TMD distributions of twist-three: definition, evolution, properties

based on [Simone Rodini, AV, 2204.03856]

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HRT

Ignazio's talk ⇒ **Cross-section & coefficient functions**
now ⇒ **TMDs of twist-three**

Generic TMDs of twist-three were introduced already in [Tangerman, Mulders, 95].
Generic TMDs are *ill-defined* objects
Theoretically well-defined objects is TMDs with a specific TMD-twist

Strategy: TMD with a with a specific TMD-twist ⇒ physical/generic TMDs

Outline

- Definite-TMD-twist distributions and their evolution
- Interpretation and support
- TMD-distributions with definite T-parity
- Singularities in TMD-distributions ⇒ physical TMD-distributions
- Generic TMD distributions and their evolution

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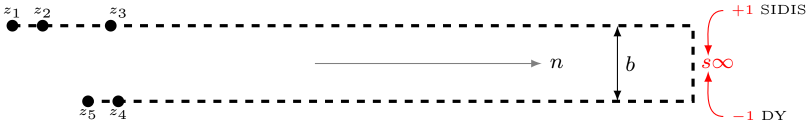
TMD operators and their divergences

Any TMD operator is the product of two *semi-compact* operators

$$\mathcal{O}_{NM}(\{z_1,...,z_n\},b) = U_N(\{z_1,...\},b)U_M(\{...,z_n\},b)$$



$$\mathcal{O}^{\text{bare}}_{NM}(\{z_1,...,z_n\},b) = R(b^2)Z_{U_N}(\{z_1,...\}) \otimes Z_{U_M}(\{...,z_n\}) \otimes \mathcal{O}_{NM}(\mu,\zeta)$$

- UV divergence for U_N
- UV divergence for U_M
- Rapidity divergence

Three independent divergences
Three renormalization constants
Three anomalous dimensions

Computation ⇒ see talk by *Ignazio*

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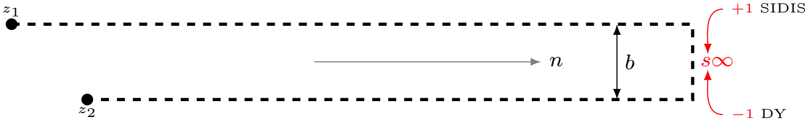
TMD-twist-(1,1)

Usual TMDs

$U_1 = [..]\xi =$ good-component of quark field (twist-1)

$$\widetilde{\Phi}_{11}^{[\Gamma]}(\{z_1, z_2\}, b) = \langle p, s|\bar{\xi}(z_1 n + b)..\frac{\Gamma}{2}..\xi(z_2 n)|p, s\rangle$$



$$\mu^2 \frac{d}{d\mu^2}\widetilde{\Phi}_{11}(\{z_{1,2}\}, b; \mu, \zeta) = (\widetilde{\gamma}_1(z_1, \mu, \zeta) + \widetilde{\gamma}_1(z_2, \mu, \zeta))\,\widetilde{\Phi}_{11}(\{z_{1,2}\}, b; \mu, \zeta)$$

$$\zeta \frac{d}{d\zeta}\widetilde{\Phi}_{11}(\{z_{1,2}\}, b; \mu, \zeta) = -\mathcal{D}(b, \mu)\widetilde{\Phi}_{11}(\{z_{1,2}\}, b; \mu, \zeta)$$

- $\gamma_1 =$ anomalous dimension of U_1
- $\mathcal{D} =$ CS kernel

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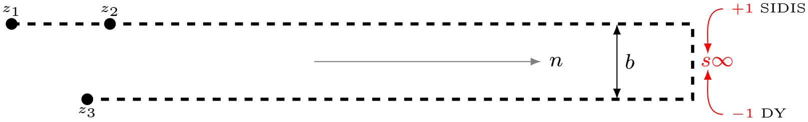
TMD-twist-(2,1)

Appear at NLP

$U_1 = [..]\xi$ = good-component of quark field (twist-1)
$U_2 = [..]F_{\mu+}[..]\xi$ = good-components of gluon and quark fields (twist-2)

$$\widetilde{\Phi}_{21}^{[\Gamma]}(\{z_1, z_2, z_3\}, b) = \langle p, s|\bar{\xi}(z_1 n + b)..F_{\mu+}(z_2 n + b)..\frac{\Gamma}{2}..\xi(z_3 n)|p, s\rangle$$



$$\mu^2 \frac{d}{d\mu^2}\widetilde{\Phi}_{21}(\{z_{1,2,3}\}, b; \mu, \zeta) = (\widetilde{\gamma}_2(z_1, z_2, \mu, \zeta) + \widetilde{\gamma}_1(z_3, \mu, \zeta))\,\widetilde{\Phi}_{21}(\{z_{1,2,3}\}, b; \mu, \zeta)$$

$$\zeta \frac{d}{d\zeta}\widetilde{\Phi}_{21}(\{z_{1,2,3}\}, b; \mu, \zeta) = -\mathcal{D}(b, \mu)\widetilde{\Phi}_{21}(\{z_{1,2,3}\}, b; \mu, \zeta)$$

