Soft-collinear gravity beyond leading power

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Soft-collinear gravity beyond leading power

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Based on

[2110.02969]

[2112.04983]

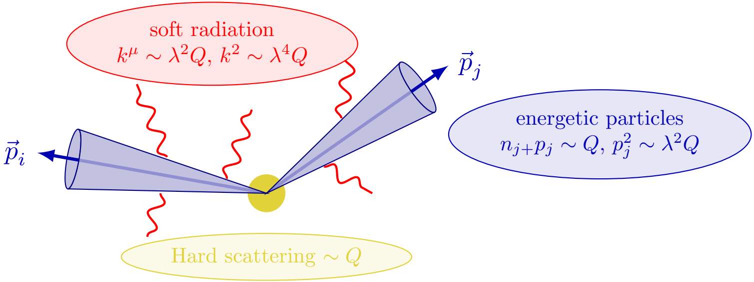
in collaboration with

Martin Beneke (TU München), Robert Szafron (BNL)

SCET Gravity at NLP?

- Similarities and differences between gauge theory and gravity.
- Recent interest in a systematic construction.
  [Beneke, Kirilin 1207.4926; Okui, Yunesi 1710.07685, + Chakraborty 1910.10738]
- So far no rigorous derivation of the subleading Lagrangian.
- Soft gravity is similar to gauge theory: Soft theorem, LBK
- Collinear gravity is different: collinear divergences are absent

Kinematics



soft radiation
$k^\mu \sim \lambda^2 Q$, $k^2 \sim \lambda^4 Q$
$\vec{p}_j$
$\vec{p}_i$
energetic particles
$n_{j+}p_j \sim Q$, $p_j^2 \sim \lambda^2 Q$
Hard scattering $\sim Q$

How to construct SCET

- Introduce the mode split, implement gauge symmetry,

$$A_\mu = A_{c\mu} + A_{s\mu} \,, \quad \phi = \phi_c + W Z^\dagger \phi_s$$

- Control leading building blocks

$$n_+ A_c \sim 1 \,.$$

- Multipole-expand the soft fields in soft-collinear products,

$$A_s(x) = A_s(x_-) + (x - x_-) \cdot \partial A_s + \dots$$

- Ensure gauge symmetry respects multipole expansion.

$$\phi_c(x) \to U_s(x_-) \phi_c(x) \,.$$

Gauge Transformations

- Collinear fluctuations on top of a soft background field:

$$\begin{aligned}
&\text{collinear:} \quad && A_c \to U_c A_c U_c^\dagger + \frac{i}{g} U_c \left[D_s \,, U_c^\dagger \right] && \Phi_c \to U_c \Phi_c \\
& && A_s \to A_s && \Phi_s \to \Phi_s \\
&\text{soft:} && A_c \to U_s A_c U_s^\dagger && \Phi_c \to U_s \Phi_c \\
& && A_s \to U_s A_s U_s^\dagger + \frac{i}{g} U_s \left[\partial \,, U_s^\dagger \right] && \Phi_s \to U_s \Phi_s
\end{aligned}$$

- Soft gluon appears in collinear transformation.
- Soft fields do not transform under collinear gauge symmetry.
- Collinear fields transform as ordinary matter fields under soft gauge.
- Soft fields have standard transformation under soft gauge.

Large collinear gluon component

- Collinear gluon contains $n_+ A_c \sim 1$.
- To get finite operator basis, this component must be under control.
- Use collinear Wilson line W_c to define $\mathcal{A}_c$ satisfying $n_+ \mathcal{A}_c = 0$:

$$g\mathcal{A}_{c\mu} = W_c^\dagger \left[i\hat{D}_{c\mu} W_c \right]$$

- $\mathcal{A}_c$ is manifestly collinear gauge-invariant and can be used in operator basis.
- $n_+ A_c$ only appears in the collinear Wilson line W_c

$$W_c(x) = P \exp \left(ig \int_{-\infty}^{0} ds \, n_+ \hat{A}_c(x + s n_+) \right) .$$

Multipole Expansion

- Soft fields must be multipole expanded around $x_-^\mu = n_+x\frac{n_-^\mu}{2}$.
  - Gauge-symmetry must respect multipole expansion.
  - Redefinition of collinear fields necessary

$$\hat{A}_c \to U_c\hat{A}_cU_c^\dagger + \frac{i}{g}U_c\left[\hat{D}_s, U_c^\dagger\right],$$

    where $\hat{D}_s^\mu = \partial^\mu - ig\frac{n_+^\mu}{2}n_-A_s(x_-)$.
  - This is equivalent to the soft field A_s in fixed-line gauge $(x-x_-)^\mu A_{s\mu}(x) = 0$.

