The photon energy spectrum in $B \to X_s\gamma$ at $\mathrm{N^3LL'}$

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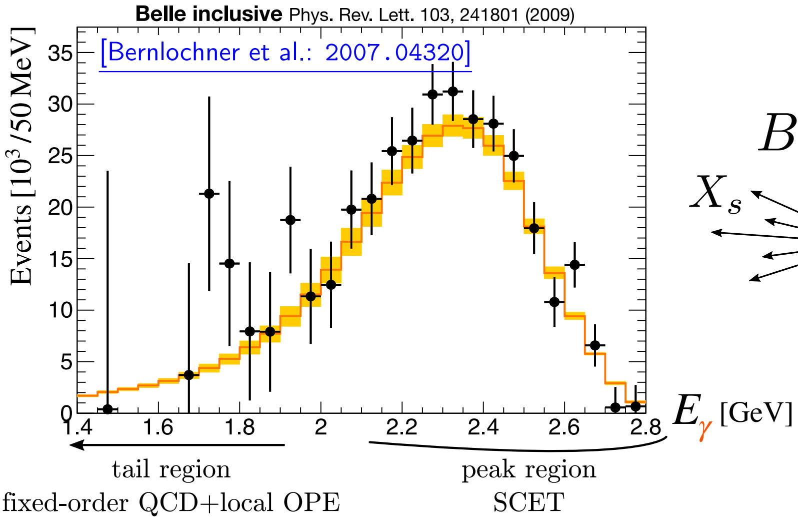
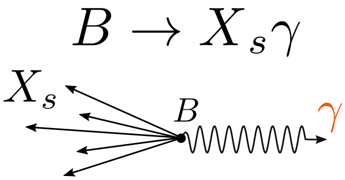
Outline

- How one can use SCET to understand $B \to X_s\gamma$, and why one would want to.
- We provide predictions for the photon energy spectrum in $B \to X_s\gamma$ at the 3-loop order, parameterizing the unknown 3-loop ingredients, which improves theoretical uncertainty.
- Impact of short-distance mass schemes, and why for $B \to X_s\gamma$ the MSR scheme is more appropriate than 1S scheme.

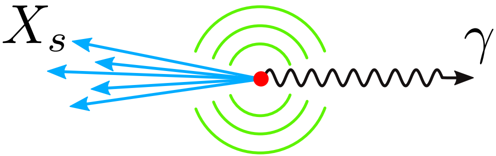
$B \to X_s\gamma$ spectrum





Measurements of $B \to X_s\gamma$ spectrum are most precise in the peak region $E_\gamma \sim m_b/2$, which is described by SCET.

$$B \to X_s\gamma$$



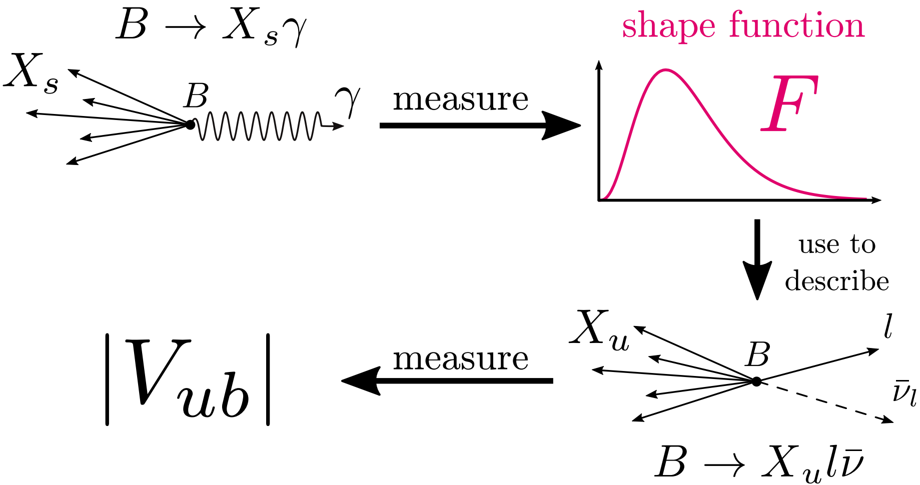
$$\mathrm{d}\Gamma \propto \overbrace{H \times J \otimes S}^{\text{perturbative}} \otimes F$$

nonperturbative, but universal

$$\mu_H \sim m_b \qquad \mu_J \sim \sqrt{m_b \Lambda_{\mathrm{QCD}}} \qquad \mu_S \sim \Lambda_{\mathrm{QCD}}$$

In the peak region leading-power SCET allows us to factorize $B \to X_s\gamma$ spectrum into perturbative hard, jet, (partonic) soft functions, and nonperturbative shape function.

