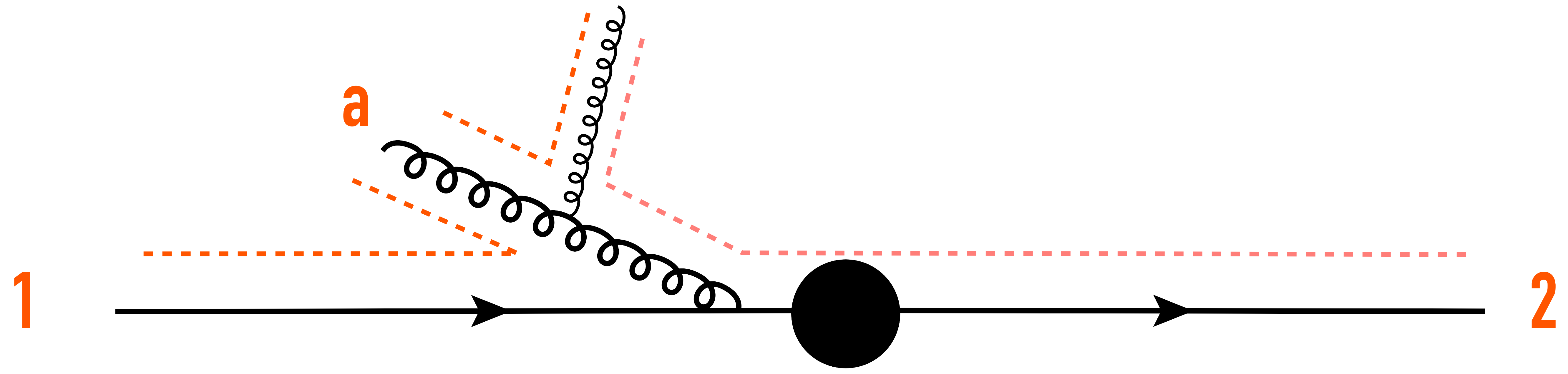

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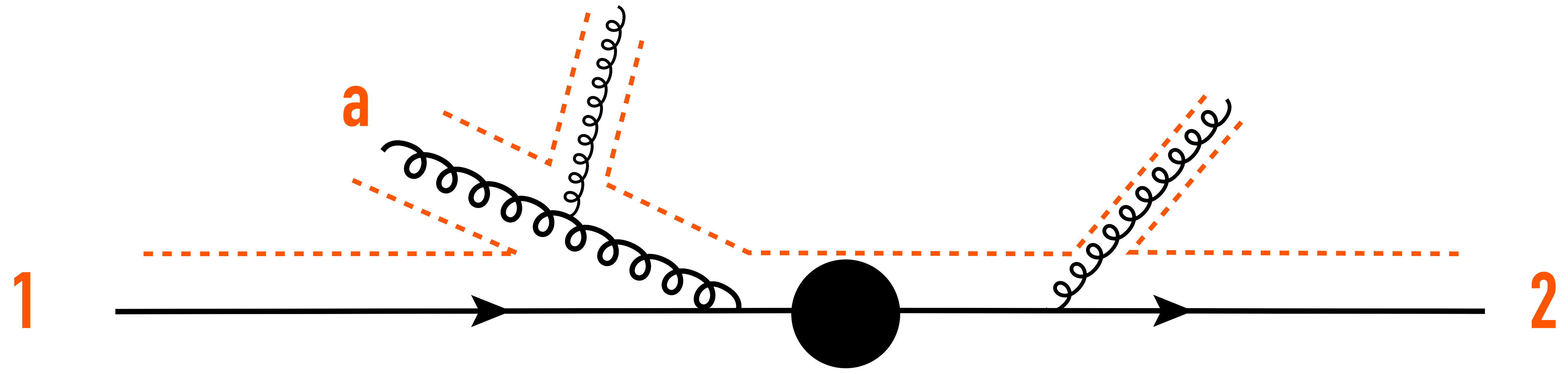


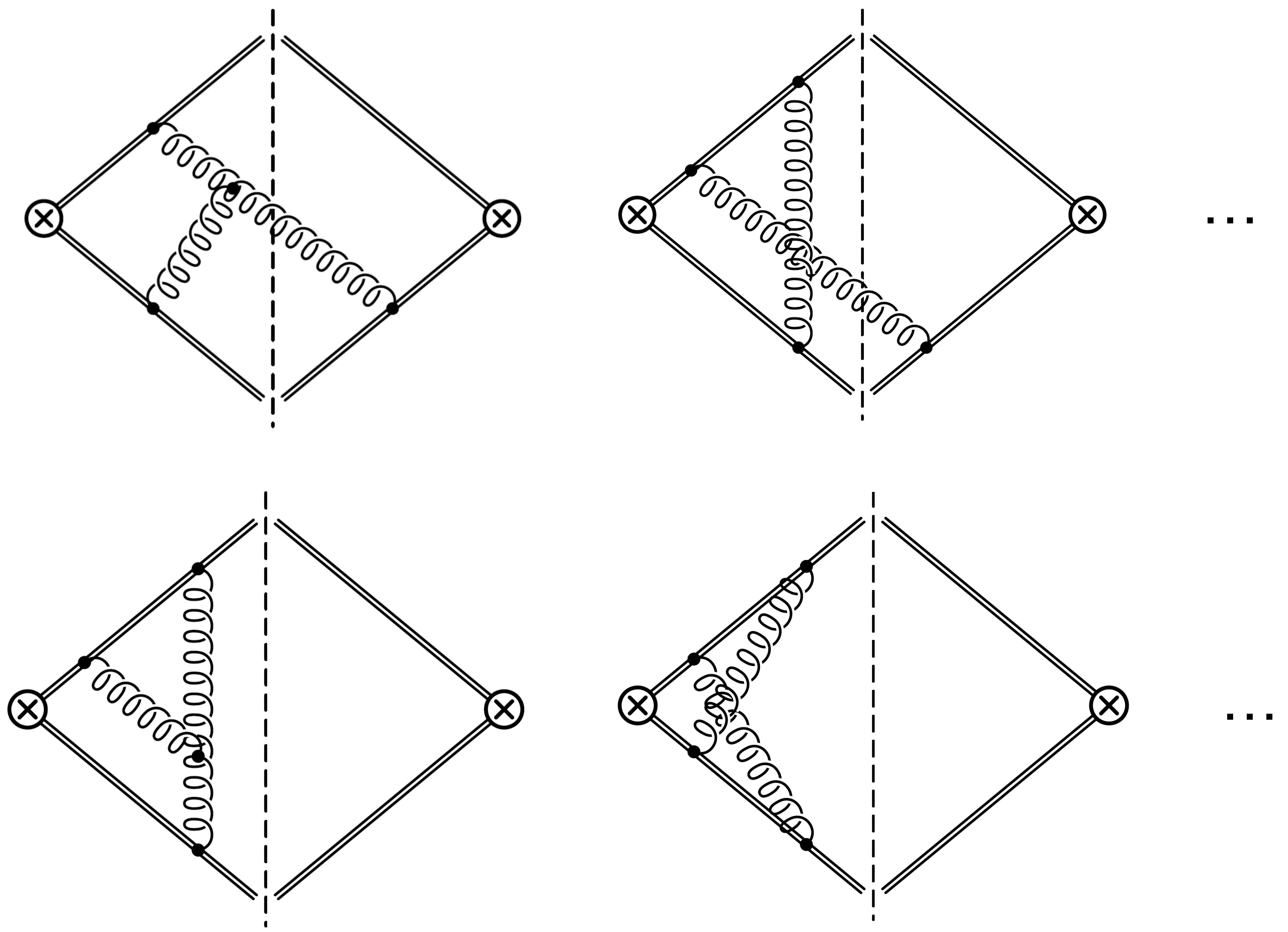
Symmetries of squared amplitude allow for an iterative reconstruction (strongly) ordered in dipole k_t

Integral evolution equations: NLL (two loop kernel)

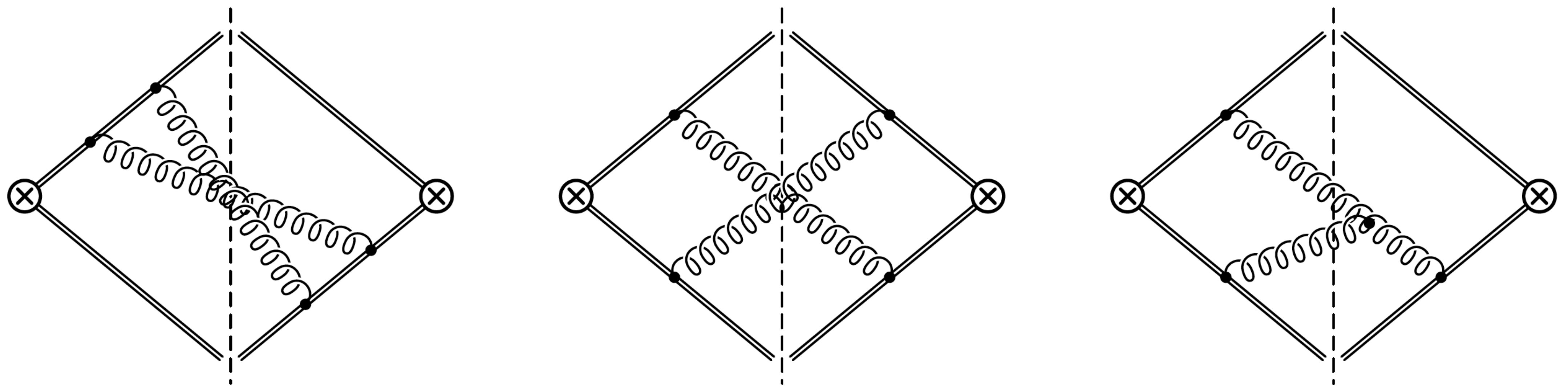
[Banfi, Dreyer, PM '21]

$$Z_{12}[Q;\{u\}] = \mathbb{K}_{\rm int}^{\rm RV+VV}[Z[Q;u],u] + \mathbb{K}_{\rm int}^{\rm RR}[Z[Q;u],u] - \mathbb{K}_{\rm int}^{\rm DC}[Z[Q;u],u]$$

subtraction of iteration of LL kernel
(no double counting)



two unordered real gluons
(planar limit of double soft squared gauge current)