- Subleading interactions expressed via field-strength tensor, e.g.

$$x_\perp^\mu n_-^\nu\left[\partial_\mu A_{s\nu}\right](x_-)n_+J_c(x) \to x_\perp^\mu n_-^\nu F_{\mu\nu}^s n_+J_c(x)\,.$$

  - Compare to non-relativistic theory: $n_-^\mu \to \delta_0^\mu$

$$x_\perp^\mu F_{\mu-}J_+ \to x^iF^{i0}J^0 \sim \vec{x}\cdot\vec{E}\,J^0\,.$$

    $\to$ Dipole interaction.

Key Observation

By homogenising the gauge symmetry, we find a theory that is covariant with respect to the emergent background field $n_-A_s(x_-)$. All interactions due to multipole expansion are expressed via field-strength tensor. This gives a very transparent structure how gauge symmetry is organised in EFT.

Soft-collinear matter Lagrangian takes the simple structure

$$\mathcal{L}^{(0)} = \frac{1}{2} n_+ D_c \bar{\phi} n_- D \phi + \frac{1}{2} D_{c\perp} \bar{\phi} D_{c\perp} \phi + \text{h.c.}$$

$$\mathcal{L}^{(1)} = \frac{1}{2} x_\perp^\mu n_-^\nu g F^s_{\mu\nu} n_+ J_c$$

$$\mathcal{L}^{(2)} = x_\perp^\mu g F^s_{\mu\nu} J^\nu_{c\perp} + \frac{1}{4} n_- x n_+^\mu n_-^\nu g F^s_{\mu\nu} n_+ J_c + \frac{1}{4} x_\perp^\mu x_\perp^\rho n_-^\nu g \left[D^s_\rho, F^s_{\mu\nu} \right] n_+ J_c$$

with $J_c^\mu = \bar{\phi}_c \overleftarrow{D}_c^\mu \phi_c + \bar{\phi}_c \overrightarrow{D}_c^\mu \phi_c$.

QCD vs Gravity

	QCD	Gravity
Fundamental Degree of Freedom	$A_\mu \sim p_\mu$	$h_{\mu\nu} \sim \frac{p_\mu p_\nu}{\lambda}$
Field-strength / curvature	$F_{\mu\nu} \sim \partial A$	$R^\mu{}_{\nu\alpha\beta} \sim \partial^2 h$
Gauge Symmetry	$SU(3)$	$\text{Diff}(M)$
Coupling Dimensionful?	no	yes

Two Sources of Inhomogeneity

- In full theory: gauge charges P^{μ} and coupling κ are inhomogeneous in λ.
  - Leads to relations for higher-order terms to form geometric objects.
  - This is different from QCD – gauge charges have no scaling in λ.
- From multipole expansion: evaluate soft fields at x_-.
  - Conceptually the same as in gauge theory.
  - Deal with it in similar fashion.

SCET Gravity Construction

- Minimally-coupled scalar field

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \sqrt{-g}\frac{\lambda_\varphi^4}{4!}\varphi^4 \,.$$

- Perform κ expansion in collinear sector $g_{\mu\nu} = g_{s\mu\nu} + \kappa h_{\mu\nu}$.
  - Collinear graviton $h_{\mu\nu}$ in presence of soft dynamical background $g_{s\mu\nu}$.
  - Duplicate soft and collinear gauge symmetry, not yet homogeneous.
- Field content: $h_{\mu\nu}$, $g_{s\mu\nu} = \eta_{\mu\nu} + \kappa s_{\mu\nu}$, φ_c, φ_s.
- Scaling: $h_{\mu\nu} \sim \frac{p_\mu p_\nu}{\lambda}$, $s_{\mu\nu} \sim \lambda^2$.

Gauge Transformations

- Collinear fluctuations on top of a soft background field:

$$
\begin{aligned}
\text{collinear:} \quad & h_{\mu\nu} \to \left[U_c \left({U_{c\mu}}^{\alpha} {U_{c\nu}}^{\beta} (g_{s\alpha\beta} + h_{\alpha\beta})\right)\right] - g_{s\mu\nu} & \varphi_c \to [U_c \varphi_c] \\
& g_{s\mu\nu} \to g_{s\mu\nu} & \varphi_s \to \varphi_s \\
\text{soft:} \quad & h_{\mu\nu} \to \left[U_s \left({U_{s\mu}}^{\alpha} {U_{s\nu}}^{\beta} h_{\alpha\beta}\right)\right] , & \varphi_c \to [U_s \varphi_c] \\
& g_{s\mu\nu} \to \left[U_s \left({U_{s\mu}}^{\alpha} {U_{s\nu}}^{\beta} g_{s\alpha\beta}\right)\right] & \varphi_s \to [U_s \varphi_s]
\end{aligned}
$$

- ▶ $U = 1 + \varepsilon^{\alpha} \nabla_{\alpha} + \ldots$ translation, ${U_{\mu}}^{\alpha}$ inverse Jacobian.