Motivation



The shape function can be extracted from $B \to X_s\gamma$ spectrum and used to describe other B-meson decays, for example the $B \to X_u l\bar{\nu}$, which is sensitive to $|V_{ub}|$.

singular
(resummed)

$$\mathrm{d}\Gamma \propto (W_{\mathrm{resum}} + W_{\mathrm{ns}}) \otimes F$$

nonsingular
(fixed-order)

We match nonsingular contributions to reproduce full fixed-order QCD in the tail region, when resummation is turned off

Known perturbative ingredients

The only missing 3-loop ingredients are the nonsingular contributions $W_{\rm ns}$ and the hard function coefficient $H^{(3)}$

recently calculated by R.Brüser, Z.L.Liu, M.Stahlhofen 1804.09722, 1911.04494

R.Brüser, A.Grozin, J.M.Henn, M.Stahlhofen 1902.05076

ingredient	known at
$W_{\rm ns}$	2-loop
H	2-loop
J	3-loop
S	3-loop
γ_H	3-loop
γ_J	3-loop
γ_S	3-loop
$\Gamma_{\rm cusp}$	4-loop
$\delta m^{1S/\rm MSR}$	3-loop

nonsingular is derived by matching to fixed-order QCD

hard anomalous dimension is known from the consistency relation

$$\gamma_H + \gamma_J + \gamma_S = 0$$

(references in backup)

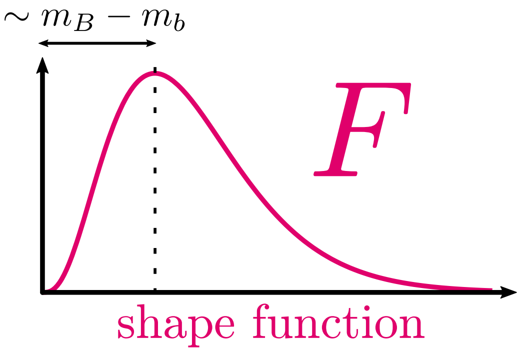
We implemented the $B \to X_s\gamma$ spectrum in `SCETlib` C++ library at $\mathrm{N^3LL'{+}N^3LO}$, parameterizing unknown 3-loop ingredients in terms of nuisance parameters normalization: $$\int_0^\infty F(k)\mathrm{d}k = 1$$

first moment: $$\int_0^\infty F(k)k\mathrm{d}k = m_B - m_b$$

B-meson mass

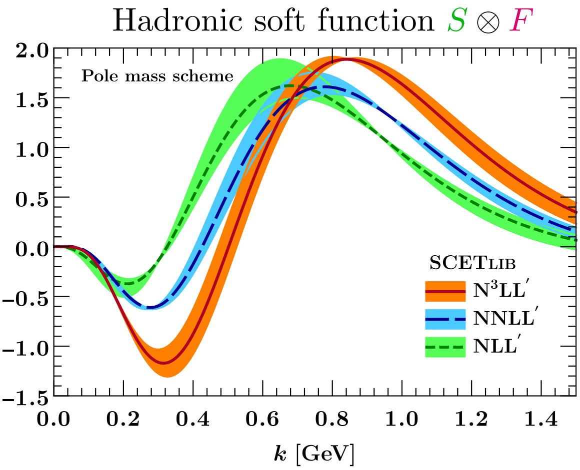
b-quark mass

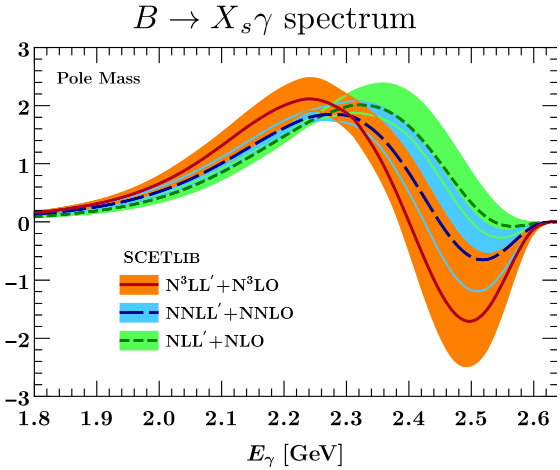
(in which scheme?)



