Dissecting the collinear structure of quark splitting at NNLL

Basem Kamal El-Menoufi - The University of Manchester

Work based on 2109.07496 with **M. Dasgupta** (see also 2007.10355)

SCET2022 - University of Bern

A Bird's-Eye view

♠ (Semi)-analytic resummation has achieved an impessive accuracy (NNLL and N^{3}LL) over previous decades.

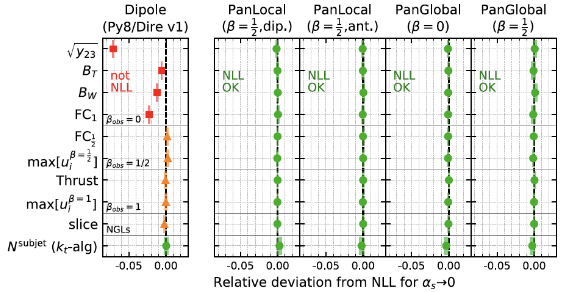
$1-T$	0803.0342,1006.3080,1105.4560
ρ_H	1005.1644
B_T, B_W	1210.0580
C-parameter	1411.6633
EEC	hep-ph/0407241,1708.04093,1801.02627
Angularities	1806.10622,1807.11487
D-parameter	1912.09341

♠ Parton showers (PS) have not kept up with such progress.

♠ PS are essential due to their versatility: It is much more efficient to simulate QCD dynamics than to resum a specific observable.

Motivation: Recent progress in NLL accurate PS

♠ The PanScales family of PS has been able to achieve NLL accuracy for any recursive IRC safe observable:[1]



[1] Dasgupta et. al. (2002.11114), color and spin (2011.10054,2103.16526,2111.01161), G. Salam "The power and limits of parton showers" (https://gsalam.web.cern.ch/gsalam/talks/repo/202109-SLAC-seminar-SLAC-panscales-seminar.pdf)

Outline

1. What do we need to achieve NNLL? Introduction to B_2^q

Outline

1. What do we need to achieve NNLL? Introduction to B_2^q
2. Triple-collinear splitting functions

Outline

1. What do we need to achieve NNLL? Introduction to B_2^q
2. Triple-collinear splitting functions
3. The physical coupling beyond the soft limit

Outline

1. What do we need to achieve NNLL? Introduction to B_2^q
2. Triple-collinear splitting functions
3. The physical coupling beyond the soft limit
4. Extracting the differential $\mathcal{B}_2^q(z)$

Outline

1. What do we need to achieve NNLL? Introduction to B_2^q
2. Triple-collinear splitting functions
3. The physical coupling beyond the soft limit
4. Extracting the differential $\mathcal{B}_2^q(z)$
5. Outlook

Look back at NLL

- Over 30 years ago Catani, Marchesini & Webber introduced the notion of a soft physical coupling:

$$\mathrm{d}\mathcal{P}_{\mathrm{sc}} = C_i \frac{\alpha_s^{\mathrm{phys}}}{\pi} \frac{\mathrm{d}k_t^2}{k_t^2} \frac{\mathrm{d}z}{1-z}, \quad \alpha_s^{\mathrm{phys}} = \alpha_s(k_t^2)\left(1 + K_{\mathrm{CMW}} \frac{\alpha_s(k_t^2)}{2\pi}\right)$$

- The CMW coupling represents the intensity of soft gluon radiation.

$$K_{\mathrm{CMW}} = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_F$$

- For showers that interwine real and virtual corrections through unitarity, specifying the (CMW) scheme and scale of the coupling is the sole NLO ingredient to achieve NLL accuracy.

Questions for NNLL PS

- ♠ What is the scale of the coupling beyond the soft limit?

$$k_t^2 \to k_t^2 * f(z), \quad f(z) = ?$$

- ♠ The *inclusive* limit of the double-soft function defines the CMW coupling. Can we furnish a commensurate understanding of the triple-collinear splitting functions?
- ♠ What is the underlying physics of the coefficient B_2^q? Can we define a suitable differential version thereof?
- ♠ Can we extend the notion of the *web* beyond the soft limit?