- Soft background appears in collinear transformation.

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- Soft fields have standard transformation under soft gauge.

Power-enhanced component

- Collinear graviton scales $h_{++} \sim \lambda^{-1}$, $h_{+\perp} \sim 1$.
- Solved like in QCD, using collinear light-cone gauge.
  [Beneke, Kirilin 1207.4926; Okui, Yunesi 1710.07685, + Chakraborty 1910.10738]
- Use collinear "Wilson line" to define $\mathfrak{h}_{\mu\nu}$ satisfying $\mathfrak{h}_{\mu+} = 0$:

$$\mathfrak{h}_{\mu\nu} = W_c{}^{\alpha}{}_{\mu} W_c{}^{\beta}{}_{\nu} \left[W_c^{-1} (\eta_{\alpha\beta} + h_{\alpha\beta}) \right] - \eta_{\mu\nu} \,.$$

- $\mathfrak{h}_{\mu\nu}$ is manifestly collinear gauge-invariant, $h_{\mu+}$ appears only in Wilson line.
- Similarity at linear level

$$\mathcal{A}_{c\mu} = A_{c\mu} - \partial_\mu \frac{n_+ A_c}{n_+ \partial} + \ldots$$

$$\mathfrak{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu \left(\frac{h_{\nu+}}{n_+\partial} - \frac{1}{2} \partial_\nu \frac{h_{++}}{(n_+\partial)^2} \right) - \partial_\nu \left(\frac{h_{\mu+}}{n_+\partial} - \frac{1}{2} \partial_\mu \frac{h_{++}}{(n_+\partial)^2} \right) + \ldots$$

Soft Multipole-expansion

- Soft fields must be multipole-expanded about $x_- = n_+ x \frac{n_-}{2}$.
- Gauge symmetry is already inhomogeneous in λ due to scaling of the charges.

What is the guiding principle?

Soft Multipole-expansion

- Soft fields must be multipole-expanded about $x_- = n_+ x \frac{n_-}{2}$.
- Gauge symmetry is already inhomogeneous in λ due to scaling of the charges.

What is the guiding principle?

Recall QCD:

- Homogeneous gauge symmetry respects the multipole expansion.
- Moves the soft field into fixed-line gauge.
- Subleading interactions expressed via field-strength tensor.

A simpler example

- Consider multipole-expansion about $x = 0$:
  - In gauge theory: fixed-point gauge $x^\mu A_\mu(x) = 0$ is convenient choice
  - Gauge field expressed as

$$A_\nu(x) = \int_0^1 ds\, s x^\mu F_{\mu\nu}(sx) = x^\mu F_{\mu\nu} + \mathcal{O}(x^2)$$

A simpler example

- Consider multipole-expansion about $x = 0$:
  - In gauge theory: fixed-point gauge $x^\mu A_\mu(x) = 0$ is convenient choice
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$$A_\nu(x) = \int_0^1 ds\, s x^\mu F_{\mu\nu}(sx) = x^\mu F_{\mu\nu} + \mathcal{O}(x^2)$$

- Gravitational analogue of fixed-point gauge: Riemann normal coordinates
  - Metric tensor expressed as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} x^\alpha x^\beta R_{\mu\alpha\nu\beta} + \mathcal{O}(x^3)$$

  - Gauge condition: $x^\alpha x^\beta \Gamma^\mu_{\alpha\beta}(x) = 0$

Fixed-line Normal Coordinates

- Generalisation in QCD: $(x - x_-)^\mu A_\mu(x) = 0$.
- Construct normal coordinates only in the transverse directions $x_\perp$ and $n_- x$.
- In RNC: Can choose $g_{\mu\nu}(0) = \eta_{\mu\nu}$. This is not possible in fixed-line case.
- Do the best alternative: $g_{\mu_\perp \nu_\perp}(x_-) = \eta_{\mu_\perp \nu_\perp}$, $g_{++}(x_-) = g_{+\perp}(x_-) = 0$.
- Coordinate transformation:

$$x^\mu = \underbrace{\check{x}^\mu + ({E^\mu}_\alpha - \delta^\mu_\alpha)(\check{x} - \check{x}_-)^\alpha}_{\text{Change } g_{\perp\perp} \to \eta_{\perp\perp}} - \underbrace{\frac{1}{2}(\check{x} - \check{x}_-)^\rho (\check{x} - \check{x}_-)^\sigma {E^\alpha}_\rho {E^\beta}_\sigma {\Gamma^\mu}_{\alpha\beta}}_{\text{Normal coordinates in transverse } x - x_- \text{ part}}$$

