Accelerator Physics for Diagnostics

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In this lecture, we

- focus on beam characterization
- beam based correction of beam dynamics

we will

- not derive beam dynamics from scratch
- but review concepts relevant to diagnostics

- specifity
  - ion beams
  - proton beams
  - electron beams

- all charged particles follow same beam dynamics
  - difference in magnitude of coefficients
  - difference in radiation behavior

- most of beam diagnostics receives the signal from a charged particle beam, the **signal source**
- we need to understand the nature of the signal
- particle beams come in many forms and shapes
- specifically, the time structure is most relevant for diagnostics

- beam current in RF-accelerators
  - particles are concentrated in **bunches** separated by RF-wavelength
    - these bunches are called **microbunches** and
    - the instantaneous current is called **peak current**

      $\hat{I} = Q/\delta\ell$ (Q = charge, $\delta\ell$ = duration per microbunch)
  - a series of bunches make a beam **pulse** and
    - the current averaged over the pulse is the **pulse current**
  - **average beam current** is defined for a long time – e.g. per second

time structure defines electronics of diagnostics

- diagnostics deals with signals from bunches or beam
- diagnostics does not deal with signal from single particle (unless the whole beam is made up of only a few particles)

- we want to detect the dynamics of bunches and beam
- therefore we need to know the dynamics of bunches/beam

Many Particle Beam Dynamics

betatron functions

linear equation of motion

$$x'' + k(z)\,x = 0 \qquad \textbf{Hill's Equation}$$

solution?

if $k = \text{const}$ solution would be $x(z) = a\cos\left(\sqrt{k}\,z\right)$

but $k = k(z)$

try "variation of integration constants":

$$x_i(z) = a_i\sqrt{\beta(z)}\cos\left[\psi(z) - \psi_i\right]$$

i is the index for individual particles insert $x_i(z) = a_i \sqrt{\beta(z)} \cos(\psi(z) - \varphi_i)$

into eq. of motion $x'' + k(z)\, x = 0$

coefficients of sin- and cos-terms must be zero separately

we get two conditions:

(1) betatron function: $\frac{\mathrm{d}^2}{\mathrm{d}z^2} \sqrt{\beta(z)} + k(z) \sqrt{\beta(z)} - \frac{1}{\beta^{3/2}(z)} = 0$

(2) betatron phase: $\psi'(z) = \frac{1}{\beta(z)}$

or after integration $\psi(z) = \int_0^z \frac{\mathrm{d}\zeta}{\beta(\zeta)}$

betatron function

- two different betatron function are defined, one for horizontal and one for the vertical plane
- for a particular set of initial values of the betatron function, there is only one solution per plane for betatron function
- for circular lattices the initial value is equal to the final value at the end of one turn (periodic solution)
- the value of the betatron function is always positive
- the square root of the betatron function can be positive and negative to represent both sides of a beam

