



SCET 2022, Bern

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Power Counting

Energy Flow Polynomials

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Based on: 2204.xxxxx

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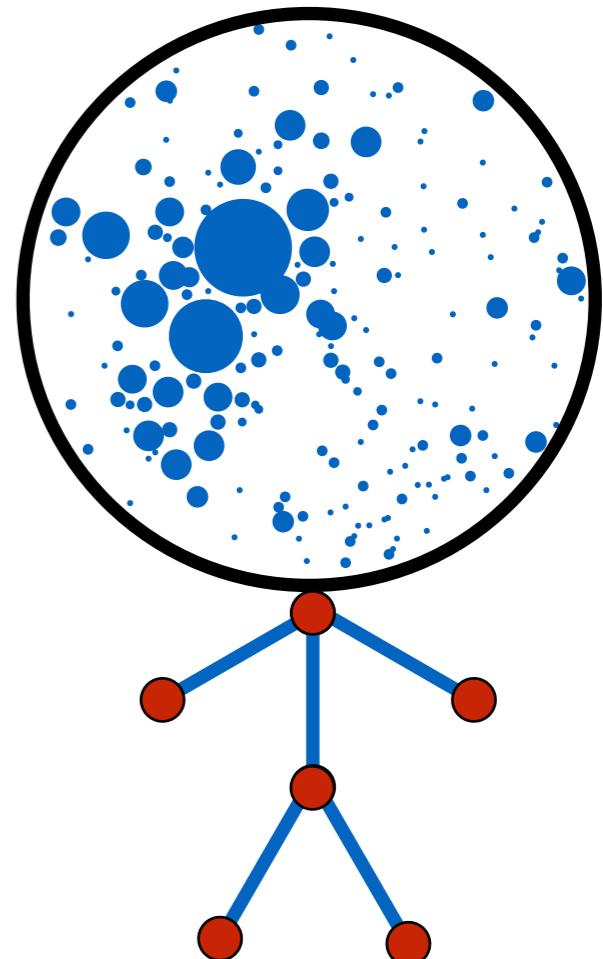
Outline

- **Intro**

- ▶ Energy Flow Polynomials (EFPs)
 - ▶ The power of power counting

- **Power counting Energy Flow Polynomials**

- ▶ Strongly ordered approximation
 - ▶ 1-collinear approximation
 - ▶ 2-collinear approximation
 - ▶ The bases
 - ▶ Logistic regression for quark/gluon classification



- **Conclusions**

Intro

Energy Flow Polynomials

- Energy Flow Polynomials (EFPs): Discrete linear basis for all infrared- and collinear-safe observables

Komiske, Metodiev, Thaler '17, '19

- Can be represented in graph form:

$$\bullet \bullet \bullet = \sum_{i_1=1}^M \sum_{i_2=1}^M \sum_{i_3=1}^M \sum_{i_4=1}^M z_{i_1} z_{i_2} z_{i_3} z_{i_4} \theta_{i_1 i_2} \theta_{i_2 i_3}^2 \theta_{i_3 i_4}$$

- How this works:

$$\bullet_i = \sum_{i=1}^M z_i$$



Sum over particles in the jet

$$j \quad k = \theta_{jk}$$



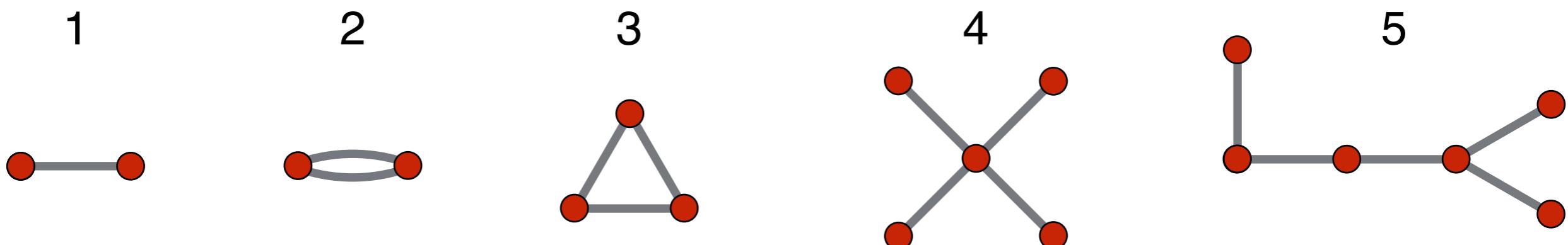
Angular distance between
particles j and k

Energy Flow Polynomials

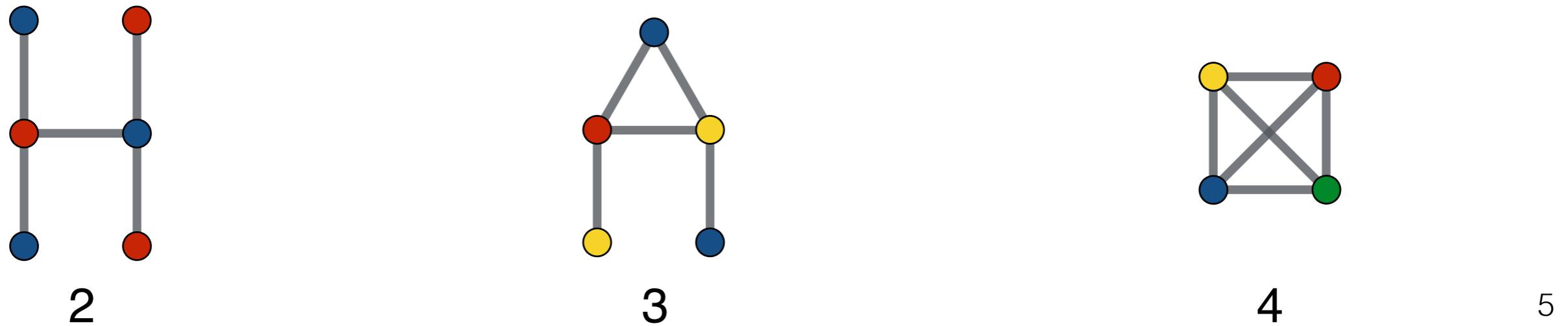
- One more:

$$= \sum_{i_1, i_2, i_3, i_4=1}^M z_{i_1} z_{i_2} z_{i_3} z_{i_4} \theta_{i_1 i_2} \theta_{i_2 i_3} \theta_{i_3 i_4}^2 \theta_{i_4 i_2}$$

- Degree: how many lines in the EFP



- Chromatic number: how many particles needed to have non-vanishing EFP



Energy Flow Polynomials

All prime (connected) EFPs up to degree 5

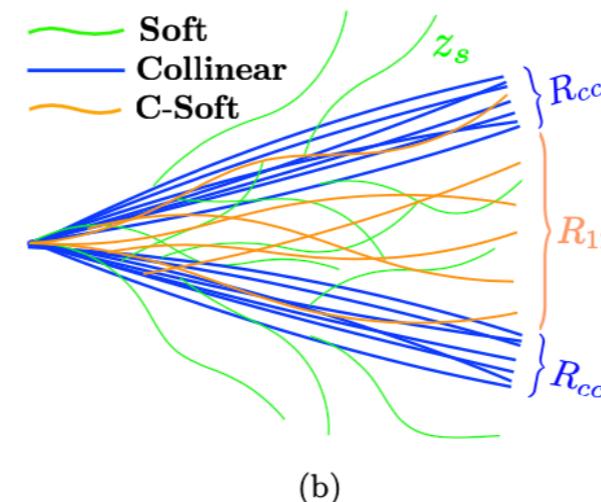
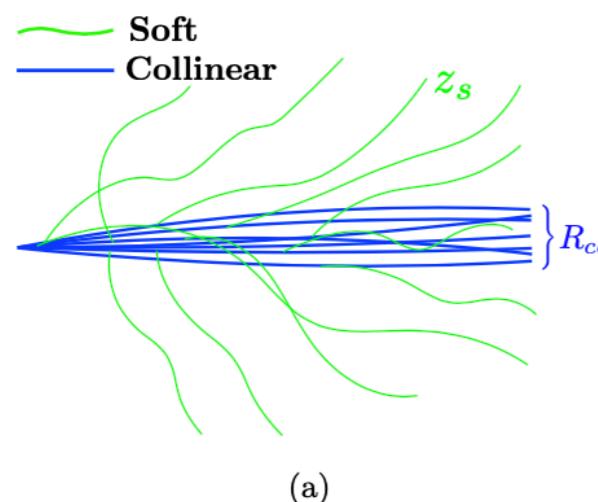
Figure from:

Komiske, Metodiev,
Thaler '17

Degree	Connected Multigraphs
$d = 0$	
$d = 1$	
$d = 2$	
$d = 3$	
$d = 4$	
$d = 5$	

The power of Power Counting

- Parametric power counting of the dynamics of QCD can be used to design substructure observables
- E.g. : QCD/Boosted-object jet discrimination **Larkoski, Moult, Neill '14**



$$R_{cc} \ll 1 , \quad R_{cs} \sim 1$$

$$\implies D_2^{(\beta)} = \frac{e_3^{(\beta)}}{(e_2^{(\beta)})^3} .$$

- We will use one-prong power counting - not to design observables - but to better understand relations between observables

This talk

- Power counting provides relations that greatly reduce the number of independent EFPs at a given level of accuracy
- Using the smaller EFP basis does not affect tagging performance

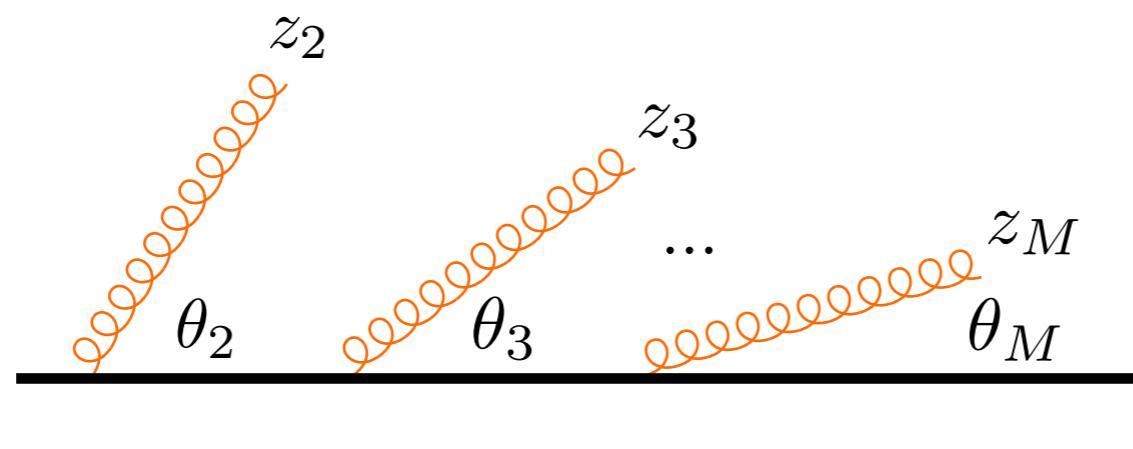
Degree		0	1	2	3	4	5	6
All EFPs	by degree	1	1	3	8	23	66	212
	cumulative	1	2	5	13	36	102	314
SO basis	by degreee	1	1	2	4	7	12	22
	cumulative	1	2	4	8	15	26	49
2-collinear basis	by degree	1	1	2	4	8	16	36
	cumulative	1	2	4	8	16	32	68

Power Counting EFPs

Strongly ordered

- We start our investigation by applying a very restrictive power counting:
strong **energy and angular** ordering

$$z_{i+1} \gg z_i \quad \text{and} \quad \theta_{i+1} \gg \theta_i$$



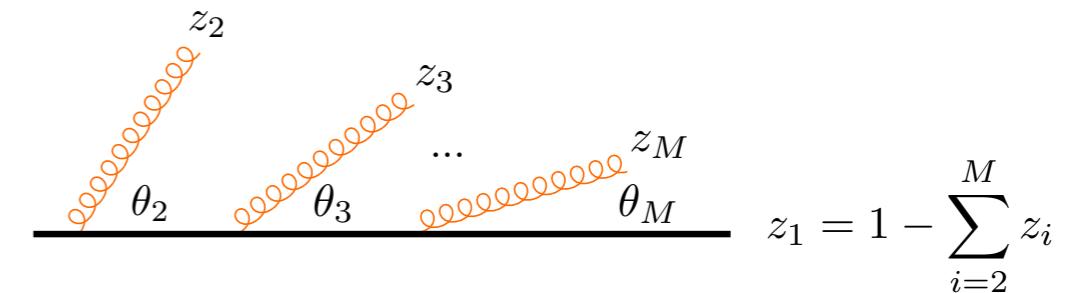
$$z_1 = 1 - \sum_{i=2}^M z_i$$

Strongly ordered

- Use this power counting to perform expansion

Full expression:

$$\bullet - \bullet = \sum_{i,j=1}^M z_i z_j \theta_{ij} \quad \xrightarrow{\text{Sum over all particles}}$$



Strongly ordered (SO) expansion:

$$\bullet - \bullet \stackrel{\text{SO}}{=} 1 - z_2 + z_2 - 1 + \mathcal{O}(z_3 \theta_3)$$

$$\stackrel{\text{SO}}{=} 2z_2 \theta_2 + \mathcal{O}(z_3 \theta_3)$$

Angle between dominant emission and hard prong

Strongly ordered

$$\bullet \quad \text{dumbbell EFP} \stackrel{\text{SO}}{=} 2z_2\theta_2 + \mathcal{O}(z_3\theta_3)$$