$$+ \underbrace{\frac{1}{6}(\check{x} - \check{x}_-)^\rho (\check{x} - \check{x}_-)^\sigma (\check{x} - \check{x}_-)^\kappa {E^\alpha}_\rho {E^\beta}_\sigma {E^\nu}_\kappa \left(2{\Gamma^\mu}_{\alpha\lambda}{\Gamma^\lambda}_{\beta\nu} - \left[\partial_\nu {\Gamma^\mu}_{\alpha\beta}\right]\right)}_{\text{Subleading normal coordinates in transverse coordinate}}$$

$$+ \mathcal{O}(\check{x}^4)\,.$$

Impact on metric tensor

- Can identify a residual dynamic soft background metric $\hat{g}_{s\mu\nu}$:

$$\begin{aligned}
\hat{g}_{s-+} &= e_{-+} - (x-x_-)^\alpha \left[\Omega_-\right]_{\alpha+} \\
\hat{g}_{s-\mu_\perp} &= e_{-\mu_\perp} - (x-x_-)^\alpha \left[\Omega_-\right]_{\alpha\mu_\perp} \\
\hat{g}_{s--} &= \left(e_-{}^\alpha - (x-x_-)^\rho \left[\Omega_-\right]_\rho{}^\alpha\right)\left(e_-{}^\beta - (x-x_-)^\sigma \left[\Omega_-\right]_\sigma{}^\beta\right)\eta_{\alpha\beta} \\
\hat{g}_{s\mu_\perp\nu_\perp} &= \eta_{\mu_\perp\nu_\perp}
\end{aligned}$$

- Determined in terms of vierbein $e_-{}^\alpha(x_-)$ and spin-connection $\left[\Omega_-\right]_{\alpha\beta}(x_-)$.
- These fields are independent when viewed from the EFT.
- Can be arranged in a soft-covariant derivative

$$n_- D_s = \hat{g}_s^{-\mu}\partial_\mu$$

- Quadrupole and higher-pole terms: expressed via Riemann tensor

Main Takeaway

"Homogeneous" symmetry in Gravity consists of local Poincaré transformations. This implies a covariant derivative that contains the momentum as well as the angular momentum. All other interaction terms due to multipole-expansion are expressed via Riemann tensor and its derivative.

Schematically, the scalar-soft graviton Lagrangian takes the form

$$\mathcal{L}_{\varphi\varphi s} = \frac{1}{2} n_+\partial\varphi n_- D_s \varphi + \frac{1}{2}\partial_\perp\varphi\partial_\perp\varphi - \frac{\kappa}{8} x_\perp^\alpha x_\perp^\beta R_{\alpha-\beta-} n_+\varphi n_+\varphi + \mathcal{O}(\lambda^3)\,,$$

where

$$n_- D_s = n_-\partial - \underbrace{\frac{\kappa}{2} s_{-\alpha}\partial^\alpha}_{\text{from vierbein}} - \underbrace{\frac{\kappa}{4}(\partial_\alpha s_{\beta-} - \partial_\beta s_{\alpha-})}_{\text{from spin-connection}} J^{\alpha\beta} + \mathcal{O}(s^2)$$

$$J^{\alpha\beta} = (x - x_-)^\alpha \partial^\beta - (x - x_-)^\beta \partial^\alpha$$

This is the transparent form we wanted, similar to QCD.

The SCET Gravity Lagrangian

$$\mathcal{L}^{(0)}_{D_s} = \frac{1}{2}\partial_+\hat{\varphi}_c D_-\hat{\varphi}_c + \frac{1}{2}\partial_{\alpha_\perp}\hat{\varphi}_c\partial^{\alpha_\perp}\hat{\varphi}_c - \frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4$$

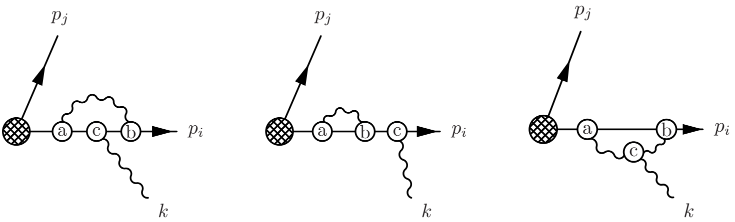
$$\mathcal{L}^{(1)}_{D_s} = -\frac{1}{2}\hat{h}^{\mu\nu}\partial_\mu\hat{\varphi}_c\partial_\nu\hat{\varphi}_c + \frac{1}{4}\hat{h}^{\beta_\perp}{}_{\beta_\perp}\left(\partial_+\hat{\varphi}_c D_-\hat{\varphi}_c + \partial_{\alpha_\perp}\hat{\varphi}_c\partial^{\alpha_\perp}\hat{\varphi}_c\right) - \frac{1}{2}\hat{h}^{\alpha_\perp}{}_{\alpha_\perp}\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4$$