- γ_1 = anomalous dimension of U_1
- γ_2 = anomalous dimension of U_2
- $\mathcal{D}$ = CS kernel

Similar for TMD-twist-(1,2)

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Anatomy of anomalous dimension

$$\widetilde{\gamma}_1(z,\mu,\zeta) = a_s(\mu)C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)+2\ln\left(\frac{q^+}{-s\partial_z^+}\right)\right)+\mathcal{O}(a_s^2),$$

$$\widetilde{\gamma}_2(z_2,z_3,\mu,\zeta) = a_s(\mu)\Big\{\mathbb{H}_{z_2z_3}+C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)\right)$$
$$+C_A\ln\left(\frac{q^+}{-s\partial_{z_2}^+}\right)+2\left(C_F-\frac{C_A}{2}\right)\ln\left(\frac{q^+}{-s\partial_{z_3}^+}\right)\Big\}+\mathcal{O}(a_s^2),$$

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Anatomy of anomalous dimension

quark AD + cusp

$$\widetilde{\gamma}_1(z,\mu,\zeta) = a_s(\mu) C_F \boxed{\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)\right.} \left. + 2\ln\left(\frac{q^+}{-s\partial_z^+}\right)\right) + \mathcal{O}(a_s^2),$$

$$\widetilde{\gamma}_2(z_2,z_3,\mu,\zeta) = a_s(\mu)\Big\{\mathbb{H}_{z_2 z_3} + \boxed{C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)\right)}$$

$$+C_A \ln\left(\frac{q^+}{-s\partial_{z_2}^+}\right) + 2\left(C_F - \frac{C_A}{2}\right)\ln\left(\frac{q^+}{-s\partial_{z_3}^+}\right)\Big\} + \mathcal{O}(a_s^2),$$

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Anatomy of anomalous dimension

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$$\widetilde{\gamma}_1(z,\mu,\zeta) = a_s(\mu)C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)+2\ln\left(\frac{q^+}{-s\partial_z^+}\right)\right)+\mathcal{O}(a_s^2),$$

$$\widetilde{\gamma}_2(z_2,z_3,\mu,\zeta) = a_s(\mu)\Big\{\mathbb{H}_{z_2z_3}+C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)\right)$$

BFLK

quasi-partonic-kernel

[Bukhvostov, Frolov, Lipatov, Kuraev, 1985]

$$+C_A\ln\left(\frac{q^+}{-s\partial_{z_2}^+}\right)+2\left(C_F-\frac{C_A}{2}\right)\ln\left(\frac{q^+}{-s\partial_{z_3}^+}\right)\Big\}+\mathcal{O}(a_s^2),$$

$$\mathbb{H}_{z_2z_3}\widetilde{\Phi}^{[\Gamma]}_{\mu,12}(z_1,z_2,z_3) = \tag{2.19}$$

$$C_A\int_0^1\frac{d\alpha}{\alpha}\left(\bar\alpha^2\widetilde{\Phi}^{[\Gamma]}_{\mu,12}(z_1,z_{23}^\alpha,z_3)+\bar\alpha\widetilde{\Phi}^{[\Gamma]}_{\mu,12}(z_1,z_2,z_{32}^\alpha)-2\widetilde{\Phi}^{[\Gamma]}_{\mu,12}(z_1,z_2,z_3)\right)$$

$$+C_A\int_0^1 d\alpha\int_0^{\bar\alpha}d\beta\,\bar\alpha\widetilde{\Phi}^{[\Gamma\gamma_\mu\gamma^\nu]}_{\nu,12}(z_1,z_{23}^\alpha,z_{32}^\beta)-2\left(C_F-\frac{C_A}{2}\right)\int_0^1 d\alpha\int_{\bar\alpha}^1 d\beta\,\bar\alpha\widetilde{\Phi}^{[\Gamma\gamma_\mu\gamma^\nu]}_{\nu,12}(z_1,z_{23}^\alpha,z_{32}^\beta)$$

$$+\left(C_F-\frac{C_A}{2}\right)\int_0^1 d\alpha\,\bar\alpha\widetilde{\Phi}^{[\Gamma\gamma^\nu\gamma_\mu]}_{\nu,12}(z_1,z_{32}^\alpha,z_2),$$