- Let's have a look at a more complicated EFP: 4-dots

$$\bullet \quad \text{4-dots EFP} \stackrel{\text{SO}}{=} \frac{1}{z_2} \bullet - z_2 \bullet + \frac{1}{z_2} \bullet - z_2 \bullet + \frac{z_2}{z_3} \bullet - \frac{1}{z_2} \bullet + \frac{z_2}{z_3} \bullet - \frac{1}{z_2} \bullet + \mathcal{O}(z_2z_3\theta_2^2\theta_3)$$
$$\stackrel{\text{SO}}{=} 2z_2^2\theta_2^3 + \mathcal{O}(z_2z_3\theta_2^2\theta_3)$$

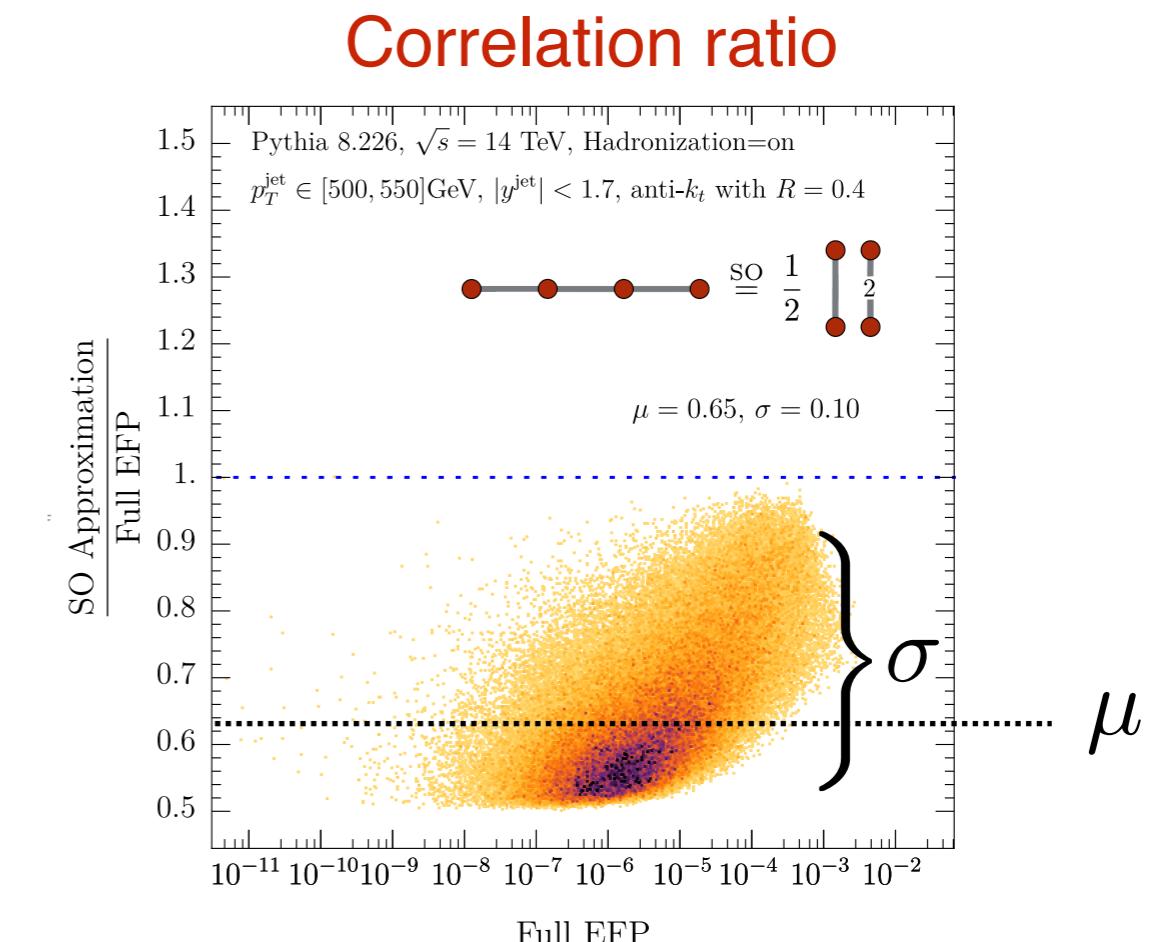
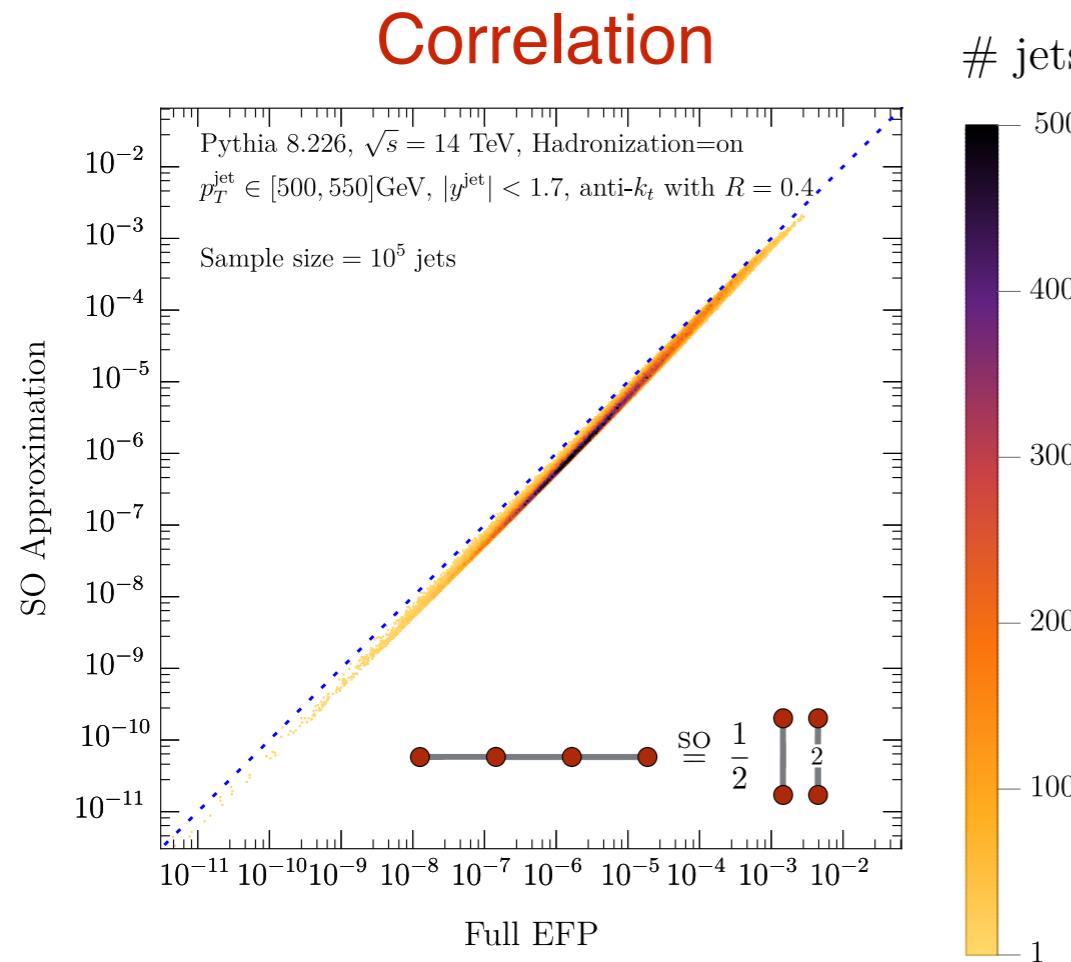
- In the SO limit we can write 4-dots in terms of the dumbbell EFP

$$\bullet \quad \text{4-dots EFP} \stackrel{\text{SO}}{=} \frac{1}{2} \bullet - \bullet - \bullet - \bullet + \mathcal{O}(z_2z_3\theta_2^2\theta_3)$$

Strongly ordered

- How does this relation hold in Pythia?

$$\text{SO} \stackrel{\cong}{=} \frac{1}{2} \quad \text{2}$$

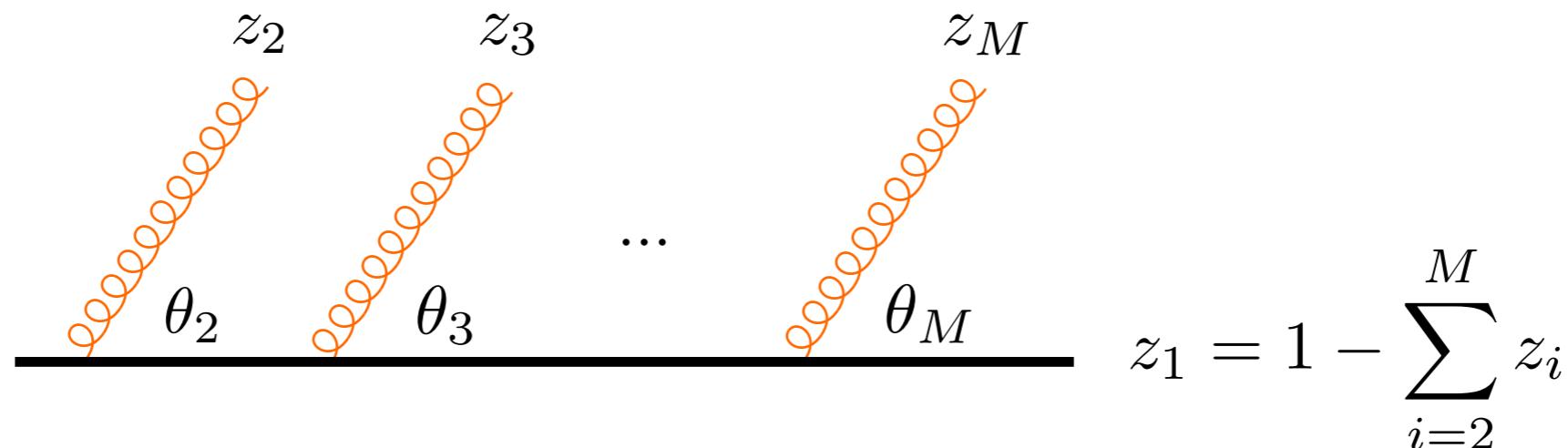


- Correlation is clearly present but expression is off by an overall factor:

$$\mu = 0.65, \quad \sigma = 0.1$$

1-collinear

- Let's perform a better expansion: the **1-collinear expansion**



- What do we know in the **1-collinear** approximation?

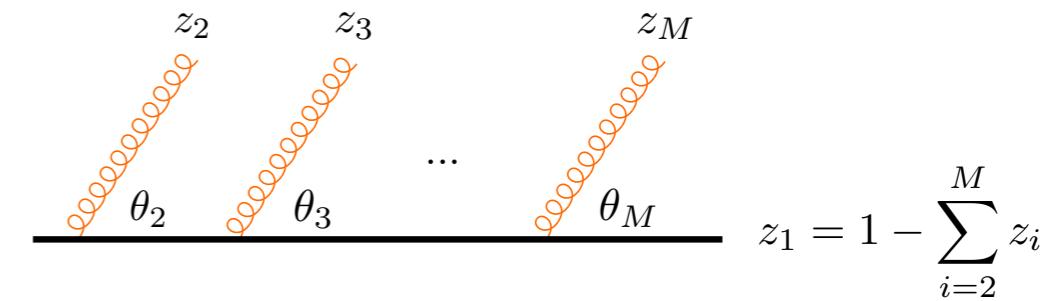
- There is **one hard prong**
- All are emissions are **collinear-soft**: $z_i \ll 1$ $\theta_i \ll 1$
- We assume no strong ordering

1-collinear

- Use this power counting to perform expansion

Full expression:

$$\bullet - \bullet = \sum_{i,j=1}^M z_i z_j \theta_{ij} \quad \xrightarrow{\text{Sum over all particles}}$$



1-collinear expansion: maximize the number of times the hard prong is assigned

$$\bullet - \bullet \stackrel{1c}{=} \frac{1}{\bullet} - \frac{z_i}{\bullet} + \frac{z_i}{\bullet} - \frac{1}{\bullet} + \mathcal{O}(z^2)$$

$$\stackrel{1c}{=} 2 \sum_{i=2}^M z_i \theta_i + \mathcal{O}(z^2)$$

Sum over collinear-soft particles

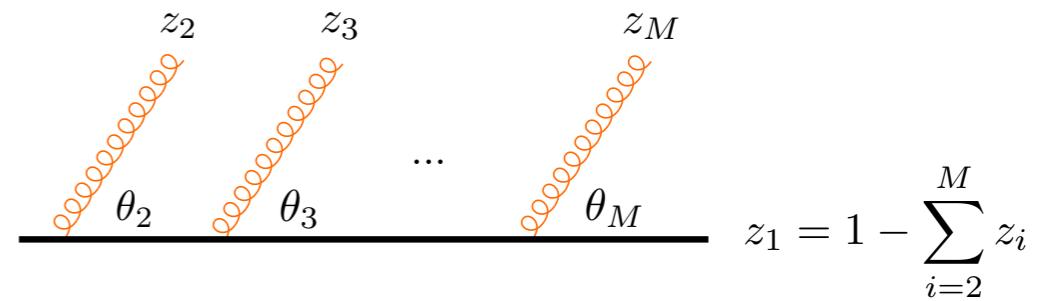
$\theta_i \equiv$ Angle between particle i
and hard prong

1-collinear

- Use this power counting to perform expansion

Full expression:

$$\text{Diagram of a triangle with three red vertices} = \sum_{i,j,k=1}^M z_i z_j z_k \theta_{ij} \theta_{jk} \theta_{ki}$$



$$z_1 = 1 - \sum_{i=2}^M z_i$$

1-collinear expansion:

$$\text{Diagram of a triangle with three red vertices} \stackrel{1c}{=} \text{Diagram of a triangle with one black vertex at top, labeled } 1 \text{, and two orange vertices at bottom, labeled } z_i \text{ and } z_j + \text{Diagram of a triangle with one orange vertex at top, labeled } z_i \text{, and two black vertices at bottom, labeled } z_j \text{ and } 1 + \text{Diagram of a triangle with one black vertex at top, labeled } 1 \text{, and two orange vertices at bottom, labeled } z_i \text{ and } z_j + \mathcal{O}(z^3)$$

$$\stackrel{1c}{=} 3 \sum_{i,j=2}^M z_i z_j \theta_i \theta_j \theta_{ij} + \mathcal{O}(z^3)$$

