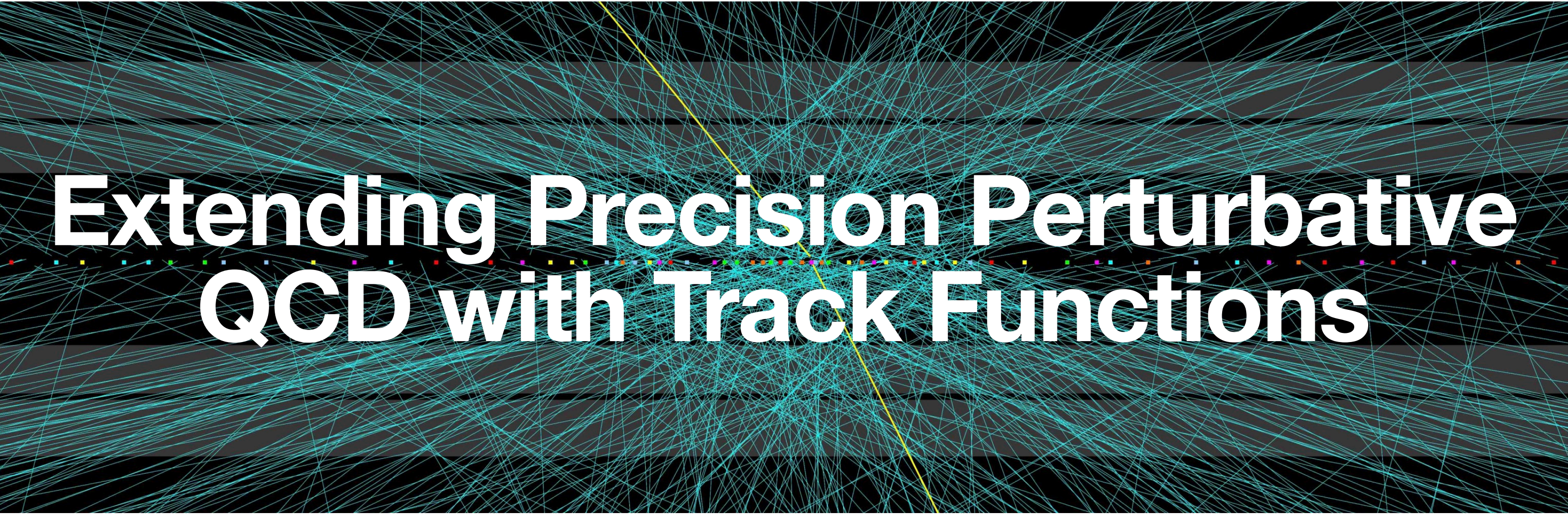
# Extending Precision Perturbative QCD with Track Functions



**Yibei Li**

**Zhejiang University**

Based on:

[YL, Ian Moult, Solange Schrijnder van Velzen, Wouter Waalewijn, HuaXing Zhu: arXiv:2108.01674]
[Max Jaarsma, YL, Ian Moult, Wouter Waalewijn, HuaXing Zhu: arXiv:2201.05166]

# Motivation

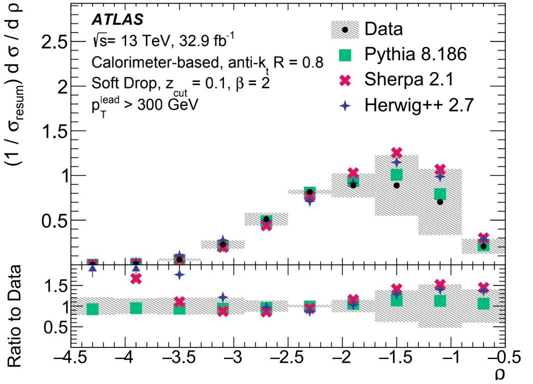
Track-based measurements offer:

- Superior angular resolution
- Pileup mitigation
- One problem: Track-based calculations are not IR safe in perturbation theory.

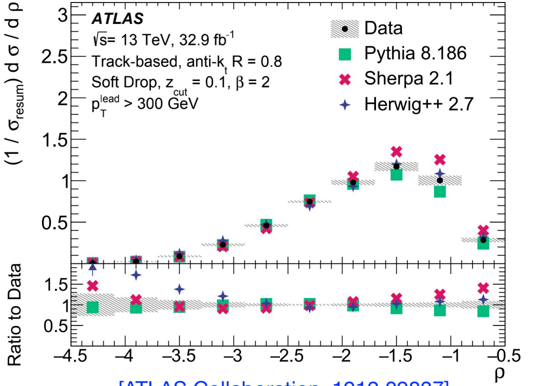
**Track Functions**

- IR divergences are absorbed into **universal** non-perturbative functions.

calorimeter-based
(all-particle)




track-based
(charged-particle)




[ATLAS Collaboration, 1912.09837]

✓ Track functions introduced and studied at $\mathcal{O}(\alpha_s)$.

[H. Chang, M. Procura, J. Thaler, W. Waalewijn, 1303.6637, 1306.6630]

- But complicated:

observables. For all of these observables, the uncertainties for the track-based observables are significantly smaller than those for the calorimeter-based observables, particularly for higher values of $\beta$, where more soft radiation is included within the jet. However, since no track-based calculations exist at the present time, calorimeter-based measurements are still useful for precision QCD studies.

[ATLAS Collaboration, 1912.09837]

the selection of charged particle jets. Note that track-based observables are IRC-unsafe. In general, nonperturbative track functions can be used to directly compare track-based measurements to analytical calculations [67–69]; however, such an approach has not yet been developed for jet angularities. Two techniques are used, described in the following subsections, to apply the nonperturbative corrections.

[ALICE Collaboration, 2107.11303]

only the charged constituents in anti-$k_T$ algorithm ("charged"). While observables computed with both charged and neutral constituents can be described more easily from first-principle calculations, the charged variants can be measured with a better resolution as a result of the high efficiency and precision of the tracking detector.

[CMS Collaboration, 2109.03340]

✓ This talk: Track function formalism beyond leading order.

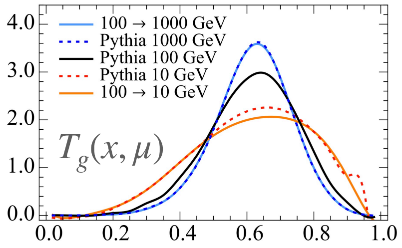
- Energy correlators are much simpler to interface with track functions.
- Moments of track functions have simple evolution.

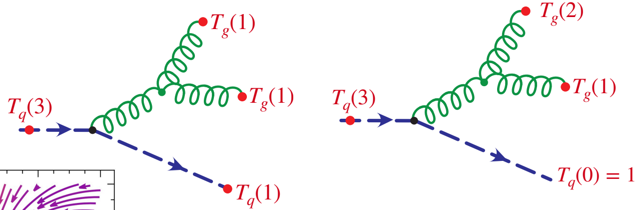
→ higher order calculation

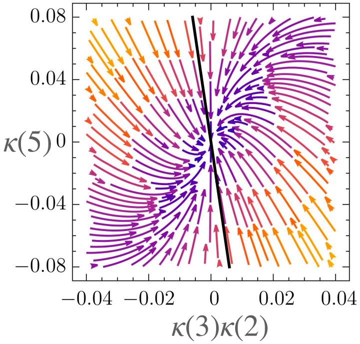
✦ New: Preliminary results for the nonlinear $x$-space evolution at $\mathcal{O}(\alpha_s^2)$.

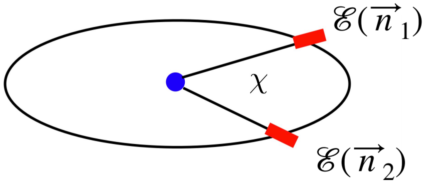
# Outline

- Introduction to Track Functions
    - Definition
    - Incorporating tracks in observables
- Track Function Evolution in Moment Space
    - Results at $\mathcal{O}(\alpha_s^2)$
    - RG flows for the moments
- Predictions for Track EECs
- Full nonlinear $x$-Space Evolution