The first moment of the shape function depends on the b-quark mass m_b. In order to get stable predictions it is essential to define m_b in a suitable short-distance mass scheme. At $\mathrm{N^3LO}$ the mass correction $\delta m = m^{\mathrm{pole}} - m^{\mathrm{short-distance}}$ must be calculated up to α_s^3.

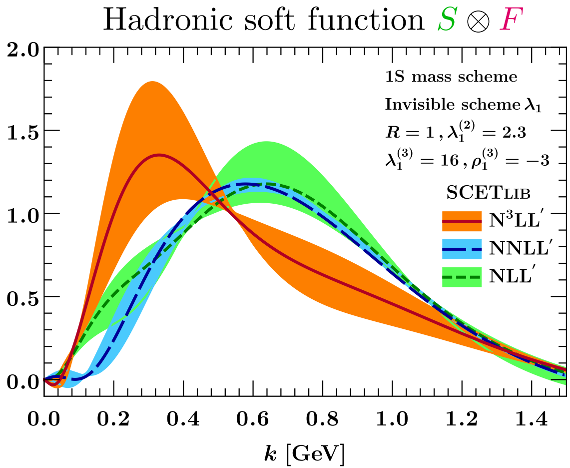
Pole mass scheme

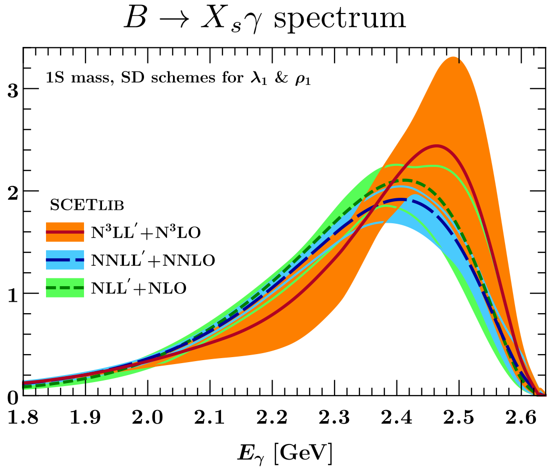




Pole mass scheme suffers from a renormalon ambiguity, and predictions in this scheme are not stable.

1S mass scheme





However, the 1S mass scheme, which has been used in the $\mathrm{N^2LL'+NNLO}$ shape function fit in [Bernlochner et al.: 2007.04320], starts to break down at $\mathrm{N^3LO}$

$$m_b^{\text{pole}} - m_b^{1S} =: \delta m_b^{1S}(\mu) = R^{1S}(\mu)\left(\sum_{k=1}^{3}\left(\frac{\alpha_s(\mu)}{\pi}\right)^k a_{1S}^{(k)} + \text{terms with } \ln\frac{\mu}{R^{1S}(\mu)} + \mathcal{O}(\alpha_s^4)\right)$$

intrinsic scale of 1S mass scheme

[K.Melnikov, A.Yelkhovsky: hep-ph/9805270]

[A.Pineda, F.J.Ynduráin: hep-ph/9711287]

$$R^{1S}(\mu) = m_b^{1S}\alpha_s(\mu)C_F$$

$$R^{1S}(m_b^{1S}) \approx R^{1S}(4.75) \approx 1.36 \sim \mu_S \qquad \text{ok!}$$

$$R^{1S}(\mu_S) \approx R^{1S}(1.3) \approx 2.4 \gg \mu_S \qquad \text{too large!}$$

(all values in GeV)

This is because the intrinsic scale of $1S$ scheme R^{1S} is small at the hard scale, but becomes too large at the soft scale

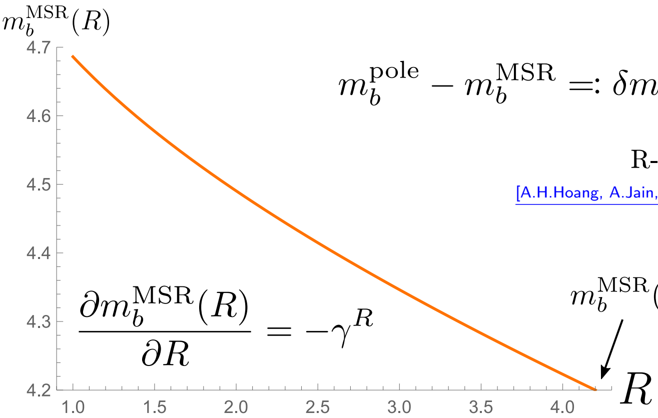
MSR mass scheme

$$m_b^{\text{pole}} - m_b^{\text{MSR}} =: \delta m_b^{\text{MSR}}(R) = R \sum_{k=1} \left(\frac{\alpha_s(R)}{\pi} \right)^k a_k^{\text{MSR}}$$