$$\begin{aligned}\mathcal{L}^{(2)}_{D_s} = &-\frac{1}{8}x_\perp^\alpha x_\perp^\beta R^s_{\alpha-\beta-}(\partial_+ W_c^{-1}\hat{\varphi}_c)^2 \\ &+ \frac{1}{2}\hat{h}^{\mu\alpha}\hat{h}^\nu_\alpha\partial_\mu\hat{\varphi}_c\partial_\nu\hat{\varphi}_c - \frac{1}{4}\hat{h}^{\alpha_\perp}{}_{\alpha_\perp}\hat{h}^{\mu\nu}\partial_\mu\hat{\varphi}_c\partial_\nu\hat{\varphi}_c \\ &+ \frac{1}{16}\left((\hat{h}^{\alpha_\perp}{}_{\alpha_\perp})^2 - 2\hat{h}^{\alpha\beta}\hat{h}_{\alpha\beta}\right)\left(\partial_+\hat{\varphi}_c D_-\hat{\varphi}_c + \partial_{\mu_\perp}\hat{\varphi}_c\partial^{\mu_\perp}\hat{\varphi}_c\right) \\ &- \left(\frac{1}{8}(\hat{h}^{\alpha_\perp}{}_{\alpha_\perp})^2 - \frac{1}{4}\hat{h}^{\mu\nu}\hat{h}_{\mu\nu}\right)\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4\end{aligned}$$

Application: Loop corrections to the soft theorem

[Bern, Davies, Nohle 1405.1015]

- Gravity differs from gauge theory:
  - Purely collinear: κ-expansion corresponds to λ-expansion
  - Purely soft: κ-expansion corresponds to λ^2-expansion
- Collinear loops:

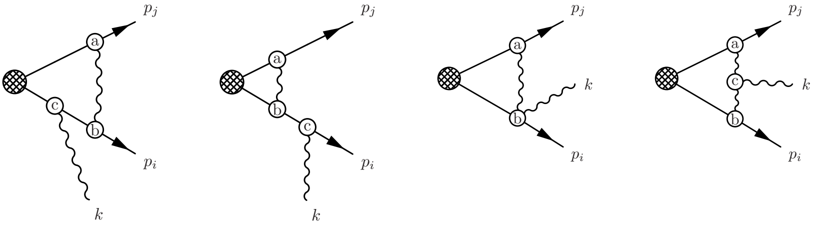


- Must be attached via subleading collinear Lagrangian
- One-loop: at least λ^2, two-loop: at least λ^4

Application: Loop corrections to the soft theorem

[Bern, Davies, Nohle 1405.1015]

- Gravity differs from gauge theory:
  - Purely collinear: κ-expansion corresponds to λ-expansion
  - Purely soft: κ-expansion corresponds to λ^2-expansion
- Soft loops:



- Soft loop vanishes unless soft emission is directly from soft graviton
- One loop: at least λ^2, two-loop: at least λ^4

Conclusion

- Derived rigorously SCET for Gravity to subleading order.
- Effective theory is covariant with respect to an emergent gauge symmetry
- Leading interactions can be cast into a covariant derivative
  - In QCD: described by $n_- A_s(x_-)$
  - In gravity: described by $e_-{}^{\alpha}$ and $\Omega_{-\alpha\beta}$
- Interactions differ in power-counting
  - Leading-power soft-collinear interactions
  - Purely-collinear: subleading, enter at $\mathcal{O}(\lambda)$
  - Purely-soft: subleading in k_s, enter at $\mathcal{O}(\lambda^2)$
- Immediately derived the form of loop corrections of the soft theorem

Auxiliary Slides

QCD Wilson lines

- To control large gluon component: W_c

$$W_c(x) = P \exp\left(ig \int_{-\infty}^{0} ds\, n_+ \hat{A}_c(x + s n_+) \right) .$$

- To implement redefinition: R

$$R(x) = P \exp\left(ig \int_{0}^{1} ds\, (x - x_-)^{\mu} A_{s\mu}\, (x + s(x - x_-)) \right) .$$

Redefinitions: QCD

$$\phi_c = R\underbrace{W_c^\dagger \hat{\phi}_c}_{\hat{\chi}_c},$$

$$A_{\perp c} = R\underbrace{\left(W_c^\dagger \hat{A}_{c\perp} W_c + \frac{i}{g} W_c^\dagger \left[\partial_\perp\, , W_c\right]\right)}_{=\hat{\mathcal{A}}_{\perp c}} R^\dagger\,,$$

$$n_- A_c = R\underbrace{\left(W_c^\dagger n_- \hat{A}_c W + \frac{i}{g} W_c^\dagger \left[n_- D_s(x_-)\, , W_c\right]\right)}_{=n_- \hat{\mathcal{A}}_c} R^\dagger\,.$$