Introduction into B_2^q

- So what exactly is B_2^q?
- Let us take an example from the transverse momentum distribution in hadronic collisions:[2]

$$\frac{\mathrm{d}\sigma_{ab\to F}}{\mathrm{d}p_t^2} = \frac{1}{2}\int b\,\mathrm{d}b J_0(bp_t)\, W_{ab}^F(s,Q,b)$$

- The interesting piece is the function $W_{ab}^F(s,Q,b)$, which includes the quark/gluon *form factor*:

$$S_{q/g}(Q,b) = \exp\left(-\int_{b_0^2/b^2}^{Q^2} \frac{\mathrm{d}q^2}{q^2}\left[A_{q/g}(\alpha_s)\ln\frac{Q^2}{q^2} + B_{q/g}(\alpha_s)\right]\right)$$

[2] de Florian & Grazzini `hep-ph/0108273` (see also the references therein)

Introduction into B_2^q

♠ Each function has a perturbative expansion. The A function has soft origin, while the B function has a hard-collinear origin.

$$A_{q/g} = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n A_{(n)}^{q/g}, \quad B_{q/g} = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n B_{(n)}^{q/g}$$

♠ Let us focus on the B series. Going back to direct space, one finds a "hard-collinear" logarithm:

$$\left(\frac{\alpha_s}{2\pi}\right) B_1^{q/g} \quad || \quad \left(\frac{\alpha_s}{2\pi}\right)^2 B_2^{q/g}$$

This talk is about B_2^q and a suitably defined differential version $\mathcal{B}_2^q(z)$.

Introduction into B_2^q

- What do we know about the structure of B_2^q?
- In $e^+e^- \to$ hadrons, there exists a complete framework to resum any recursive IRC (global) observable up to NNLL accuracy - ARES.[2]
- For any such observable, we have:[3]

$$B_2^q = -\gamma_q^{(2)} + C_F b_0 X_v, \quad b_0 = \frac{11}{6} C_A - \frac{2}{3} T_R n_f$$

- We have two pieces. First, an observable-dependent constant, X_v, that comes multiplied by b_0. The other piece, $\gamma_q^{(2)}$, is universal and represents the endpoint contribution, i.e. $\delta(1-x)$, to the NLO non-singlet DGLAP kernel obtained from sum rules.[4]

---

[2] Banfi, BKE & Monni 1807.11487, Banfi et. al. 1412.2126

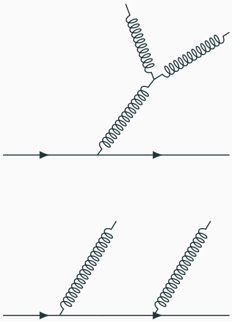
[3] See also `hep-ph/0407241, Davies & Striling Nucl.Phys.B 244 (1984)`

[4] Ellis et. al. "QCD and Collider Physics"

Triple collinear splitting functions

At NLO, we have four different splittings:[5]

- ♠ $q \to g_1 g_2 q_3$



[5] Catani & Grazzini hep-ph/9810389

Triple collinear splitting functions

♠ Therefore, we end up with abelian, C_F^2, and non-abelian, $C_F C_A$, pieces:

$$\langle \hat{P}_{g_1 g_2 q_3} \rangle = C_F^2 \langle \hat{P}^{(\mathrm{ab})}_{g_1 g_2 q_3} \rangle + C_F C_A \langle \hat{P}^{(\mathrm{nab})}_{g_1 g_2 q_3} \rangle$$

♠ These are functions of the invariant masses $s_{ij} \simeq z_i \, z_j \, \theta_{ij}$, where z_i is the light-cone momentum fraction of parton i.

$$\begin{aligned}
\langle \hat{P}^{(\mathrm{ab})}_{g_1 g_2 q_3} \rangle = \Bigg\{ & \frac{s_{123}^2}{2 s_{13} s_{23}} z_3 \left[\frac{1+z_3^2}{z_1 z_2} - \epsilon \frac{z_1^2 + z_2^2}{z_1 z_2} - \epsilon(1+\epsilon) \right] \\
& + \frac{s_{123}}{s_{13}} \left[\frac{z_3(1-z_1) + (1-z_2)^3}{z_1 z_2} + \epsilon^2 (1+z_3) - \epsilon (z_1^2 + z_1 z_2 + z_2^2) \frac{1-z_2}{z_1 z_2} \right] \\
& + (1-\epsilon) \left[\epsilon - (1-\epsilon) \frac{s_{23}}{s_{13}} \right] \Bigg\} + (1 \leftrightarrow 2)
\end{aligned}$$