1-collinear

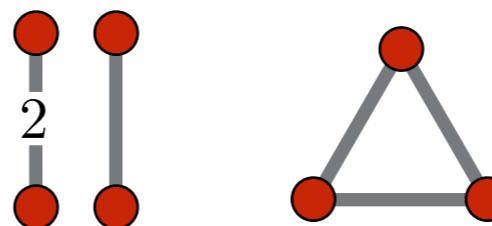
- So far, at 1c:

$$\begin{array}{ccc} \text{Diagram: two red dots connected by a horizontal gray line} & \stackrel{1c}{=} & 2 \sum_{i=2}^M z_i \theta_i \\ & & \text{Diagram: three red dots forming an equilateral triangle} & \stackrel{1c}{=} & 3 \sum_{i,j=2}^M z_i z_j \theta_i \theta_j \theta_{ij} \end{array}$$

- What happens to 4-dots now?

$$\begin{array}{ccccccccc} \text{Diagram: four red dots connected by a horizontal gray line} & \stackrel{1c}{=} & \text{Diagram: four dots (black, orange, black, orange) connected by a horizontal gray line} & + & \text{Diagram: four dots (orange, black, orange, black) connected by a horizontal gray line} \\ & & z_i & & z_j & & z_i & & z_j \\ & & 1 & & 1 & & 1 & & 1 \\ & & & & & & & & \\ \text{new} & \leftarrow + & \text{Diagram: four dots (black, orange, orange, black) connected by a horizontal gray line} & + \mathcal{O}(z^3) & & & & & \\ & & z_i & & z_j & & 1 & & \\ & & 1 & & z_i & & 1 & & \\ & & & & & & & & \end{array}$$

$$\stackrel{1c}{=} \sum_{i,j=2}^M z_i z_j (2\theta_i^2 \theta_j + \theta_i \theta_j \theta_{ij}) + \mathcal{O}(z^3)$$



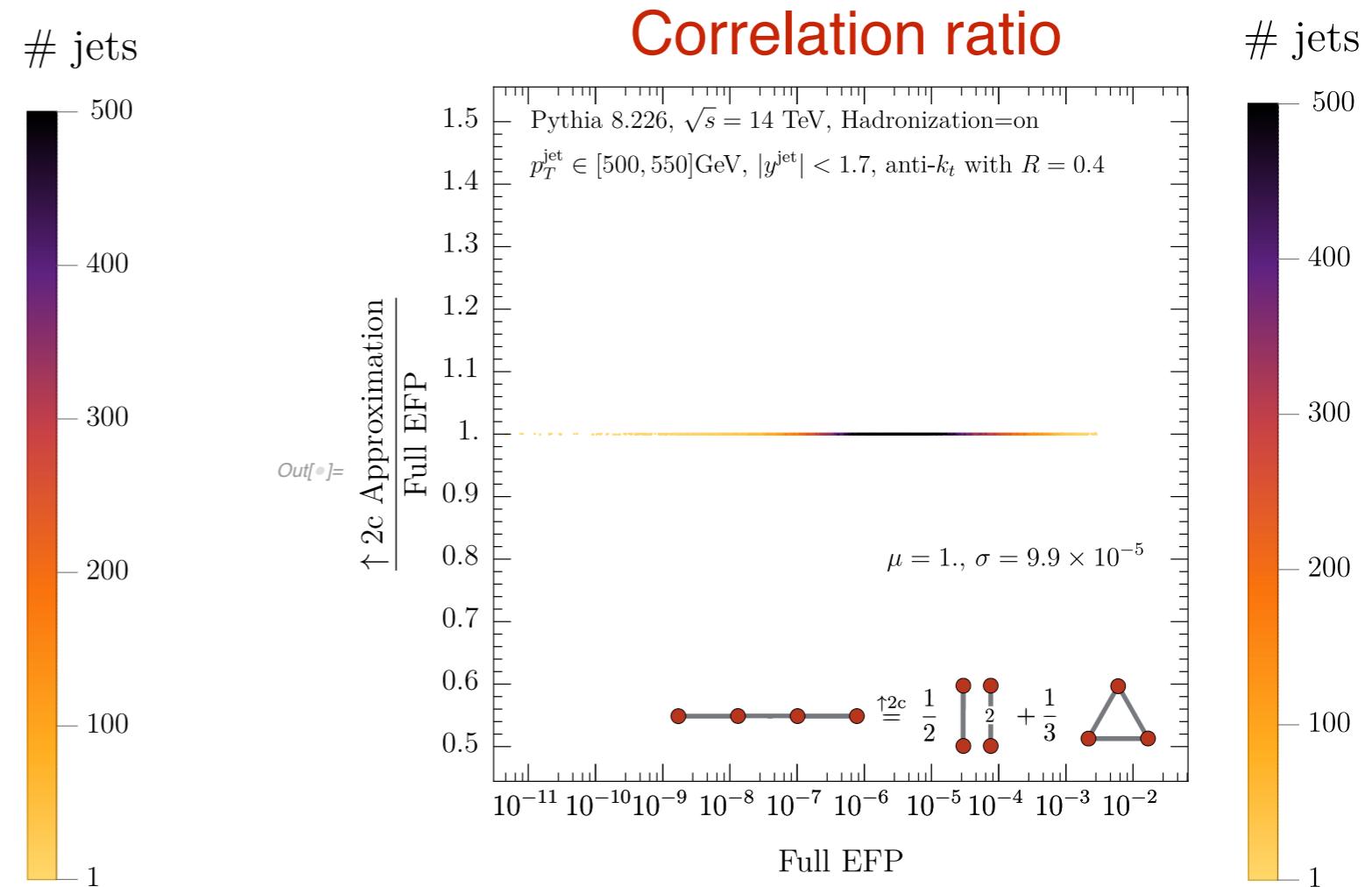
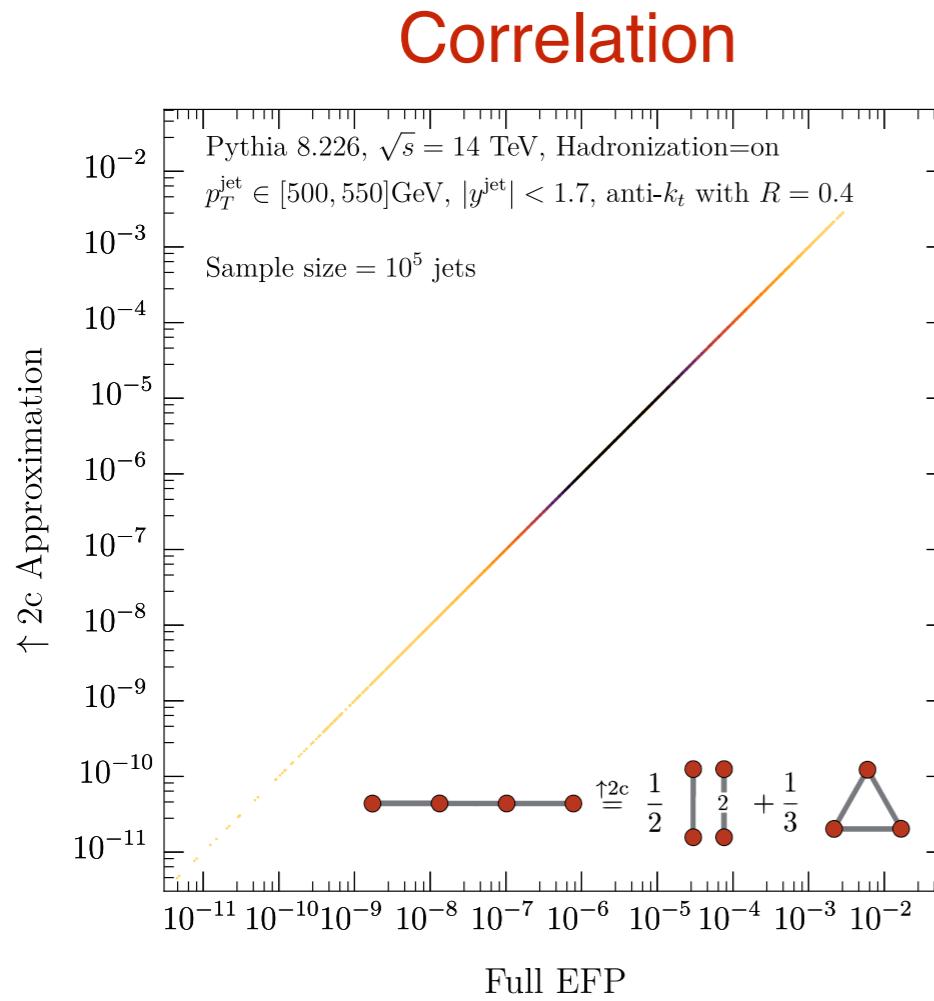
1-collinear

- At 1-collinear accuracy we obtain the relation:

$$\text{Diagram: } \text{Four red dots connected by a horizontal line.} \stackrel{1c}{=} \frac{1}{2}$$

new compared to SO

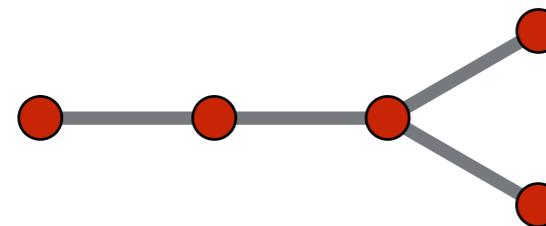
$$+ \frac{1}{3} \text{Diagram: } \text{Three red dots forming an equilateral triangle with gray edges.} + \mathcal{O}(z^3)$$



$$\mu = 1.0000, \quad \sigma = 1 \times 10^{-5}$$

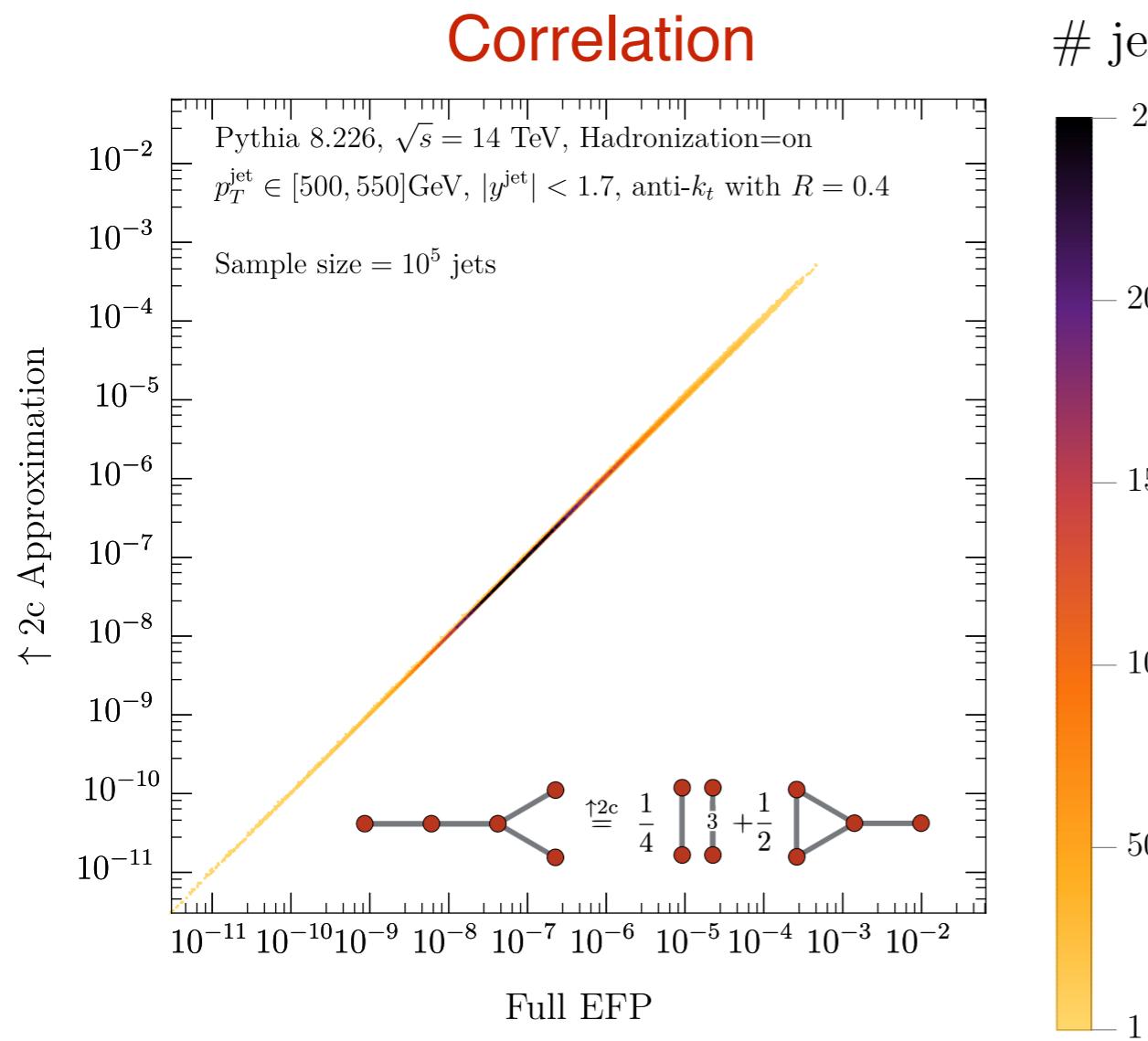
1-collinear

- Another 1-collinear example:

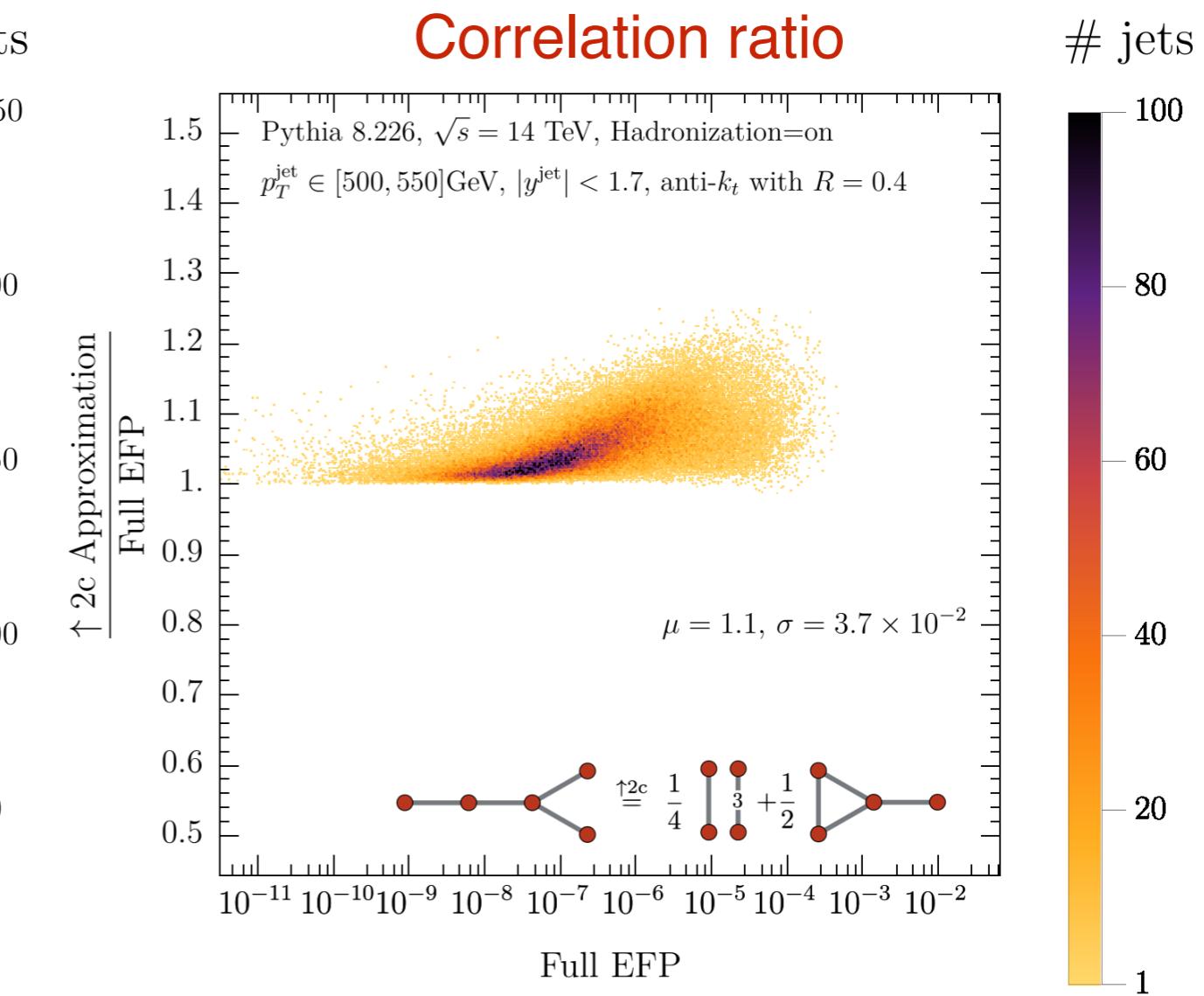


$$\stackrel{1c}{=} \frac{1}{4}$$

$$+ \frac{1}{2} \text{ new compared to SO} + \mathcal{O}(z^3)$$



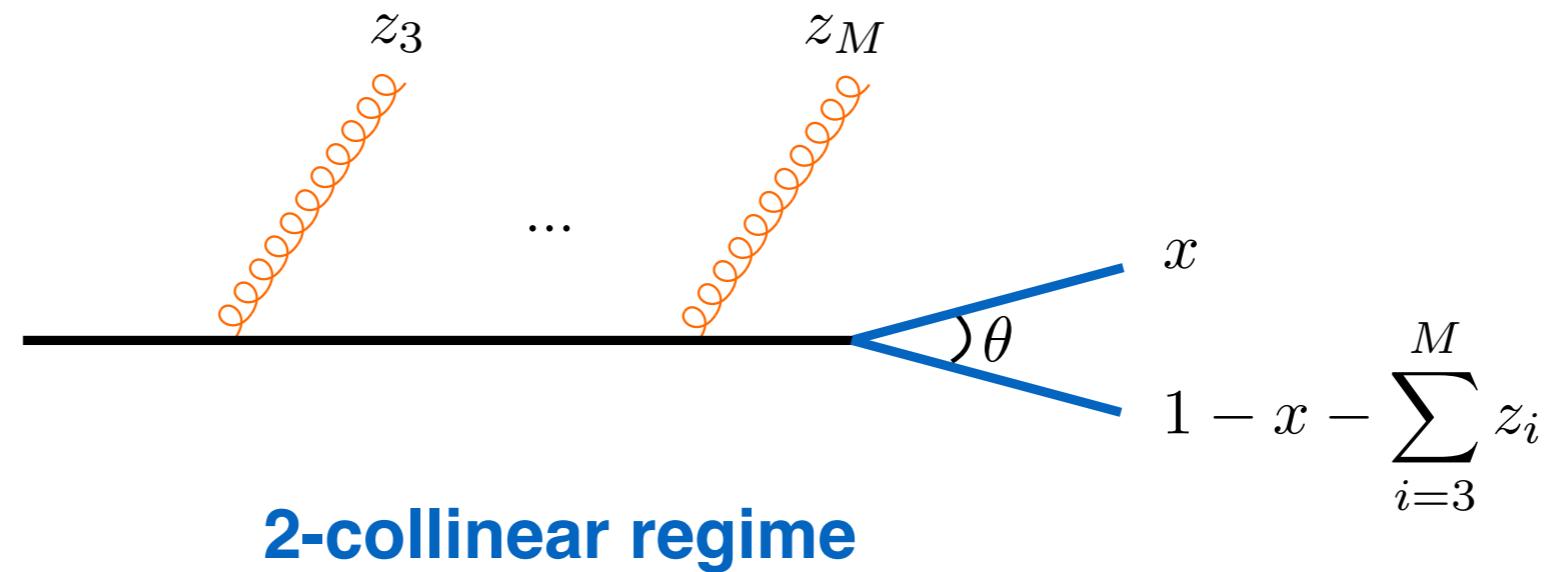
$$\mu = 1.1, \quad \sigma = 0.037$$



Cal, Thaler, Waalewijn '22 (to appear)

2-collinear

- An even better expansion: **2-collinear expansion**
- **2-collinear approximation:** 2 collinear particles, the remaining are **collinear-soft**



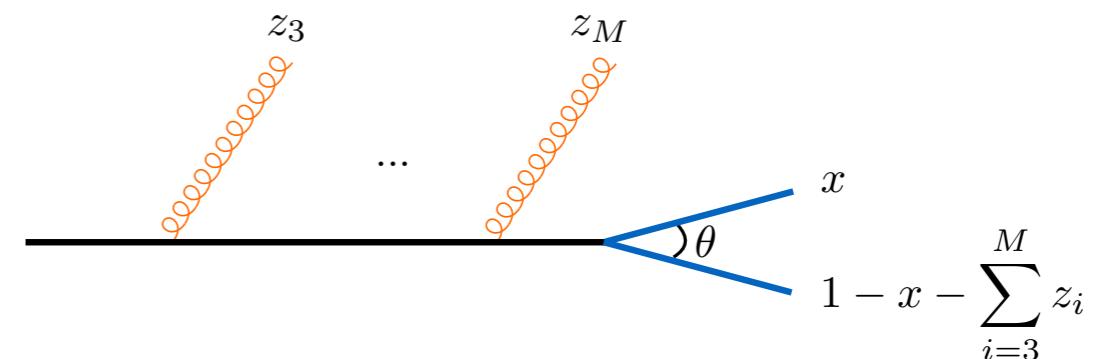
- Note: the general solution must combine the individual 1- and 2-collinear solutions (see how later). We call this the **up-to-2-collinear solution** ($\uparrow 2c$)

2-collinear

- 1-collinear solution:

$$\begin{array}{c} \text{Diagram: } \text{Two horizontal red lines with 4 red dots each, connected by a vertical gray line between the second and third dots from the left.} \\ \stackrel{1c}{=} \frac{1}{2} \begin{array}{c} \text{Diagram: } \text{A triangle with vertices labeled 1, 2, and 3. Vertex 1 is at the top, vertex 2 is at the bottom-left, and vertex 3 is at the bottom-right. Edges are gray.} \end{array} + \mathcal{O}(z^3) \end{array}$$

- In the 2-collinear regime:



$$\begin{array}{c} \text{Diagram: } \text{Two horizontal red lines with 4 red dots each, connected by a vertical gray line between the second and third dots from the left.} \\ \stackrel{2c}{=} \begin{array}{c} \text{Diagram: } \text{Two horizontal blue lines with 4 blue dots each. The top line has dots labeled x, 1-x, x from left to right. The bottom line has dots labeled 1-x, x, 1-x from left to right. They are connected by a vertical gray line between the second and third dots from the left.} \end{array} + \begin{array}{c} \text{Diagram: } \text{Two horizontal blue lines with 4 blue dots each. The top line has dots labeled 1-x, x, 1-x from left to right. The bottom line has dots labeled x, 1-x, x from left to right. They are connected by a vertical gray line between the second and third dots from the left.} \end{array} + \mathcal{O}(z) \\ \stackrel{2c}{=} 2x^3(1-x)^3\theta^5 + \mathcal{O}(z) \end{array}$$

2-collinear

- The dumbbell EFP expansion in the 2-collinear approximation

$$\text{Diagram: two red circles connected by a horizontal line} \stackrel{2c}{=} x \text{ } \begin{matrix} 1-x \\ \text{Diagram: two blue circles connected by a horizontal line} \end{matrix} + \begin{matrix} 1-x \\ \text{Diagram: two blue circles connected by a horizontal line} \end{matrix} x + \mathcal{O}(z) = 2x(1-x)\theta + \mathcal{O}(z)$$

allows us to write:

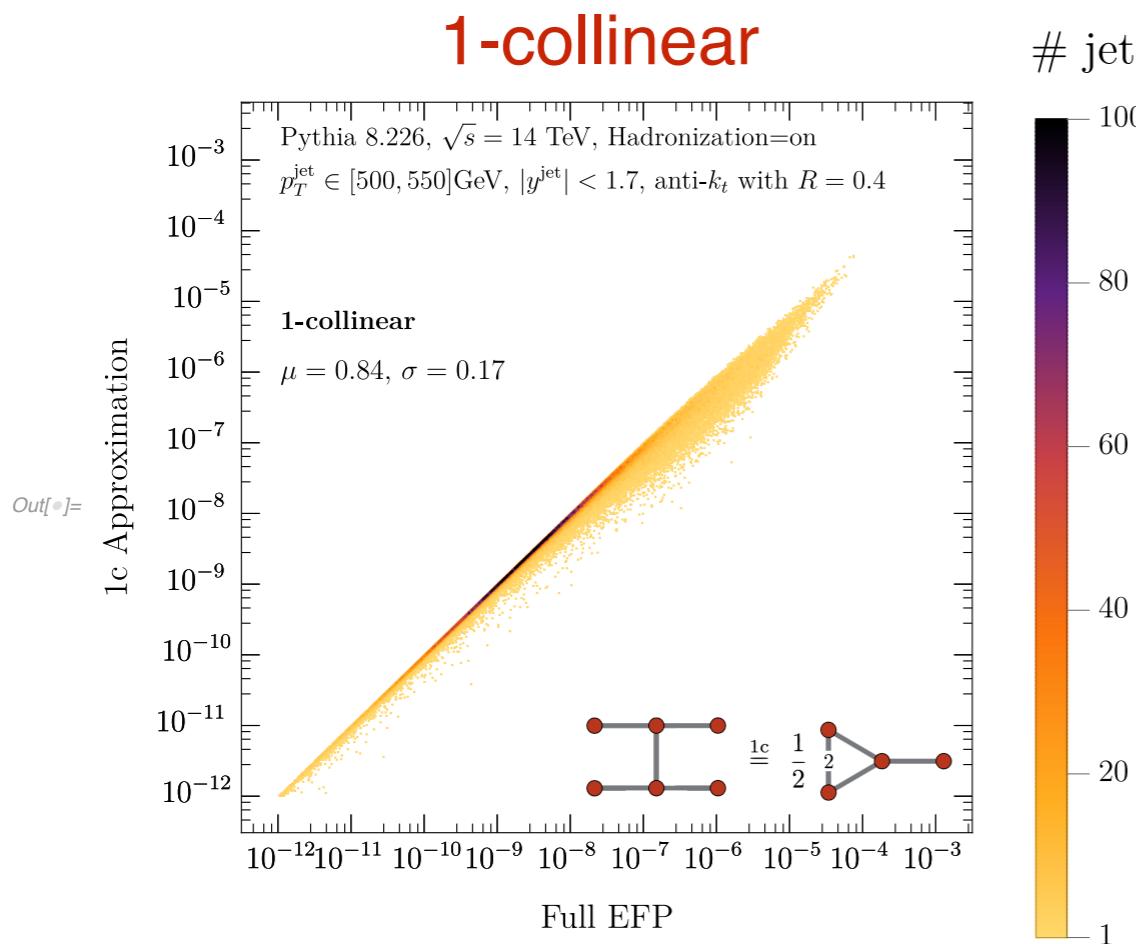
$$\text{Diagram: a square with four red circles at vertices and midpoints} \stackrel{2c}{=} \frac{1}{4} \text{ } \begin{matrix} \text{Diagram: two red circles connected by a horizontal line} \\ (\text{Diagram: two red circles connected by a horizontal line})^2 \end{matrix} + \mathcal{O}(z)$$