Gravity "Wilson lines"

- To control $h_{+\mu}$: W_c

$$W_c^{-1} = T_{\theta_{\rm LC}} = 1 + \theta^\alpha_{\rm LC}\partial_\alpha + \mathcal{O}(\lambda^2)\,,$$

$$\theta^\mu_{\rm LC} = -\frac{1}{(n_+\partial)^2}\hat{\Gamma}^\mu_{++} + \frac{1}{(n_+\partial)^2}\left(2\hat{\Gamma}^\mu_{\tau+}\frac{1}{n_+\partial}\hat{\Gamma}^\tau_{++} + \partial_\nu\hat{\Gamma}^\mu_{++}\frac{1}{(n_+\partial)^2}\hat{\Gamma}^\nu_{++}\right) + \ldots$$

- To implement multipole expansion: R

$$R^{-1} \equiv T_{\theta_{\rm FLNC}} = 1 + \theta^\alpha_{\rm FLNC}\partial_\alpha + \mathcal{O}(\lambda^2)\,.$$

$$\theta^\mu_{\rm FLNC} = ({E^\mu}_\rho - \delta^\mu_\rho)(x-x_-)^\rho - \frac{1}{2}(x-x_-)^\rho(x-x_-)^\sigma {E^\alpha}_\rho {E^\beta}_\sigma {\Gamma^\mu}_{\alpha\beta} + \ldots$$

Redefinitions: Gravity

$$\varphi_c = \big[R \underbrace{W_c^{-1}\hat{\varphi}_c}_{=\hat{\chi}_c}\big]\,,$$

$$h_{\mu\nu} = \big[R R_\mu{}^\alpha R_\nu{}^\beta \underbrace{\big(W^\rho{}_\alpha\, W^\sigma{}_\beta\,\Big[W_c^{-1}(\hat{h}_{\rho\sigma} + \hat{g}_{s\rho\sigma})\Big] - \hat{g}_{s\alpha\beta}\big)}_{=\hat{\mathfrak{h}}_{\mu\nu}}\big]\,.$$

Full Lagrangian

$$\mathcal{L}^{(0)} = \frac{1}{2}\partial_+\hat{\varphi}_c\partial_-\hat{\varphi}_c + \frac{1}{2}\partial_{\alpha_\perp}\hat{\varphi}_c\partial^{\alpha_\perp}\hat{\varphi}_c - \frac{\kappa}{8}s_{--}(\partial_+\hat{\varphi}_c)^2 - \frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4$$

$$\mathcal{L}^{(1)} = -\frac{\kappa}{8}\left[\partial_\alpha s_{--} - \partial_- s_{\alpha -}\right] x_\perp^\alpha (\partial_+\hat{\varphi}_c)^2 - \frac{\kappa}{4}s_{\mu_\perp -}\partial^{\mu_\perp}\hat{\varphi}_c\partial_+\hat{\varphi}_c$$
$$- \frac{\kappa}{2}\left(\hat{h}^{\mu_\perp\nu_\perp}\partial_{\mu_\perp}\hat{\varphi}_c\partial_{\nu_\perp}\hat{\varphi}_c + \hat{h}^{\mu_\perp -}\partial_{\mu_\perp}\hat{\varphi}_c\partial_+\hat{\varphi}_c + \frac{1}{4}\hat{h}^{--}(\partial_+\hat{\varphi}_c)^2\right)$$
$$+ \frac{\kappa}{4}\hat{h}^{\alpha_\perp}{}_{\alpha_\perp}\left(\partial_+\hat{\varphi}_c\partial_-\hat{\varphi}_c - \frac{\kappa}{4}s_{--}(\partial_+\hat{\varphi}_c)^2 + \partial_{\alpha_\perp}\hat{\varphi}_c\partial^{\alpha_\perp}\hat{\varphi}_c\right) - \frac{\kappa}{2}\hat{h}^{\alpha_\perp}_{\alpha_\perp}\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4$$

$$\mathcal{L}^{(1)}_{\varphi_s} = -\frac{\lambda_\varphi}{3!}\hat{\varphi}_c^3\varphi_s$$

$$\mathcal{L}^{(2)}_{\varphi_s} = -\frac{\lambda_\varphi}{4}\hat{\varphi}_c{}^2\varphi_s^2 - \frac{\lambda_\varphi}{3!}\frac{\kappa}{2}\hat{h}^{\alpha_\perp}_{\alpha_\perp}\hat{\varphi}_c^3\varphi_s$$