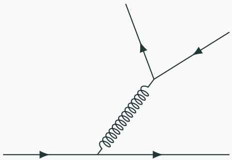
Triple collinear splitting functions

$$
\begin{aligned}
\langle \hat{P}^{(\mathrm{nab})}_{g_1 g_2 q_3} \rangle = & \Bigg\{ (1-\epsilon)\left(\frac{t^2_{12,3}}{4 s^2_{12}} + \frac{1}{4} - \frac{\epsilon}{2} \right) \\
& + \frac{s^2_{123}}{2 s_{12} s_{13}} \left[\frac{(1-z_3)^2(1-\epsilon) + 2 z_3}{z_2} + \frac{z_2^2(1-\epsilon) + 2(1-z_2)}{1-z_3} \right] \\
& - \frac{s^2_{123}}{4 s_{13} s_{23}} z_3 \left[\frac{(1-z_3)^2(1-\epsilon) + 2 z_3}{z_1 z_2} + \epsilon(1-\epsilon) \right] \\
& + \frac{s_{123}}{2 s_{12}} \left[(1-\epsilon) \frac{z_1(2 - 2 z_1 + z_1^2) - z_2(6 - 6 z_2 + z_2^2)}{z_2(1-z_3)} + 2\epsilon \frac{z_3(z_1 - 2 z_2) - z_2}{z_2(1-z_3)} \right] \\
& + \frac{s_{123}}{2 s_{13}} \Bigg[(1-\epsilon) \frac{(1-z_2)^3 + z_3^2 - z_2}{z_2(1-z_3)} - \epsilon \left(\frac{2(1-z_2)(z_2 - z_3)}{z_2(1-z_3)} - z_1 + z_2 \right) \\
& - \frac{z_3(1-z_1) + (1-z_2)^3}{z_1 z_2} + \epsilon(1-z_2)\left(\frac{z_1^2 + z_2^2}{z_1 z_2} - \epsilon \right) \Bigg] \Bigg\} + (1 \leftrightarrow 2)
\end{aligned}
$$

Triple collinear splitting functions

At NLO, we have four different splittings:

♠ $q \to q'_1 \bar{q}'_2 q_3$ ♠ $q \to q_1 \bar{q}_2 q_3$



Triple collinear splitting functions

♠ Therefore, we end up with two structures. Summing over flavours:

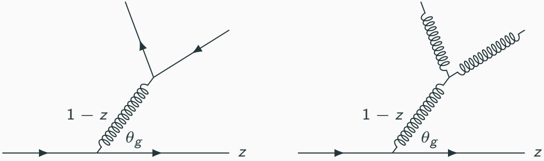
$$\sum_f \langle \hat{P}_{q_1^f \bar{q}_2^f q_3} \rangle = n_f \langle \hat{P}_{q_1' \bar{q}_2' q_3} \rangle + \langle \hat{P}^{(\mathrm{id})}_{q_1 \bar{q}_2 q_3} \rangle$$

$$\langle \hat{P}_{q_1' \bar{q}_2' q_3} \rangle = \frac{1}{2} C_F T_R \frac{s_{123}}{s_{12}} \left[-\frac{t_{12,3}^2}{s_{12} s_{123}} + \frac{4z_3 + (z_1 - z_2)^2}{z_1 + z_2} + (1 - 2\epsilon) \left(z_1 + z_2 - \frac{s_{12}}{s_{123}} \right) \right]$$