Mixes Lorentz structures Γ

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Anatomy of anomalous dimension

quark AD + cusp

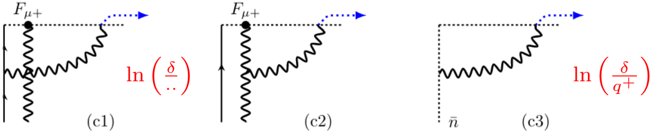
$$\widetilde{\gamma}_1(z,\mu,\zeta) = a_s(\mu) C_F \left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right) + 2\ln\left(\frac{q^+}{-s\partial_z^+}\right)\right) + \mathcal{O}(a_s^2),$$

$$\widetilde{\gamma}_2(z_2,z_3,\mu,\zeta) = a_s(\mu)\Big\{\mathbb{H}_{z_2z_3} + C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)\right)$$

BFLK
quasi-partonic-kernel

$$+C_A\ln\left(\frac{q^+}{-s\partial_{z_2}^+}\right) + 2\left(C_F-\frac{C_A}{2}\right)\ln\left(\frac{q^+}{-s\partial_{z_3}^+}\right)\Big\} + \mathcal{O}(a_s^2),$$

Remnants of collinear divergences
(canceled with SF)



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To momentum-fraction space

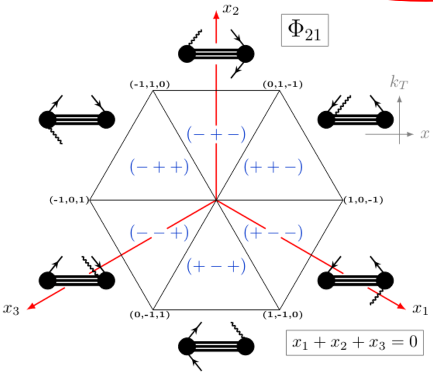
$$\widetilde{\Phi}^{[\Gamma]}_{11}(z_1,z_2,b) = p^+ \int_{-1}^{1} dx e^{ix(z_1-z_2)p^+} \Phi^{[\Gamma]}_{11}(x,b),$$

$$\widetilde{\Phi}^{[\Gamma]}_{\mu,21}(z_1,z_2,z_3,b) = (p^+)^2 \int [dx] e^{-i(x_1z_1+x_2z_2+x_3z_3)p^+} \Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b),$$

$$\widetilde{\Phi}^{[\Gamma]}_{\mu,12}(z_1,z_2,z_3,b) = (p^+)^2 \int [dx] e^{-i(x_1z_1+x_2z_2+x_3z_3)p^+} \Phi^{[\Gamma]}_{\mu,12}(x_1,x_2,x_3,b),$$

$$\Phi(x) = \theta(x)q(x) - \theta(-x)\bar{q}(-x)$$

$$\int [dx] = \int_{-1}^{1} dx_1 \int_{-1}^{1} dx_2 \int_{-1}^{1} dx_3\, \delta(x_1+x_2+x_3)$$



Support domain $|x_i| < 1$
momentum-fractions
could be **positive or negative**

important for factorization

- divergences-cancelation
- agreement with collinear evolution
- evolution mixture

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Evolution equations in the momentum-fraction space

has involved structure

$$\mu^2 \frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_1x_2x_3} + 2\pi i\, s\,\Theta_{x_1x_2x_3}\right)\Phi^{[\Gamma]}_{\mu,21} + \mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21} + \mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},$$

$$\mu^2 \frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_3x_2x_1} + 2\pi i\, s\,\Theta_{x_3x_2x_1}\right)\Phi^{[\Gamma]}_{\mu,12} + \mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12} + \mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},$$

Rapidity evolution is the same

Γ_{cusp}-part is the same

$$\zeta\frac{d}{d\zeta}\Phi^{[\Gamma]}_{\mu,12}(x_1,x_2,x_3,b;\mu,\zeta) = -\mathcal{D}(b,\mu)\Phi^{[\Gamma]}_{\mu,12}(x_1,x_2,x_3,b;\mu,\zeta),$$

$$\zeta\frac{d}{d\zeta}\Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b;\mu,\zeta) = -\mathcal{D}(b,\mu)\Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b;\mu,\zeta).$$

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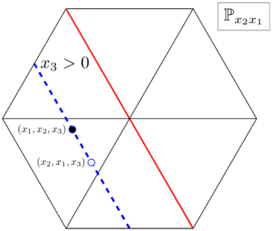
Evolution equations in the momentum-fraction space

has involved structure

$$\mu^2 \frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_1x_2x_3} + 2\pi i\, s\,\Theta_{x_1x_2x_3}\right)\Phi^{[\Gamma]}_{\mu,21}$$
$$\boxed{+\mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21} + \mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},}$$
$$\mu^2 \frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_3x_2x_1} + 2\pi i\, s\,\Theta_{x_3x_2x_1}\right)\Phi^{[\Gamma]}_{\mu,12}$$
$$\boxed{+\mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12} + \mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},}$$