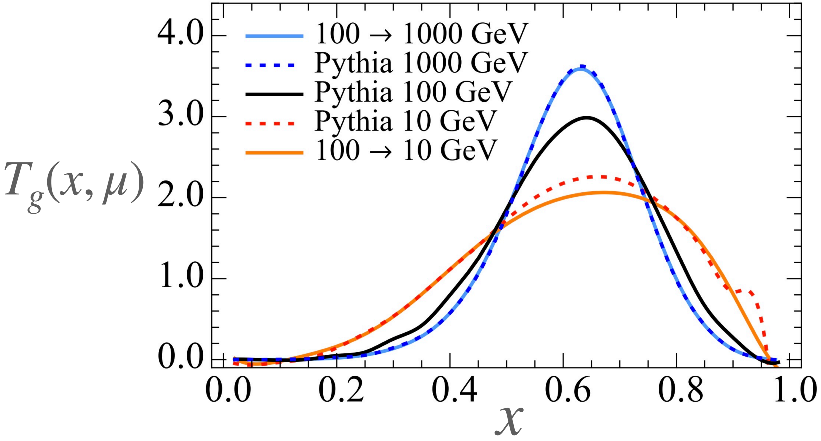


# Introduction to Track Functions



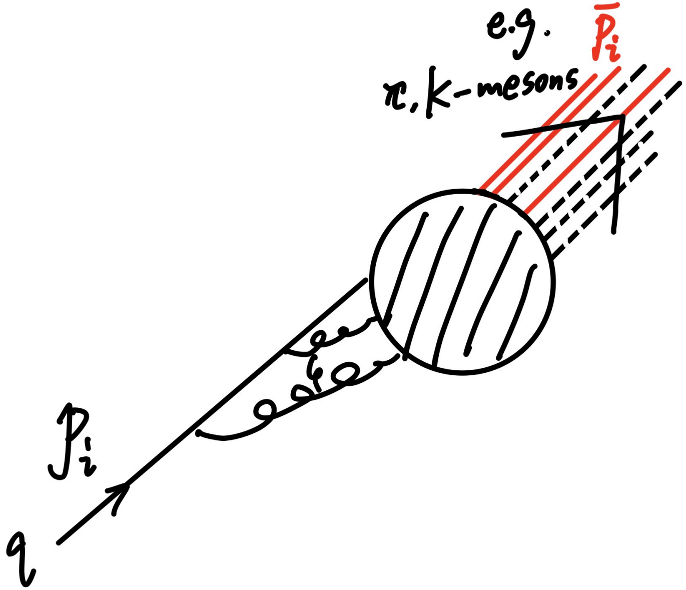
# Track Functions $T_i(x, \mu)$ [H. Chang, M. Procura, J. Thaler, W. Waalewijn, 1303.6637, 1306.6630]

## Definition

- The track function $T_i(x, \mu)$ describes the total momentum fraction $x$ of **all** charged particles (tracks) in a jet initiated by a hard parton $i$.

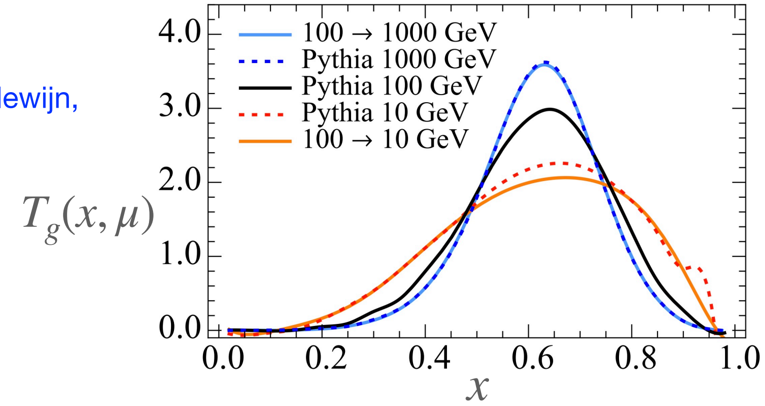
$$\bar{p}_i^{\mu} = x p_i^{\mu} + O(\Lambda_{\text{QCD}}) \;, (0 \leq x \leq 1) \;.$$

- This formalism applies to other subsets of particles (positively-charged, strange, etc).



# Track Functions

## Features [H. Chang, M. Procura, J. Thaler, W. Waalewijn, 1303.6637, 1306.6630]




- A generalization of the fragmentation function (FF).
    - Independent of hard process.
    - Fundamentally non-perturbative, with a calculable scale ($\mu$) dependence.
    - Incorporating correlations between final-state hadrons, like multi-hadron FFs.
    - Sum rule: $\displaystyle\int_0^1 dx\ T_i(x, \mu) = 1\ .$

- The (single-hadron) fragmentation function:
    - The probability of a parton to produce a single-hadron state considered.
    - The momentum sum rule:

$$\sum_h \int_0^1 dz\ z\ D_{i\to h}(z, \mu) = 1\ .$$

# Incorporating Tracks

[1303.6637]

- For a $\delta$-function type observable $e$ measured using partons:

$$\frac{d\sigma}{de} = \sum_N \int d\Pi_N \frac{d\sigma_N}{d\Pi_N} \delta\left[e - \hat{e}(p_i^\mu)\right]$$

$$\Big\downarrow \text{tracks}$$

full functional form of T

$$\frac{d\sigma}{d\bar{e}} = \sum_N \int d\Pi_N \frac{d\bar{\sigma}_N}{d\Pi_N} \int \prod_{i=1}^{N} dx_i T_i(x_i) \delta\left[\bar{e} - \hat{e}(x_i p_i^\mu)\right]$$

- For correlations of energy flow: $k$-point correlation functions

$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\cdots\mathcal{E}(\vec{n}_k)\rangle$$

- An energy flow operator that measures energy flow on a restricted set $R$ of final states: $\mathcal{E}_R$ e.g. charged hadrons
- Then, the $k$-point correlator is

$$\langle \mathcal{E}_R(\vec{n}_1)\mathcal{E}_R(\vec{n}_2)\cdots\mathcal{E}_R(\vec{n}_k)\rangle$$

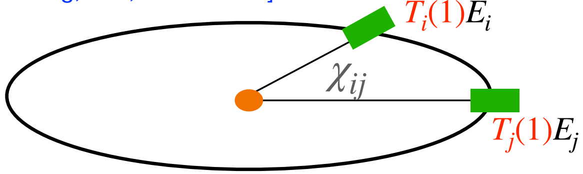
  - This can be related to the partonic-level correlation functions by a factorization formula:

$$\boxed{\begin{aligned}&\langle \mathcal{E}_R(\vec{n}_1)\mathcal{E}_R(\vec{n}_2)\cdots\mathcal{E}_R(\vec{n}_k)\rangle \\ &= \sum_{i_1, i_2, \cdots i_k} T_{i_1}(1)\cdots T_{i_k}(1)\langle \mathcal{E}_{i_1}(\vec{n}_1)\mathcal{E}_{i_2}(\vec{n}_2)\cdots\mathcal{E}_{i_k}(\vec{n}_k)\rangle \\ &\quad + \text{contact terms}\end{aligned}}$$

with dependence on higher moments of T

# Incorporating Tracks

[Chen, Moult, Zhang, Zhu, 2004.11381]




[1303.6637]

- For a $\delta$-function type observable $e$ measured using partons:

$$\frac{d\sigma}{de} = \sum_N \int d\Pi_N \frac{d\sigma_N}{d\Pi_N} \delta\left[e - \hat{e}(p_i^\mu)\right]$$

$\downarrow$ tracks

$$\frac{d\sigma}{d\bar{e}} = \sum_N \int d\Pi_N \frac{d\bar{\sigma}_N}{d\Pi_N} \int \prod_{i=1}^{N} dx_i T_i(x_i) \delta\left[\bar{e} - \hat{e}(x_i p_i^\mu)\right]$$

full functional form of T

- E.g., 2-point correlator (EEC)

$$\frac{d\Sigma}{d\cos\chi} = \sum_{i,j} \int \frac{E_i E_j}{Q^2} \delta\left(\cos\chi - \cos\chi_{ij}\right) d\sigma$$

$$E_i^n \to \int dx_i T_i(x_i) x_i^n E_i^n$$

$$= T_i(n) E_i^n$$

**Mellin moments**

only moments of T

$$\left(\frac{d\Sigma}{d\cos\chi}\right)_{\text{tr}} = \sum_{i\neq j} T_i(1) T_j(1) \int \frac{E_i E_j}{Q^2} \delta\left(\cos\chi - \cos\chi_{ij}\right) d\bar{\sigma}$$

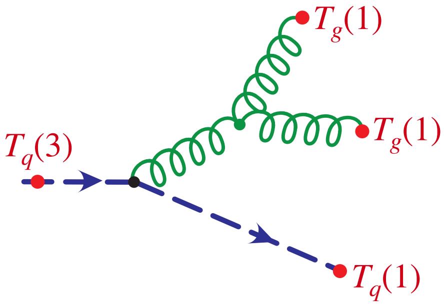
$$+ \sum_k T_k(2) \int \frac{E_k^2}{Q^2} \delta\left(\cos\chi - 1\right) d\bar{\sigma}$$

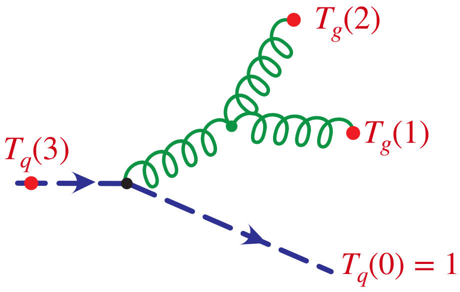
**Track EEC**

- **Energy correlators**: tracking easily included and can use modern fixed-order techniques.