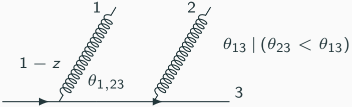
$$\begin{aligned}
\langle \hat{P}^{(\mathrm{id})}_{q_1 \bar{q}_2 q_3} \rangle = C_F \left(C_F - \frac{1}{2} C_A \right) \Bigg\{ & (1-\epsilon) \left(\frac{2 s_{23}}{s_{12}} - \epsilon \right) + \frac{s_{123}}{s_{12}} \left[\frac{1 + z_1^2}{1 - z_2} - \frac{2 z_2}{1 - z_3} \right. \\
& \left. - \epsilon \left(\frac{(1 - z_3)^2}{1 - z_2} + 1 + z_1 - \frac{2 z_2}{1 - z_3} \right) - \epsilon^2 (1 - z_3) \right] \\
& - \frac{s_{123}^2}{s_{12} s_{13}} \frac{z_1}{2} \left[\frac{1 + z_1^2}{(1 - z_2)(1 - z_3)} - \epsilon \left(1 + 2 \frac{1 - z_2}{1 - z_3} \right) - \epsilon^2 \right] \Bigg\} + (2 \leftrightarrow 3)
\end{aligned}$$

Road map

- What variables do we fix?
- Gluon decay:



- Gluon emission:

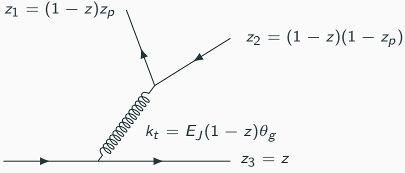


Gluon decay: web variables

♠ To obtain an analytic handle on the integrals, we express the triple collinear phase space as follows:

$$d\Phi_{1\to 3}^{\text{web}} = \frac{(4\pi)^{2\epsilon}}{256\pi^4} \frac{2z^{1-2\epsilon}dz}{1-z} \frac{1}{\Gamma(1-\epsilon)} \frac{d^{2-2\epsilon}k_\perp}{\Omega_{2-2\epsilon}} \frac{ds_{12}}{(s_{12})^\epsilon} \frac{dz_p}{(z_p(1-z_p))^\epsilon} \frac{1}{\Gamma(1-\epsilon)} \frac{d\Omega_{2-2\epsilon}}{\Omega_{2-2\epsilon}}$$

♠ The meaning of different variables is as follows:



♠ The invariant masses (s_{13}, s_{23}) can be readily expressed in terms of these variables.

The θ_g distribution: $C_F T_R n_f$

♠ Using the web variables the computation is quite manageable:

$$\left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\theta_g^2\,dz}\right)^{C_F T_R n_f} = C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 z^{-3\epsilon}\left((1-z)^2\theta_g^2\right)^{-2\epsilon} \left(-\frac{2}{3\epsilon}p_{qq}(z,\epsilon) - \frac{10}{9}p_{qq}(z) - \frac{2}{3}(1-z)\right)$$

♠ Due to the angular ordering property built into the splitting function, we can send the invariant mass to infinity:

$$\text{max.}\{s_{12}\} \to \infty$$

♠ The virtual corrections of $1 \to 2$ splitting is quite simple for this colour structure:

$$\left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}_{\text{virt.}}}{d\theta_g^2\,dz}\right)^{C_F T_R n_f} = C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 z^{-2\epsilon}(1-z)^{-2\epsilon}\left(\theta_g^2\right)^{-\epsilon}\left(\frac{2}{3\epsilon}p_{qq}(z,\epsilon)\right)$$

The θ_g distribution: $C_F T_R n_f$

♠ The final result then reads:

$$\left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\theta_g^2\, dz}\right)^{C_F T_R n_f} = C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{1+z^2}{1-z}\left(\frac{2}{3}\ln\left(z(1-z)^2\theta_g^2\right) - \frac{10}{9}\right) - \frac{2}{3}(1-z)\right)$$

♠ One can also compute the ρ distribution ($\rho = s_{123}/E^2$):

$$\left(\frac{\rho}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\rho\, dz}\right)^{C_F T_R n_f} = C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{1+z^2}{1-z}\left(\frac{2}{3}\ln\left((1-z)\rho\right) - \frac{10}{9}\right) - \frac{2}{3}(1-z)\right)$$

♠ We immediately observe a remarkable property. One can move between both distributions using the LO relation:

$$\rho = z(1-z)\theta_g^2$$

Extracting $\mathcal{B}_2^q(z)$: $C_F T_R n_f$

♠ To zoom on the NNLL structure, we need to subtract off the LL & NLL (soft-enhanced) structures:

$$C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 \left[\frac{2}{1-z}\left(\frac{2}{3}\ln\left((1-z)^2\theta_g^2\right) - \frac{10}{9}\right) - \frac{2}{3}(1+z)\ln\theta_g^2\right]$$