- We obtain the **up-to-2-collinear solution** ($\uparrow 2c$) by taking the intersection (including degeneracies) of the 1- and 2-collinear solutions

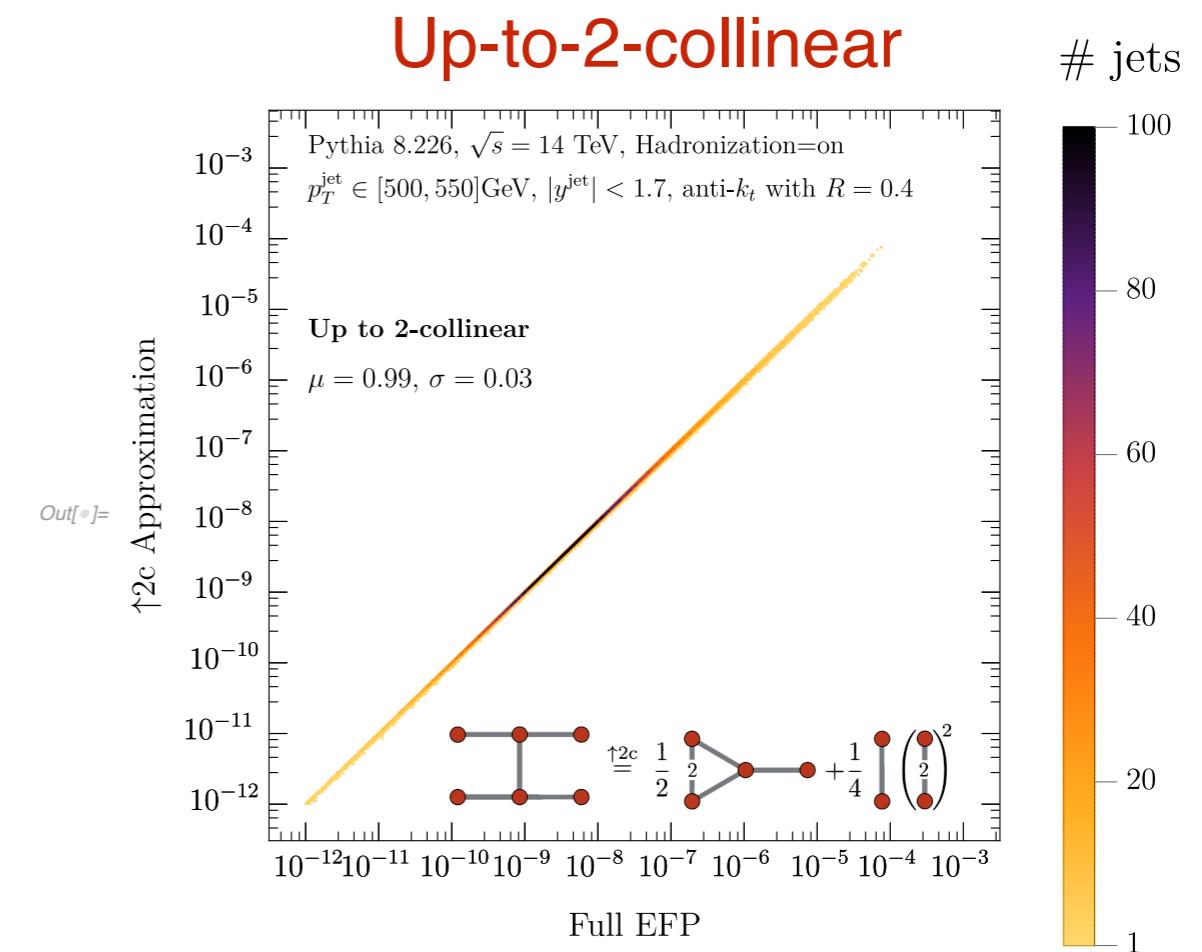
$$\text{Diagram: a square with four red circles at vertices and midpoints} \stackrel{\uparrow 2c}{=} \underbrace{\frac{1}{2} \text{ } \begin{matrix} \text{Diagram: two red circles connected by a horizontal line} \\ \text{Diagram: two red circles connected by a diagonal line} \end{matrix}}_{\substack{1c : \mathcal{O}(z^2) \\ 2c : \mathcal{O}(z)}} + \underbrace{\frac{1}{4} \text{ } \begin{matrix} \text{Diagram: two red circles connected by a horizontal line} \\ (\text{Diagram: two red circles connected by a horizontal line})^2 \end{matrix}}_{\substack{1c : \mathcal{O}(z^3) \\ 2c : \mathcal{O}(1)}} + \text{subleading}$$

2-collinear

$$\begin{array}{c} \text{Diagram showing two horizontal lines with red dots at both ends, connected by a vertical line between the middle points.} \\ \stackrel{\uparrow 2c}{=} \end{array}
 \quad
 \begin{array}{c} \text{Diagram showing a triangle with vertices labeled 1, 2, and 3. Vertex 1 is at the top, vertex 2 is at the bottom left, and vertex 3 is at the bottom right. Edges connect (1,2), (2,3), and (1,3).} \\ \frac{1}{2} \end{array}
 \quad
 + \frac{1}{4} \quad
 \begin{array}{c} \text{Diagram showing a horizontal line with red dots at both ends, with a bracket below it labeled } (2-2) \\ (2-2)^2 \end{array}
 + \text{subleading}$$



$$\mu = 0.84, \sigma = 0.17$$

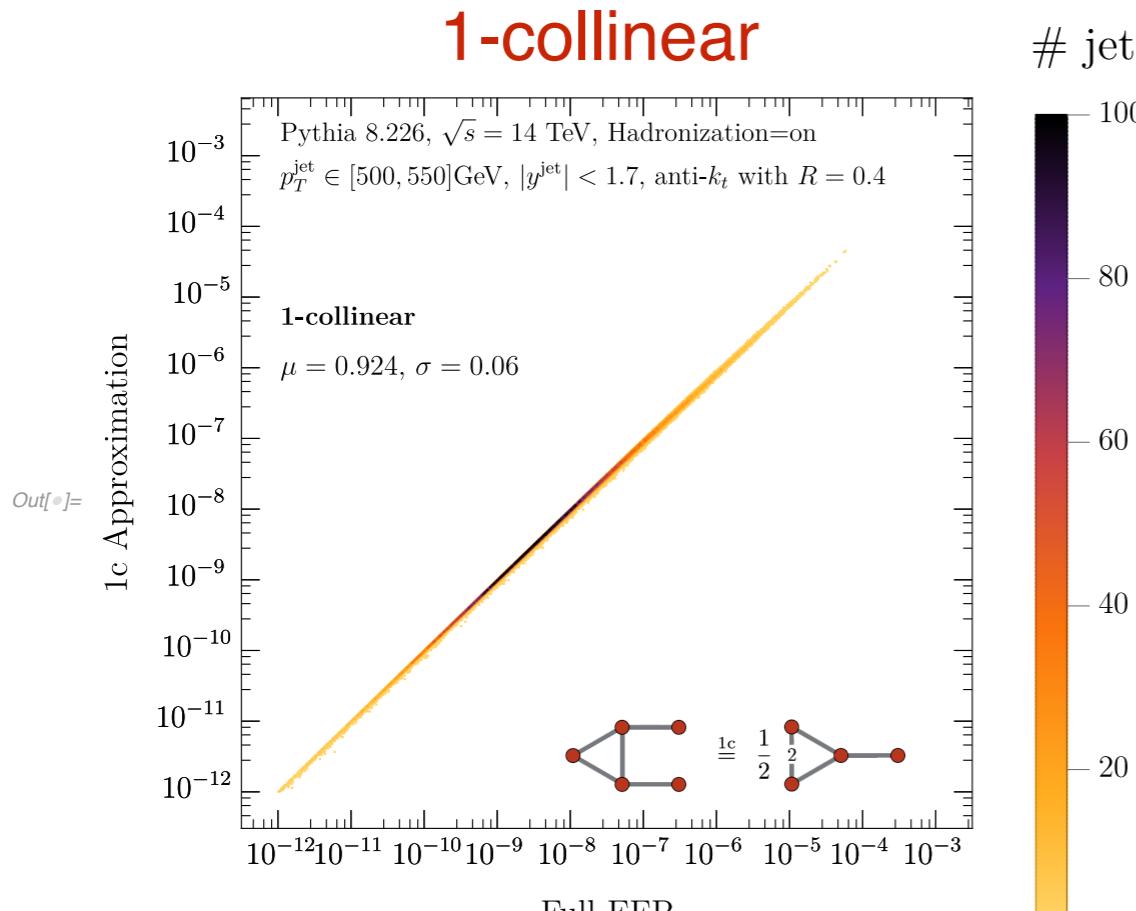


$$\mu = 0.99, \sigma = 0.03$$

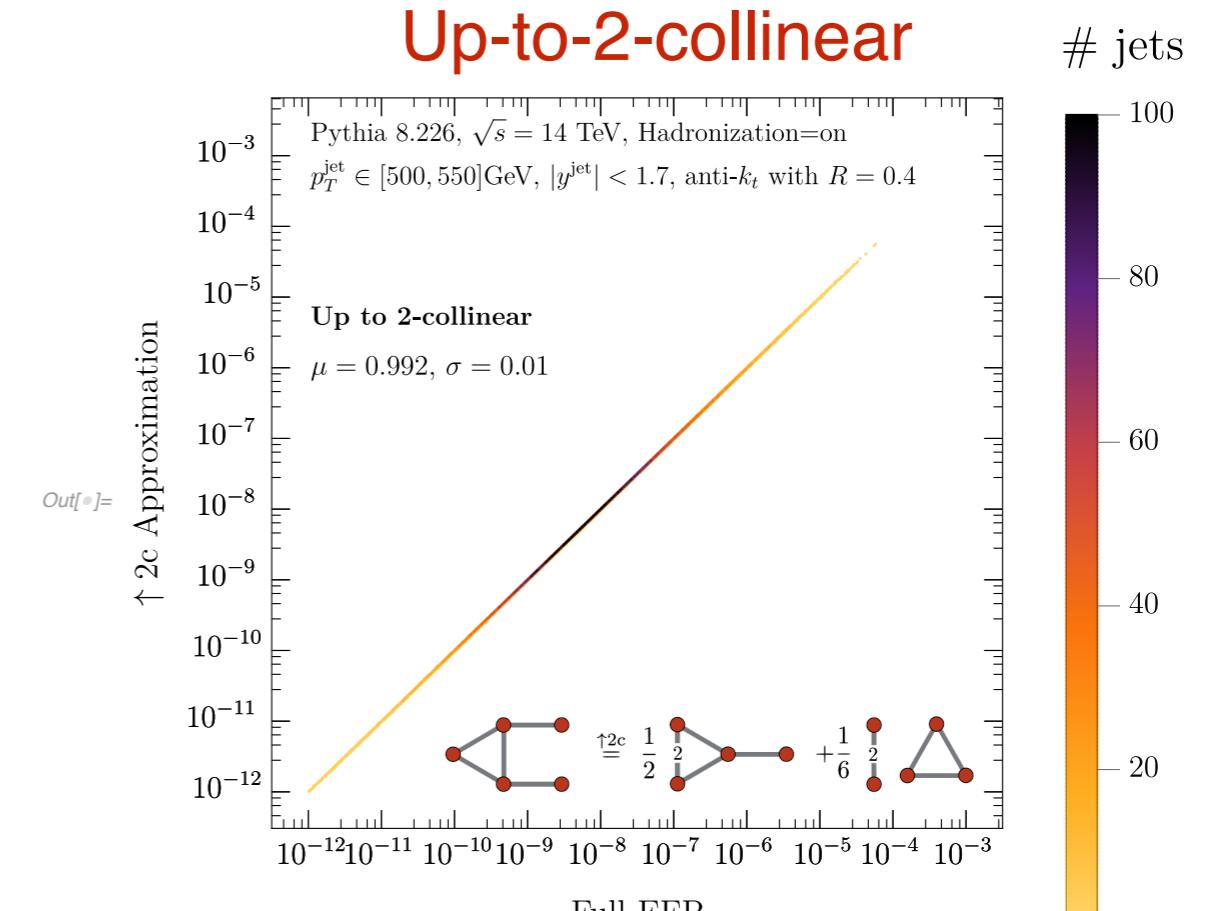
2-collinear

- What about a 3-color EFP? Same reasoning

$$\begin{array}{c} \text{Diagram of a 2-collinear loop with 6 red vertices and 8 gray edges.} \\ \stackrel{\uparrow 2c}{=} \frac{1}{2} \begin{array}{c} \text{Diagram of a 2-collinear loop with 4 red vertices and 6 gray edges, with a factor } \frac{1}{2} \text{ below it.} \end{array} + \frac{1}{6} \begin{array}{c} \text{Diagram of a 2-collinear loop with 4 red vertices and 4 gray edges, with a factor } \frac{1}{6} \text{ below it.} \end{array} + \text{subleading} \end{array}$$



$$\mu = 0.92, \sigma = 0.06$$



$$\mu = 0.99, \sigma = 0.01$$

The collinear basis

- We can use these relations to create a **basis** of linearly independent EFPs at a given accuracy
- We do so by looking at all possible terms that can appear at leading power