$\longrightarrow$ Study of RG equations for moments of track functions

# Track Function Evolution

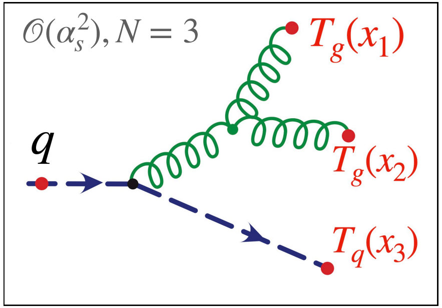


# Track Function Evolution

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = \sum_N \sum_{\{i_f\}} \left[\prod_{m=1}^{N}\int_0^1 \mathrm{d}z_m\right]\delta\left(1-\sum_{m=1}^{N} z_m\right)P_{i\to\{i_f\}}(\{z_f\})$$
$$\times \left[\prod_{m=1}^{N}\int_0^1 \mathrm{d}x_m\, T_{i_m}(x_m)\right]\delta\left(x-\sum_{m=1}^{N} z_m x_m\right)$$

$(i, i_f = g, u, \bar{u}, d, \cdots)$



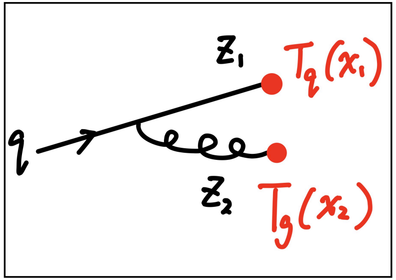

- Nonlinear, involving contributions from all branches of splittings.

- While for fragmentation functions:
  Only one branch observed → Linearity

$$\frac{d}{d\ln\mu^2}D_{i\to h}(z,\mu) = \sum_j D_{j\to h}\otimes P_{ji}^T(z,\mu)$$

- LO evolution:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x,\mu) = a_s(\mu)\sum_{\{jk\}}\int \mathrm{d}z\; P^{(0)}_{i\to jk}(z_1,z_2)\delta(1-z_1-z_2)$$
$$\times \int \mathrm{d}x_1\mathrm{d}x_2 T_j(x_1,\mu)T_k(x_2,\mu)\delta[x-z_1x_1-z_2x_2]\ .$$




Involving contributions from both the branches of the splitting.

# Track Function Evolution

## In Mellin Space

$$\int_0^1 \mathrm{d}x\ x^n \longrightarrow$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = \sum_N \sum_{\{i_f\}} \left[\prod_{m=1}^N \int_0^1 \mathrm{d}z_m\right] \delta\Big(1 - \sum_{m=1}^N z_m\Big) P_{i\to\{i_f\}}(\{z_f\})$$
$$\times \left[\prod_{m=1}^N \int_0^1 \mathrm{d}x_m\, T_{i_m}(x_m)\right] \delta\Big(x - \sum_{m=1}^N z_m x_m\Big)$$

- RG equations for $\mathbf{T} =$

$$\{T_i(n), \cdots, T_{i_1}(k)T_{i_2}(n-k), \cdots, T_{i_1}(1)\cdots T_{i_n}(1)\}^{\mathrm{t}}$$

  - Matrix form: $\dfrac{d}{d\ln\mu^2}\mathbf{T} = \mathbb{R}\mathbf{T}$
  - $\mathbb{R}$: related to moments of timelike splitting functions.

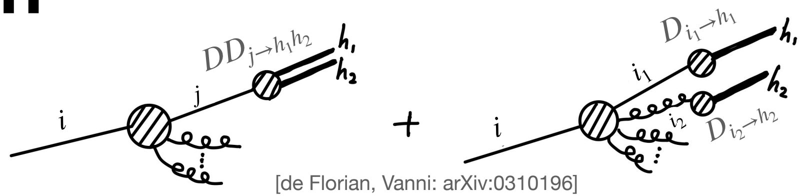
- For fragmentation functions:

$$\frac{d}{d\ln\mu^2}D_{i\to h}(n) = -\sum_j D_{j\to h}(n)\gamma^T_{ji}(n+1)$$

- $\dfrac{d}{d\ln\mu^2}T_i(n) = -\sum_j T_j(n)\ \gamma^T_{ji}(n+1)$+terms of products of lower moments

# Track Function Evolution

## In Mellin Space




[de Florian, Vanni: arXiv:0310196]

- Taking the $n$th moment sets a cutoff at the number of the branches observed, because

$$\int_0^1 dx\ T_i(x) = 1$$

related to

- The evolution equations for $\leq n$-hadron FFs.

- The evolution for $T_i(1)$:

$$\frac{d}{d\ln\mu^2}T_i(1) = -\sum_j T_j(1)\ \gamma^T_{ji}(2)$$

- The evolution for $T_i(2)$:

$$\frac{d}{d\ln\mu^2}T_i(2) = -\sum_j T_j(2)\gamma^T_{ji}(3) + \sum_{i_1,i_2}\mathbb{R}_{i_1,i_2}T_{i_1}(1)T_{i_2}(1)$$

- For single-hadron FFs:

$$T_i(1) \to D_{i\to h}(1)\,,$$

$$\frac{d}{d\ln\mu^2}D_{i\to h}(1) = -\sum_j D_{j\to h}(1)\gamma^T_{ji}(2)$$

- With di-hadron FFs,

$$T_i(2) \to \frac{1}{2}\Big[D_{i\to h_1}(2) + D_{i\to h_2}(2) + 2\ DD_{i\to h_1h_2}(1,1)\Big]$$

# A Surprising Symmetry:

For fragmentation functions:

$$\frac{d}{d\ln\mu^2} D_{i\to h}(z,\mu) = \sum_j D_{j\to h} \otimes P_{ji}(z,\mu)$$

- Scale invariant $D(y) \to D(ay)$ .

- Energy conservation implies the evolution is **shift-symmetric**: $x \to x + a$

$$\frac{d}{d\ln\mu^2} T_i(x+a) = \sum_X \int \left( \prod_m dx_m dz_m T_{i_m}(x_m + a) \right) P_{i\to i_1\cdots i_m\cdots}(\{z_m\})\, \delta\left(1 - \sum_m z_m\right) \delta\left(x - \sum_m x_m z_m\right)$$

- This uniquely fixes the form of the evolution of the first three moments:

$$\frac{d}{d\ln\mu^2}\Delta = [-\gamma_{qq}(2) - \gamma_{gg}(2)]\,\Delta \ ,$$

$$\frac{d}{d\ln\mu^2}\begin{bmatrix}\sigma_g(2)\\ \sigma_q(2)\end{bmatrix} = \begin{bmatrix}-\gamma_{gg}(3) & -\gamma_{qg}(3)\\ -\gamma_{gq}(3) & -\gamma_{qq}(3)\end{bmatrix}\begin{bmatrix}\sigma_g(2)\\ \sigma_q(2)\end{bmatrix} + \begin{bmatrix}\gamma_g^{\Delta^2}\\ \gamma_q^{\Delta^2}\end{bmatrix}\Delta^2 \ ,$$

$$\frac{d}{d\ln\mu^2}\begin{bmatrix}\sigma_g(3)\\ \sigma_q(3)\end{bmatrix} = \begin{bmatrix}-\gamma_{gg}(4) & -\gamma_{qg}(4)\\ -\gamma_{gq}(4) & -\gamma_{qq}(4)\end{bmatrix}\begin{bmatrix}\sigma_g(3)\\ \sigma_q(3)\end{bmatrix} + \begin{bmatrix}\gamma_{gg}^{\sigma\Delta} & \gamma_{qg}^{\sigma\Delta}\\ \gamma_{gq}^{\sigma\Delta} & \gamma_{qq}^{\sigma\Delta}\end{bmatrix}\begin{bmatrix}\sigma_g(2)\\ \sigma_q(2)\end{bmatrix}\Delta + \begin{bmatrix}\gamma_g^{\Delta^3}\\ \gamma_q^{\Delta^3}\end{bmatrix}\Delta^3$$

**shift-invariant** objects:

$$\Delta := T_q(1) - T_g(1)$$
$$\sigma_i(2) := T_i(2) - T_i(1)^2$$
$$\sigma_i(3) := T_i(3) - 3T_i(2)T_i(1) + 2T_i(1)^3$$

Here $\gamma_{ji}(n) = -\int_0^1 dz\ z^{n-1} P_{ji}(z, a_s)$ where $P_{ji}$ denotes the singlet timelike splitting function.

# Methods of Calculation

- **Two independent approaches to extracting the evolution at $\mathcal{O}(\alpha_s^2)$:**

  - Calculating two IR-safe observables modified to measure on tracks. When computed on tracks, they have collinear divergences.
    absorbed by the track functions

  - Guideline: By computing the collinear divergences, we can extract the RG evolution for track functions.

○ (Projected) Energy Correlators ⟷ Agree ⟷ ○ Jet Functions

[Ritzmann, Waalewijn,1407.3272]

**(Projected) Energy Correlators**

- $n$-point correlators on tracks involves moments up to $T(n)$.
- $n$-point track correlator $\xrightarrow[\text{in collinear limit}]{\text{pole cancellation}}$ Evolution for $T(n,\mu)$
  - The evolution for the lower moments can be checked at wide-angle region.