BFLK kernels in momentum space are quite cumbersome

- non-analytic
- continious
- mix-sectors
- longish
- for "$x_i > 0$" region agrees with [Beneke, et al, 17]

$$\begin{aligned}
\mathbb{P}^A_{x_2x_1}\otimes\Phi(x_1,x_2,x_3) = -\frac{a_s}{2}\Big\{&\delta_{x_20}\,C_A\Phi(x_1,0,x_3) \\
&+C_A\int_{-\infty}^{\infty}dv\Big[\frac{x_2}{v}[(v+x_2)\Phi(x_1,x_2,x_3)-x_2\Phi(x_1-v,x_2+v,x_3)]\frac{\theta(v,x_2)-\theta(-v,-x_2)}{(v+x_2)^2} \\
&\quad+\frac{x_1}{v}(\Phi(x_1,x_2,x_3)-\Phi(x_1-v,x_2+v,x_3))\frac{\theta(v,-x_1)-\theta(-v,x_1)}{v-x_1}\Big] \\
&-C_A\int_{-\infty}^{\infty}dv\Big[\frac{x_2^2(v+2x_2+x_1)}{(x_1+x_2)^2}\frac{\theta(v,x_2)-\theta(-v,-x_2)}{(v+x_2)^2} \\
&\quad+\frac{x_1(2x_2+x_1)}{(x_1+x_2)^2}\frac{\theta(v,-x_1)-\theta(-v,x_1)}{v-x_1}\Big]\Phi(x_1-v,x_2+v,x_3) \\
&+2\left(C_F-\frac{C_A}{2}\right)\int_{-\infty}^{\infty}dv\Big[\frac{-x_2^2}{(x_1+x_2)^2}\frac{\theta(v,x_2)-\theta(-v,-x_2)}{v+x_2} \\
&\quad+\frac{x_1(x_1x_2-2vx_2-vx_1)}{(x_1+x_2)^2}\frac{\theta(v,-x_1)-\theta(-v,x_1)}{(v-x_1)^2}\Big]\Phi(x_2+v,x_1-v,x_3)\Big\} \\
&+\mathcal{O}(a_s^2),
\end{aligned} \tag{1}$$

$$\begin{aligned}
\mathbb{P}^B_{x_2x_1}\otimes\Phi(x_1,x_2,x_3) = -\frac{a_s}{2}\Big\{&\delta_{x_20}\,2(C_A-C_F)\Phi(x_1,0,x_3) \\
&+C_A\int_{-\infty}^{\infty}dv\Big[\frac{x_2}{v}[(v+x_2)\Phi(x_1,x_2,x_3)-x_2\Phi(x_1-v,x_2+v,x_3)]\frac{\theta(v,x_2)-\theta(-v,-x_2)}{(v+x_2)^2} \\
&\quad+\frac{x_1}{v}(\Phi(x_1,x_2,x_3)-\Phi(x_1-v,x_2+v,x_3))\frac{\theta(v,-x_1)-\theta(-v,x_1)}{v-x_1}\Big] \\
&+2\left(C_F-\frac{C_A}{2}\right)\int_{-\infty}^{\infty}dv\,x_1\Phi(x_2+v,x_1-v,x_3)\frac{\theta(v,-x_1)-\theta(-v,x_1)}{(v-x_1)^2}\Big\}+\mathcal{O}(a_s^2).
\end{aligned} \tag{2}$$

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Evolution equations in the momentum-fraction space

has involved structure

$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_1x_2x_3} + 2\pi i\, s\,\Theta_{x_1x_2x_3}\right)\Phi^{[\Gamma]}_{\mu,21} + \mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21} + \mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},$$

$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_3x_2x_1} + 2\pi i\, s\,\Theta_{x_3x_2x_1}\right)\Phi^{[\Gamma]}_{\mu,12} + \mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12} + \mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},$$

$$\Upsilon_{x_1x_2x_3} = a_s\left[3C_F + C_A\ln\left(\frac{|x_3|}{|x_2|}\right) + 2\left(C_F - \frac{C_A}{2}\right)\ln\left(\frac{|x_3|}{|x_1|}\right)\right] + \mathcal{O}(a_s^2).$$