Beam Envelope

$$x_i(z) = a_i \sqrt{\beta(z)} \cos(\psi(z) + \varphi_i)$$

look for particle with maximum amplitude a_i

and phase $\cos(\psi(z) + \varphi_i) = \pm 1$

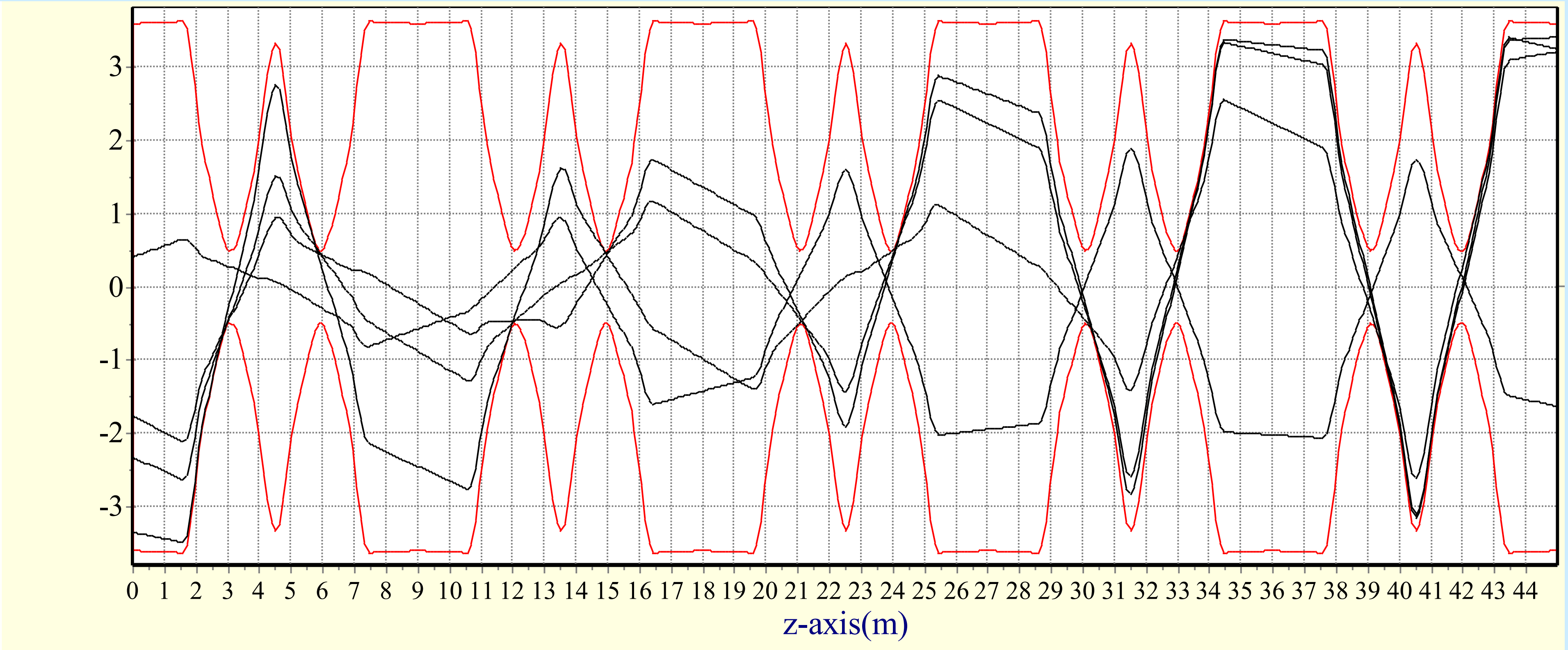
The envelope of a beam along a transport line is

$$E_{x,y}(z) = \pm\sqrt{\epsilon_{x,y}\,\beta_{x,y}(z)} \qquad \epsilon_{x,y} = \text{const.}$$