$$
\begin{aligned}
\mathcal{L}^{(2)} = &-\frac{\kappa}{16}\left[\partial_{[+}s_{-]-}\right]n_- x(\partial_+\hat{\varphi}_c)^2 - \frac{\kappa}{4}\left[\partial_{[\alpha_\perp}s_{\mu_\perp]-}\right]x_\perp^\alpha \partial^{\mu_\perp}\hat{\varphi}_c\partial_+\hat{\varphi}_c \\
&+\frac{\kappa^2}{32}s_{--}s_{+-}(\partial_+\hat{\varphi}_c)^2 + \frac{\kappa^2}{32}s_{-\alpha_\perp}s_-^{\alpha_\perp}(\partial_+\hat{\varphi}_c)^2 - \frac{1}{8}x_\perp^\alpha x_\perp^\beta R_{\alpha-\beta-}(\partial_+\hat{\varphi}_c)^2 \\
&+\frac{\kappa}{8}s_{+-}\partial_{\alpha_\perp}\hat{\varphi}_c\partial^{\alpha_\perp}\hat{\varphi}_c - \frac{\kappa}{4}s_{+-}\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4 \\
&+\frac{\kappa^2}{2}\left(\hat{h}^{\mu_\perp\alpha_\perp}\hat{h}^{\nu_\perp}_{\alpha_\perp}\partial_{\mu_\perp}\hat{\varphi}_c\partial_{\nu_\perp}\hat{\varphi}_c + \hat{h}^{\mu_\perp\alpha_\perp}\hat{h}_{\alpha_\perp -}\partial_{\mu_\perp}\hat{\varphi}_c\partial_+\hat{\varphi}_c + \frac{1}{4}\hat{h}^{-\alpha_\perp}\hat{h}_{\alpha_\perp -}(\partial_+\hat{\varphi}_c)^2\right) \\
&-\frac{\kappa^2}{4}\hat{h}^{\alpha_\perp}{}_{\alpha_\perp}\left(\hat{h}^{\mu_\perp\nu_\perp}\partial_{\mu_\perp}\hat{\varphi}_c\partial_{\nu_\perp}\hat{\varphi}_c + \hat{h}^{\mu_\perp -}\partial_{\mu_\perp}\hat{\varphi}_c\partial + \hat{\varphi}_c + \frac{1}{4}\hat{h}_{--}\partial_+\hat{\varphi}_c\partial_+\hat{\varphi}_c\right) \\
&+\frac{\kappa^2}{16}\left((\hat{h}^{\alpha_\perp}_{\alpha_\perp})^2 - 2\hat{h}^{\alpha_\perp\beta_\perp}\hat{h}_{\alpha_\perp\beta_\perp}\right)\left(\partial_+\hat{\varphi}_c\partial_-\hat{\varphi}_c - \frac{\kappa}{4}s_{--}(\partial_+\hat{\varphi}_c)^2 + \partial_{\mu_\perp}\hat{\varphi}_c\partial^{\mu_\perp}\hat{\varphi}_c\right) \\
&+\frac{\kappa^2}{4}\hat{h}^{\mu_\perp\alpha_\perp}s_{\alpha_\perp -}\partial_+\hat{\varphi}_c\partial_{\mu_\perp}\hat{\varphi}_c + \frac{\kappa^2}{8}\hat{h}^{-\alpha_\perp}s_{\alpha_\perp -}(\partial_+\hat{\varphi}_c)^2 \\
&-\frac{\kappa^2}{8}\hat{h}^{\alpha_\perp}_{\alpha_\perp}s_{\mu_\perp -}\partial_+\hat{\varphi}_c\partial^{\mu_\perp}\hat{\varphi}_c - \frac{\kappa^2}{16}\hat{h}^{\alpha_\perp}_{\alpha_\perp}\left[\partial_{[\mu_\perp}s_{-]-}\right]x_\perp^\mu(\partial_+\hat{\varphi}_c)^2 \\
&-\kappa^2\frac{\lambda_\varphi}{4!}\hat{\varphi}_c{}^4\left((\hat{h}^{\alpha_\perp}_{\alpha_\perp})^2 - \frac{1}{4}\hat{h}^{\alpha_\perp\beta_\perp}\hat{h}_{\alpha_\perp\beta_\perp}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{EH}}^{(0)} = {} & \frac{1}{2}\partial_\mu \hat{h}_{\alpha_\perp\beta_\perp}\partial^\mu \hat{h}^{\alpha_\perp\beta_\perp} - \frac{1}{2}\partial_\mu \hat{h}\partial^\mu \hat{h} \\
& + \left(\partial_{\alpha_\perp}\hat{h}^{\alpha_\perp\beta_\perp}\partial_{\beta_\perp}\hat{h} + \frac{1}{2}\partial_{\alpha_\perp}\hat{h}^{\alpha_\perp -}\partial_+\hat{h} + \frac{1}{2}\partial_+\hat{h}^{-\beta_\perp}\partial_{\beta_\perp}\hat{h} + \frac{1}{4}\partial_+\hat{h}_{--}\partial_+\hat{h}\right) \\