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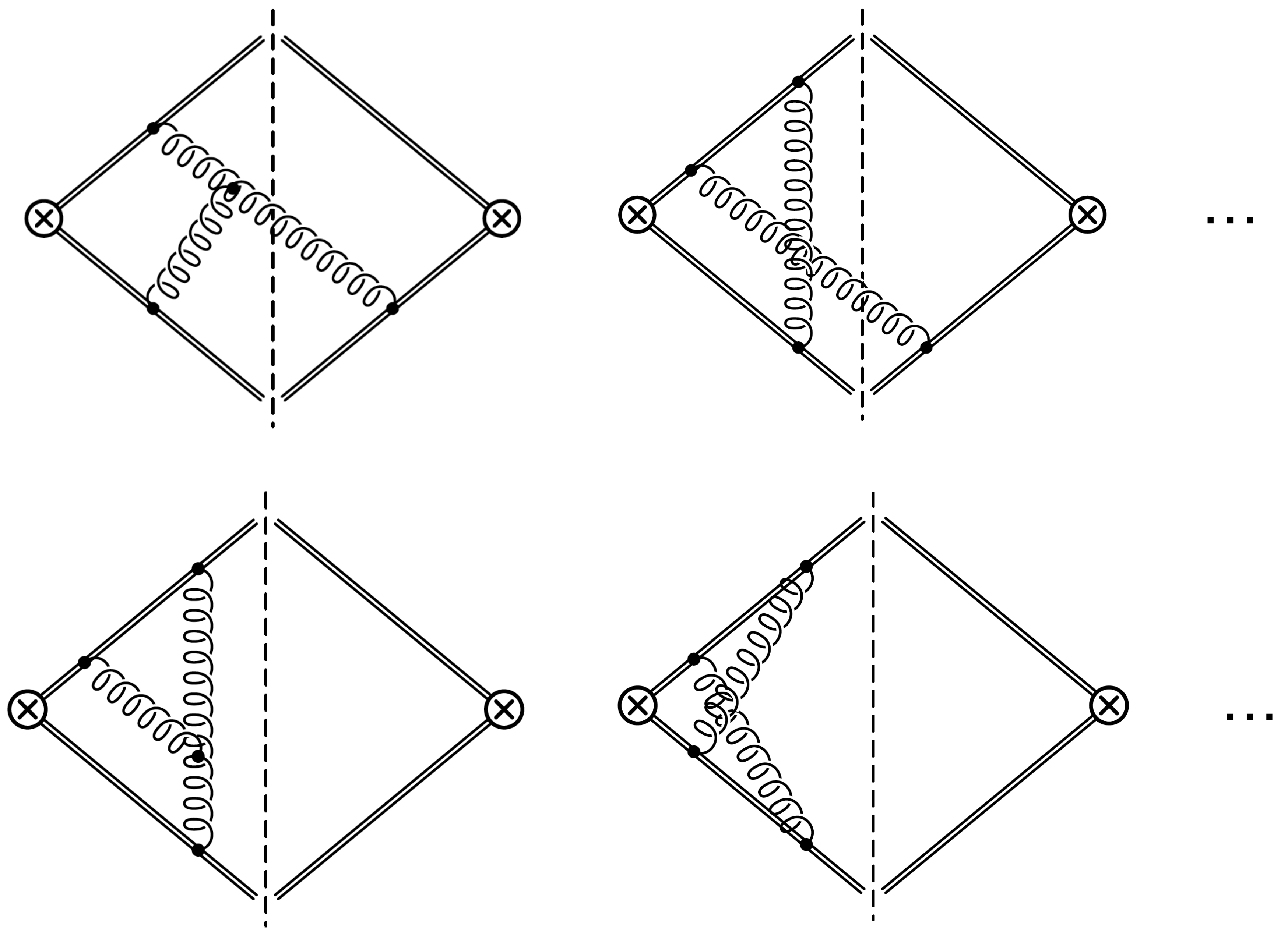
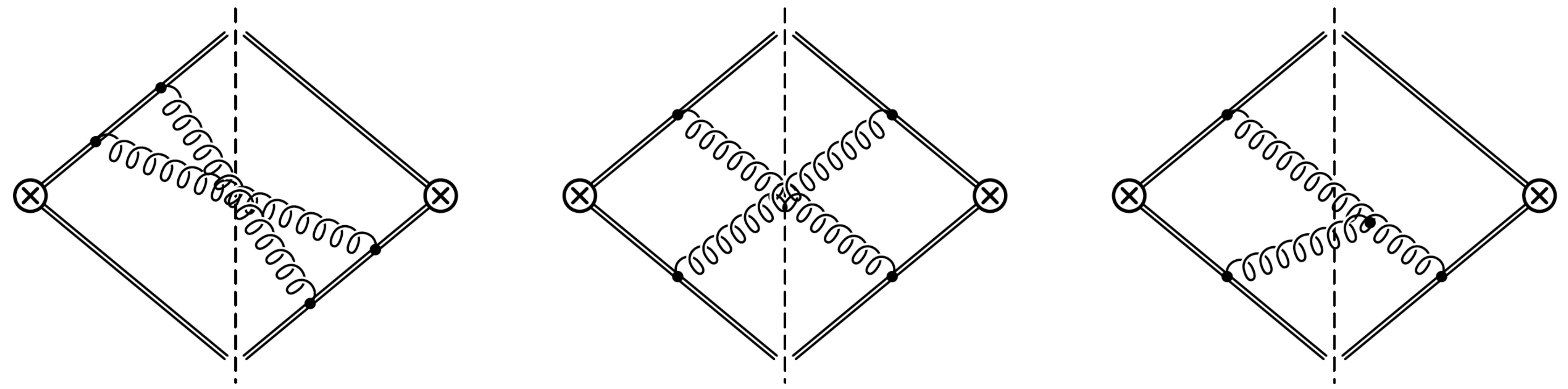




- local counter-term

+ integrated counter-terms

Introduce IRC counter-term to make each term manifestly finite in 4 dimensions

Integral evolution equations: new structures at NLL

[Banfi, Dreyer, PM ’21]

$$Z_{12}[Q;\{u\}] = \mathbb{K}_{\rm int}^{\rm RV+VV}[Z[Q;u],u] + \mathbb{K}_{\rm int}^{\rm RR}[Z[Q;u],u] - \mathbb{K}_{\rm int}^{\rm DC}[Z[Q;u],u]$$

$\supset Z_{12} \to Z_{1i} Z_{i2}$

Same structure as in LL kernel

(1→2 dipole branching)

Easy to iterate with Monte Carlo methods

$\supset Z_{12} \to Z_{1i} Z_{ij} Z_{j2}$

New structure of double real radiation

(1→2 dipole branching)

Hard to iterate with Monte Carlo methods

[Full kernels in backup slides]

The hard factors

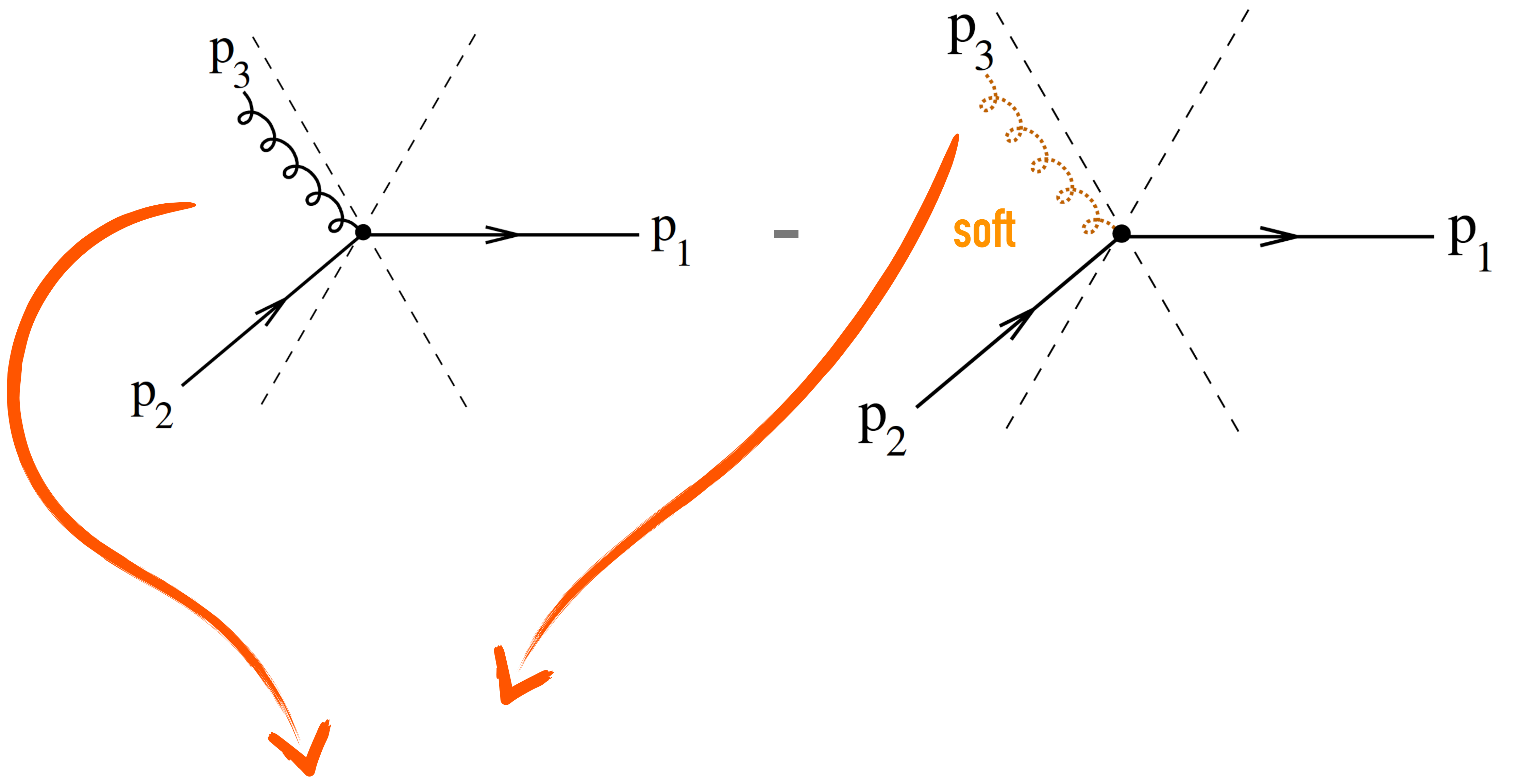
Hard factor with 3 legs at NLL: H_3

Recall:
$$\mathcal{H}_n \otimes S_n(v) = \int \left(\prod_{i=1}^{n} d^2\Omega_i\right) \mathcal{H}_{1\ldots n} \times S_{1\ldots n}(v)$$

[Banfi, Dreyer, PM '21]

- Computed by matching the soft theory to full QCD
- Cancellation of collinear divergences between H_2 and H_3 (only combination is scheme independent)

e.g. H_3



$$\Rightarrow \quad \mathcal{H}_3 \otimes S_3(v) = \Sigma^{(3),\mathrm{sub}}(v) - \Sigma^{(3),\mathrm{sub}}_{\mathrm{soft}}(v)$$

All particles required to be outside the interjet gap

Remaining collinear singularity subtracted with standard methods, here a generalisation at all-orders of Projection-to-Born (integrated counter-term to be added back to H_2)

Hard factor with 2 legs at NLL: H_2

Recall:
$$\mathcal{H}_n \otimes S_n(v) = \int \left(\prod_{i=1}^{n} d^2\Omega_i\right) \mathcal{H}_{1\ldots n} \times S_{1\ldots n}(v)$$

[Banfi, Dreyer, PM '21]

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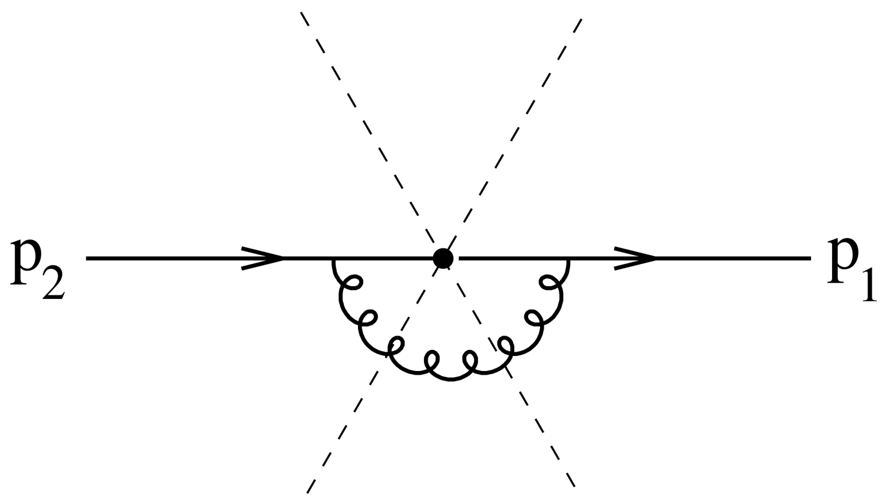
e.g. H_2:



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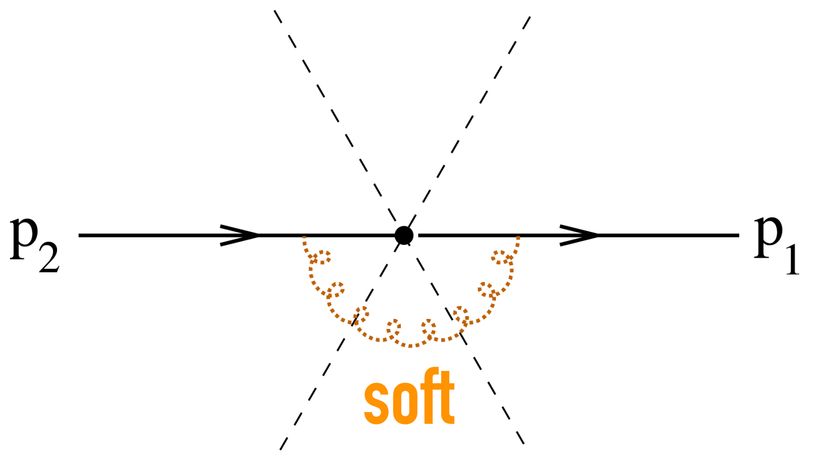