	Degree 1	Degree 2	Degree 3	
Expression	$\sum_{i=2}^M z_i \theta_i$	$\sum_{i=2}^M z_i \theta_i^2$	$\sum_{i=2}^M z_i \theta_i^3$	$\sum_{i,j=2}^M z_i z_j \theta_i \theta_j \theta_{ij}$
EFP term	$\frac{1}{2} \bullet - \bullet$	$\frac{1}{2} \bullet - 2 - \bullet$	$\frac{1}{2} \bullet - 3 - \bullet$	$\frac{1}{3} \bullet \triangle \bullet$

- What about terms like $\sum_{i,j=2}^M z_i z_j \theta_{ij}$? \longrightarrow There is no EFP for which it is leading power

$$\bullet - \bullet \stackrel{1c}{=} 2 \sum_{i=2}^M z_i \theta_i + \sum_{i,j=2}^M z_i z_j \theta_{ij}$$

LP **NLP**

The collinear basis

- We can proceed with this exercise to arbitrarily high degree

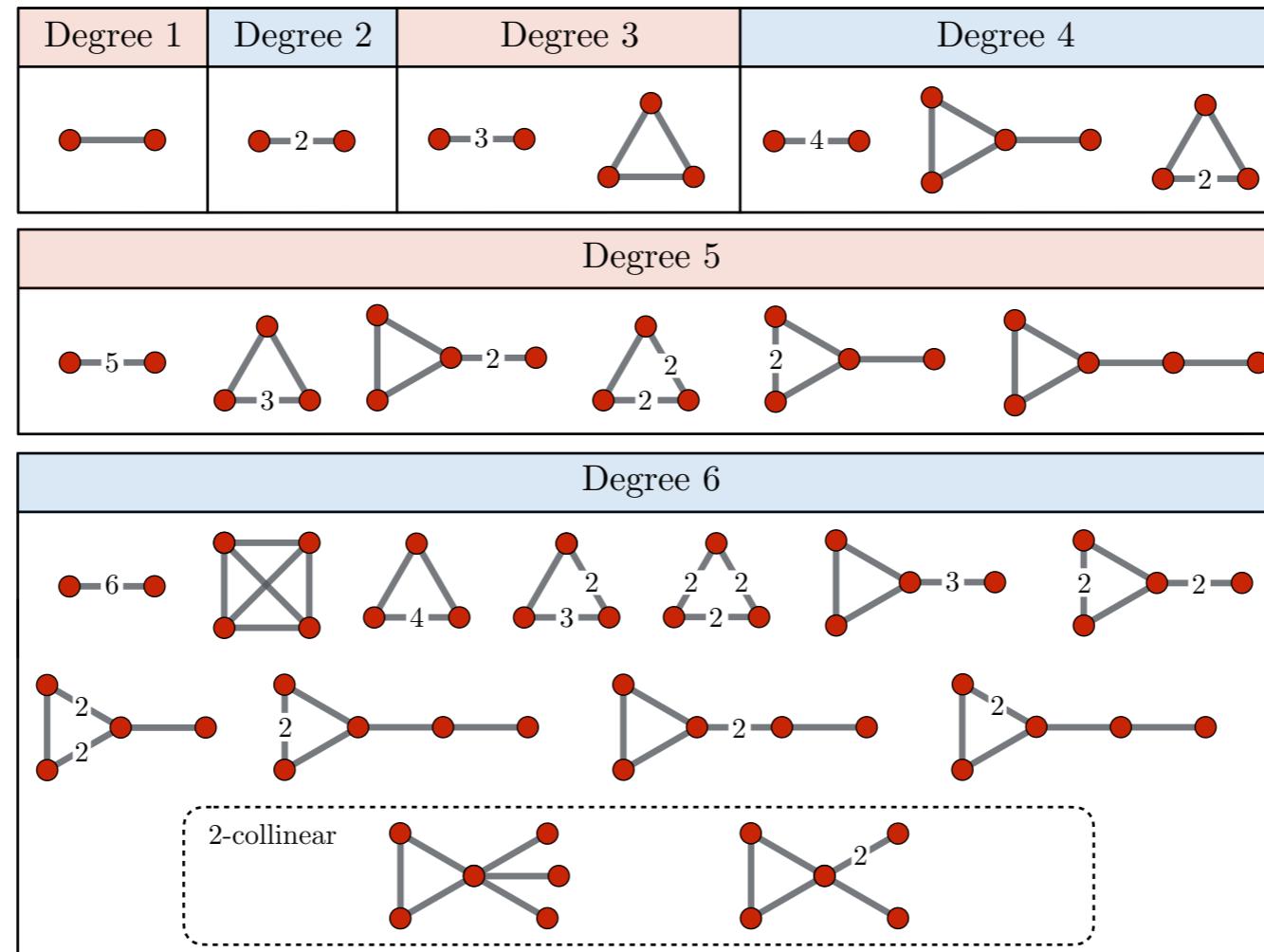
	Degree 1	Degree 2	Degree 3	
Expression	$\sum_{i=2}^M z_i \theta_i$	$\sum_{i=2}^M z_i \theta_i^2$	$\sum_{i=2}^M z_i \theta_i^3$	$\sum_{i,j=2}^M z_i z_j \theta_i \theta_j \theta_{ij}$
EFP term	$\frac{1}{2} \bullet - \bullet$	$\frac{1}{2} \bullet - 2 - \bullet$	$\frac{1}{2} \bullet - 3 - \bullet$	$\frac{1}{3} \bullet \triangle \bullet$

	Degree 4		
Expression	$\sum_{i=2}^M z_i \theta_i^4$	$\sum_{i,j=2}^M z_i z_j \theta_i^2 \theta_j \theta_{ij}$	$\sum_{i,j=2}^M z_i z_j \theta_i \theta_j \theta_{ij}^2$
EFP term	$\frac{1}{2} \bullet - 4 - \bullet$	$\frac{1}{2} \bullet \triangle \bullet - \bullet$	$\bullet - 2 - \bullet$

...

The collinear basis

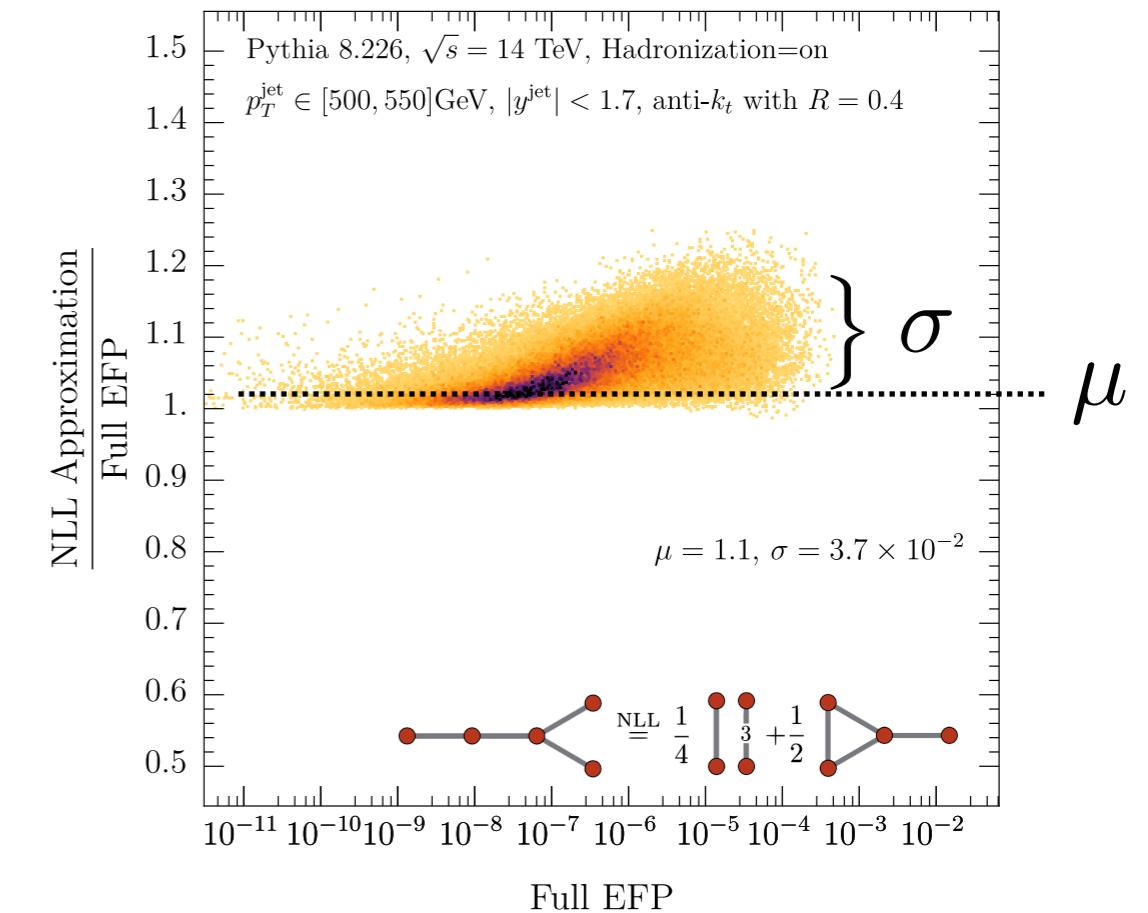
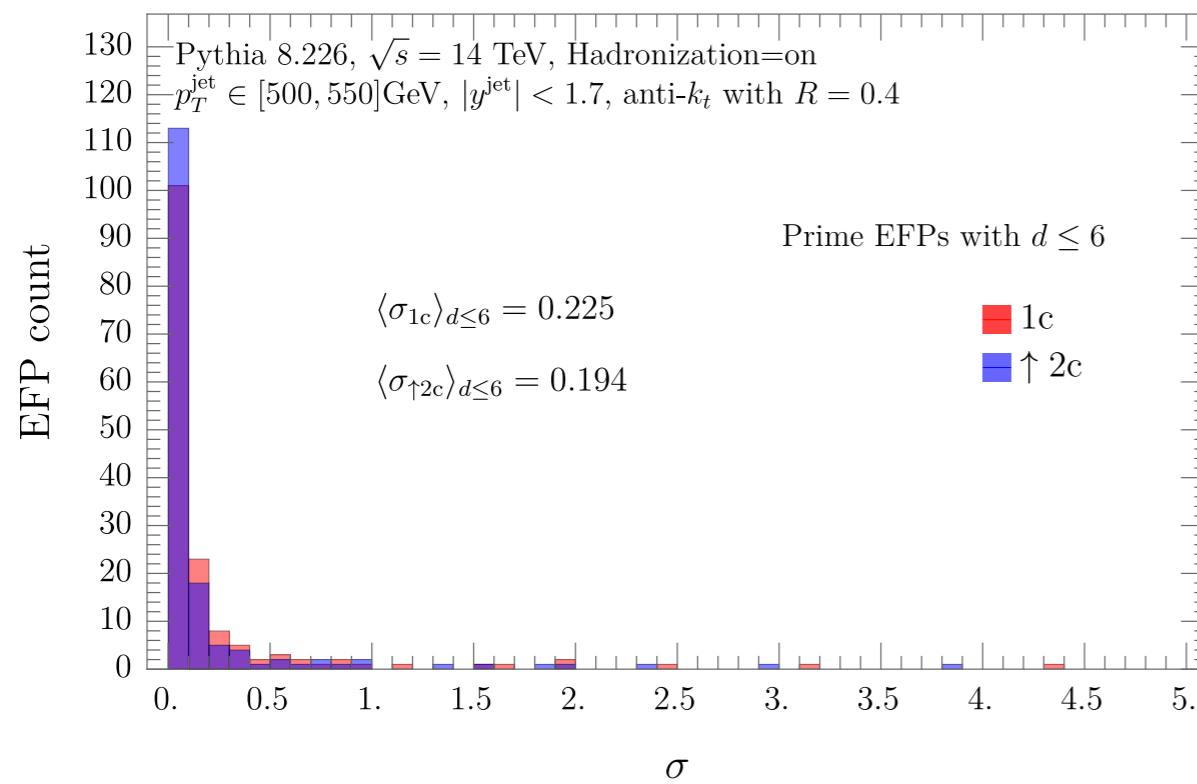
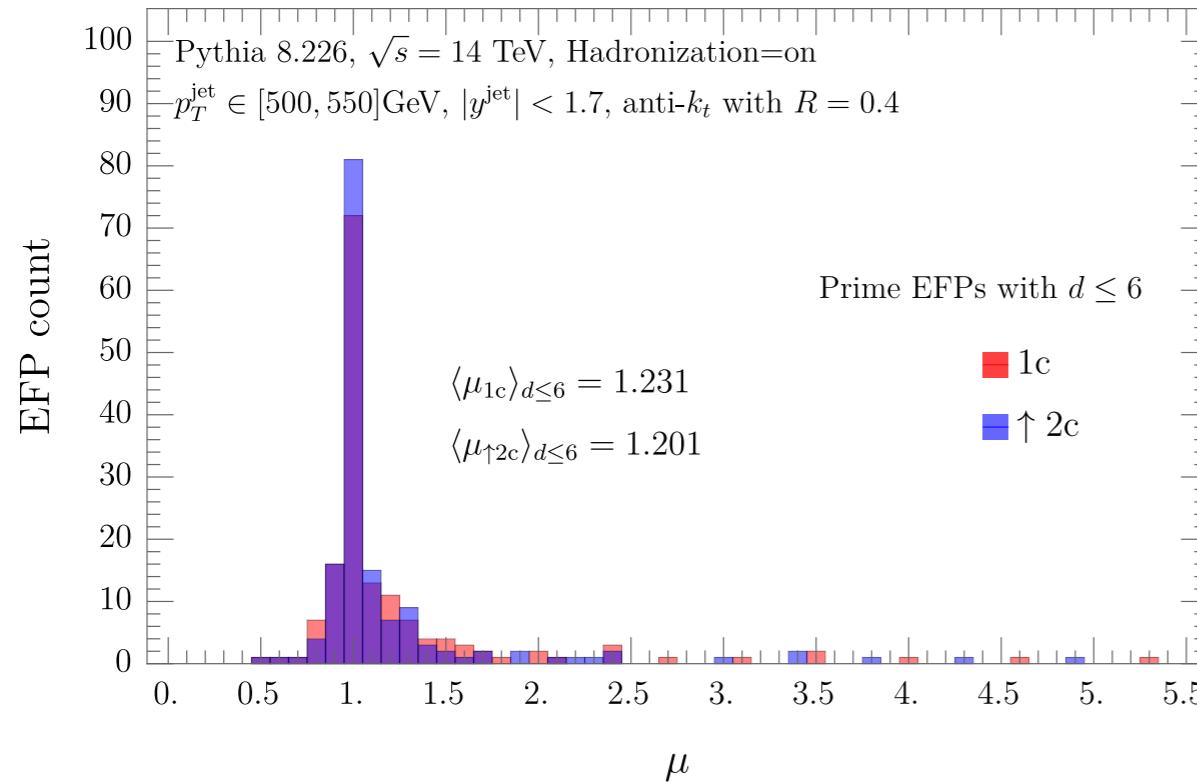
- We dub this the **2-collinear basis**