**Jet Functions**

- Directly calculating track jet functions $J(s,x)$ .
- Taking $n$-th moments to extract the evolution for $T(n,\mu)$.

✓ **check** on the track function formalism

# Track Function Evolution at $\mathcal{O}(\alpha_s^2)$

## In Mellin Space

$$\boxed{\Delta_{q_i} = T_{q_i}(1) - T_g(1)}$$

$$\frac{\mathrm{d}\Delta_q}{\mathrm{d}\ln\mu^2} = -\left[\gamma_{qq}(2) + \gamma_{gg}(2)\right]\Delta_q$$

$$\frac{\mathrm{d}\sigma_g(2)}{\mathrm{d}\ln\mu^2} = -\gamma_{gg}^{(1)}(3)\sigma_g(2) + \sum_i \Big\{-\gamma_{qg}^{(1)}(3)\big(\sigma_{q_i}(2) + \sigma_{\bar{q}_i}(2) + \Delta_{q_i}^2 + \Delta_{\bar{q}_i}^2\big)$$
$$+ T_F\Big[\Big(\frac{12413}{1350} - \frac{52}{45}\pi^2\Big)C_A + \frac{1528}{225}C_F - \frac{16}{25}n_f T_F\Big]\Delta_{q_i}\Delta_{\bar{q}_i}\Big\},$$

$$\frac{\mathrm{d}\sigma_g(3)}{\mathrm{d}\ln\mu^2} = -\gamma_{gg}^{(1)}(4)\sigma_g(3) + \sum_i \Big\{-\gamma_{qg}^{(1)}(4)\big(\sigma_{q_i}(3) + \sigma_{\bar{q}_i}(3) + 3\sigma_{q_i}(2)\Delta_{q_i} + 3\sigma_{\bar{q}_i}(2)\Delta_{\bar{q}_i} + \Delta_{q_i}^3 + \Delta_{\bar{q}_i}^3\big)$$
$$+ T_F\Big[\Big(-\frac{638}{45} + \frac{8}{3}\pi^2\Big)C_A - \frac{3803}{250}C_F\Big]\sigma_g(2)(\Delta_{q_i} + \Delta_{\bar{q}_i})$$
$$+ T_F\Big[\Big(\frac{5321}{3000} - \frac{2}{5}\pi^2\Big)C_A + \frac{1523}{240}C_F - \frac{12}{25}n_f T_F\Big]\big(\sigma_{q_i}(2)\Delta_{\bar{q}_i} + \sigma_{\bar{q}_i}(2)\Delta_{q_i} + \Delta_{q_i}^2\Delta_{\bar{q}_i} + \Delta_{\bar{q}_i}^2\Delta_{q_i}\big)\Big\}$$

➡ For the higher moments, there're **three** parts: a linear part fixed by DGLAP, corrections proportional to powers of $\Delta$, and nonlinear terms that are not proportional to powers of $\Delta$.

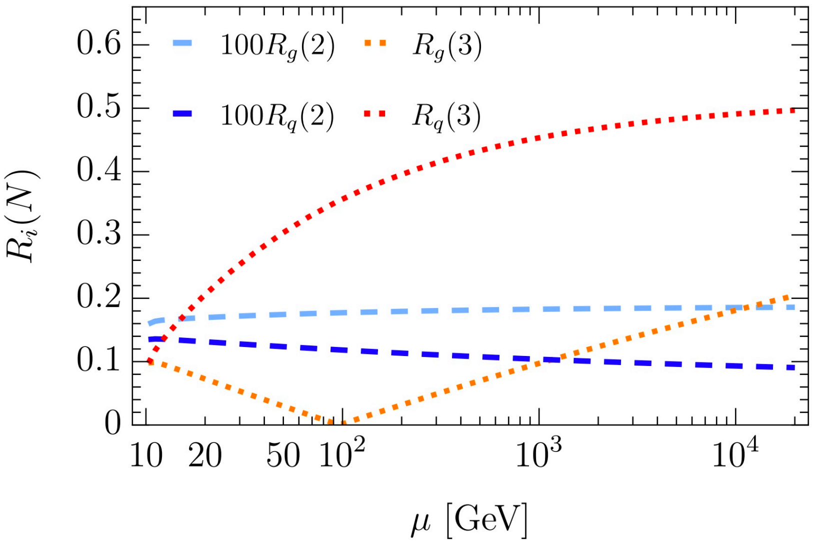
- The evolution for $\Delta$ is fixed by DGLAP to all orders.
- For the second and the third moments, the evolution can be divided into **two** parts: a linear part fixed by DGLAP, and corrections proportional to powers of $\Delta$.

# The Size of $\Delta$ in QCD

For the evolution of $\sigma_i(n)$ $(i = q, g\,;\, n = 2{,}3)$,

$$R_i(n) = \left| \frac{\text{effect of } \Delta\text{-terms at NLO}}{\text{effect of the DGLAP part at NNLO}} \right|$$

- The effect of $\Delta$ on the evolution of the second central moment is much smaller than for the higher moments.
- The $\Delta$ terms are effectively suppressed by one order in the perturbative expansion.




→ **NNLO**

# Non-Linearities
## in the 4th and 5th Moments

**QCD:**

Notation: $\vec{\sigma}(n) = \begin{pmatrix} \sigma^g(n) \\ \sigma^q(n) \end{pmatrix}$ $\hat{\gamma}(n) = \begin{pmatrix} \gamma_{gg}(n) & \gamma_{qg}(n) \\ \gamma_{gq}(n) & \gamma_{qq}(n) \end{pmatrix}$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\vec{\sigma}(4) = -\hat{\gamma}(5)\vec{\sigma}(4) + \hat{\gamma}_{\sigma_2\sigma_2}\left[\vec{\sigma}(2)\cdot\vec{\sigma}^T(2)\right] + \hat{\gamma}_{\sigma_3\Delta}\vec{\sigma}(3)\Delta + \hat{\gamma}_{\sigma_2\Delta^2}\vec{\sigma}(2)\Delta^2 + \vec{\gamma}_{\Delta^4}\Delta^4$$

linear term fixed by DGLAP

**nonlinear terms not**
proportional to powers of $\Delta$

corrections proportional to powers of $\Delta$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\vec{\sigma}(5) = -\hat{\gamma}(6)\vec{\sigma}(5) + \hat{\gamma}_{\sigma_3\sigma_2}\left[\vec{\sigma}(3)\cdot\vec{\sigma}^T(2)\right]$$

$$+ \hat{\gamma}_{\sigma_4\Delta}\vec{\sigma}(4)\Delta + \hat{\gamma}_{\sigma_2^2\Delta}\left[\vec{\sigma}(2)\cdot\vec{\sigma}^T(2)\right]\Delta + \hat{\gamma}_{\sigma_3\Delta^2}\vec{\sigma}(3)\Delta^2 + \hat{\gamma}_{\sigma_2\Delta^3}\vec{\sigma}(2)\Delta^3 + \vec{\gamma}_{\Delta^5}\Delta^5$$

# Non-Linearities
## in the 4th and 5th Moments

**Pure Yang-Mills theory:**

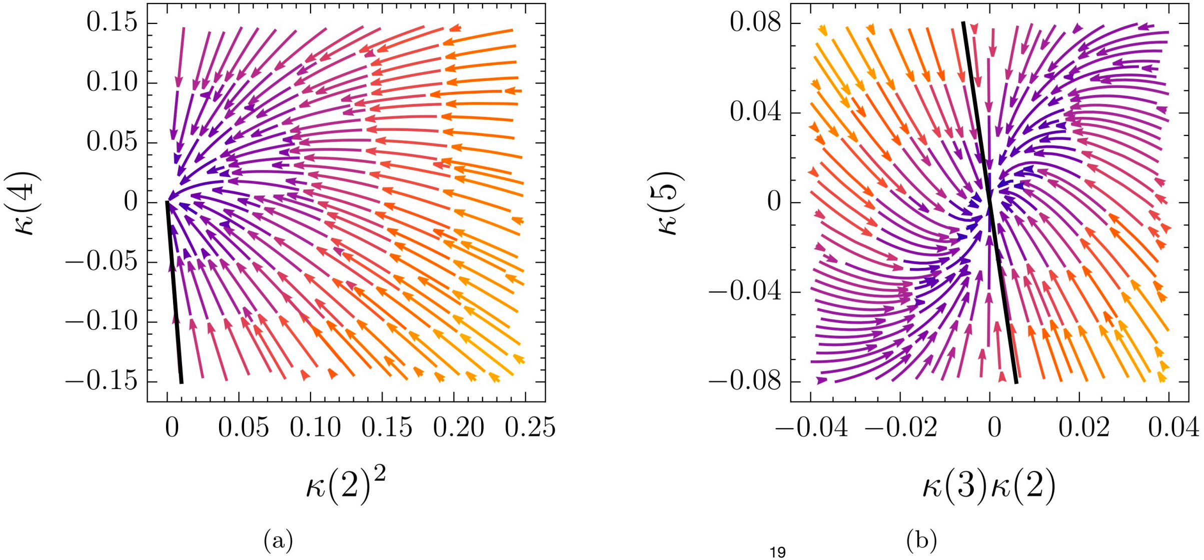
$\kappa$: cumulants

$\kappa(4) = \sigma(4) - 3\sigma^2(2)$

$\kappa(5) = \sigma(5) - 10\sigma(3)\sigma(2)$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\kappa(4) = -\gamma_{gg}(5)\kappa(4) + \gamma_{\kappa_2\kappa_2}\kappa^2(2)$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\kappa(5) = \underbrace{-\gamma_{gg}(6)\kappa(5)}_{\text{linear term fixed by DGLAP}} + \underbrace{\gamma_{\kappa_3\kappa_2}\kappa(3)\kappa(2)}_{\text{nonlinear term}}$$