R-scale is a parameter

[A.H.Hoang, A.Jain, C.Lepenik, V.Mateu, M.Preisser, I.Scimemi, I.W.Stewart: 1704.01580]

[A.H.Hoang, A.Jain, I.Scimemi, I.W.Stewart: 0803.4214]



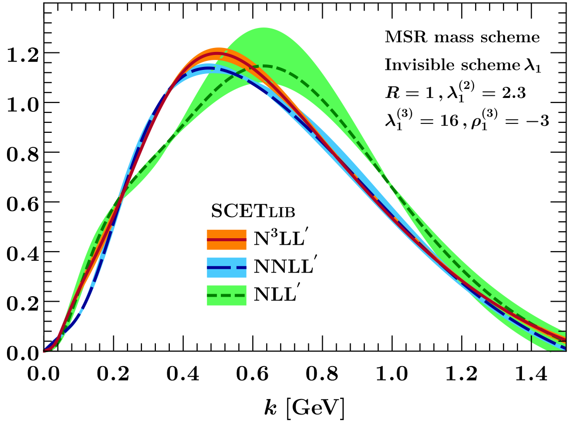
$$\frac{\partial m_b^{\text{MSR}}(R)}{\partial R} = -\gamma^R$$

$$m_b^{\text{MSR}}(\bar{m}_b) = \bar{m}_b(\bar{m}_b)$$

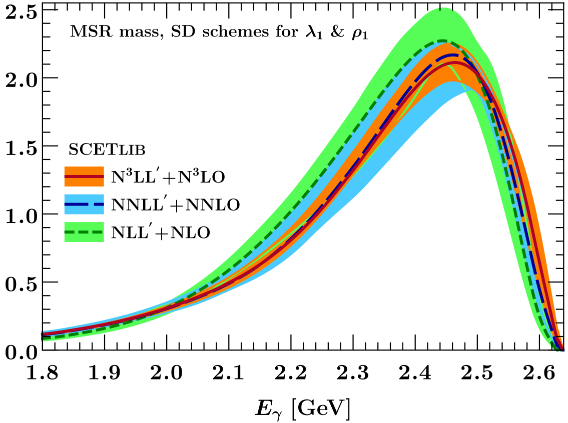
The MSR mass $m_b^{\text{MSR}}(R)$ depends on scale R as a parameter. Masses at different R-scales are related by the R-evolution equation.

The MSR mass is a natural extension of the $\overline{\text{MS}}$ mass for scales below the mass of the quark.

Hadronic soft function $S \otimes F$



$B \to X_s\gamma$ spectrum



The MSR scheme yields much more stable results because we can pick the R-scale $R \sim \mu_S$

1S vs MSR numerically

$$R^{\text{MSR}} = \mu \qquad\qquad \sim \alpha_s \quad \sim \alpha_s^2 \quad \sim \alpha_s^3$$

$$(m_b^{\text{MSR}} - m_b^{1S})(\mu = 4.2) \approx -0.35 - 0.12 - 0.04 \leftarrow \text{converges}$$

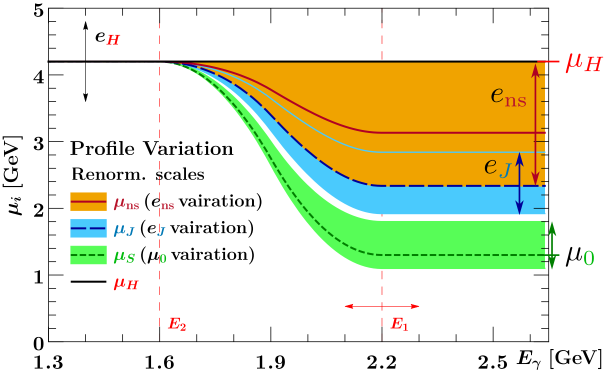
$$(m_b^{\text{MSR}} - m_b^{1S})(\mu = 1.93) \approx -0.15 - 0.06 + 0.02 \leftarrow \text{converges}$$

$$(m_b^{\text{MSR}} - m_b^{1S})(\mu = 1.3) \approx -0.06 - 0.06 + 0.10 \leftarrow \text{does not converge}$$