+ integrated counterterms from H_3

$c \equiv \cos(\theta_{\text{jet}})$

$$\mathcal{H}_2^{(1)} = \frac{C_F}{2\left(1-c^2\right)^2}\Bigg(4\left(1-c^2\right)^2\left(\mathrm{Li}_2\left(\frac{1+c}{2}\right)-\mathrm{Li}_2\left(\frac{1-c}{2}\right)\right)$$
$$-2\left(1-c^2\right)^2\log^2(1+c)+16c\left(3+c^2\right)\ln(2)-(1-c^2)(c(16+3c)-3)$$
$$+2\ln(1-c)\left(-2\left(1+c^4\right)\log(2)-4c\left(3+c^2\right)+\left(1-c^2\right)^2\ln(1-c)\right)$$
$$+\left(4\left(1+c^4\right)\ln(2)-8c\left(3+c^2\right)\right)\ln(1+c)-4\left(-3c^4+2c^2(9+2\ln(2))+1\right)\tanh^{-1}(c)\Bigg).$$

Perturbative solution & results

$$Z_{12}[Q;\{u\}] = \mathbb{K}^{\mathrm{RV+VV}}_{\mathrm{int}}[Z[Q;u],u] + \mathbb{K}^{\mathrm{RR}}_{\mathrm{int}}[Z[Q;u],u] - \mathbb{K}^{\mathrm{DC}}_{\mathrm{int}}[Z[Q;u],u]$$

$\supset Z_{12} \to Z_{1i} Z_{i2}$ $\supset Z_{12} \to Z_{1i} Z_{ij} Z_{j2}$

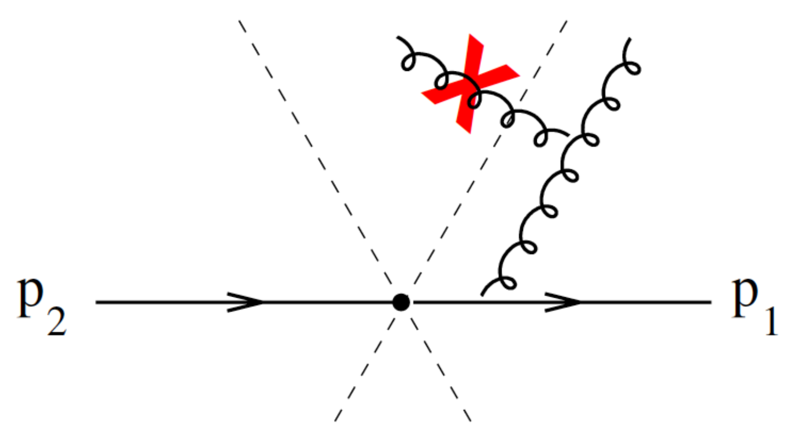
Generating functionals

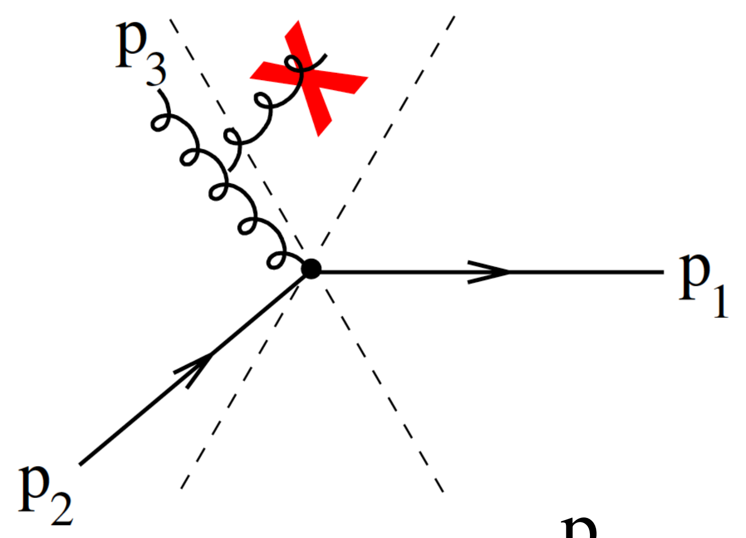
- Recast evolution equations in terms of generating functionals (→ calculation of probabilities → Monte Carlo)

see e. g. [Konishi, Ukawa, Veneziano '79; Dokshitzer, Khoze, Mueller, Troyan '91]

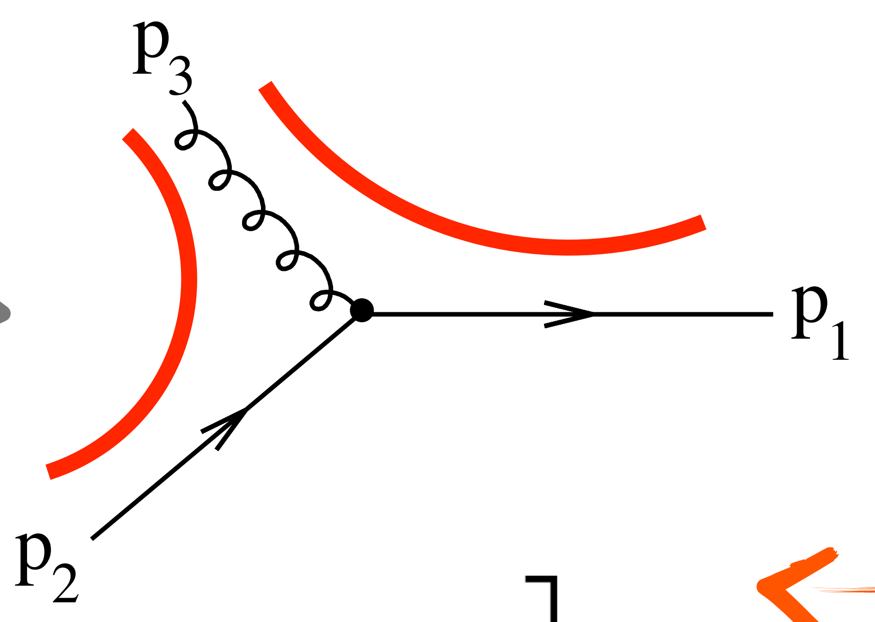
$$dP_n = \aleph(n) \prod_{i=1}^{n} [dk_i] \frac{\delta}{\delta u(k_i)} Z_{12}[Q; \{u\}] \Bigg|_{\{u\}=0} , \quad \frac{\delta}{\delta u(k_i)} u(k) \equiv \bar{\delta}(k - k_i)$$

- Cross section becomes:





$N_c \gg 1$



$$\Sigma(v) = \mathcal{H}_2 \otimes \left[\sum_{i=0}^{\infty} \int dP_i^{\{12\}} \Theta(v - V(\{k_i\})) \right]$$
$$+ \mathcal{H}_3 \otimes \left[\left(\sum_{i=0}^{\infty} \int dP_i^{\{13\}} \right) \left(\sum_{j=0}^{\infty} \int dP_j^{\{23\}} \right) \Theta(v - V(\{k_i\}, \{k_j\})) \right] + \mathcal{O}(\text{NNLL})$$

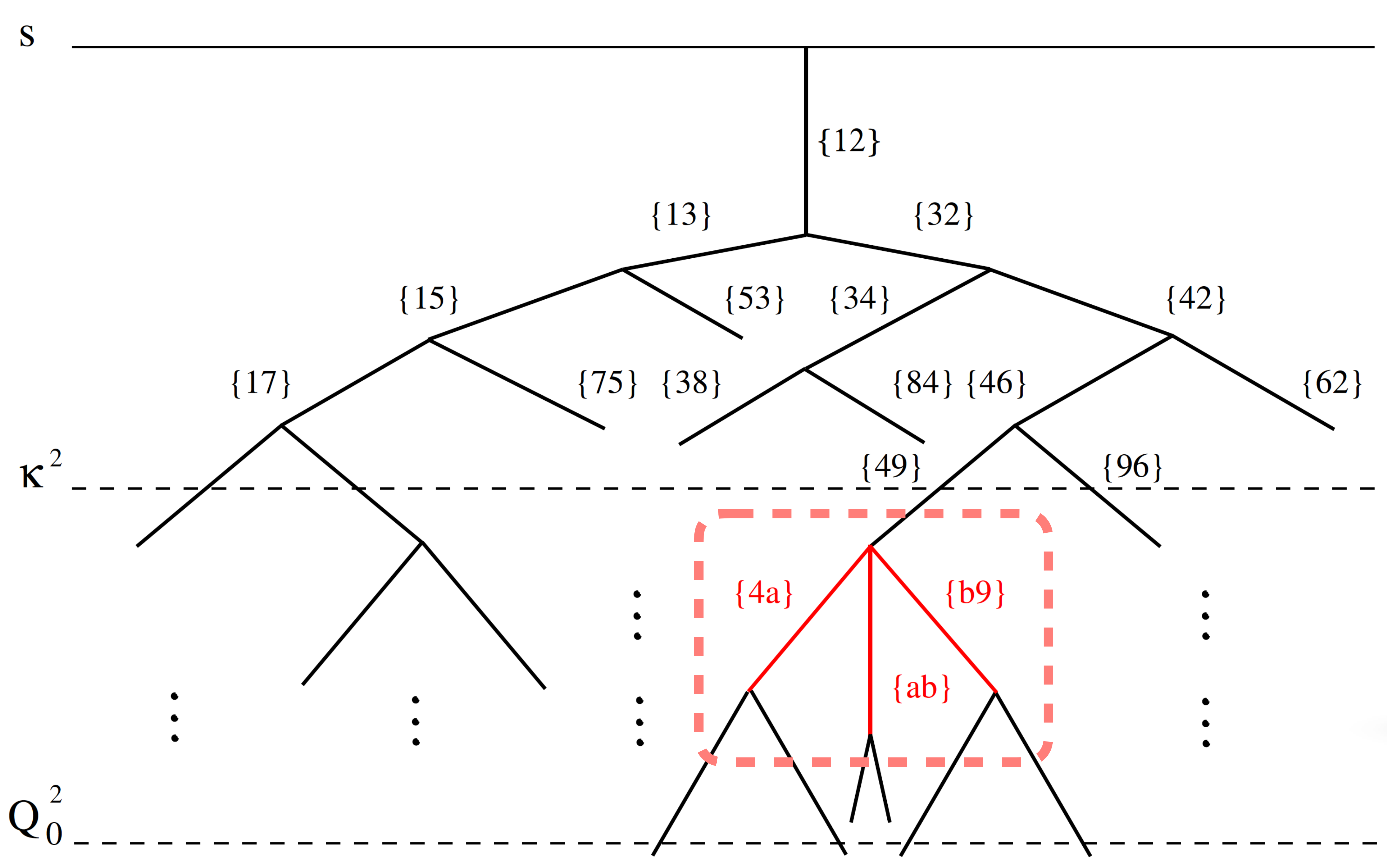
Perturbative solution of the NLL evolution equation

[Banfi, Dreyer, PM '21]

- All-order solution can be formulated in a perturbative form, i.e.

$$Z_{12}[Q;\{u\}] = Z_{12}^{(0)}[Q;\{u\}] + Z_{12}^{(1)}[Q;\{u\}] \quad \text{with} \quad Z_{12}^{(0)}[Q;\{u\}] = \mathbb{K}_{\text{int}}^{\text{RV+VV}}[Z^{(0)}[Q;u],u]$$

- Linearise evolution equation in $Z^{(1)}$ by neglecting $(Z^{(1)})^2$ corrections (NNLL and higher)