Real-part of collinear logarithms

- Singular at $x_i = 0$
- Integrable
- Checked by NLP coeff.function

q^+ is as in fact. theorem

$$q^+ = \begin{cases} |p_{\bar q}| = |p_q|, & \text{for } \widetilde{\Phi}^{[\Gamma]}_{11}, \\ |p_{\bar q}|, & \text{for } \widetilde{\Phi}^{[\Gamma]}_{12}, \\ |p_q|, & \text{for } \widetilde{\Phi}^{[\Gamma]}_{21}, \end{cases}$$

Evolution equations in the momentum-fraction space

has involved structure

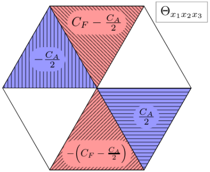
$$\mu^2 \frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_1x_2x_3} + \boxed{2\pi i\, s\,\Theta_{x_1x_2x_3}}\right)\Phi^{[\Gamma]}_{\mu,21} + \mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21} + \mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},$$

$$\mu^2 \frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_3x_2x_1} + \boxed{2\pi i\, s\,\Theta_{x_3x_2x_1}}\right)\Phi^{[\Gamma]}_{\mu,12} + \mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12} + \mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},$$

Imaginary-part of collinear logarithms

- Discontinious
- Process dependent!
- OMG!

$$\Theta_{x_1x_2x_3} = a_s \times \begin{cases} \frac{C_A}{2} & x_{1,2,3}\in(+,-,-), \\ -\left(C_F-\frac{C_A}{2}\right) & x_{1,2,3}\in(+,-,+), \\ 0 & x_{1,2,3}\in(-,-,+), \\ -\frac{C_A}{2} & x_{1,2,3}\in(-,+,+), \\ C_F-\frac{C_A}{2} & x_{1,2,3}\in(-,+,-), \\ 0 & x_{1,2,3}\in(+,+,-), \end{cases} + \mathcal{O}(a_s^2),$$



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Evolution equations in the momentum-fraction space

has involved structure

$$\mu^2 \frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_1x_2x_3} + 2\pi i\, s\,\Theta_{x_1x_2x_3}\right)\Phi^{[\Gamma]}_{\mu,21}$$
$$+\mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21} + \mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},$$
$$\mu^2 \frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_3x_2x_1} + 2\pi i\, s\,\Theta_{x_3x_2x_1}\right)\Phi^{[\Gamma]}_{\mu,12}$$
$$+\mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12} + \mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},$$

- Complex
- Discontinious
- Singular

Live is not that bad!

Complex evolution for complex functions!

Transformation properties

No definite complexity

$$[\Phi^{[\Gamma]}_{\mu,12}(x_1,x_2,x_3,b)]^* = \Phi^{[\gamma^0\Gamma^\dagger\gamma^0]}_{\mu,21}(-x_3,-x_2,-x_1,-b),$$
$$[\Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b)]^* = \Phi^{[\gamma^0\Gamma^\dagger\gamma^0]}_{\mu,12}(-x_3,-x_2,-x_1,-b).$$

No definite T-parity

$$\mathcal{PT}\Phi^{[\Gamma]}_{\mu,12}(x_1,x_2,x_3,b;s,L)(\mathcal{PT})^{-1} = -\Phi^{[\gamma^0T\Gamma^*T^{-1}\gamma^0]}_{\mu,21}(-x_3,-x_2,-x_1,-b;-s,-L),$$
$$\mathcal{PT}\Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b;s,L)(\mathcal{PT})^{-1} = -\Phi^{[\gamma^0T\Gamma^*T^{-1}\gamma^0]}_{\mu,12}(-x_3,-x_2,-x_1,-b;-s,-L).$$

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T-parity-definite combinations

$$\Phi^{[\Gamma]}_{\mu,\oplus}(x_1,x_2,x_3,b) = \frac{\Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b)+\Phi^{[\Gamma]}_{\mu,12}(-x_3,-x_2,-x_1,b)}{2},$$
$$\Phi^{[\Gamma]}_{\mu,\ominus}(x_1,x_2,x_3,b) = i\frac{\Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b)-\Phi^{[\Gamma]}_{\mu,12}(-x_3,-x_2,-x_1,b)}{2}$$

Real functions, with real evolution
Price: evolution mixes $\oplus$ and $\ominus$ sectors

Definite complexity

$$[\Phi^{[\Gamma]}_{\mu,\oplus}(x_1,x_2,x_3,b)]^* = \Phi^{[\gamma^0\Gamma^\dagger\gamma^0]}_{\mu,\oplus}(x_1,x_2,x_3,-b),$$
$$[\Phi^{[\Gamma]}_{\mu,\ominus}(x_1,x_2,x_3,b)]^* = \Phi^{[\gamma^0\Gamma^\dagger\gamma^0]}_{\mu,\ominus}(x_1,x_2,x_3,-b)$$