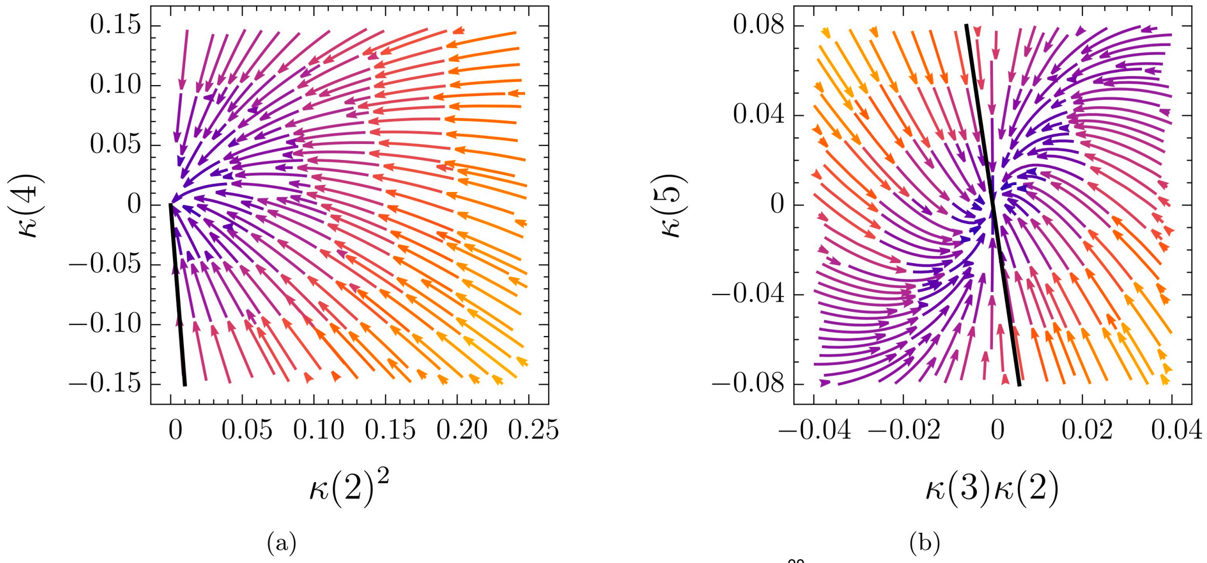


# Non-Linearities

## in the 4th and 5th Moments

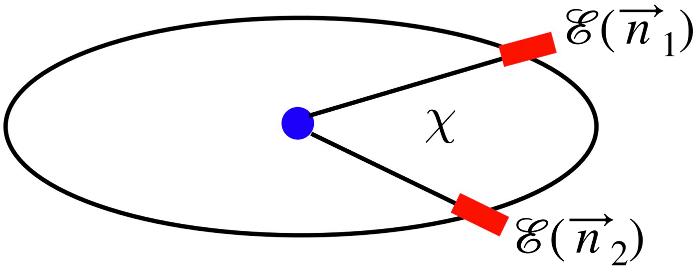
**Pure Yang-Mills theory:**

- Arrows denote the direction of the derivatives with respect to $\ln \mu$.
- A single fixed-point in the evolution at the origin, corresponding to the trivial fixed point where all cumulants vanish.



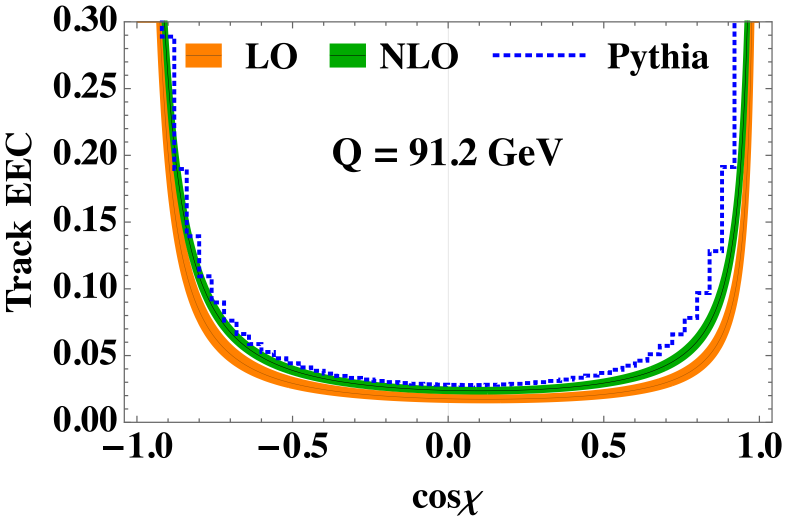

- $\kappa(5) \to -\kappa(5)$, $\kappa(3) \to -\kappa(3)$ symmetry

# Predictions for Track EECs

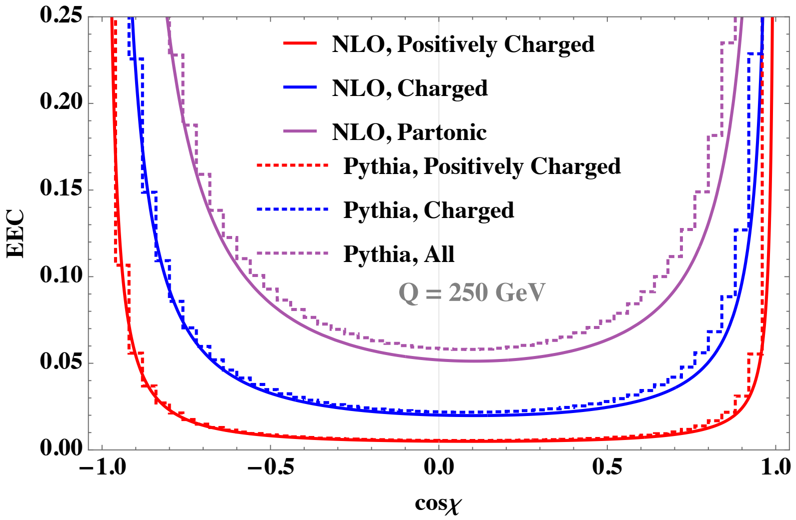


# Track EEC for $e^+e^-$ annihilation

- **First NLO ($\mathcal{O}(\alpha_s^2)$) calculations for track-based observables**
- **Results are available in completely analytical form.**




- Track function formalism can be applied to other subsets of hadrons specified by their quantum numbers.



# Jet Substructure

In the collinear limit:

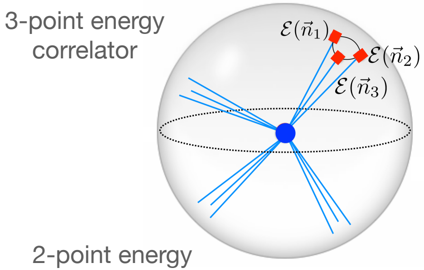
- The energy correlator is a jet observable: $\Sigma(x_L) = \vec{J} \otimes \vec{H}$.
- Jet functions for projected energy correlators on tracks, $\vec{J}_{\mathrm{tr}}\left(\ln \frac{x_L Q^2}{\mu^2}, a_s(\mu)\right)$: Moments $T_i(n,\mu)$ appear as the coefficients.
- The jet function constants (the jet functions with the logarithmic dependence excluded): e.g. for track EECs, up to $\mathcal{O}(\alpha_s^2)$

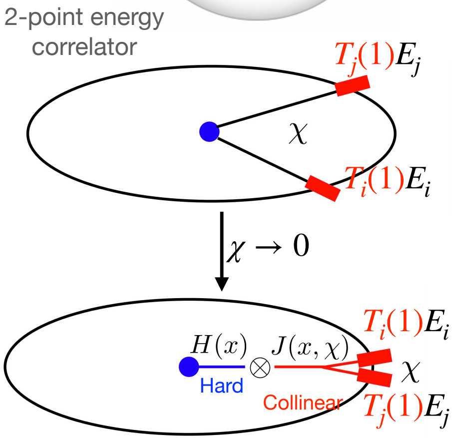
$$j^g = \frac{1}{4}T_g(2) + a_s\left\{T_g(1)T_g(1)C_A\left(-\frac{449}{150}\right) + \sum_q T_q(1)T_{\bar{q}}(1)T_F\left(-\frac{7}{25}\right)\right\}$$

$$+a_s^2\left\{T_g(1)T_g(1)\left\{C_A^2\left(-\frac{527\zeta_3}{10}+\frac{133639871}{3240000}-\frac{2159\pi^2}{1800}+\frac{19\pi^4}{90}\right)+C_A n_f T_F\frac{139}{270}\right\}+\sum_q T_q(1)T_{\bar{q}}(1)\cdots\right\}$$