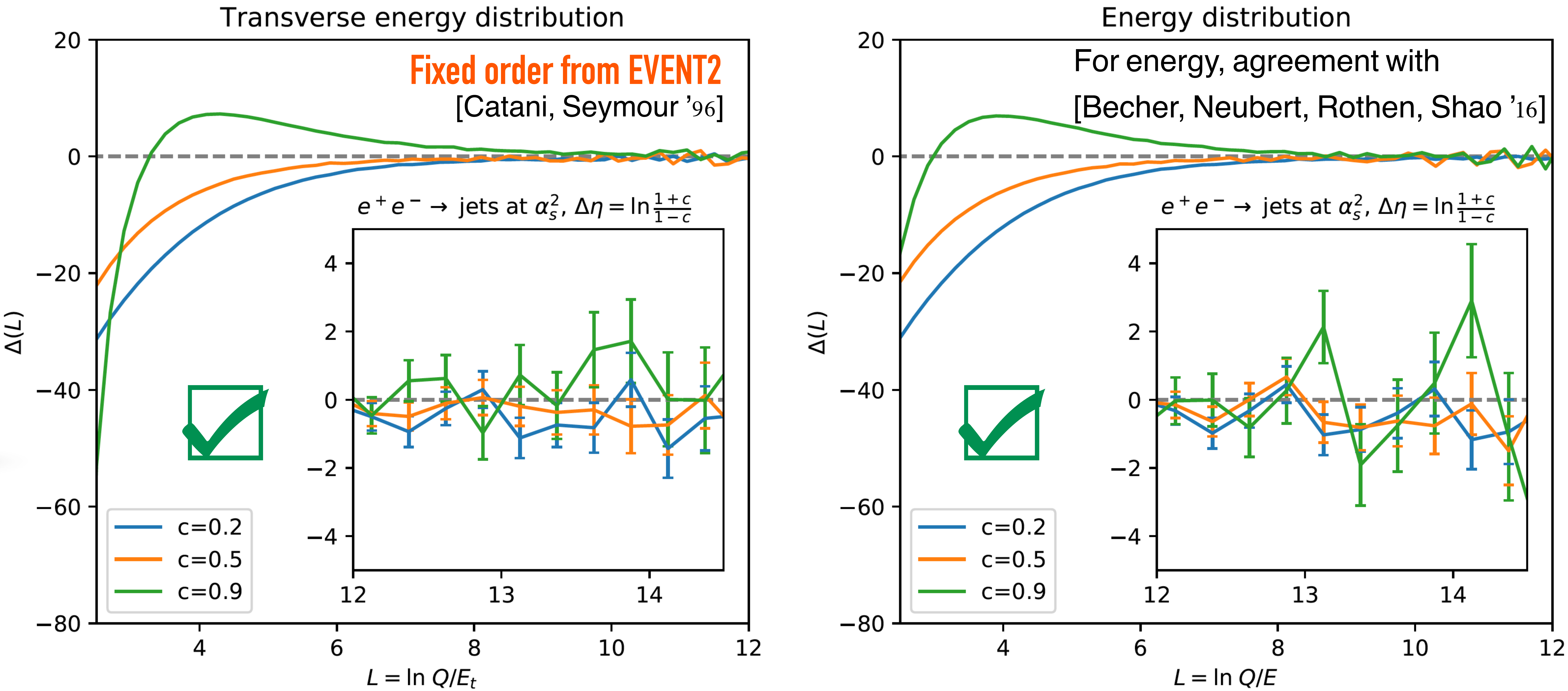
All-order iteration of $Z^{(0)}$ and a single insertion of $Z^{(1)}$ at <u>any scale</u> in the evolution graph (truncated shower).
Structure emerges from the ev. eqn.

Fixed order checks

- $O(\alpha_S^2)$ expansion expected to reproduce the logarithmic structure of QCD

$$\Delta(L) := \frac{1}{\sigma_0}\left(\frac{d\Sigma^{\rm NLO}}{dL} - \frac{d\Sigma^{\rm EXP.}}{dL}\right) \qquad \text{expect} \qquad \lim_{L\to\infty} \Delta(L) = 0$$

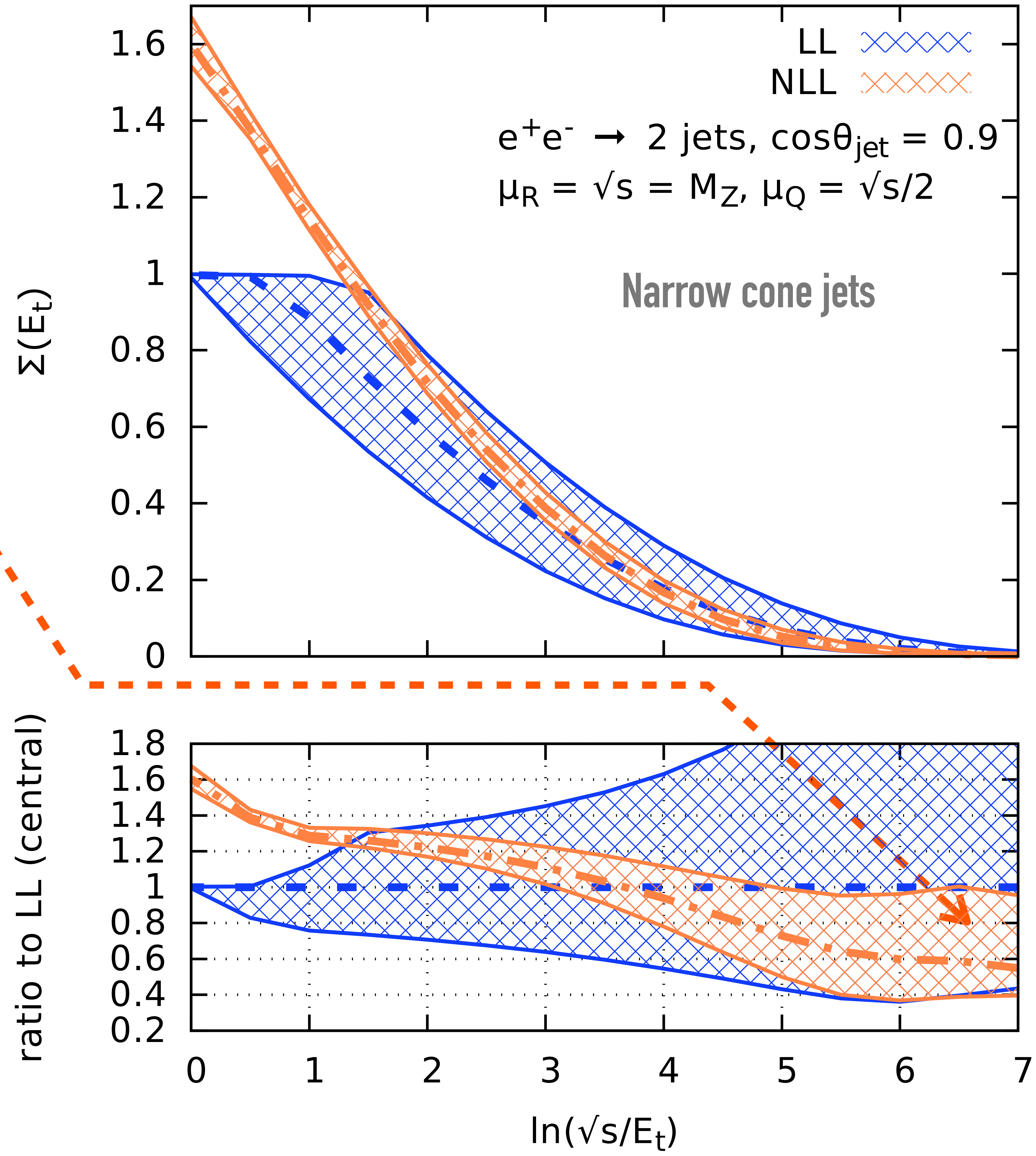
OK for different jet-cone sizes



All order results at NLL: narrow cone jets

NLL corrections sizeable (up to ~40%), significant (~50%) reduction of perturbative uncertainty

$$\Sigma(v) := \frac{1}{\sigma_0}\int_0^v \frac{d\sigma}{dv'}dv'$$



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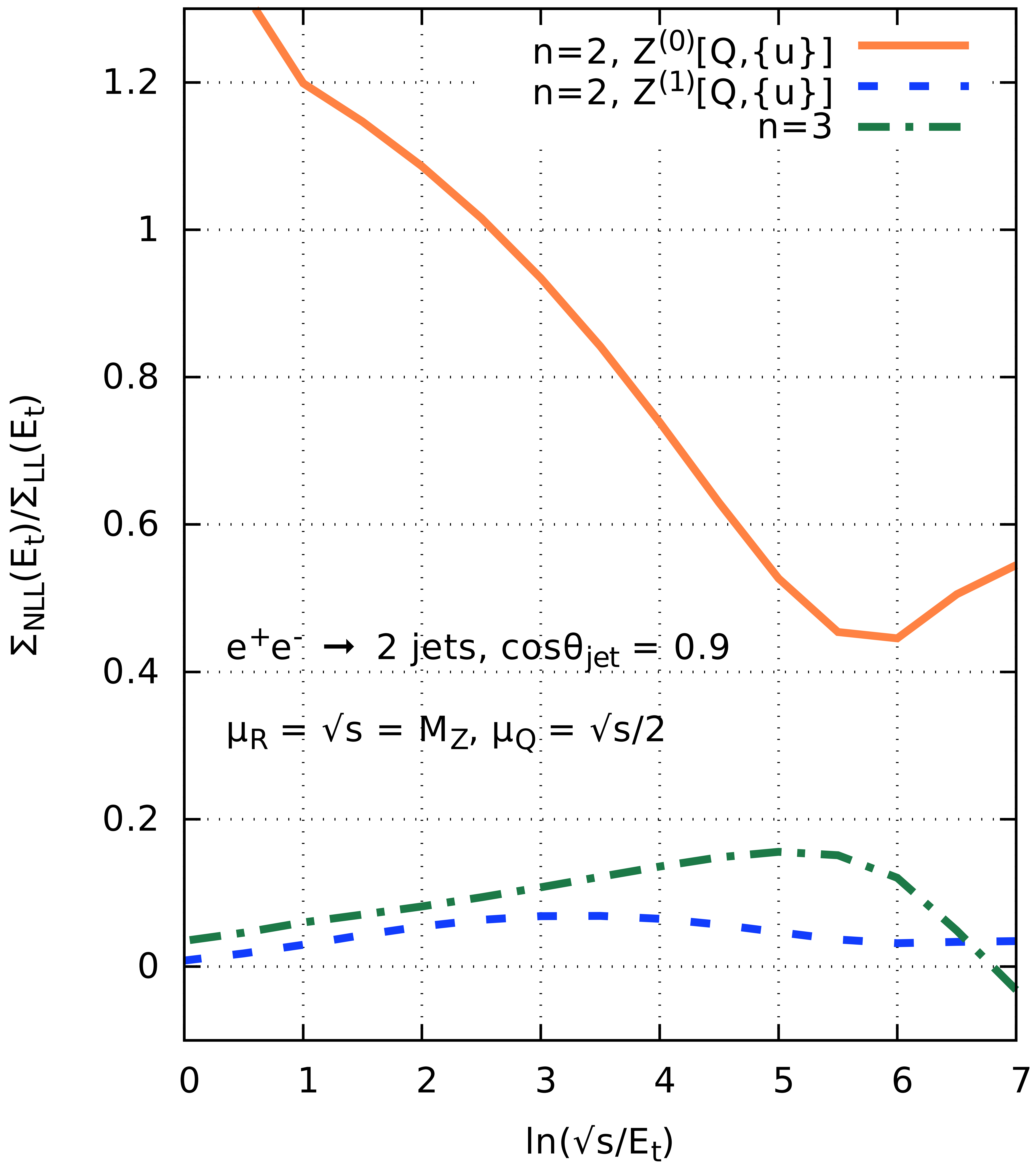