- The 1- and 2-collinear basis are identical apart from two EFPs at degree 6
- With these elements we can describe any EFP at up-to-2-collinear accuracy

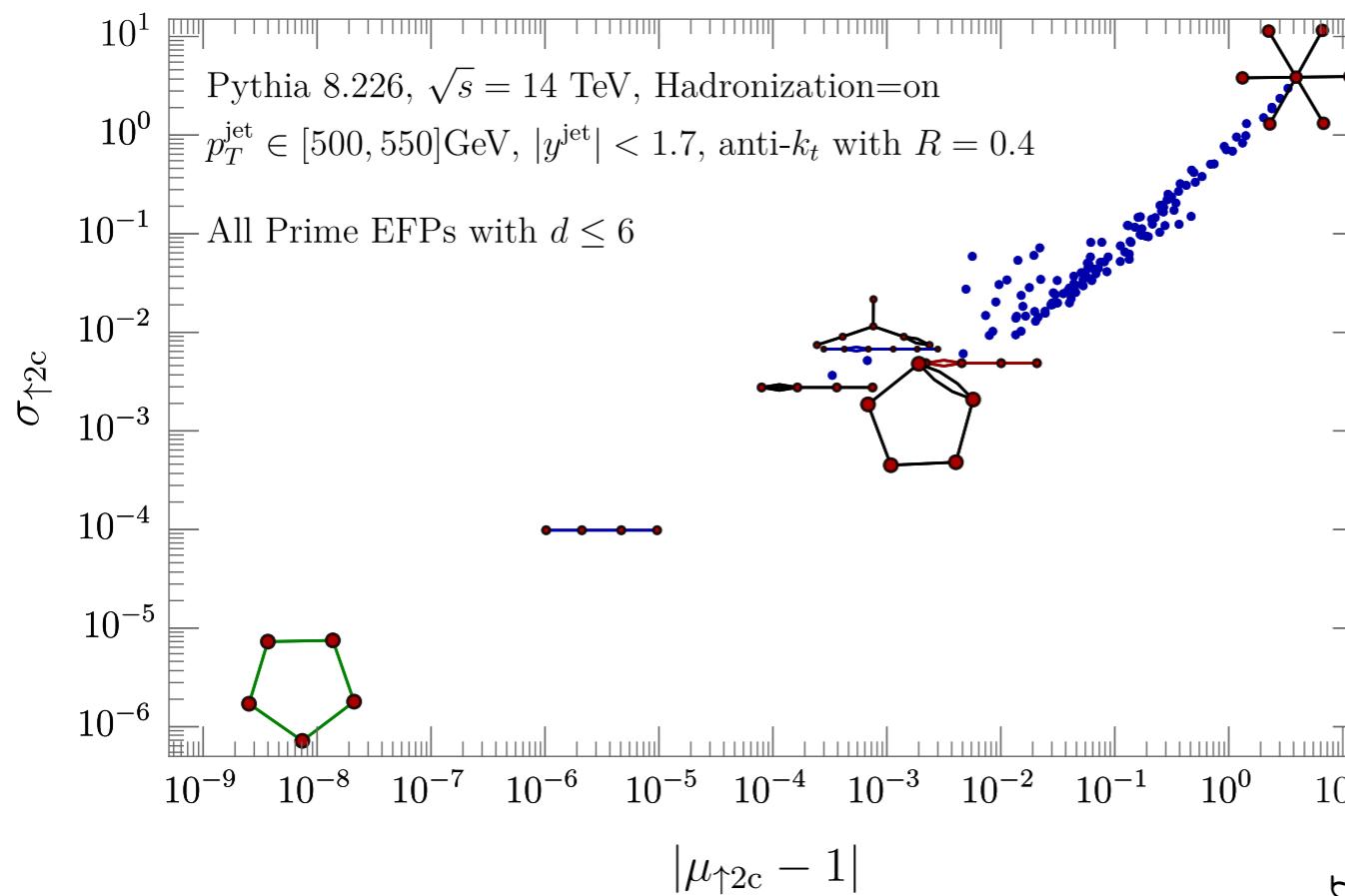
The collinear basis

- All prime EFPs up to degree 6

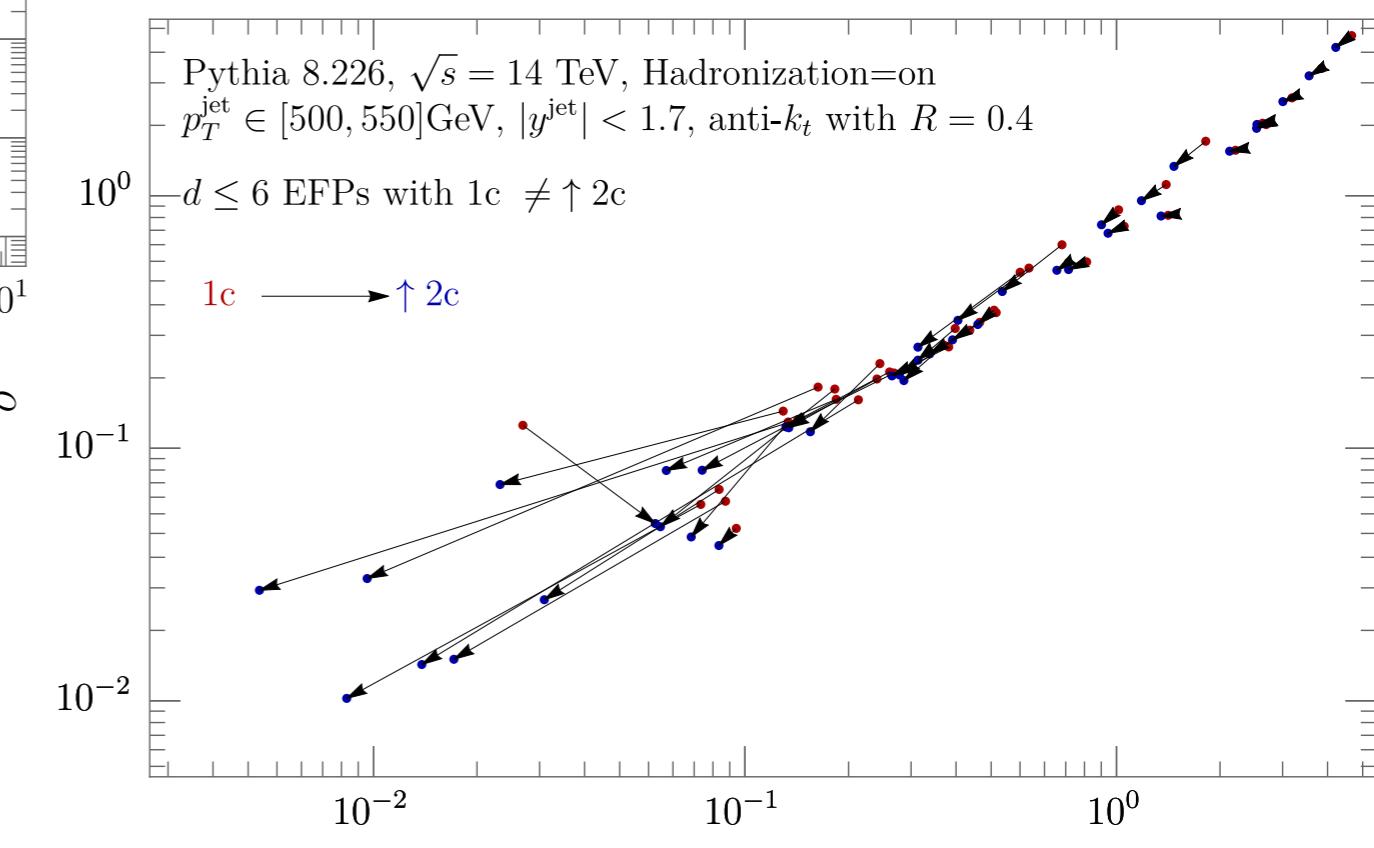


The collinear basis

- For all EFPs, going from 1-collinear to up-to-2-collinear improves the correlation between the full EFP and its approximation

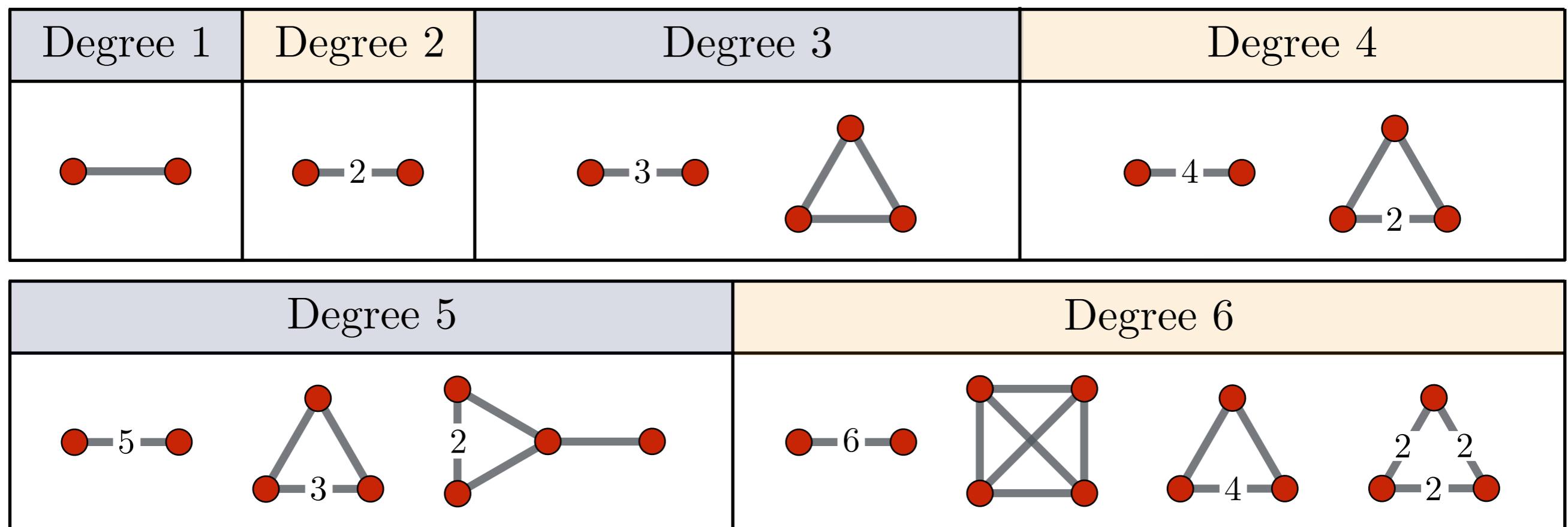


Cal, Thaler, Waalewijn '22 (to appear)



The SO basis

- We can also obtain the **SO basis** by performing the strongly ordered expansion on the 1-collinear basis