♠ Now we have a purely collinear object:

$$\mathcal{B}_2^{q,n_f}(z;\theta_g^2) = \left(\frac{1+z^2}{1-z}\,\frac{2}{3}\ln z - (1+z)\left(\frac{2}{3}\ln(1-z)^2 - \frac{10}{9}\right) - \frac{2}{3}(1-z)\right)$$

♠ Integrating over z one finds:

$$B_2^{q,\theta_g^2,n_f} = C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 \int_0^1 dz\, \mathcal{B}_2^{q,n_f}(z;\theta_g^2) = -\gamma_q^{(2,n_f)} + C_F b_0^{(n_f)} X_{\theta_g^2}$$

♠ One can surely play the same game with the ρ distribution:

$$X_\rho = \frac{\pi^2}{3} - \frac{7}{2}, \quad X_{\theta_g^2} = \frac{2\pi^2}{3} - \frac{13}{2}$$

The θ_g distribution: $C_F(C_F - C_A/2)$

- Here, the full structure contributes at NNLL.
- The web variables allows an analytic evaluation:

$$\left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\theta_g^2\,dz}\right)^{(\mathrm{id.})} = C_F\left(C_F - \frac{C_A}{2}\right)\left(\frac{\alpha_s}{2\pi}\right)^2 \left[\left(4z - \frac{7}{2}\right) + \frac{5z^2-2}{2(1-z)}\ln z + \frac{1+z^2}{1-z}\left(\frac{\pi^2}{6} - \ln z\ln(1-z) - \mathrm{Li}_2(z)\right)\right]$$

- Thus it is straightforward to extract $\mathcal{B}_2^q(z)$:

$$\mathcal{B}_2^{q,(\mathrm{id.})}(z) = \left(4z - \frac{7}{2}\right) + \frac{5z^2-2}{2(1-z)}\ln z + \frac{1+z^2}{1-z}\left(\frac{\pi^2}{6} - \ln z\ln(1-z) - \mathrm{Li}_2(z)\right)$$

- This function is regular as $z \to 1$, and its integral reads:

$$\int_0^1 dz\,\mathcal{B}_2^{q,(\mathrm{id.})}(z) = \frac{13}{4} - \frac{\pi^2}{2} + 2\zeta_3$$

The θ_g distribution: non-abelian channel

- The non-abelian channel is the most tedious to compute. The web variables allow for an anlaytic computation:

$$\left(\frac{\rho}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\rho\, dz}\right)^{\text{nab.}} = C_F C_A \left(\frac{\alpha_s}{2\pi}\right)^2 \left[\left(\frac{1+z^2}{1-z}\right)\left(-\frac{11}{6}\ln\left(\rho\,(1-z)\right) + \frac{67}{18} - \frac{\pi^2}{6} + \ln^2 z + \mathrm{Li}_2\left(\frac{z-1}{z}\right) + 2\,\mathrm{Li}_2(1-z)\right) + \frac{3}{2}\frac{z^2\ln z}{1-z} + \frac{1}{6}(8-5z)\right]$$

- We can now obtain the θ_g distribution using the LO replacement:

$$\mathcal{B}_2^{q,(\text{nab.})}(z;\theta_g^2) = -\frac{1+z^2}{1-z}\frac{11}{6}\ln z + (1+z)\left(\frac{11}{6}\ln(1-z)^2 - \frac{67}{18} + \frac{\pi^2}{6}\right) + \frac{11}{6}(1-z) + \frac{2z-1}{2} + \frac{1+z^2}{1-z}\left(\ln^2 z + \mathrm{Li}_2\left(\frac{z-1}{z}\right) + 2\mathrm{Li}_2(1-z)\right)$$

The θ_g distribution: non-abelian channel

♠ To find the $C_F C_A$ color structure of B_2^q, we must not forget the identical fermions interference term:

$$\begin{aligned} B_2^{q,\theta_g^2,C_F C_A} &= C_F C_A \left(\frac{\alpha_s}{2\pi}\right)^2 \int_0^1 dz \, \left(\mathcal{B}_2^{q,(\mathrm{nab.})}(z;\theta_g^2) - \frac{1}{2}\mathcal{B}_2^{q,(\mathrm{id.})}(z;\theta_g^2)\right) \\ &= -\gamma_q^{(2,C_A)} + C_F b_0^{(C_A)} X_{\theta_g^2} \end{aligned}$$

♠ Same consideration holds for the ρ distribution with $X_{\theta_g^2} \to X_\rho$.