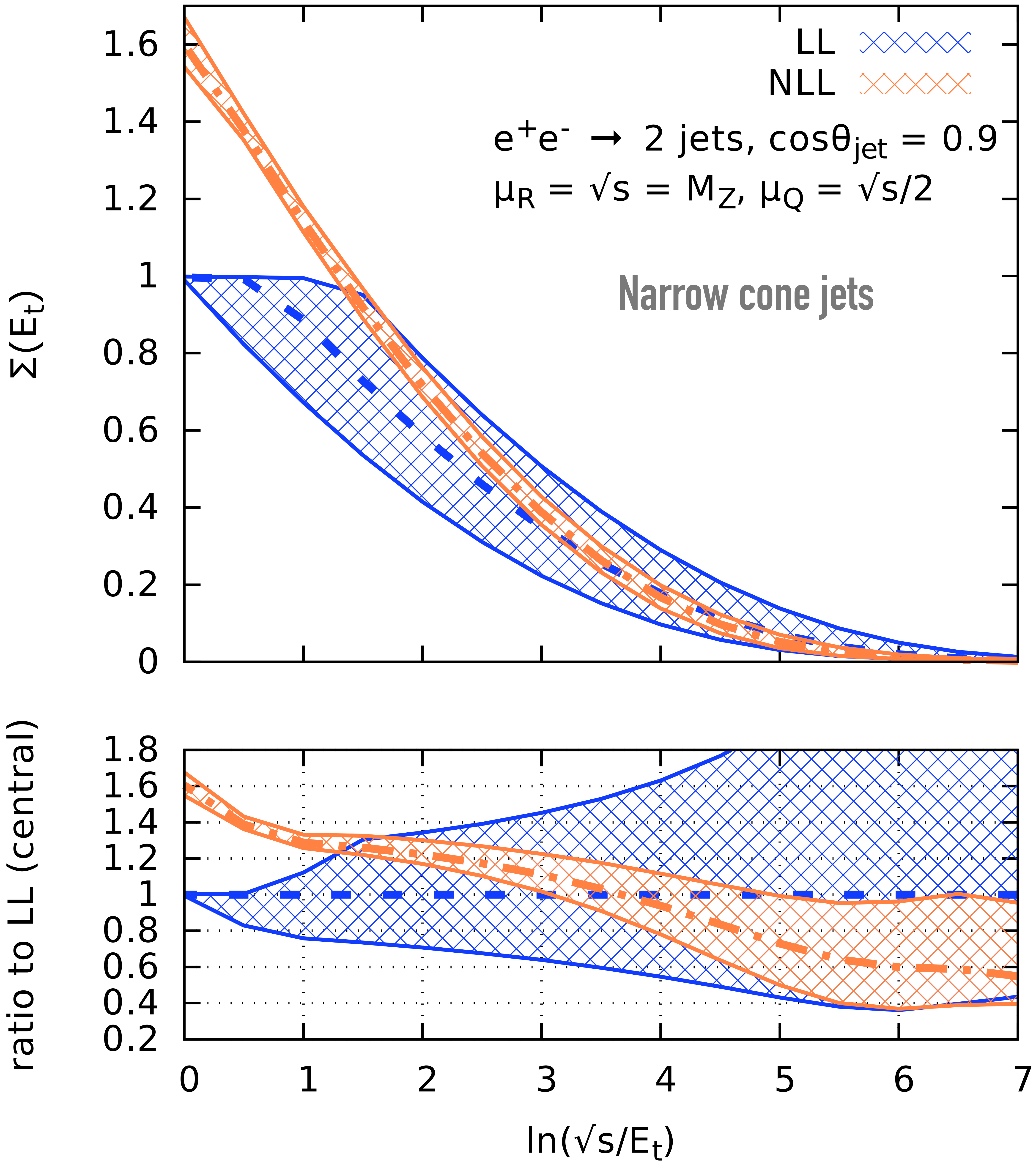

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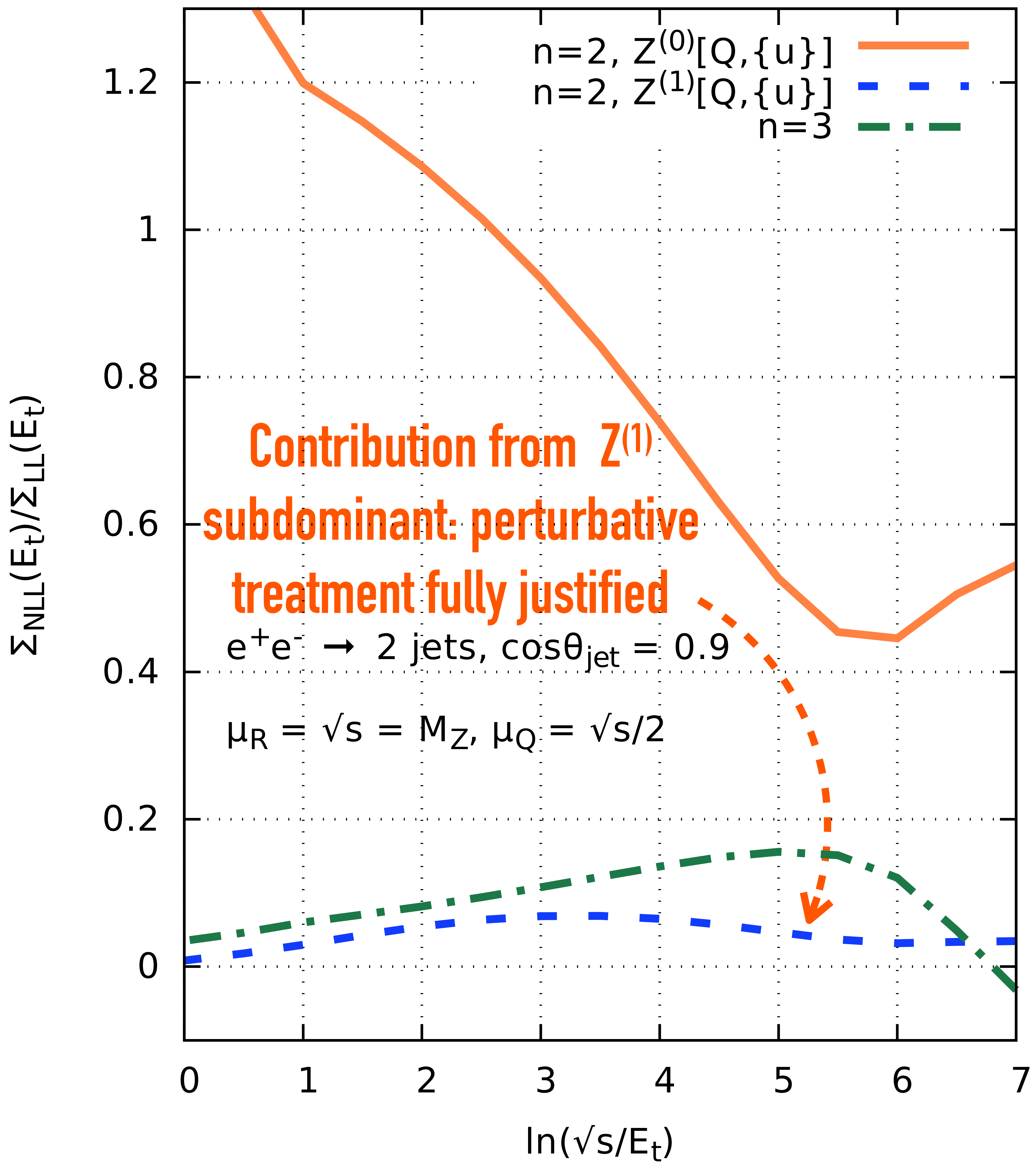
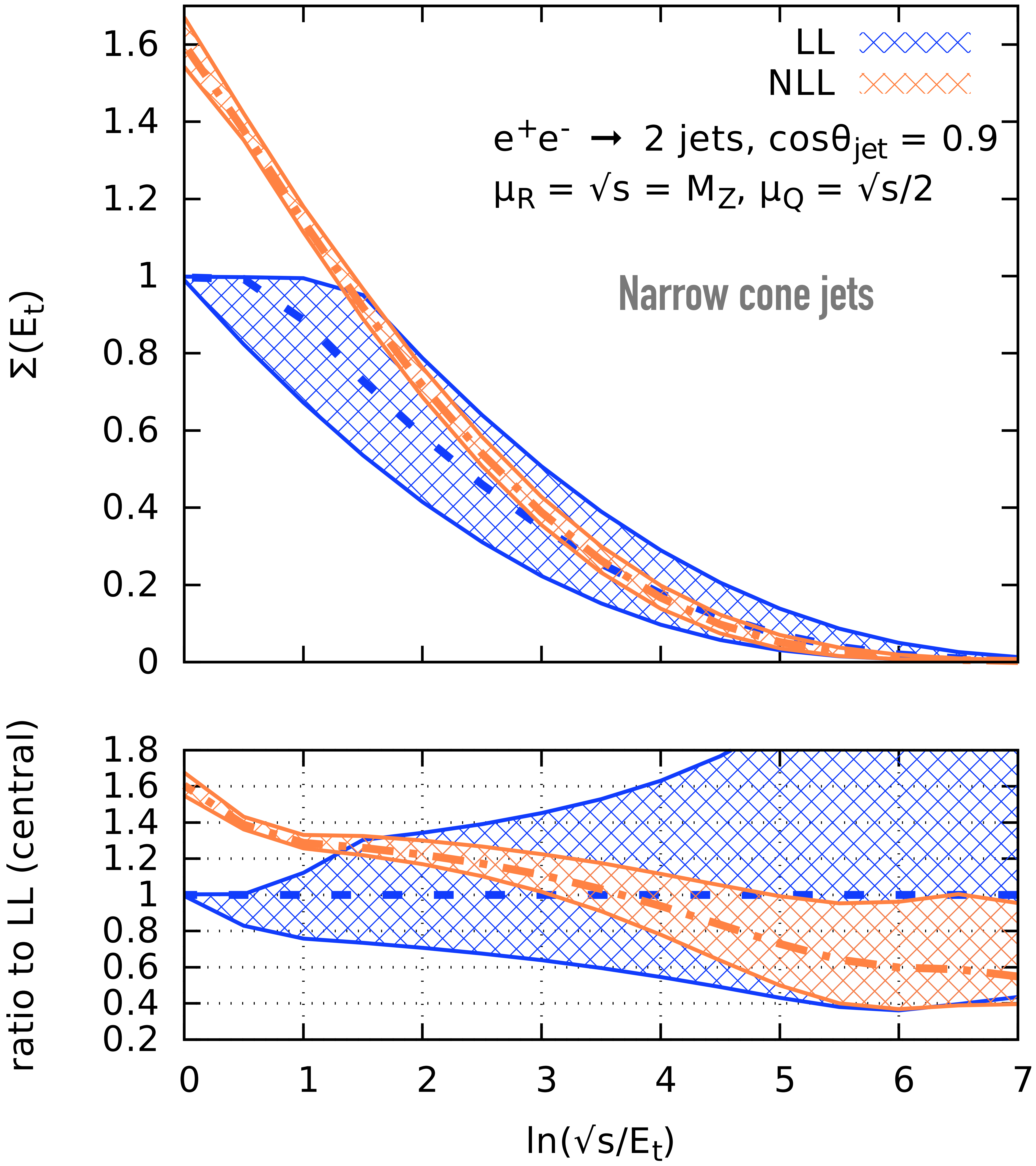
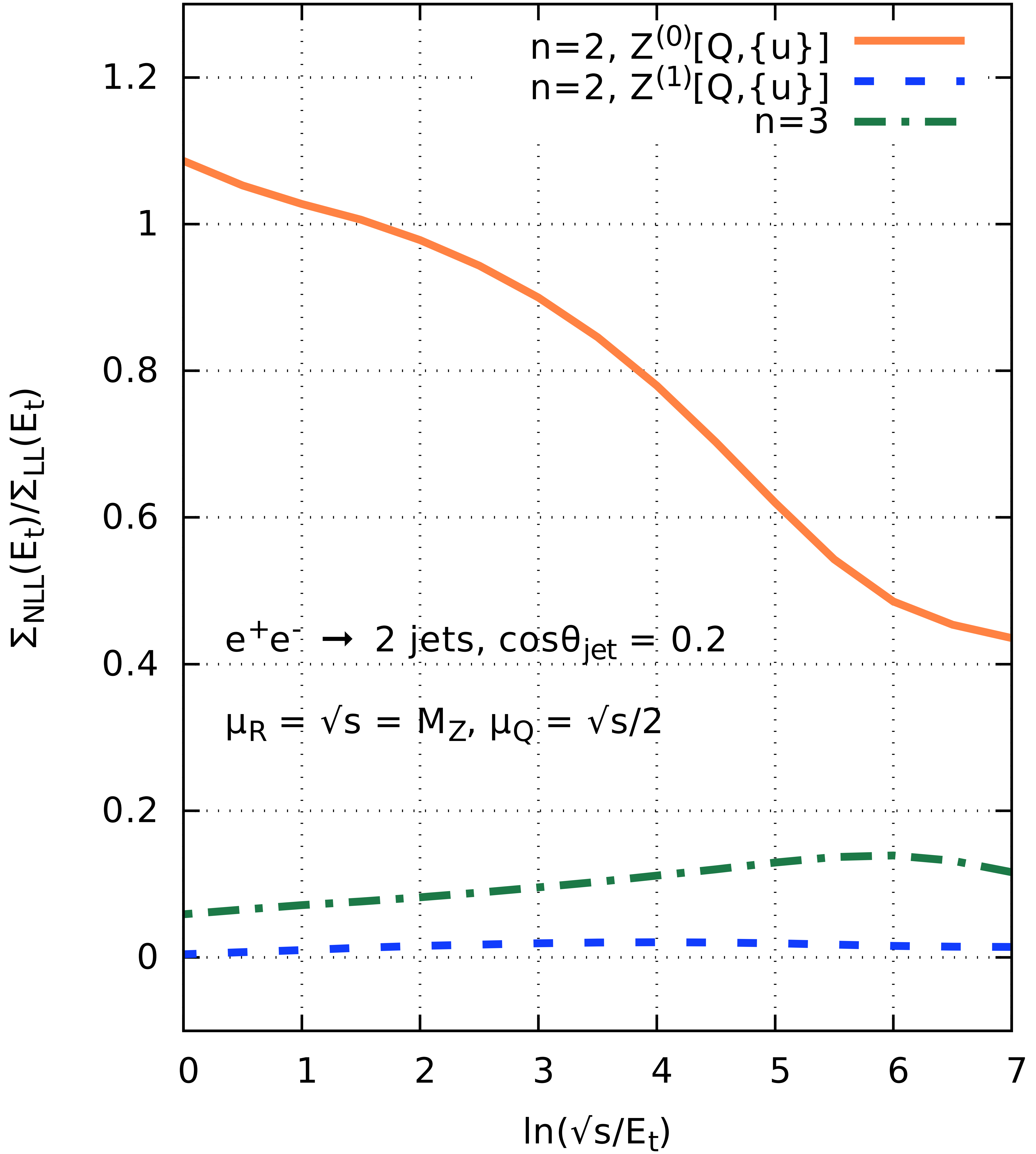
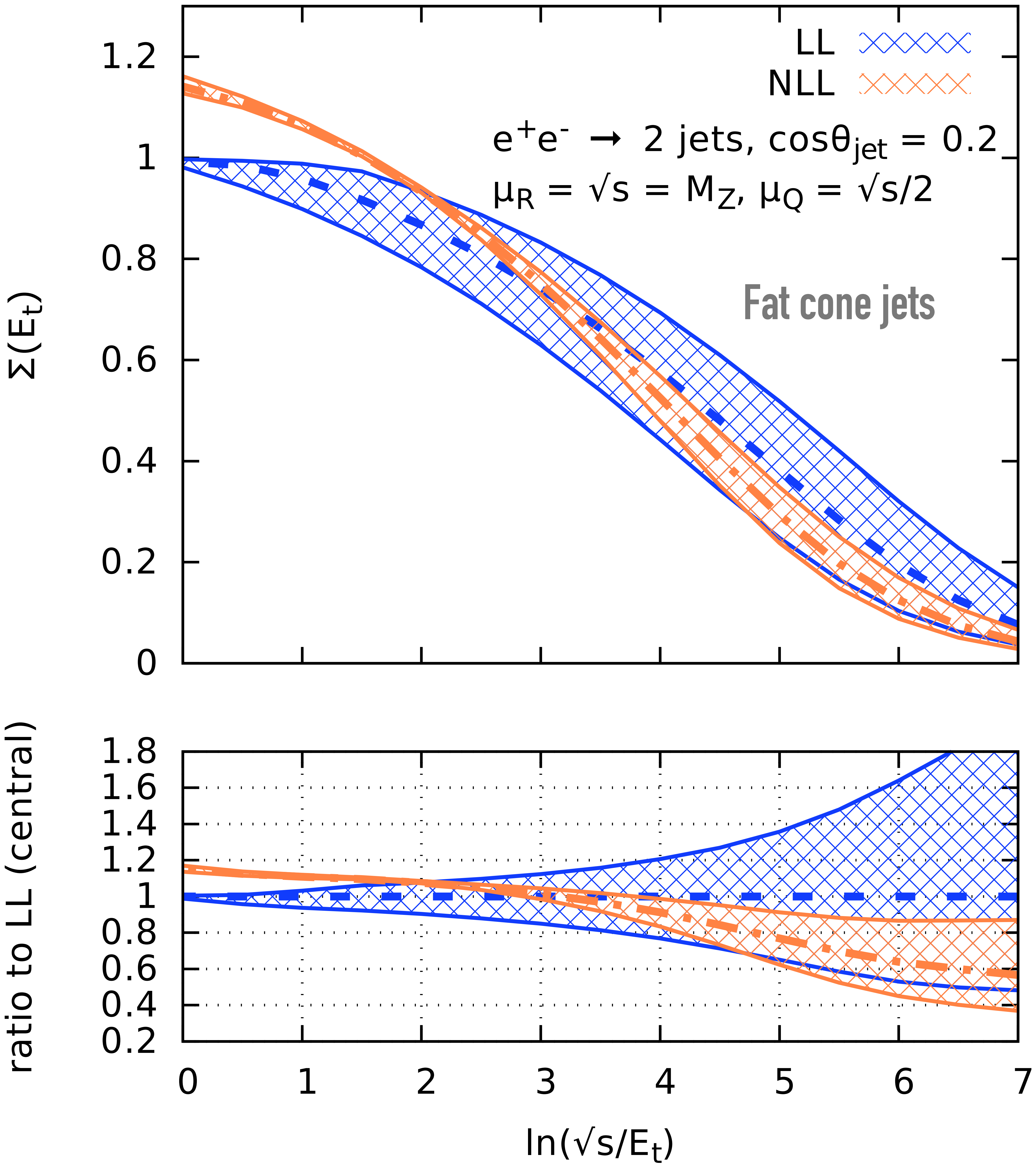