Definite T-parity

$$\mathcal{PT}\Phi^{[\Gamma]}_{\mu,\oplus}(x_1,x_2,x_3,b;s,L)(\mathcal{PT})^{-1} = -\Phi^{[\gamma^0 T\Gamma^* T^{-1}\gamma^0]}_{\mu,\oplus}(x_1,x_2,x_3,-b;-s,-L),$$
$$\mathcal{PT}\Phi^{[\Gamma]}_{\mu,\ominus}(x_1,x_2,x_3,b;s,L)(\mathcal{PT})^{-1} = +\Phi^{[\gamma^0 T\Gamma^* T^{-1}\gamma^0]}_{\mu,\ominus}(x_1,x_2,x_3,-b;-s,-L).$$

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At this stage we can introduce the parametrization

- Three Γ-structures $\{\gamma^+, \gamma^+\gamma^5, i\sigma^{\alpha+}\gamma^5\}$
- In the tensor case, one can sort $F_{\mu+}\sigma^{\alpha+}$-tensors into $J = 0, 1, 2$ cases.
- **32 distributions** ($\oplus$ and $\ominus$)
- **16 T-odd** and **16 T-even**

Example

$$\Phi_\bullet^{\mu[\gamma^+]}(x_{1,2,3}, b) = \epsilon^{\mu\nu} s_{T\nu} M f_{\bullet T}(x_{1,2,3}, b) + i b^\mu M^2 f_\bullet^\perp(x_{1,2,3}, b) + i\lambda\epsilon^{\mu\nu} b_\nu M^2 f_{\bullet L}^\perp(x_{1,2,3}, b) + b^2 M^3 \epsilon_T^{\mu\nu}\left(\frac{g_{T,\nu\rho}}{2} - \frac{b_\nu b_\rho}{b^2}\right) s_T^\rho f_{\bullet T}^\perp(x_{1,2,3}, b)$$

$$f_{\oplus,T;DY} = f_{\oplus,T;SIDIS}, \qquad f_{\oplus;DY}^\perp = -f_{\oplus;SIDIS}^\perp$$

	U	L	$T_{J=0}$	$T_{J=1}$	$T_{J=2}$
U	$f_\bullet^\perp$	$g_\bullet^\perp$		$h_\bullet$	$h_\bullet^\perp$
L	$f_{\bullet L}^\perp$	$g_{\bullet L}^\perp$	$h_{\bullet L}$		$h_{\bullet L}^\perp$
T	$f_{\bullet T}$, $f_{\bullet T}^\perp$	$g_{\bullet T}$, $g_{\bullet T}^\perp$	$h_{\bullet T}^{D\perp}$	$h_{\bullet T}^{A\perp}$	$h_{\bullet T}^{S\perp}$, $h_{\bullet T}^{T\perp}$

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Evolution equations split into two cases:
Evolution with kernels $\mathbb{P}^A$ or $\mathbb{P}^B$

Example $\mathbb{P}^A$

$$\mu^2\frac{d}{d\mu^2}\begin{pmatrix} H^A_\oplus \\ H^A_\ominus \end{pmatrix} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\Upsilon_{x_1x_2x_3}\right)\begin{pmatrix} H^A_\oplus \\ H^A_\ominus \end{pmatrix} + \begin{pmatrix} 2\mathbb{P}^A_{x_2x_1} & 2\pi s\Theta_{x_1x_2x_3} \\ -2\pi s\Theta_{x_1x_2x_3} & 2\mathbb{P}^A_{x_2x_1} \end{pmatrix}\begin{pmatrix} H^A_\oplus \\ H^A_\ominus \end{pmatrix},$$

$$\begin{pmatrix} f^\perp_\oplus + g^\perp_\ominus \\ f^\perp_\ominus - g^\perp_\oplus, \end{pmatrix}, \begin{pmatrix} f^\perp_{\oplus,L} + g^\perp_{\ominus,L} \\ f^\perp_{\ominus,L} - g^\perp_{\oplus,L} \end{pmatrix}, \qquad \begin{pmatrix} f_{\oplus,T} + g_{\ominus,T} \\ f_{\ominus,T} - g_{\oplus,T} \end{pmatrix}, \begin{pmatrix} f^\perp_{\oplus,T} + g^\perp_{\ominus,T} \\ f^\perp_{\ominus,T} - g^\perp_{\oplus,T} \end{pmatrix},$$

$$\begin{pmatrix} h_\oplus \\ h_\ominus \end{pmatrix}, \begin{pmatrix} h_{\oplus,L} \\ h_{\ominus,L} \end{pmatrix}, \qquad \begin{pmatrix} h^{A\perp}_{\oplus,T} \\ h^{A\perp}_{\ominus,T} \end{pmatrix}, \begin{pmatrix} h^{D\perp}_{\oplus,T} \\ h^{D\perp}_{\ominus,T} \end{pmatrix}.$$