(all values in GeV)

The perturbative series of correction between MSR and 1S schemes seems to start diverging as we approach the soft scale.

$$\mathrm{d}\Gamma \propto \left(W_{\text{resum}}(\mu_H, \mu_J, \mu_S) + W_{\text{ns}}(\mu_{\text{ns}})\right) \otimes F$$



$$\mu_H = e_H m_b$$

$$\mu_{\text{ns}}(E_\gamma) = \mu_S(E_\gamma)^{\frac{1-e_{\text{ns}}}{4}} \mu_H^{\frac{3+e_{\text{ns}}}{4}}$$

$$\mu_J(E_\gamma) = \mu_S(E_\gamma)^{\frac{1-e_J}{2}} \mu_H^{\frac{1+e_J}{2}}$$

$\mu_S(E_\gamma)$ interpolates between μ_0 and μ_H

We use the so-called profile functions to smoothly turn off the resummation away from the peak region by setting all scales to the same value. The profile functions depend on parameters e_H, e_{ns}, e_J, μ_0, E_1, which are varied to estimate the perturbative uncertainty the only unknown term

fixed by RGE $\dfrac{\partial H(\mu)}{\partial \ln \mu} = \gamma_H \times H$

$$H(\mu) = 1 + \frac{\alpha_s(\mu)}{\pi} H^{(1)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 H^{(2)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 H^{(3)} + \text{terms with logs } \ln\frac{\mu}{m_b} + \mathcal{O}(\alpha_s^4)$$

$$\frac{(H^{(2)})^2}{H^{(1)}} \approx \frac{19.3^2}{4.55} \approx 80 \implies H^{(3)} = 0 \pm 80$$

For the hard function H the only missing piece is the 3-loop constant $H^{(3)}$.
We set its central value to 0 and use Padé approximation to estimate its possible magnitude.

singular without resummation

$$x = 1 - \frac{2E_\gamma}{m_b} \in [0,1]$$

$$W_\mathrm{s}(x) = \frac{\alpha_s}{\pi}\sigma_\mathrm{s}^{(1)}(x) + \left(\frac{\alpha_s}{\pi}\right)^2 \sigma_\mathrm{s}^{(2)}(x) + \left(\frac{\alpha_s}{\pi}\right)^3 \sigma_\mathrm{s}^{(3)}(x) + \text{terms with } \ln\frac{\mu}{m_b} + \mathcal{O}(\alpha_s^4)$$

$$W_\mathrm{ns}(x) = \frac{\alpha_s}{\pi}\sigma_\mathrm{ns}^{(1)}(x) + \left(\frac{\alpha_s}{\pi}\right)^2 \sigma_\mathrm{ns}^{(2)}(x) + \left(\frac{\alpha_s}{\pi}\right)^3 \sigma_\mathrm{ns}^{(3)}(x) + \text{terms with } \ln\frac{\mu}{m_b} + \mathcal{O}(\alpha_s^4)$$

nonsingular

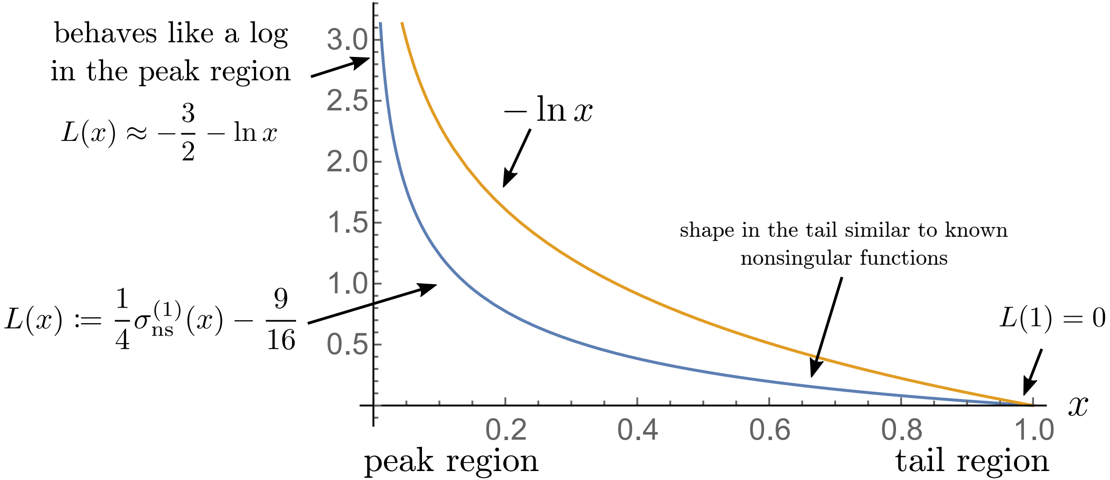
unknown function

$$\text{parameterize } \sigma_\mathrm{ns}^{(3)}(x) = -\sigma_\mathrm{s}^{(3)}(1) + \sum_{k=0}^{5} c_k L^k(x)$$