All order results at NLL: fat cone jets





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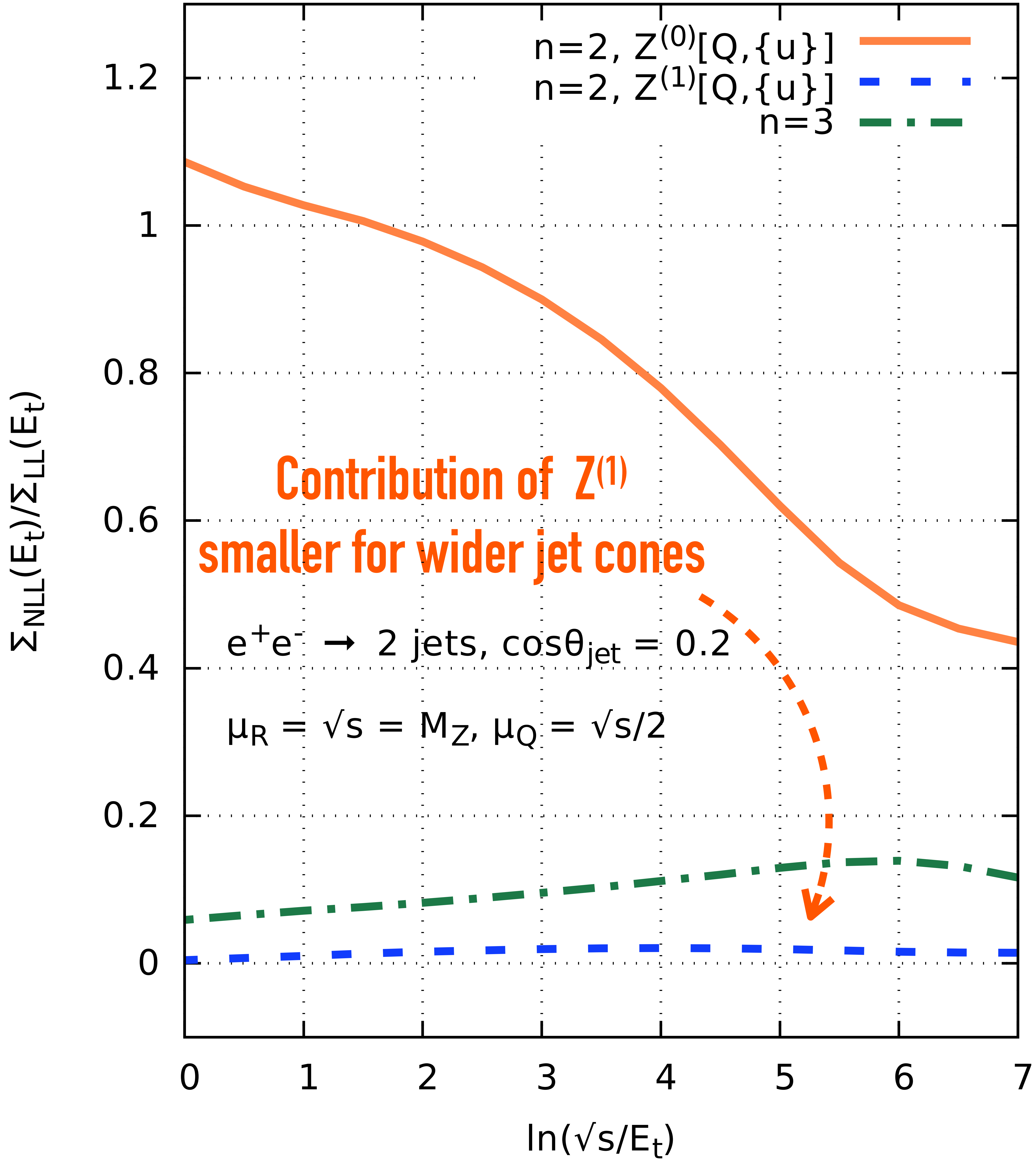
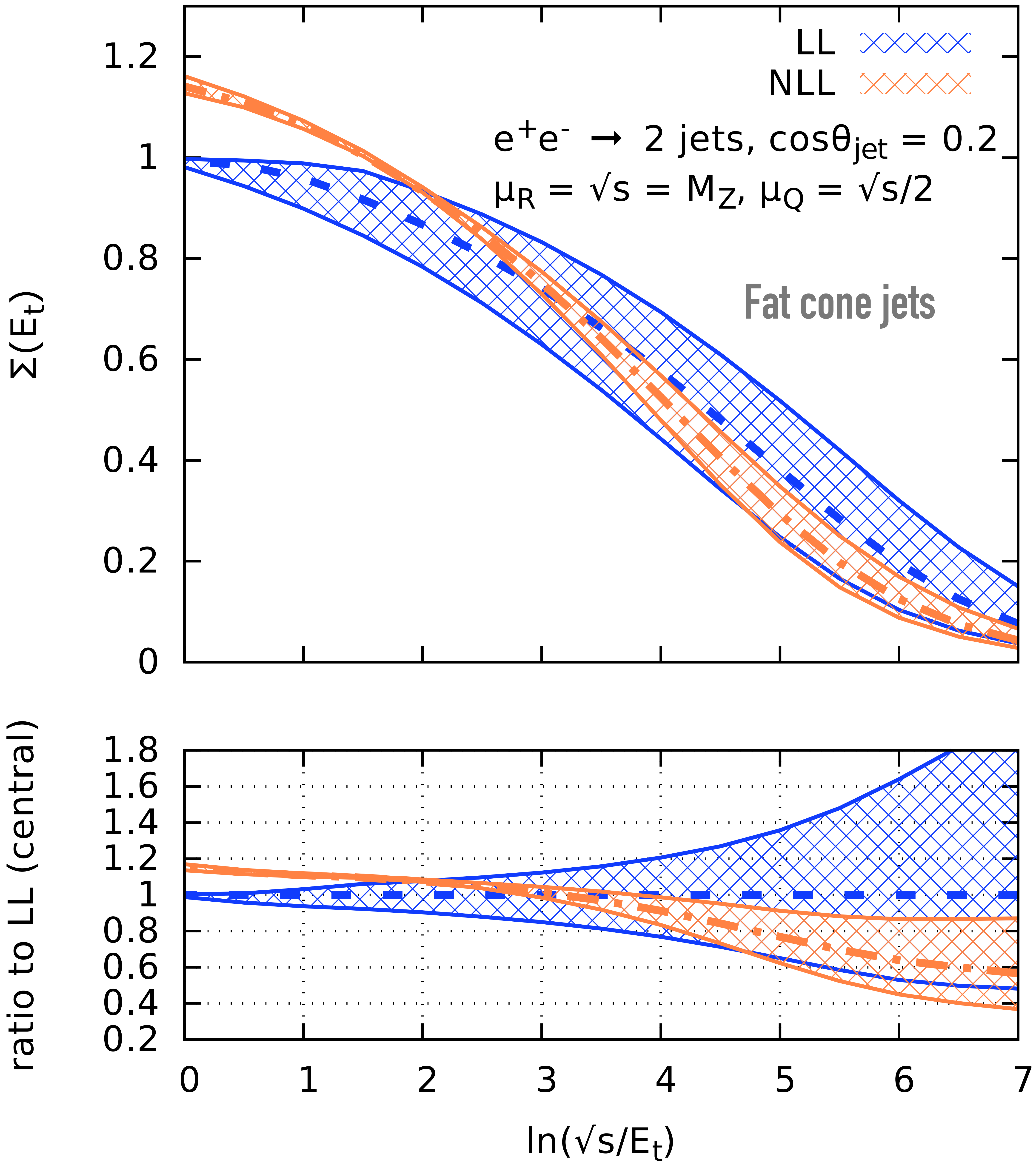