- Real functions = real evolution
- Mixes T-odd and T-even distributions
- Mixing is proportional to s, so T-parity is preserved, and distributions are universal

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TMD distributions of twist-three are *generalized functions*
No definite value at $x_i = 0$, but definite integrals

A typical term in the cross-section

$$\int [dx]\delta(x - x_3)\frac{\Phi_{12}(x_{1,2,3}, b)}{x_2 - is0}\Phi_{11}(-\tilde{x}, -b) + \int [dx]\delta(\tilde{x} - \tilde{x}_1)\Phi_{11}(x, b)\frac{\Phi_{21}(\tilde{x}_{1,2,3}, -b)}{\tilde{x}_2 - is0}$$

- The integral is divergent since $\Phi_\bullet$ is discontinuous at $x_2 = 0$
  - Important: integral from [-1,1], otherwise it would be the end-point divergence
- In fact, divergences cancel
- Let us subtract the divergences and define **PHYSICAL** TMD distribution

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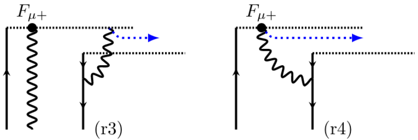
$$\int [dx]\delta(x - x_3)\frac{\Phi_{12}(x_{1,2,3}, b)}{x_2 - is0} \longleftrightarrow \int_{s\infty}^{y} d\sigma\ \Phi_{12}(\{y, \sigma, 0\}, b)$$

- It is the rapidity divergence
- It can be computed

$$\int \frac{\Phi_{12}}{x_2} \sim \ln(\delta^+)\, \partial_\mu \mathcal{D}\, \Phi_{11}$$



Physical TMD distributions of twist-three

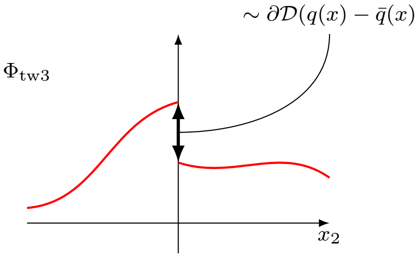
$$\mathbf{\Phi}^{[\Gamma]}_{21,\mu}(x_{1,2,3},b) = \Phi^{[\Gamma]}_{21,\mu}(x_{1,2,3},b) - [\mathcal{R}_{21}\otimes\Phi_{11}]^{[\Gamma]}_{\mu}(x_{1,2,3},b)$$

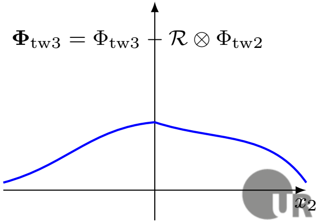
similar for Φ_{12}

- Subtraction term cancels in-between parts of factorized expression
- Obeys same evolution equations
- $\mathcal{R}$ is know at $\mathcal{O}(\alpha_s)$

$$[\mathcal{R}_{21}\otimes\Phi_{11}]^{[\Gamma]}_{\mu}(x_1,x_2,x_3,b) = i\partial_\mu\mathcal{D}(b)\,\Phi^{[\Gamma]}_{11}(-x_1,b)(\theta(x_2,x_3)-\theta(-x_2,-x_3)) + \mathcal{O}(a_s^2),$$

- Produce a *term-by-term-finite* cross-section





Generic distributions of twist-three

good- and bad- component of quark fields

$$\Phi_{\bar{q}q}^{[\Gamma]}(x,b) = \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-ixzp^+} \langle p,s|\bar{q}(zn+b)...\frac{\Gamma}{2}...q(0)\}|p,s\rangle,$$

$\Gamma \in \{\gamma^+, \gamma^+\gamma^5, \sigma^{\alpha+}\}$

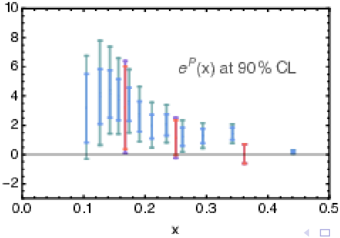
twist-(1,1)

$\Gamma \in \{1, \gamma^5, \gamma_T^\alpha, \gamma_T^\alpha\gamma^5, \sigma^{\alpha\beta}, \sigma^{+-}\}$

indefinite-twist
using EOMs rewrite via
twist-(1,1), twist-(2,1), twist-(1,2)

These are mere **structure functions**
not preserved beyond tree-order, often used by experimentalists