Bases vs. All EFPs

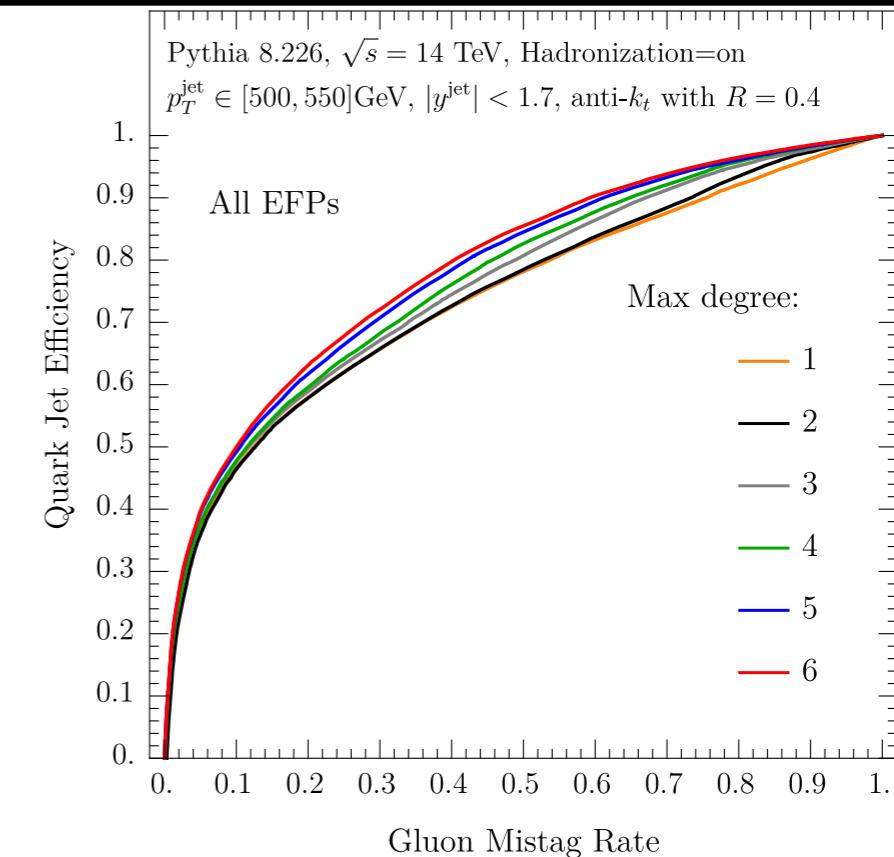
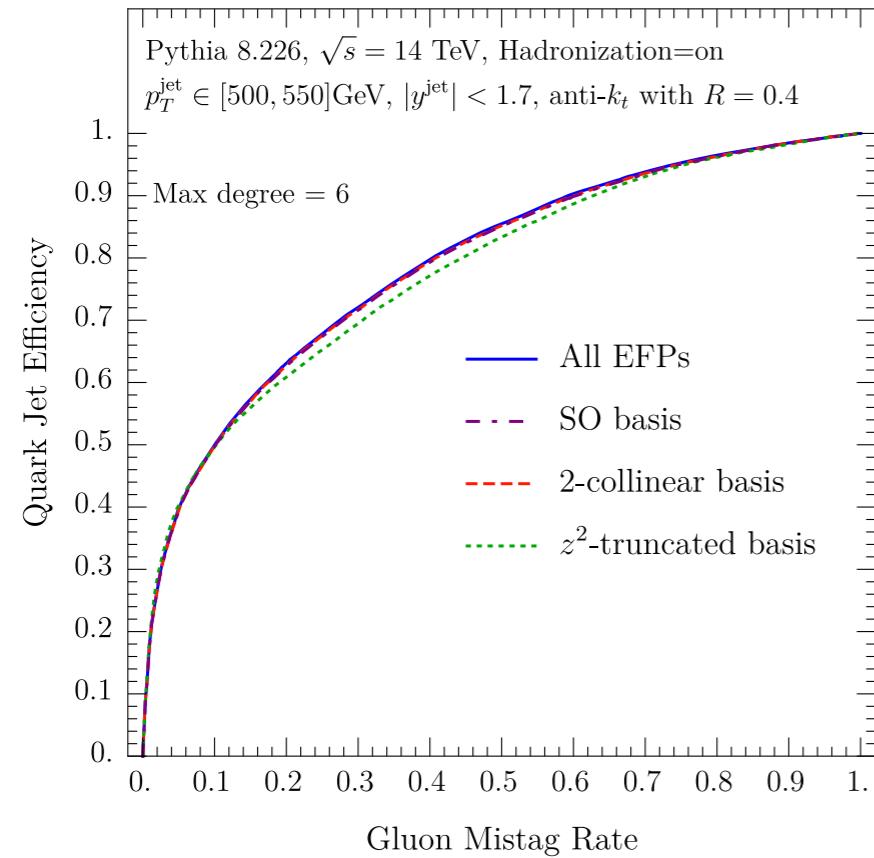
- The size of the bases is much smaller compared to all possible EFPs

Degree		0	1	2	3	4	5	6
All EFPs	by degree	1	1	3	8	23	66	212
	cumulative	1	2	5	13	36	102	314
SO basis	by degreee	1	1	2	4	7	12	22
	cumulative	1	2	4	8	15	26	49
2-collinear basis	by degree	1	1	2	4	8	16	36
	cumulative	1	2	4	8	16	32	68

- Let's study the differences in quark/gluon tagging performance when using 3 different sets of inputs:

- 1) All EFPs
- 2) The SO basis
- 3) The 2-collinear basis

Logistic regression



- Area under the curve (AUC) - measure of classification performance

Max degree	All EFPs	SO	2-collinear
1	0.741 ± 0.003	0.741 ± 0.003	0.741 ± 0.003
2	0.745 ± 0.003	0.740 ± 0.003	0.741 ± 0.003
3	0.761 ± 0.003	0.755 ± 0.003	0.755 ± 0.003
4	0.770 ± 0.003	0.765 ± 0.003	0.766 ± 0.003
5	0.784 ± 0.003	0.781 ± 0.003	0.782 ± 0.003
6	0.792 ± 0.003	0.789 ± 0.003	0.789 ± 0.003

Performance nearly identical!

Cal, Thaler, Waalewijn '22 (to appear)

Conclusions

- We characterize EFPs using power counting, obtaining relations between them that hold at a given accuracy for quark and gluon jets
- These relationships are validated by Pythia, finding excellent agreement
- We can use these relationships to substantially reduce the number of EFPs without affecting tagging performance
- Future directions:
 - ▶ Power counting for 2 and 3 prong jets
 - ▶ Analytic resummation of EFPs, looking for patterns in factorization theorems

Dankesfolie

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Backup

1-collinear

$$\begin{array}{c}
 \text{Diagram: } \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \stackrel{1c}{=} \bullet \text{---} z_j \text{---} \bullet \text{---} 1 \text{---} z_i \text{---} \bullet^1 \text{---} \bullet \text{---} 1 \text{---} \bullet^1 \\
 + \bullet \text{---} 1 \text{---} z_j \text{---} z_i \text{---} \bullet^1 \text{---} \bullet \text{---} 1 \text{---} \bullet^1 + \mathcal{O}(z^3)
 \end{array}$$

$$\stackrel{1c}{=} \sum_{i,j=2}^M z_i z_j (\theta_i^3 \theta_j + \boxed{\theta_i^2 \theta_j \theta_{ij}}) + \mathcal{O}(z^3)$$

↓

How do we get this term?

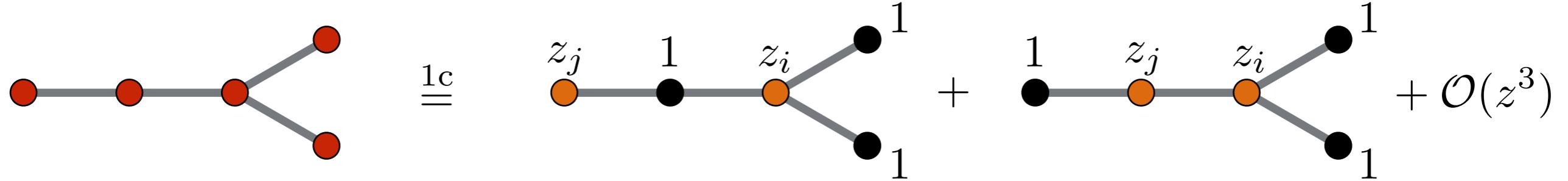
$$\begin{array}{c}
 \text{Diagram: } \bullet \text{---} \bullet \text{---} \bullet \stackrel{1c}{=} \bullet \text{---} 1 \text{---} \bullet \text{---} z_i \text{---} 2 \text{---} z_j + \bullet \text{---} z_i \text{---} \bullet \text{---} 2 \text{---} z_j \text{---} 1 + \bullet \text{---} z_i \text{---} \bullet \text{---} 1 \text{---} 2 \text{---} z_j + \mathcal{O}(z^3) \\
 \text{Red X}
 \end{array}$$

$$\stackrel{1c}{=} \sum_{i,j=2}^M z_i z_j (\theta_i \theta_j \theta_{ij}^2 + 2\theta_i^2 \theta_j \theta_{ij}) + \mathcal{O}(z^3)$$

↓ ↓

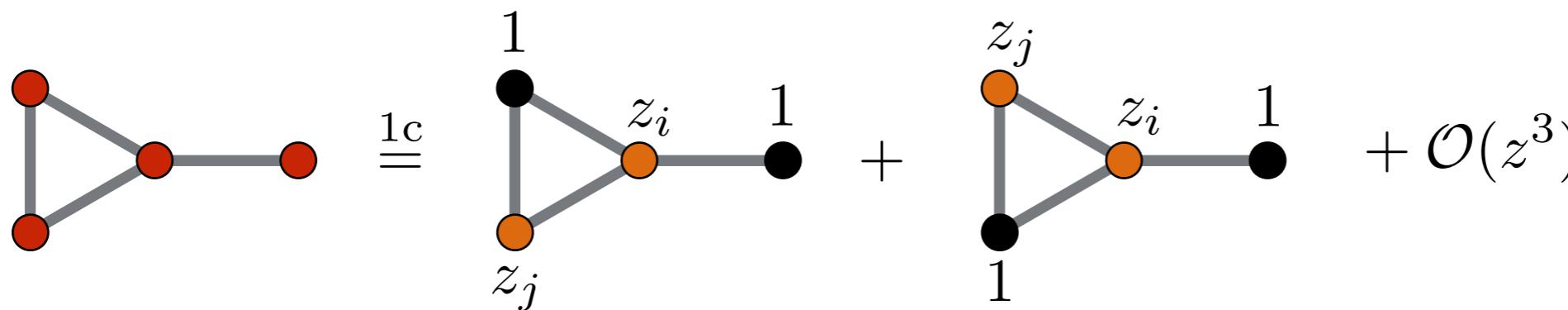
Unwanted Wanted

1-collinear



$$\stackrel{1c}{=} \sum_{i,j=2}^M z_i z_j (\theta_i^3 \theta_j + \boxed{\theta_i^2 \theta_j \theta_{ij}}) + \mathcal{O}(z^3)$$

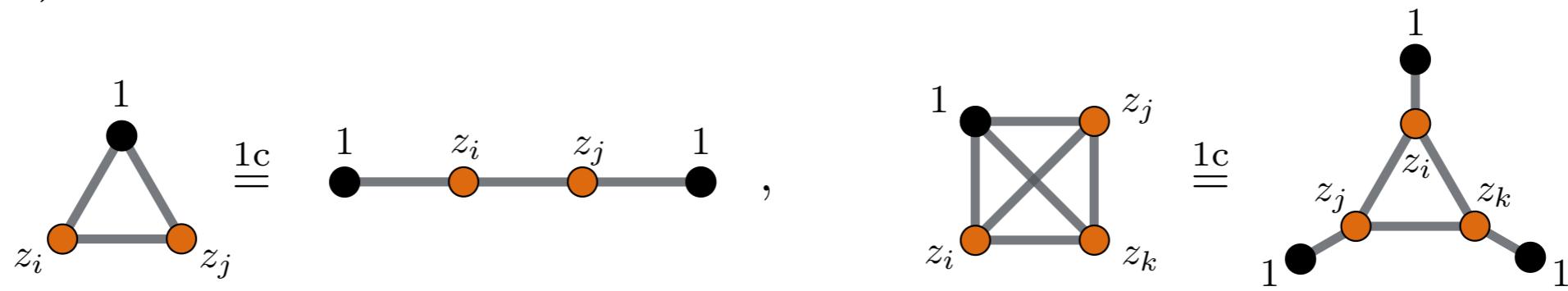
How do we get this term?



$$\stackrel{1c}{=} 2 \sum_{i,j=2}^M z_i z_j \theta_i^2 \theta_j \theta_{ij} + \mathcal{O}(z^3)$$

Color reduction for 1-collinear

- 1-collinear basis can be “color reduced”, i.e. n -color graphs can be traded for $(n - 1)$ colored, for $n \geq 3$



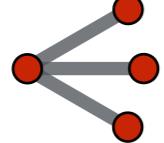
Degree 1	Degree 2	Degree 3	Degree 4
Degree 5			
Degree 6			

z^2 -truncated basis

- Another way to define leading power: **Lowest power of z across all EFPs**

Degree	EFP	$\mathcal{O}(z)$	$\mathcal{O}(z^2)$
1		$2 \sum_i z_i \theta_i$	$-2z_{\text{cs}} \sum_i z_i \theta_i + \sum_{i,j} z_i z_j \theta_{ij}$
2		$2 \sum_i z_i \theta_i^2$	$-2z_{\text{cs}} \sum_i z_i \theta_i^2 + \sum_{i,j} z_i z_j \theta_{ij}^2$
		$\sum_i z_i \theta_i^2$ 0	$-2z_{\text{cs}} \sum_i z_i \theta_i^2 + \sum_{i,j} z_i z_j (2\theta_i \theta_{ij} + \theta_i \theta_j)$ $4 \sum_{i,j} z_i z_j \theta_i \theta_j$

z^2 -truncated basis

Degree	EFP	$\mathcal{O}(z)$	$\mathcal{O}(z^2)$
3		$2 \sum_i z_i \theta_i^3$	$-2z_{\text{cs}} \sum_i z_i \theta_i^3 + \sum_{i,j} z_i z_j \theta_{ij}^3$
		$\sum_i z_i \theta_i^3$	$-2z_{\text{cs}} \sum_i z_i \theta_i^3 + \sum_{i,j} z_i z_j (\theta_i^2 \theta_{ij} + \theta_i^2 \theta_j + \theta_i \theta_{ij}^2)$
		$\sum_i z_i \theta_i^3$	$-3z_{\text{cs}} \sum_i z_i \theta_i^3 + 3 \sum_{i,j} z_i z_j \theta_i^2 \theta_{ij}$
		0	$4 \sum_{i,j} z_i z_j \theta_i^2 \theta_j$

- If we keep only $\mathcal{O}(z)$ terms, we end up with only Dumbbells in our basis
- Let's therefore find a basis that can recover all $\mathcal{O}(z^2)$

z^2 -truncated basis

- We dub this the z^2 -truncated basis

Degree	z^2 monomial basis									
1										
2										
3										
4										
5										
6										