Take home 1: We can define a suitable differential object, which gives rise to the resummation coefficient B_2^q.

Take home 2: We can move from the θ_g distribution to any other observable by using the LO relation.

The scale of the physical coupling

♠ Let us combine the $C_F T_R n_f$ and non-abelian channels with the LO distribution:

$$\left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma}{d\theta_g^2 dz}\right)^{\text{tot.}} = \frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(1)}}{d\theta_g^2\, dz} + \left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\theta_g^2\, dz}\right)^{C_F T_R n_f} + \left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\theta_g^2\, dz}\right)^{\text{nab.}}$$

$$= C_F p_{qq}(z)\left[\frac{\alpha_s\left(E^2\right)}{2\pi} + \left(\frac{\alpha_s}{2\pi}\right)^2\left(-b_0 \ln\left((1-z)^2\theta_g^2\right) + K_{\text{CMW}}\right) - \left(\frac{\alpha_s}{2\pi}\right)^2 b_0 \ln z\right]$$

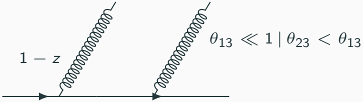
$$+ C_F b_0 \left(\frac{\alpha_s}{2\pi}\right)^2 (1-z) + \left(\frac{\alpha_s}{2\pi}\right)^2 R^{\text{nab.}}(z)$$

Take home 3: The structure of different pieces:

- Red: the usual soft physical coupling
- Blue: the scale of the coupling beyond the soft limit zk_t^2
- Orange: absorb in a new scheme of the coupling
- Black: a remainder function with a $C_F C_A$ colour factor

The abelian channel: C_F^2

- The physics of gluon emissions off the quark is quite distinct different from the gluon decay.



- To zoom in on the NNLL structure, we need to subtract the iterated $1 \to 2$ limit (strongly ordered):[5]

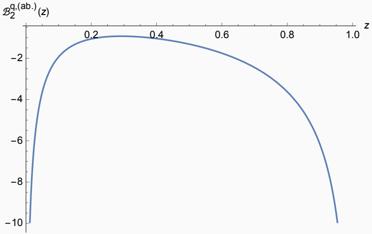
$$\boxed{\mathcal{B}_2^{q,(\mathrm{ab.})}(z;\theta^2) = \left(\frac{\theta^2}{\sigma_0}\frac{d^2\sigma}{dzd\theta^2}\right)^{\text{d-r}} - \left(\frac{\theta^2}{\sigma_0}\frac{d^2\sigma}{dzd\theta^2}\right)^{\text{s-o}} + \left(\frac{\theta^2}{\sigma_0}\frac{d^2\sigma}{dzd\theta^2}\right)^{\text{r-v}}, \quad \theta \equiv \theta_{13}}$$

[5] For uniformity, a factor of $(C_F\alpha_s/2\pi)^2$ is stripped from the RHS.

The abelian channel: C_F^2

- Unfortunately, the constraint $\theta_{23} < \theta_{13}$ renders an analytic evaluation impossible.

- Nevertheless, we were able to express the result as a 1d integral:



- We can use the PSLQ algorithm to fit the integral:[5]

$$\int_0^1 \mathrm{d}z\, \mathcal{B}_2^{q,(\mathrm{ab.})}(z;\theta^2) = \pi^2 - 8\zeta(3) - \frac{29}{8}$$

[5] We thank Pier Monni for letting us use his routine.

Outlook

- One practical side of this work is the ability to resum a host of groomed observables using a QCD-based approach (along the style of ARES).
- The work for gluon jets is underway, and one can ask the same type of questions.
- The most important application is the inclusion in PS.

THANK YOU FOR THE LISTENING!