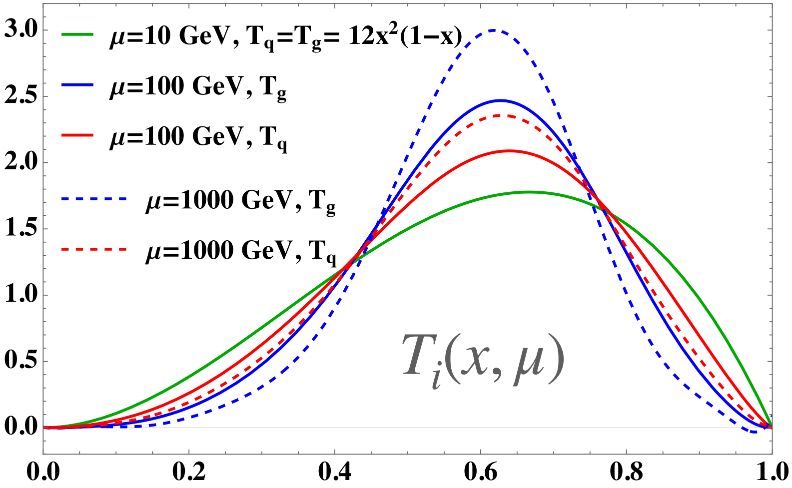
- Matches the state-of-the-art calculation for jet substructure, but now on tracks!

[Kardos, Larkoski, Trocsanyi, 2002.05730]



# Full Non-Linear x-Space Evolution



# Track Jet Functions

In DR: $T_a^{(0)} = T_a^{\text{bare}}$

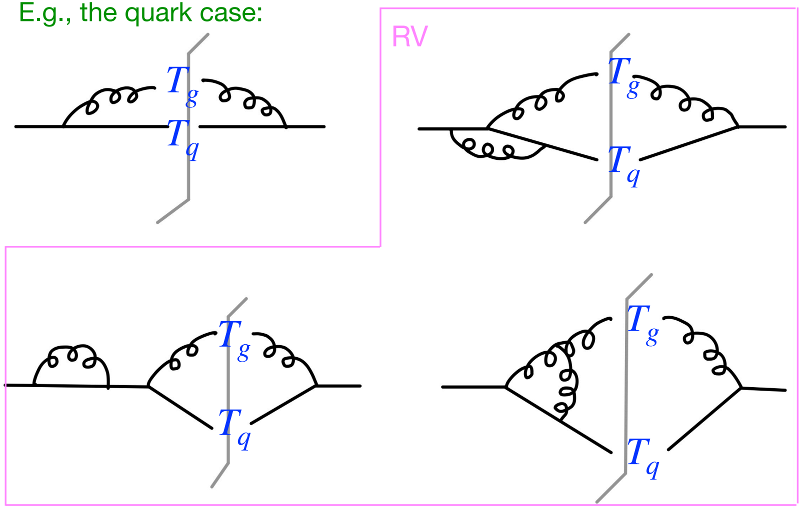
LO track jet function: $J_a^{(0)} = \delta(s) T_a^{(0)}$

## To calculate directly…

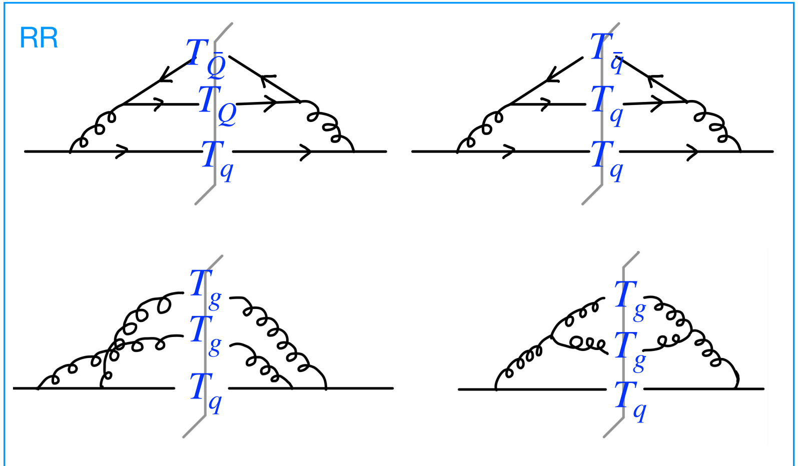
The definition for track jet functions is that

$$J_{a,\mathrm{bare}}(s,x) = \sum_N \sum_{\{a_f\}} \int \mathrm{d}\Phi_N^c\, \delta(s-s')\, \sigma^c_{a\to\{a_f\}}(\{z_f\},\{s_{ff'}\}) \int \left[\prod_{i=1}^N \mathrm{d}x_i T_{a_i}^{(0)}(x_i)\right] \delta\Big(x - \sum_{i=1}^N x_i z_i\Big)$$

E.g., the quark case:




[G. Sborlini, D. Florian, G. Rodrigo: arXiv:1310.6841]




[S. Catani, M. Grazzini: arXiv:hep-ph/9908523]

# Calculation of Track Jet Functions

After integration over angular variables,

$$J_a(s,x) \supset \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \int_0^1 \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\delta(1-z_1-z_2-z_3)P_{a\to a_1a_2a_3}(z_1,z_2,z_3)$$

have not been expanded in $\epsilon$

$$\times\, T_{a_1}^{(0)}(x_1)T_{a_2}^{(0)}(x_2)T_{a_3}^{(0)}(x_3)\delta(x-z_1x_1-z_2x_2-z_3x_3)$$

- For $z_{i_1} < z_{i_2} < z_{i_3}$ $(i_1,i_2,i_3 = 1,2,3)$, do the coordinate transformation

$$z_{i_1} \to \frac{zt}{1+z+zt},\, z_{i_2} \to \frac{z}{1+z+zt},\, z_{i_3} \to \frac{1}{1+z+zt}$$

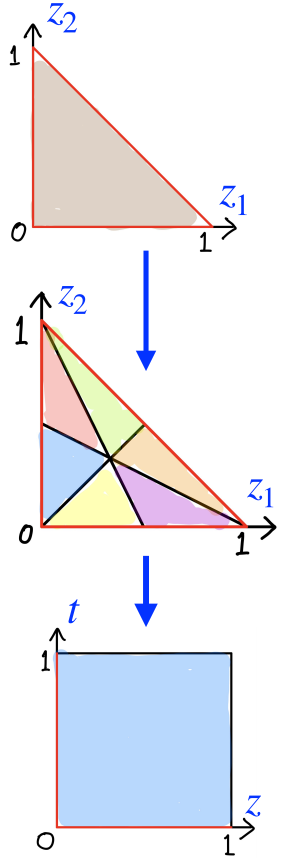
[Sector decomposition (Heinrich, arXiv:0803.4177)]

- For $1 \to n+1$ splitting $P_{1\to n+1}(z_1,z_2,\cdots,z_{n+1})$ , we can set

  $z_{i_1} < z_{i_2} < \cdots < z_{i_n} < z_{i_{n+1}}$ and $t_1 \to \frac{z_{i_1}}{z_{i_2}}, t_2 \to \frac{z_{i_2}}{z_{i_3}}, \cdots, t_n \to \frac{z_{i_n}}{z_{i_{n+1}}}$

  to divide the integration region and then separate the singularities.

  - For $1 \to 2$ splittings, $z_{i_1} \to \frac{z}{1+z}, z_{i_2} \to \frac{1}{1+z}$ for $z_{i_1} < z_{i_2}$ .