Conclusions & Outlook

- Formalism for calculation of non-global corrections at NLL in the planar limit:
  - ➡ Soft evolution solvable in terms of colour dipoles with Monte Carlo methods
  - ➡ NLL resummation for final-state radiation in e^+e^- (veto in interjet rapidity gap). NLL corrections are substantial (up to ~40%), with a considerable reduction of TH errors (~50%)

- Next steps:
  - ➡ Self-similar iteration of $Z^{(1)}$ (formally sub-leading), connection between orderings and RGE
  - ➡ Application to pp collisions (process dependence encoded in hard factors; complications arise at sub-leading N_c, e.g. SLL)

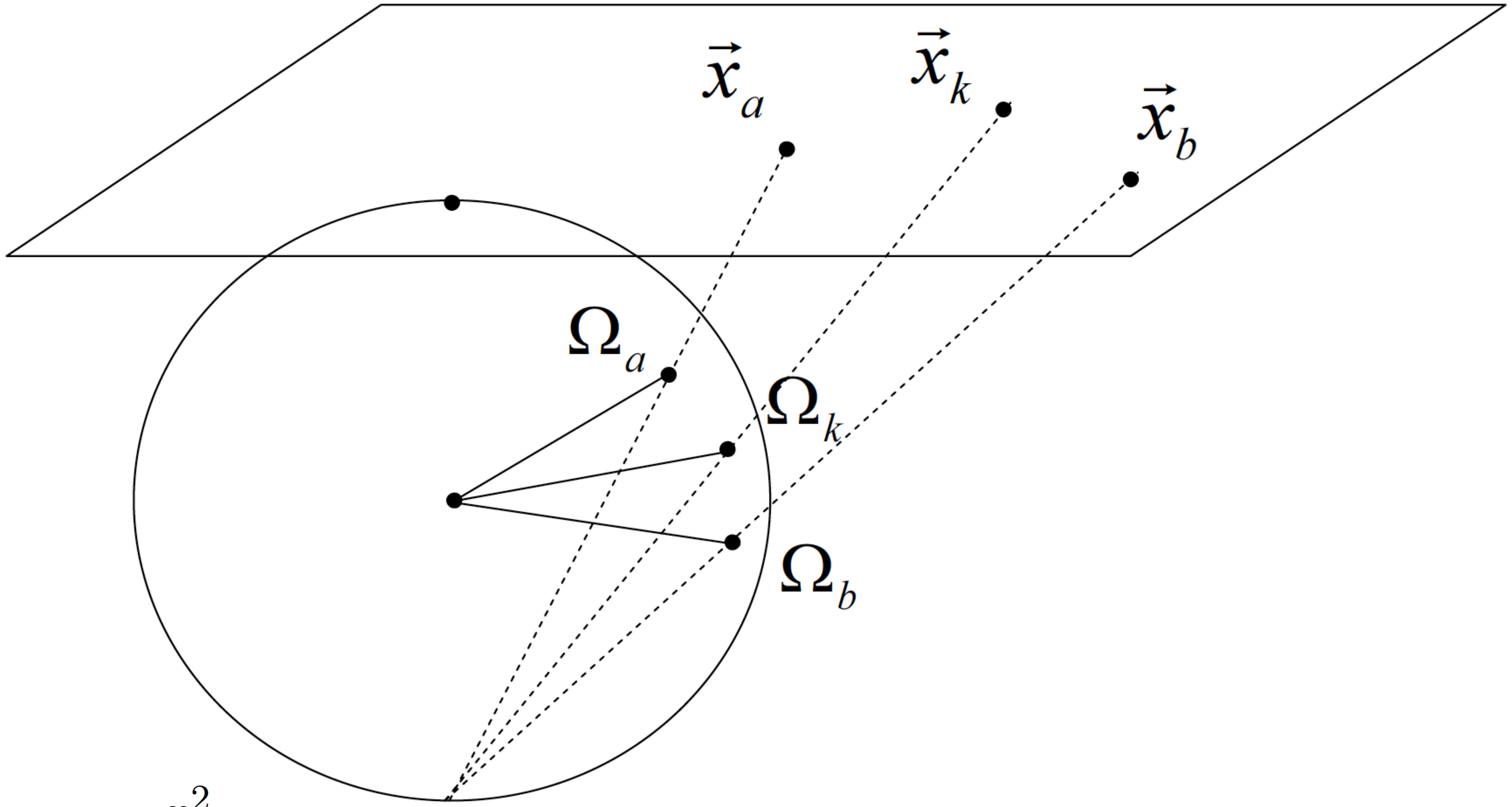
    see Matthias Neubert's talk
  - ➡ MC algorithm closely related to a parton shower: important insight on NNLL PS structure

Extra material

Non-global logarithms & BFLK

- Stereographic projection relates NGL evolution equation (BMS) to saturation dynamics in high-energy forward scattering (BK/JIMWLK) at all orders

[Weigert '03; Hatta '08; Caron-Huot '15]



$$\cos\theta = \frac{1-|\vec{x}|^2}{1+|\vec{x}|^2}, \quad \sin\theta = \frac{2|\vec{x}|}{1+|\vec{x}|^2}, \quad \cos\phi = \frac{x^1}{|\vec{x}|}, \quad \sin\phi = \frac{x^2}{|\vec{x}|}$$

$$\frac{d^2\Omega_k}{4\pi}\frac{1-\cos\theta_{ab}}{(1-\cos\theta_{ak})(1-\cos\theta_{bk})} = \frac{d^2\vec{x}_k}{2\pi}\frac{(\vec{x}_{ab})^2}{(\vec{x}_{ak})^2(\vec{x}_{bk})^2}$$

i.e. distribution of small-x gluons in the transverse plane is equivalent to angular distribution of soft gluons on the sphere at infinity

Second-order (planar) corrections to evolution kernel

$$Z_{12}[Q;\{u\}] = \mathbb{K}_{\text{int}}^{\text{RV+VV}}[Z[Q;u],u] + \mathbb{K}_{\text{int}}^{\text{RR}}[Z[Q;u],u] - \mathbb{K}_{\text{int}}^{\text{DC}}[Z[Q;u],u]$$

two-loop cusp anomalous dimension

$$\mathbb{K}_{\text{int}}^{\text{RV+VV}}[Z[Q;u],u] = \Delta_{12}(Q) + \int [dk_a]\bar{\alpha}(k_{ta})w_{12}^{(0)}(k_a)\left(1+\bar{\alpha}(k_{ta})\,\bar{K}^{(1)}\right)\frac{\Delta_{12}(Q)}{\Delta_{12}(k_{ta})}$$
$$\times Z_{1a}[k_{ta};\{u\}]Z_{a2}[k_{ta};\{u\}]u(k_a)\Theta(Q-k_{ta})$$

Same structure as LL kernel

(1 → 2 dipole branching)

<u>Easy to iterate in a MCMC</u>

Squared amplitudes from [Catani, Grazzini '00]

also [Angeles Martinez, Forshaw, Seymour '16]

Second-order (planar) corrections to evolution kernel

$$Z_{12}[Q;\{u\}] = \mathbb{K}_{\rm int}^{\rm RV+VV}[Z[Q;u],u] + \mathbb{K}_{\rm int}^{\rm RR}[Z[Q;u],u] - \mathbb{K}_{\rm int}^{\rm DC}[Z[Q;u],u]$$

$$\mathbb{K}_{\rm int}^{\rm RR}[Z[Q;u],u] = \int [dk_a] \int [dk_b]\, \bar\alpha^2(k_{t(ab)})\Theta(Q-k_{t(ab)})\Theta(k_{ta}-k'_{tb})\frac{\Delta_{12}(Q)}{\Delta_{12}(k_{t(ab)})}$$
$$\times\Big[\bar w_{12}^{(gg)}(k_b,k_a)Z_{1b}[k_{t(ab)};\{u\}]Z_{ba}[k_{t(ab)};\{u\}]Z_{a2}[k_{t(ab)};\{u\}]u(k_a)u(k_b)$$
$$+\bar w_{12}^{(gg)}(k_a,k_b)Z_{1a}[k_{t(ab)};\{u\}]Z_{ab}[k_{t(ab)};\{u\}]Z_{b2}[k_{t(ab)};\{u\}]u(k_a)u(k_b)$$
$$-\left(\bar w_{12}^{(gg)}(k_b,k_a)+\bar w_{12}^{(gg)}(k_a,k_b)\right)Z_{1(ab)}[k_{t(ab)};\{u\}]Z_{(ab)2}[k_{t(ab)};\{u\}]u(k_{(ab)})\Big]$$

New structure of real radiation

(1→3 dipole branching)

Hard to iterate in a MCMC

collinear counter-term defined on a projected pseudo-parent momentum

Squared amplitudes from [Campbell, Glover '97]

also [Gehrmann-De Ridder, Gehrmann, Glover '05]

Second-order (planar) corrections to evolution kernel

$$Z_{12}[Q;\{u\}] = \mathbb{K}^{\mathrm{RV+VV}}_{\mathrm{int}}[Z[Q;u],u] + \mathbb{K}^{\mathrm{RR}}_{\mathrm{int}}[Z[Q;u],u] - \mathbb{K}^{\mathrm{DC}}_{\mathrm{int}}[Z[Q;u],u]$$

$$\begin{aligned}
\mathbb{K}^{\mathrm{DC}}_{\mathrm{int}}[Z[Q;u],u] = & \int [dk_a] \int [dk_b] \bar{\alpha}^2(k_{ta}) \Theta(Q-k_{ta}) \Theta(k_{ta}-k_{tb}) \frac{\Delta_{12}(Q)}{\Delta_{12}(k_{ta})} \\
& \times \Bigg[w^{(0)}_{12}(k_a) \left(w^{(0)}_{1a}(k_b) - \frac{1}{2} w^{(0)}_{12}(k_b) \right) Z_{1b}[k_{ta};\{u\}] Z_{ba}[k_{ta};\{u\}] Z_{a2}[k_{ta};\{u\}] u(k_a) u(k_b) \\
& + w^{(0)}_{12}(k_a) \left(w^{(0)}_{a2}(k_b) - \frac{1}{2} w^{(0)}_{12}(k_b) \right) Z_{1a}[k_{ta};\{u\}] Z_{ab}[k_{ta};\{u\}] Z_{b2}[k_{ta};\{u\}] u(k_a) u(k_b) \\
& - w^{(0)}_{12}(k_a) \left(w^{(0)}_{1a}(k_b) + w^{(0)}_{a2}(k_b) - w^{(0)}_{12}(k_b) \right) Z_{1a}[k_{ta};\{u\}] Z_{a2}[k_{ta};\{u\}] u(k_a) \Bigg]
\end{aligned}$$