model cancellation between singular and nonsingular in the tail region $(x \to 1)$

$$L(x) := \frac{1}{4}\sigma_\mathrm{ns}^{(1)}(x) - \frac{9}{16}$$

For the nonsingular terms W_ns only the 3-loop function $\sigma_\mathrm{ns}^{(3)}(x)$ is unknown. We parameterize it using six parameters $c_0 \ldots c_5$.



The function $L(x)$ used in the parameterization is similar to $-\ln x$, but has a more realistic shape in the transition region $0 < x < 1$.

Nonsingular terms

In the peak region, where $x \to 0$:

$$4x\sigma_{\mathrm{s}}^{(1)}(x) \approx -7.0 - 4\ln x$$

$$\sigma_{\mathrm{ns}}^{(1)}(x) \approx -3.8 - 4\ln x$$

similar

$$4x\sigma_{\mathrm{s}}^{(2)}(x) \approx 28.8 + 46.7\ln x + 26.5\ln^2 x + 2.7\ln^3 x$$

$$\sigma_{\mathrm{ns}}^{(2)}(x) \approx 16.1 + 33.9\ln x + 25\ln^2 x + 2.7\ln^3 x$$

similar

$$\begin{aligned} 4x\sigma_{\mathrm{s}}^{(3)}(x) &\approx 406.3 + 142.5\ln x - 113.2\ln^2 x - 125.5\ln^3 x - 21.7\ln^4 x - 0.9\ln^5 x \\ &\approx 258.5 + 95.0L(x) + 189.0L^2(x) + 15.5L^3(x) - 15L^4(x) + 0.9L^5(x) \end{aligned}$$

$$\sigma_{\mathrm{ns}}^{(3)}(x) = c_0 + c_1 L(x) + c_2 L^2(x) + c_3 L^3(x) + c_4 L^4(x) + c_5 L^5(x) - \sigma_{\mathrm{s}}^{(3)}(1)$$

$c_0 = 0 \pm 20$ (estimated differently) $\quad c_1 = 0 \pm 100 \quad c_2 = 0 \pm 80 \quad c_3 = 0 \pm 10 \quad c_4 = 0 \pm 5 \quad c_5 = 0 \pm 0.9$

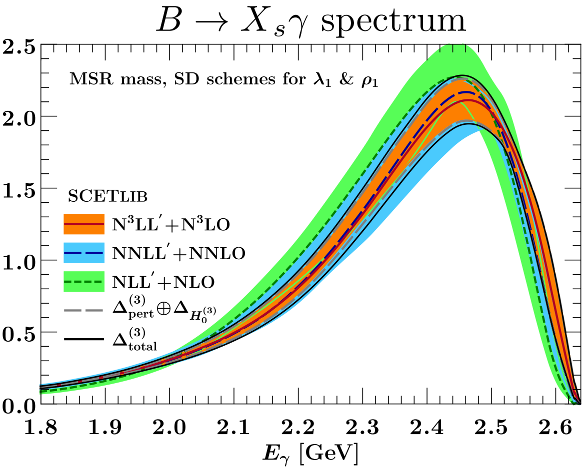
The asymptotics of nonsingular functions $\sigma_{\mathrm{ns}}^{(k)}(x)$ are similar to asymptotics of $4x\sigma_{\mathrm{s}}^{(k)}$.
We exploit this to estimate the possible magnitude of model coefficients c_k.