& - \left(\partial_{\alpha_\perp}\hat{h}^{\alpha_\perp\mu_\perp}\partial^{\beta_\perp}\hat{h}_{\beta_\perp\mu_\perp} + \partial_+\hat{h}^{-\mu_\perp}\partial^{\beta_\perp}\hat{h}_{\beta_\perp\mu_\perp} + \frac{1}{4}\partial_+\hat{h}^{-\mu_\perp}\partial_+\hat{h}_{-\mu_\perp}\right) \\
& - \frac{\kappa}{8}s_{--}\partial_+\hat{h}_{\alpha\beta}\partial_+\hat{h}^{\alpha\beta} + \frac{\kappa}{8}s_{--}\partial_+\hat{h}\partial_+\hat{h}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{EH}}^{(1)} = & \frac{\kappa}{4}\hat{h}^{\alpha\beta}(-2\partial_\mu \hat{h}^\mu_\alpha \partial_\beta \hat{h} + 4\partial_\mu \hat{h}_{\alpha\nu}\partial_\beta \hat{h}^{\mu\nu} + 2\partial_\nu \hat{h}^{\mu\nu}\partial_\mu \hat{h}_{\alpha\beta} - 2\partial_\alpha \hat{h}^\mu_\beta \partial_\mu \hat{h} \\
& \quad + \partial^\mu \hat{h}_{\alpha\nu}\partial^\nu \hat{h}_{\beta\mu} - 2\partial^\mu \hat{h}_{\beta\nu}\partial_\mu \hat{h}_{\alpha\nu}) \\
& + \frac{\kappa}{8}\partial_\alpha(\hat{h}^2)\partial^\alpha \hat{h} + \frac{\kappa}{4}\partial_\alpha(\hat{h}^{\mu\nu}\hat{h}_{\mu\nu})\partial^\alpha \hat{h} - \frac{\kappa}{8}\partial_\alpha(\hat{h}^2)\partial_\beta \hat{h}^{\alpha\beta} - \frac{3\kappa}{4}\partial_\alpha(\hat{h}^{\mu\nu}\hat{h}_{\mu\nu})\partial_\beta \hat{h}^{\alpha\beta} \\
& + \frac{\kappa}{2}\partial_\alpha(\hat{h}^{\mu\rho}\hat{h}^\nu_\rho)\partial^\alpha \hat{h}_{\mu\nu} - \partial_\alpha(\hat{h}^{\mu\rho}\hat{h}^\nu_\rho)\partial_\mu \hat{h}^\alpha_\nu + \frac{\kappa}{2}\partial_\mu(\hat{h}^{\mu\rho}\hat{h}^\nu_\rho)\partial_\nu \hat{h} - \frac{\kappa}{8}\hat{h}\partial_\alpha \hat{h}\partial^\alpha \hat{h} \\
& - \frac{\kappa}{8}\hat{h}\partial_\alpha \hat{h}_{\mu\nu}\partial^\alpha \hat{h}^{\mu\nu} + \frac{\kappa}{4}\hat{h}\partial_\mu \hat{h}_{\alpha\beta}\partial^\alpha \hat{h}^{\mu\beta} - \frac{\kappa}{4}s_{-\mu_\perp}\partial^{\mu_\perp}\hat{h}^{\alpha\beta}\partial_+\hat{h}_{\alpha\beta} + \frac{\kappa}{4}s_{-\mu_\perp}\partial^{\mu_\perp}\hat{h}\partial_+\hat{h} \\
& - \frac{\kappa}{8}\left[\partial_\alpha s_{--} - \partial_- s_{\alpha -}\right] x^\alpha_\perp \partial_+\hat{h}_{\mu\nu}\partial_+\hat{h}^{\mu\nu} + \frac{\kappa}{8}\left[\partial_\alpha s_{--} - \partial_- s_{\alpha -}\right] x^\alpha_\perp \partial_+\hat{h}\partial_+\hat{h} \\
& - \frac{\kappa^2}{64}s_{--}\partial_+(\hat{h}^2)\partial_+\hat{h} - \frac{\kappa^2}{32}s_{--}\partial_+(\hat{h}^{\mu\nu}\hat{h}_{\mu\nu})\partial_+\hat{h} - \frac{\kappa^2}{16}s_{--}\partial_+(\hat{h}^{\mu\rho}\hat{h}^\nu_\rho)\partial_+\hat{h}_{\mu\nu} \\
& + \frac{\kappa^2}{64}s_{--}\hat{h}\partial_+\hat{h}\partial_+\hat{h} + \frac{\kappa^2}{64}s_{--}\hat{h}\partial_+\hat{h}_{\mu\nu}\partial_+\hat{h}^{\mu\nu}
\end{aligned}
$$