# Evolution for Track Functions

## NLO, in $\mathcal{N}=4$ SYM

$a$: t' Hooft coupling constant

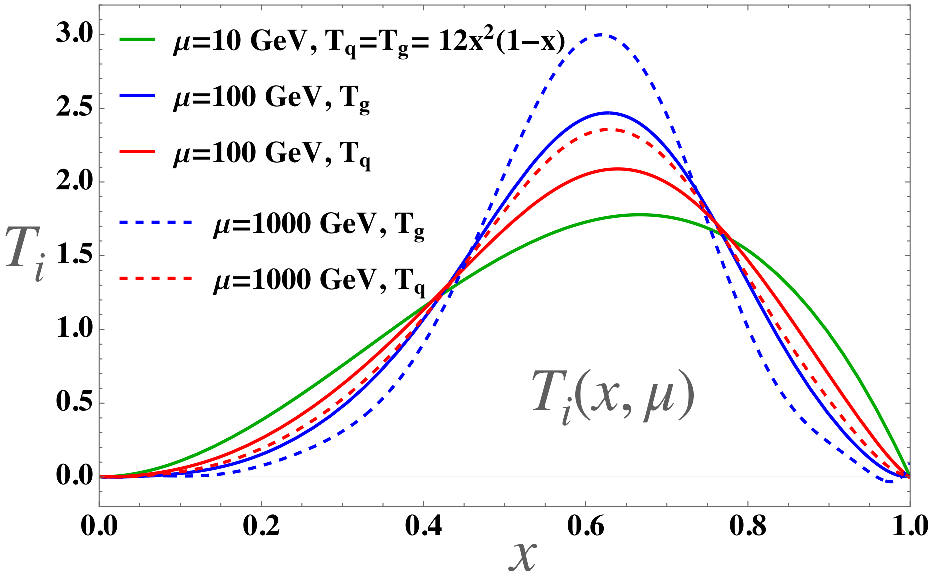
$$
\begin{aligned}
\frac{d}{d\ln\mu^2}T(x) = a^2 \Bigg\{ & \underbrace{-25\zeta_3}_{K^{(1)}_{1\to1}} T(x) + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}z\, T(x_1)T(x_2)\, \delta\left(x - x_1\frac{1}{1+z} - x_2\frac{z}{1+z}\right) \\
& \times \underbrace{\left\{\frac{8}{3}\pi^2\left[\frac{1}{z}\right]_+ + \frac{32\ln^2(z+1)}{z} - \frac{16\ln(z)\ln(z+1)}{z}\right\}}_{K^{(1)}_{1\to2}(z)} \\
& + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}t\, T(x_1)T(x_2)T(x_3) \\
& \times \delta\left(x - x_1\frac{1}{1+z+zt} - x_2\frac{z}{1+z+zt} - x_3\frac{zt}{1+z+zt}\right) \\
& \times 8\left\{\frac{4\ln(1+z)}{z}\left[\frac{1}{t}\right]_+ + \left[\frac{1}{z}\right]_+\left(4\left[\frac{\ln t}{t}\right]_+ - \frac{\ln t}{1+t} - \frac{7\ln(1+t)}{t}\right)\right. \\
& + \frac{2\left[\ln(1+tz) - \ln(1+z+tz)\right]}{(1+t)(1+z)(1+tz)} + \frac{10\left[\ln(1+z+tz) - \ln(1+z)\right]}{tz} \\
& - \frac{7\ln(1+tz)}{tz} + \frac{\ln(1+t) - \ln t}{(1+t)(1+tz)} + \frac{\ln(1+z) + \ln(1+t)}{(1+t)(1+z)} - \frac{\ln(1+z)}{(1+t)z} \\
& \left. - \frac{z\ln(1+z)}{(1+z)(1+tz)} + \frac{\ln(1+tz)}{(1+t)z(1+z)}\right\}\Bigg\}
\end{aligned}
$$

$K^{(1)}_{1\to3}(z,t)$

# Solving RGEs

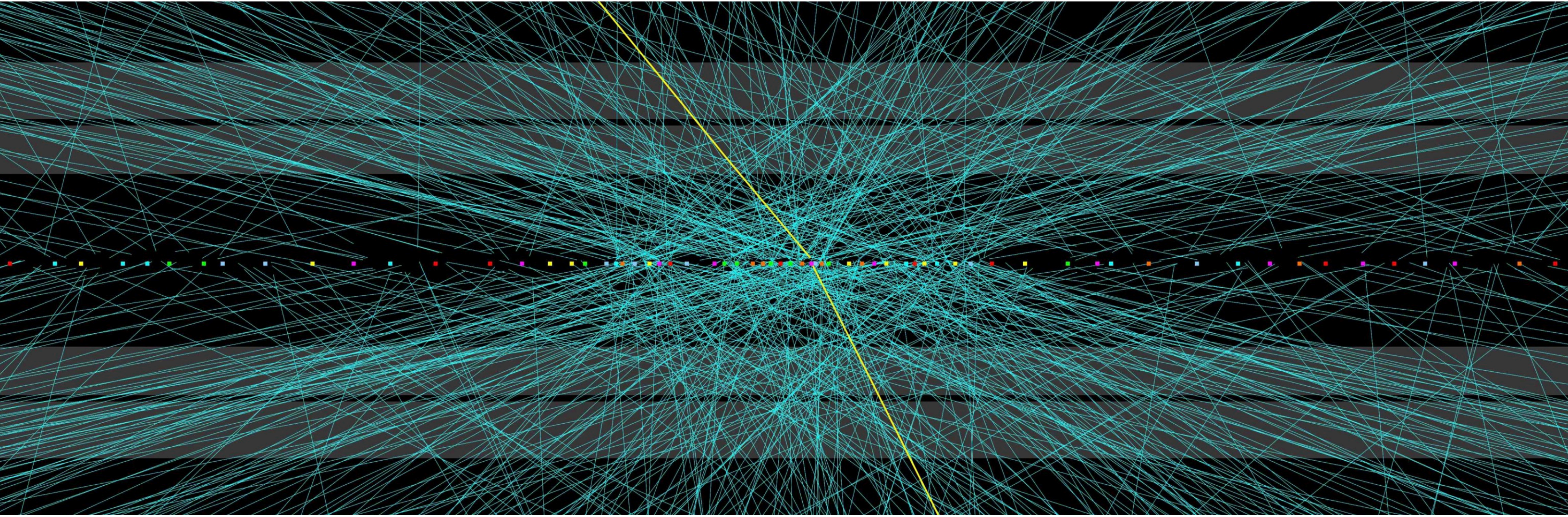
## Numerically, at LO

- **A toy model:** at $\mu = 10$ GeV, $T_i(x) = 12x^2(1-x)$ $(i = q, g)$ of which the first moment is 0.6 ~ that in real world QCD.
- Suppose that the track function at any scale, $T(x, \mu_j)$, can be well described by a polynomial of some degree. $T(x, \mu_j)$ can be restored from a finite number of its moments.



# Summary & Outlook

- Track functions offer a QFT approach to calculating track-based observables:
- Track function formalism studied beyond leading order:
    - Evolution for moments of track functions at $\mathcal{O}(\alpha_s^2)$.
    - Numerical studies on the $\Delta$-terms and the RG flows.
    - Energy correlators interface in a simple manner with tracking information through the moments, allowing for high order calculations.
- Preliminary results for the nonlinear $x$-space evolution at $\mathcal{O}(\alpha_s^2)$
    - Evolution for any moment of the track function is provided.
- This formalism allows IR-safe observables to be computed on any subset of final-state hadrons specified by some particular quantum numbers.



# Thank you!

# Backup

# A Surprising Symmetry:

- $x \to x + a$ leads to $T_j(n) \to \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} T_j(k)$ in Mellin space, e.g.,

$$T_j(1) \to T_j(1) - a \ ,$$
$$T_j(2) \to T_j(2) - 2aT_j(1) + a^2 \ ,$$
$$T_j(3) \to T_j(3) - 3aT_j(2) + 3a^2T_j(1) - a^3$$

- The **shift symmetry** requires that the evolution for moments of track function should *still* hold after the above transformation, which constrains the form of the evolution up to all loop orders: e.g.,

$$\frac{d}{d\ln\mu^2}\left[T_g(1) - a\right] = c_1\left[T_g(1) - a\right] + c_2\left[T_q(1) - a\right] \ ,$$

$$\frac{d}{d\ln\mu^2}\left[T_g(2) - 2aT_g(1) - a^2\right] = b_1\left[T_g(2) - 2aT_g(1) - a^2\right] + b_2\left[T_q(2) - 2aT_q(1) - a^2\right]$$
$$+ y_1\left[T_g(1) - a\right]^2 + y_2\left[T_g(1) - a\right]\left[T_q(1) - a\right] + y_3\left[T_q(1) - a\right]^2$$

# A Surprising Symmetry:

- $x \to x + a$ leads to $T_j(n) \to \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} T_j(k)$ in Mellin space, e.g.,

$$T_j(1) \to T_j(1) - a \ ,$$
$$T_j(2) \to T_j(2) - 2aT_j(1) + a^2 \ ,$$
$$T_j(3) \to T_j(3) - 3aT_j(2) + 3a^2T_j(1) - a^3$$

- The **shift symmetry** requires that the evolution for moments of track function should *still* hold after the above transformation, which constrains the form of the evolution up to all loop orders: e.g.,

$$c_1 + c_2 = 0 \ ,$$

$$\begin{aligned} \frac{d}{d\ln\mu^2} T_g(2) =& b_1 T_g(2) + b_2 T_q(2) + y_1 T_g(1) T_g(1) + y_2 T_g(1) T_q(1) + y_3 T_q(1) T_q(1) \\ & + a\left[(-2b_1 - 2y_1 - y_2 + 2c_1) T_g(1) + (-2b_2 - y_2 - 2y_3 + 2c_2) T_q(1)\right] \\ & + a^2 (b_1 + b_2 + y_1 + y_2 + y_3) \end{aligned}$$

$$= 0$$

# A Surprising Symmetry:

- $x \to x + a$ leads to $T_j(n) \to \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} T_j(k)$ in Mellin space, e.g.,

$$T_j(1) \to T_j(1) - a \ ,$$
$$T_j(2) \to T_j(2) - 2aT_j(1) + a^2 \ ,$$
$$T_j(3) \to T_j(3) - 3aT_j(2) + 3a^2 T_j(1) - a^3$$

- The **shift symmetry** requires that the evolution for moments of track function should *still* hold after the above transformation, which constrains the form of the evolution up to all loop orders: e.g.,

$\Rightarrow$ This implies there is redundancy for these evolution kernels.