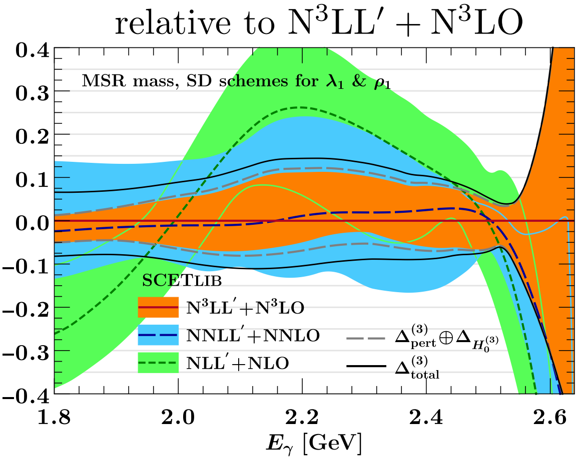
Components of perturbative uncertainty

$$\Delta = \sqrt{\Delta^2_{\text{resum}} + \Delta^2_{\text{nonsingular}} + \Delta^2_{\text{matching}} + \underbrace{\overbrace{\Delta^2_{H^{(3)}} + \sum_{k=0}^{5} \Delta^2_{c_k}}^{\text{variations of nuisance parameters}}}_{(\text{only at N}^3\text{LL}' + \text{N}^3\text{LO})}}$$

$$\begin{aligned}
\Delta_{\text{resum}} &= \text{envelope of } (e_H, e_J, \mu_0) \text{ variations} \\
&= \text{scale variations in resummed terms} \\
\Delta_{\text{nonsingular}} &= \Delta_{e_{\text{ns}}} = \text{scale variations in nonsingular terms} \\
\Delta_{\text{matching}} &= \Delta_{E_1} = \text{variations of peak region edge}
\end{aligned}$$

We add in quadrature uncertainties from variations of scales and nuisance parameters.

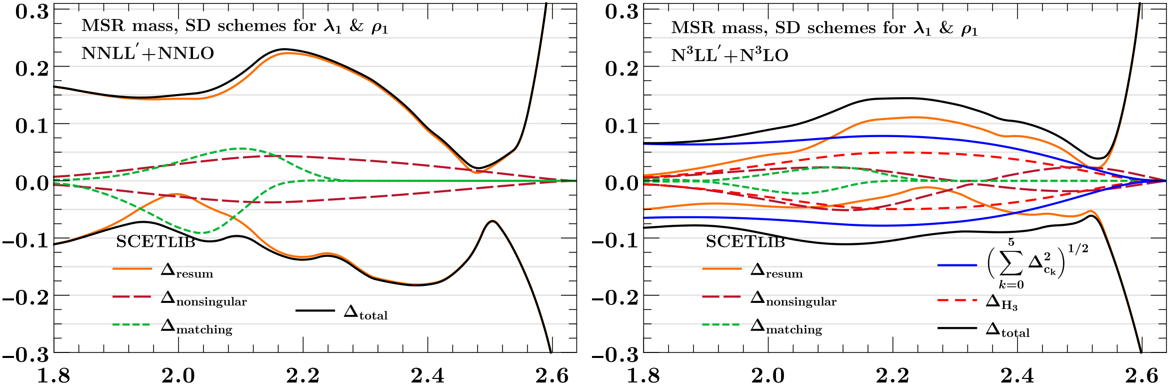




The predictions at different orders are converging well, and the uncertainties are under control.

Relative uncertainty from different sources

$$\Delta = \sqrt{\Delta^2_{\text{resum}} + \Delta^2_{\text{nonsingular}} + \Delta^2_{\text{matching}} + \Delta^2_{H^{(3)}} + \sum_{k=0}^{5} \Delta^2_{c_k}}$$



As expected, unknown 3-loop nonsingular terms are not relevant in the peak region, but increase the uncertainty towards the tail