Example of recent extraction from CLAS12 data [Courtoy, et al,2203.14975]



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From generic to genuine TMDs via Equation of motion

$$\Phi_{\bar{q}q}(x,b) \simeq \frac{i}{2xp^+}\left(\frac{\partial}{\partial b^\mu}\Phi_{\text{tw2}}(x,b) + \int [dx]\delta(x-x_3)\frac{\Phi_{\text{tw3}}(x_1,x_2,x_3,b)}{x_2 - i0s}\right)$$

- Involves positive and negative x's

Example

$$\Phi_{\bar{q}q}^{[\gamma^\alpha]}(x,b) = \frac{M}{p^+}\Big[-\epsilon_T^{\alpha\mu}s_{T\mu}f_T(x,b) + i\lambda\epsilon^{\alpha\mu}b_\mu M\, f_L^\perp(x,b) - ib^\alpha M f^\perp(x,b)$$
$$-b^2M^2\left(\frac{g_T^{\alpha\mu}}{2} - \frac{b^\alpha b^\mu}{b^2}\right)\epsilon_{T\mu\nu}s_T^\nu f_T^\perp(x,b)\Big],$$

$$x\,f_T = \mathbf{f}_{\ominus,T}^{(0)} - \mathbf{g}_{\oplus,T}^{(0)} - f_{1T}^\perp - \frac{b^2M^2}{2}\mathring{f}_{1T}^\perp\,, \qquad x\,f^\perp = \mathbf{f}_{\ominus}^{(0)\perp} - \mathbf{g}_{\oplus}^{(0)\perp} + \mathring{f}_1\,,$$

$$x\,f_L^\perp = -\mathbf{f}_{\ominus,L}^{(0)\perp} + \mathbf{g}_{\oplus,L}^{(0)\perp}\,, \qquad x\,f_T^\perp = -\mathbf{f}_{\ominus,T}^{(0)\perp} + \mathbf{g}_{\oplus,T}^{(0)\perp} + \mathring{f}_{1T}^\perp\,,$$

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The evolution equation for *generic* distributions is not closed

$$\mu^2\frac{d}{d\mu^2}\begin{pmatrix} F_+ \\ F_- \end{pmatrix} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\gamma_1\right)\begin{pmatrix} F_+ \\ F_- \end{pmatrix} - \left(\gamma_1+\frac{\gamma_V}{2}\right)\frac{1}{x}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} + \frac{1}{x}\int\frac{[dx]}{x_2}\begin{pmatrix} 2\mathbb{P} & 2\pi s\Theta \\ -2\pi s\Theta & 2\mathbb{P} \end{pmatrix}\begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix},$$

- Pair of generic distributions e.g $\{f^\perp, g^\perp\}$
- $\gamma_1 = a_s C_F + ...$
- twist-two
- mixing with genuine twist-3 **complicated expression**

The evolution equation for *generic* distributions is not closed

$$\mu^2\frac{d}{d\mu^2}\begin{pmatrix}F_+\\ F_-\end{pmatrix} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\gamma_1\right)\begin{pmatrix}F_+\\ F_-\end{pmatrix} - \left(\gamma_1+\frac{\gamma_V}{2}\right)\frac{1}{x}\begin{pmatrix}f_+\\ f_-\end{pmatrix} + \frac{1}{x}\int\frac{[dx]}{x_2}\begin{pmatrix}2\mathbb{P} & 2\pi s\Theta\\ -2\pi s\Theta & 2\mathbb{P}\end{pmatrix}\begin{pmatrix}\Phi_+\\ \Phi_-\end{pmatrix},$$

Pair of generic distributions e.g $\{f^{\perp}, g^{\perp}\}$

$\gamma_1 = a_s C_F + ...$

twist-two

mixing with genuine twist-3 **complicated expression**

$= 0$ in the large-N_c limit

In the large-N_c limit, generic distribution satisfy **closed evolution equation**

Example

$$\mu^2\frac{d}{d\mu^2}f_T = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+a_sC_F\right)f_T - \frac{2a_sC_F}{x}\left(f_{1T}^{\perp}+\frac{b^2M^2}{2}\mathring{f}_{1T}^{\perp}\right)$$

$$\mu^2\frac{d}{d\mu^2}f_L^{\perp} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+a_sC_F\right)f_L^{\perp}\ ,$$

The structure of TMDs at twist-three is totally clear

Main points

- 32 genuine TMDPDFs of twist-three
- Evolution mixes T-odd and T-even functions, in a process-independent way
- Genuine TMD distributions are *generalized functions*
- *Physical distribution* requires extra subtraction
- Evolution of generic TMD distributions is closed in the large-N_c limit.

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