Perturbative insertion of double-real corrections

$$
\begin{aligned}
Z_{12}^{(1)}[Q;\{u\}] \simeq & \int [dk_a]\bar{\alpha}(k_{ta})w_{12}^{(0)}(k_a)\,\frac{\Delta_{12}(Q)}{\Delta_{12}(k_{ta})} \\
& \times \left(Z_{1a}^{(0)}[k_{ta};\{u\}]Z_{a2}^{(1)}[k_{ta};\{u\}] + Z_{1a}^{(1)}[k_{ta};\{u\}]Z_{a2}^{(0)}[k_{ta};\{u\}]\right) u(k_a)\Theta(Q-k_{ta}) \\
& + \int [dk_a] \int [dk_b]\,\bar{\alpha}^2(k_{t(ab)})\Theta(Q-k_{t(ab)})\Theta(k_{ta}-k'_{tb})\frac{\Delta_{12}(Q)}{\Delta_{12}(k_{t(ab)})} \\
& \times \Big[\tilde{w}_{12}^{(0)}(k_b,k_a)Z_{1b}^{(0)}[k_{t(ab)};\{u\}]Z_{ba}^{(0)}[k_{t(ab)};\{u\}]Z_{a2}^{(0)}[k_{t(ab)};\{u\}]u(k_a)u(k_b) \\
& + \tilde{w}_{12}^{(0)}(k_a,k_b)Z_{1a}^{(0)}[k_{t(ab)};\{u\}]Z_{ab}^{(0)}[k_{t(ab)};\{u\}]Z_{b2}^{(0)}[k_{t(ab)};\{u\}]u(k_a)u(k_b) \\
& - \left(\tilde{w}_{12}^{(0)}(k_b,k_a) + \tilde{w}_{12}^{(0)}(k_a,k_b)\right) Z_{1(ab)}^{(0)}[k_{t(ab)};\{u\}]Z_{(ab)2}^{(0)}[k_{t(ab)};\{u\}]u(k_{(ab)})\Big] \\
& - \int [dk_a] \int [dk_b]\bar{\alpha}^2(k_{ta})\Theta(Q-k_{ta})\Theta(k_{ta}-k_{tb})\frac{\Delta_{12}(Q)}{\Delta_{12}(k_{ta})} \\
& \times \Big[w_{12}^{(0)}(k_a)w_{1a}^{(0)}(k_b)Z_{1b}^{(0)}[k_{ta};\{u\}]Z_{ba}^{(0)}[k_{ta};\{u\}]Z_{a2}^{(0)}[k_{ta};\{u\}]u(k_a)u(k_b) \\
& + w_{12}^{(0)}(k_a)w_{a2}^{(0)}(k_b)Z_{1a}^{(0)}[k_{ta};\{u\}]Z_{ab}^{(0)}[k_{ta};\{u\}]Z_{b2}^{(0)}[k_{ta};\{u\}]u(k_a)u(k_b) \\
& - w_{12}^{(0)}(k_a)\left(w_{1a}^{(0)}(k_b) + w_{a2}^{(0)}(k_b)\right) Z_{1a}^{(0)}[k_{ta};\{u\}]Z_{a2}^{(0)}[k_{ta};\{u\}]u(k_a)\Big]
\end{aligned}
$$

Fixed order expansion (full colour)

- Keep only terms up to NLL & extend to full colour (at fixed order only)

promote $(N_c)^n$ to correct Casimirs

$$
\begin{aligned}
\Sigma(v) \simeq 1 &+ \left(\frac{\alpha_s}{2\pi}\right)\left(\mathcal{H}_2^{(1)} - 4C_F \int [dk] w_{12}^{(0)}(k)\Theta_{\rm in}(k)\Theta(v(k)-v)\Theta(Q-k_t) + \mathcal{H}_3^{(1)} \otimes \mathbb{1}\right) \\
&- 4C_F \left(\frac{\alpha_s}{2\pi}\right)^2 \int [dk] w_{12}^{(0)}(k)\Theta_{\rm in}(k)\Theta(v(k)-v)\Theta(k_t-Q)\left(K^{(1)} - 4\pi\beta_0 \ln\frac{k_t}{Q}\right) \\
&+ 8C_F^2 \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\int [dk] w_{12}^{(0)}(k)\Theta_{\rm in}(k)\Theta(v(k)-v)\Theta(Q-k_t)\right)^2 \\
&- 8C_F \left(\frac{\alpha_s}{2\pi}\right)^2 \int [dk_a] \int [dk_b] \left[C_A\left(\bar{w}_{12}^{(gg)}(k_a,k_b) + \bar{w}_{12}^{(gg)}(k_b,k_a)\right)\right. \\
&\qquad \left. + n_f\left(\bar{w}_{12}^{(q\bar{q})}(k_a,k_b) + \bar{w}_{12}^{(q\bar{q})}(k_b,k_a)\right)\right] \\
&\times \Theta(Q-k_{t(ab)})\Theta(k_{ta}-k_{tb})\left\{\Theta_{\rm out}(k_{(ab)})\left[\Theta_{\rm in}(k_a)\Theta_{\rm out}(k_b)\Theta(v(k_a)-v)\right.\right. \\
&\left.\left. + \Theta_{\rm out}(k_a)\Theta_{\rm in}(k_b)\Theta(v(k_b)-v)\right] - \Theta_{\rm in}(k_{(ab)})\Theta_{\rm out}(k_a)\Theta_{\rm out}(k_b)\Theta(v(k_{(ab)})-v)\right\} \\
&- 2\left(\frac{\alpha_s}{2\pi}\right)^2 \int [dk]\Theta_{\rm in}(k)\Theta(v(k)-v)\Theta(Q-k_t) \\
&\times \left[2C_F\mathcal{H}_2^{(1)} w_{12}^{(0)}(k) + \mathcal{H}_3^{(1)} \otimes \left(C_A(w_{13}^{(0)}(k) + w_{32}^{(0)}(k)) + (2C_F - C_A)w_{12}(k)\right)\right].
\end{aligned}
$$

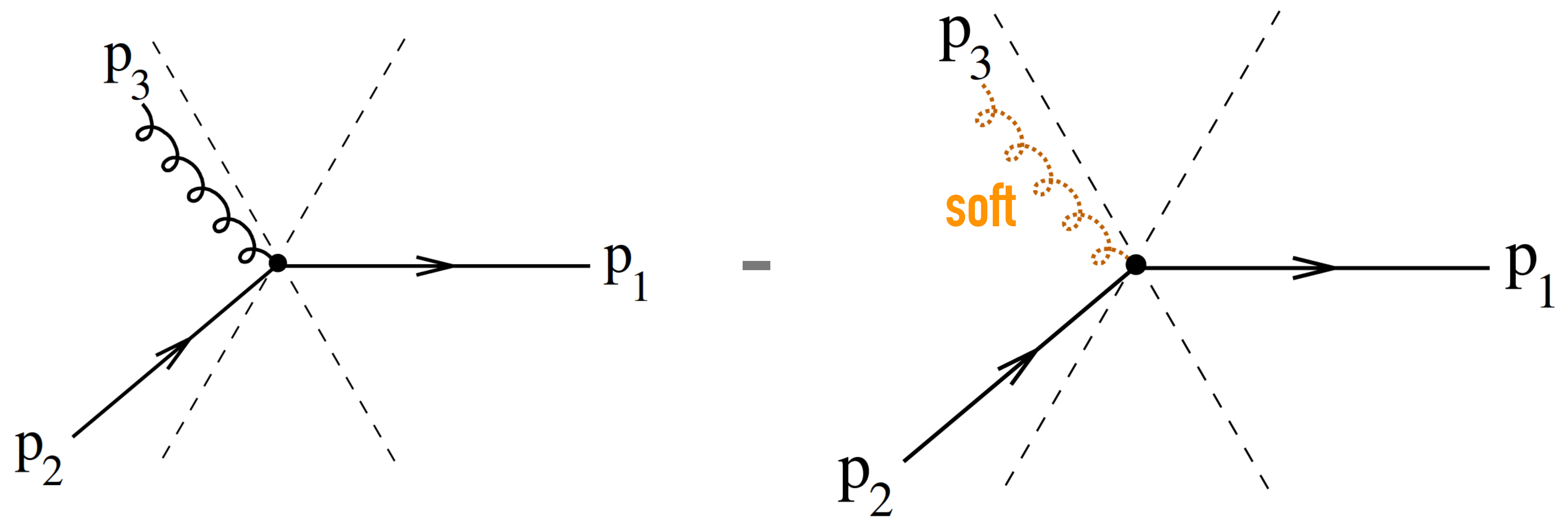
add double soft fermionic current

add subl. colour 3-jet dipole

Hard factor with 3 legs at NLL: H_3

- Computed by matching the soft theory to full QCD
- Cancellation of collinear divergences between H_2 and H_3 (only combination is scheme indep.)

e.g. H_3



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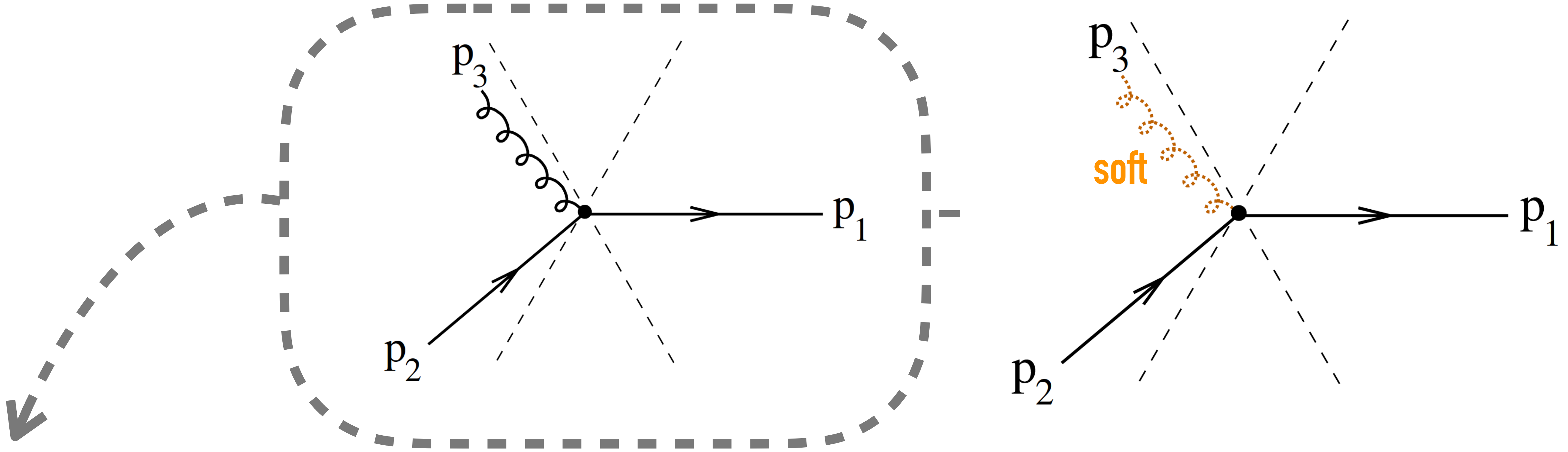

Subtract counter-term (<u>2-jet kinematics</u>) with full ME, requiring all partons to be outside the slice. Thrust axis along the hardest parton

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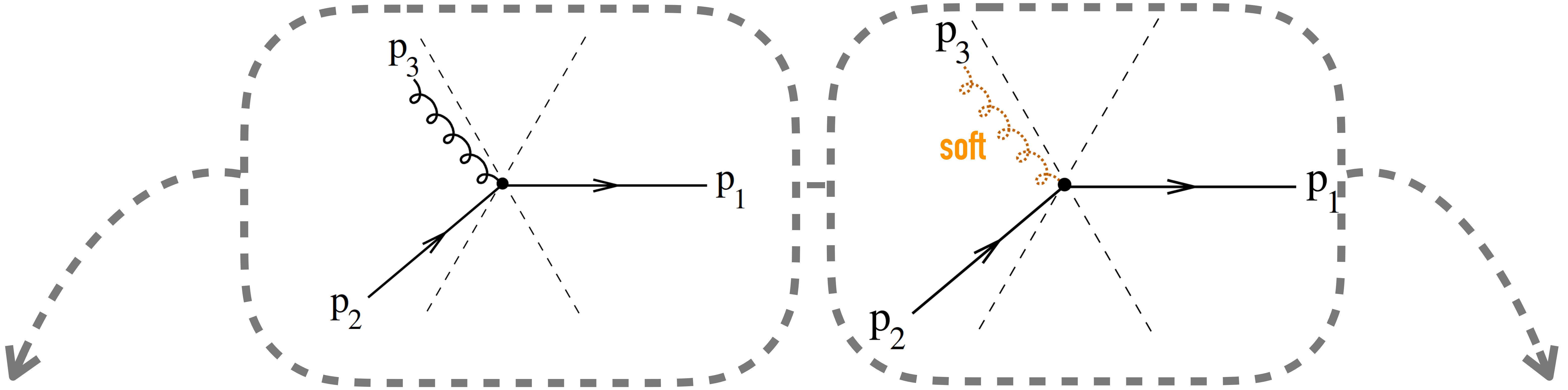

Subtract counter-term (2-jet kinematics) with full ME, requiring all partons to be outside the slice. Thrust axis along the hardest parton

Subtract soft counter-term, requiring the soft gluon to be outside the slice. Thrust axis along q (qbar) direction

Dependence on infrared freezing scale Q_0

- Mild dependence at low scales for fat cone jets, indicating sensitivity to non-perturbative corrections
- Impact is more moderate for narrower jets