Conclusions

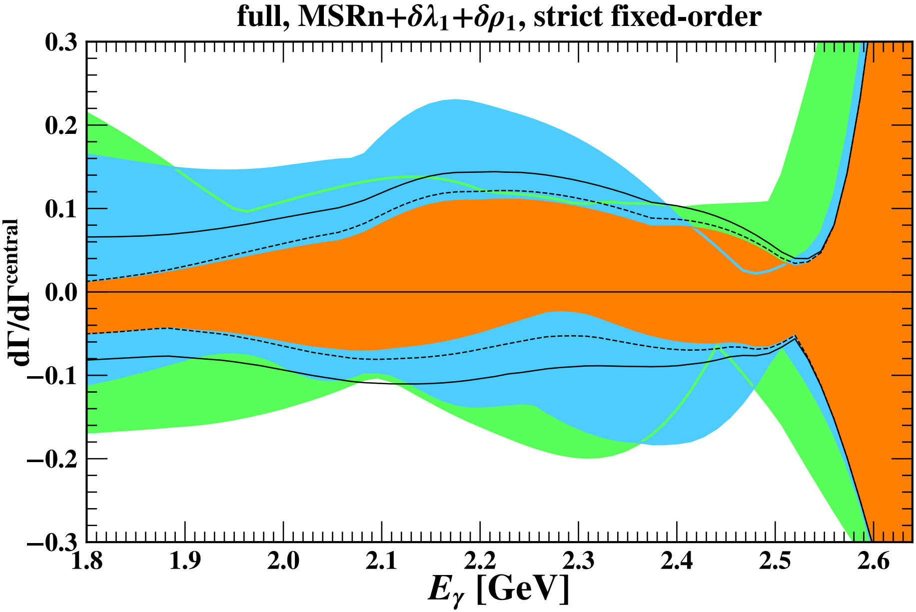
- Theoretical uncertainties are improved by extending $B \to X_s\gamma$ spectrum predictions to 3-loop level, in spite of the fact that some 3-loop ingredients are not known.
- At 3-loop order the 1S mass scheme does not work for scales much lower than m_b
- MSR mass scheme is appropriate for this process
- Not discussed: starting at 3-loop order the pole mass must be consistently expanded in hard and jet functions to resum a formally-subleading, but nevertheless singular term
- Not discussed: short-distance schemes for hadronic parameters λ_1, ρ_1

Thank you for your attention!

Backup slides

Known perturbative ingredients

- 1-loop hard function [C.W.Bauer, S.Fleming, D.Pirjol, I.W.Stewart: hep-ph/0011336]
- 1-loop jet and soft functions [C.W.Bauer, A.V.Manohar: hep-ph/0312109] [S.W.Bosch, B.O.Lange, M.Neubert, G.Paz: hep-ph/0402094]
- 2-loop full QCD [K.Melnikov, A.Mitov: hep-ph/0505097]
- 2-loop soft function [T.Becher, M.Neubert: hep-ph/0512208]
- 2-loop jet function [T.Becher, M.Neubert: hep-ph/0603140]
- 3-loop jet function [R.Brüser, Z.L.Liu, M.Stahlhofen: 1804.09722]
- 4-loop Γ_{cusp} [A.Manteuffel, E.Panzer, R.M.Schabinger: 2002.04617] and references therein
- 3-loop soft function [R.Brüser, Z.L.Liu, M.Stahlhofen: 1911.04494]


full, MSRn+δλ1+δρ1, strict fixed-order
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
1.8
1.9
2.0
2.1
2.2
2.3
2.4
2.5
2.6
Eγ [GeV]
dΓ/dΓcentral

$$\int_0^\infty F(k)\mathrm{d}k = 1$$

$$\int_0^\infty F(k)k\mathrm{d}k = m_B - m_b$$

$$\int_0^\infty F(k)k^2\mathrm{d}k = (m_B - m_b)^2 - \frac{\lambda_1}{3}$$

$$\int_0^\infty F(k)k^3\mathrm{d}k = (m_B - m_b)^3 - \lambda_1(m_B - m_b) + \frac{\rho_1}{3}$$

Hadronic soft:

$$(S \otimes F)(k) = \langle B|\bar{b}_v \delta((n, iD) - (m_B - m_b) + k) b_v|B\rangle$$

Hadronic parameters:

$$\langle B|\bar{b}_v (iD_\alpha)(iD_\mu)(iD_\beta) b_v|B\rangle = \frac{\rho_1}{3}(g_{\alpha\beta} - v_\alpha v_\beta)v_\mu$$

$$\langle B|\bar{b}_v (iD)^2 b_v|B\rangle = \lambda_1$$

$$\sigma_{\mathrm{s}}^{(1)}(1) + \sigma_{\mathrm{ns}}^{(1)}(1) = \frac{9}{4} - \frac{7}{4} = \frac{1}{2}$$

$$\sigma_{\mathrm{s}}^{(2)}(1) + \sigma_{\mathrm{ns}}^{(2)}(1) \approx 7.20 - 4.28 = 2.92$$

$$c_0 = \sigma_{\mathrm{s}}^{(3)}(1) + \sigma_{\mathrm{ns}}^{(3)}(1) \approx 101.57 + ? \quad \text{expect } c_0 \approx \frac{2.92^2}{0.5} \approx 20$$

There's a somewhat large finite cancellation between singular and nonsingular in the tail region, where $x \to 1$