- We can use shift invariant objects to reorganize the form of the evolution to avoid this redundancy. (See Page 14.)

Up to this stage we haven't used the Feynman diagram approach.

# Track EECs

- The expression for track EEC has the form:

$$\left(\frac{d\Sigma}{dz}\right)_{\mathrm{tr}} = \sum_{a,b} T_a(1,\mu) T_b(1,\mu) \left\{ \sum_{i\neq j} \delta_{a,i}\delta_{b,j} \int \frac{E_i E_j}{Q^2} \delta\left(z - \frac{1-\cos\chi_{ij}}{2}\right) d\sigma \right\}_{\text{partonic,subtracted}}$$

$$+ \sum_c T_c(2,\mu) \left\{ \sum_k \delta_{ck} \int \frac{E_k^2}{Q^2} \delta\left(z\right) d\sigma \right\}_{\text{partonic,subtracted}}$$

$$\triangleq \sum_{a,b} T_a(1,\mu) T_b(1,\mu)\ \frac{d\hat{\Sigma}_{ab}}{dz} + \sum_c T_c(2,\mu)\ \frac{d\hat{\Sigma}_{c^2}}{dz}\ .$$

# Track EECs

- The expression for track EEC has the form:

$$\left(\frac{\mathrm{d}\Sigma}{\mathrm{d}z}\right)_{\mathrm{tr}} = \frac{\mathrm{d}\vec{\Sigma}}{\mathrm{d}z} \cdot \underbrace{\overbrace{\left[\mathbf{1} + a_s \frac{\widehat{R}_2^{(1)}}{\epsilon} + \frac{1}{2} a_s^2 \left( \frac{\widehat{R}_2^{(2)}}{\epsilon} + \frac{\widehat{R}_2^{(1)} \widehat{R}_2^{(1)} - \beta_0 \widehat{R}_2^{(1)}}{\epsilon^2} \right) + \mathcal{O}(a_s^3) \right]}^{\widehat{\Gamma}_2} \overbrace{\begin{pmatrix} T_g(2) \\ T_{q_1}(2) \\ \cdots \\ T_{q_{n_f-1}}(1) T_{q_{n_f}}(1) \end{pmatrix}}^{\vec{\mathbf{T}}_2(\mu)}}_{\vec{\mathbf{T}}_{2,\mathrm{bare}}}$$

where $\hat{\Gamma}_2^{-1}$ has a similar pole structure to the partonic collinear FF.

$$\frac{d\overrightarrow{\Sigma}}{dz} = \frac{d\overrightarrow{\hat{\Sigma}}}{dz} \cdot \hat{\Gamma}_2^{-1}$$

# $x$-Space Evolution

The evolution up to NLO has the form:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = & a_s \sum_{\substack{\text{tree-level}\\ 1\to 2 \text{ splittings}}} \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}z_1\mathrm{d}z_2 K^{(0)}_{i\to jk}(z_1,z_2)T_j(x_1)T_k(x_2)\delta(1-z_1-z_2)\delta(x-x_1z_1-x_2z_2) \\
& + a_s^2\Bigg[\sum_{\substack{\text{one-loop}\\ 1\to 2 \text{ splittings}}} \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}z_1\mathrm{d}z_2 K^{(1)}_{i\to jk}(z_1,z_2)T_j(x_1)T_k(x_2)\delta(1-z_1-z_2)\delta(x-x_1z_1-x_2z_2) \\
& + \sum_{\substack{\text{tree-level}\\ 1\to 3 \text{ splittings}}} \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 K^{(1)}_{i\to jmn}(z_1,z_2,z_3)T_j(x_1)T_m(x_2)T_n(x_3) \\
& \times \delta(1-z_1-z_2-z_3)\delta(x-x_1z_1-x_2z_2-x_3z_3)\Bigg]
\end{aligned}
$$

where $i, j, k, m, n$ denote parton species (e.g., $q, g$) and $a_s = \alpha_s(\mu)/(4\pi)$ .

# $x$-Space Evolution

For brevity,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2} T_i(x) =& a_s K^{(0)}_{i\to jk} \otimes T_j T_k(x) \\
&+ a_s^2 \left[ K^{(1)}_{i\to jk} \otimes T_j T_k(x) + K^{(1)}_{i\to jmn} \otimes T_j T_m T_n(x) \right]
\end{aligned}
$$

# Track Jet Functions

E.g., The form of the quark jet function on tracks: $\delta(s)$-terms:

$\mathcal{J}$: finite matching coefficients

$$J_{\mathrm{tr},q}(s,\mu) = \delta(s) T_q(x,\mu) + a_s\, \delta(s)\left[\mathcal{J}^{(1)}_{q\to q} T_q(x,\mu) + \mathcal{J}^{(1)}_{q\to qg}\otimes T_q T_g(x,\mu)\right]$$

$$+ a_s^2\, \delta(s)\left[\mathcal{J}^{(2)}_{q\to q} T_q(x,\mu) + \mathcal{J}^{(2)}_{q\to qg}\otimes T_q T_g(x,\mu) + \mathcal{J}^{(2)}_{q\to qgg}\otimes T_q T_g T_g(x,\mu) + \cdots\right]$$

$$J_{\mathrm{tr},q}(s,\mu) = \delta(s) T_q^{(0)}(x) + a_s \delta(s)\left\{\boxed{-\frac{1}{\epsilon}} K^{(0)}_{q\to qg}\otimes T_q^{(0)} T_g^{(0)} + \mathcal{J}^{(1)}_{q\to q} T_q^{(0)} + \mathcal{J}^{(1)}_{q\to qg}\otimes T_q^{(0)} T_g^{(0)}\right\}$$

LO evolution kernel

$$+ a_s^2 \delta(s)\Big\{\mathcal{J}^{(2)}_{q\to q} T_q^{(0)} + \mathcal{J}^{(2)}_{q\to qg}\otimes T_q^{(0)} T_g^{(0)} + \mathcal{J}^{(2)}_{q\to qgg}\otimes T_q^{(0)} T_g^{(0)} T_g^{(0)} + \mathcal{J}^{(2)}_{q\to qq\bar{q}}\otimes T_q^{(0)} T_q^{(0)} T_{\bar{q}}^{(0)} + \sum_q \mathcal{J}^{(2)}_{q\to qQ\bar{Q}}\otimes T_q^{(0)} T_Q^{(0)} T_{\bar{Q}}^{(0)}$$

$$+ \boxed{\left(-\frac{1}{2\epsilon}\right)}\left\{K^{(1)}_{q\to qgg}\otimes T_q^{(0)} T_g^{(0)} T_g^{(0)} + K^{(1)}_{q\to qq\bar{q}}\otimes T_q^{(0)} T_q^{(0)} T_{\bar{q}}^{(0)} + \sum_{Q\neq q} K^{(1)}_{q\to qQ\bar{Q}}\otimes T_q^{(0)} T_Q^{(0)} T_{\bar{Q}}^{(0)} + K^{(1)}_{q\to qg}\otimes T_q^{(0)} T_g^{(0)}\right\}$$

RHS of the evolution for $T_q(x)$ at NLO with $T^{(0)}$ in place of $T(x,\mu)$

$$+ \boxed{\left(-\frac{1}{\epsilon}\right)}\Big\{\mathcal{J}^{(1)}_{q\to q} K^{(0)}_{q\to qg}\otimes T_q^{(0)} T_g^{(0)}(x) + \mathcal{J}^{(1)}_{q\to qg}\otimes \Big[T_g^{(0)}\left(K^{(0)}_{q\to qg}\otimes T_q^{(0)} T_g^{(0)}\right)$$

$$+ T_q^{(0)}\left(K^{(0)}_{g\to gg}\otimes T_g^{(0)} T_g^{(0)} + \sum_q K^{(0)}_{g\to q\bar{q}}\otimes T_q^{(0)} T_{\bar{q}}^{(0)}\right)\Big]\Big\} + \boxed{\frac{1}{2\epsilon^2}}\Big\{\underbrace{K^{(0)}_{q\to ab}\otimes\left[T_b^{(0)}\left(K^{(0)}_{a\to a_1 a_2}\otimes T_{a_1}^{(0)} T_{a_2}^{(0)}\right)\right]}_{+\,\beta_0 K^{(0)}_{q\to qg}\otimes T_q^{(0)} T_g^{(0)}}\Big\}\Big\}$$