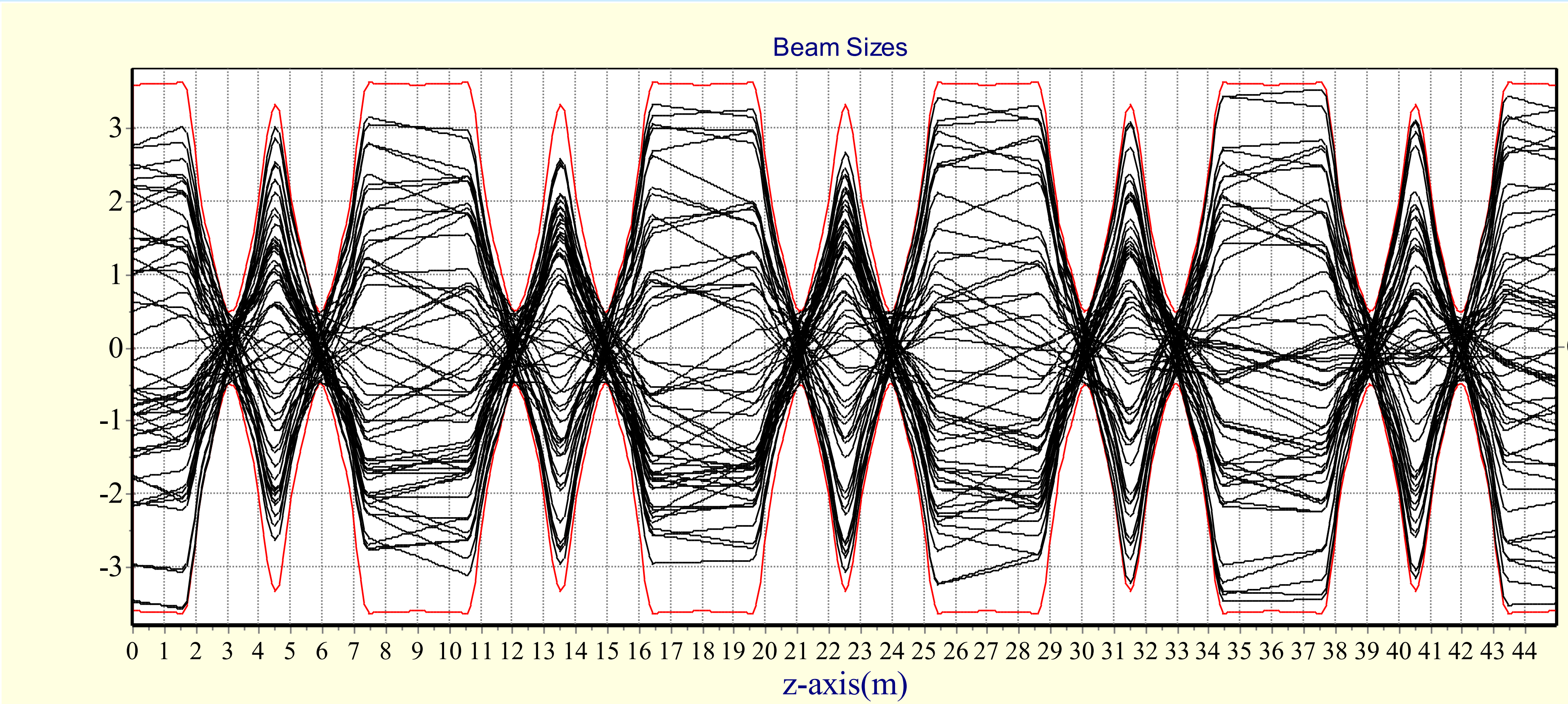
If this envelope should repeat itself from lattice unit to lattice unit, we may claim to have found a focusing structure that ensures beam stability in a circular accelerator.

We are therefore looking for **periodic solutions** of the betatron functions $\beta_{x,y}$

particles perform oscillatory motion, called **betatron oscillations**, with phase ψ



horizontal envelope (red) with 4 random trajectories (black)



MAX-II, half ring: ($\epsilon_x = \epsilon_y = 10^{-6}$ m) with the two conditions (1) and (2), the trajectory for particle i is

$$x_i(z) = a_i \sqrt{\beta(z)} \cos\left[\psi(z) + \varphi_i\right]$$

with derivative

$$x_i'(z) = a_i \frac{\beta'(z)}{2\sqrt{\beta(z)}} \cos\left[\psi(z) + \varphi_i\right] - a_i \sqrt{\beta(z)} \sin\left[\psi(z) + \varphi_i\right] \psi'(z)$$

using $\alpha(z) = -\frac{1}{2}\beta'(z)$ and $\psi'(z) = \frac{1}{\beta(z)}$

$$x_i'(z) = -a_i \frac{1}{\sqrt{\beta(z)}} \left\{\alpha(z) \cos\left[\psi(z) + \varphi_i\right] + \sin\left[\psi(z) + \varphi_i\right]\right\}$$

$$x_i(z) = a_i \sqrt{\beta(z)} \cos[\psi(z) + \varphi_i]$$

$$x_i'(z) = -\frac{a_i\,\alpha(z)}{\sqrt{\beta(z)}} \cos[\psi(z) + \varphi_i] - \frac{a_i}{\sqrt{\beta(z)}} \sin[\psi(z) + \varphi_i]$$

we eliminate the phase $\psi(z) + \varphi_i$ from these two

equations and get with $\gamma(z) = \frac{1 + \alpha^2(z)}{\beta(z)}$

the Courant-Snyder invariant

$$\gamma x_i'^2 + 2\alpha x_i x_i' + \beta x_i^2 = a_i^2$$

which is the equation of an ellipse

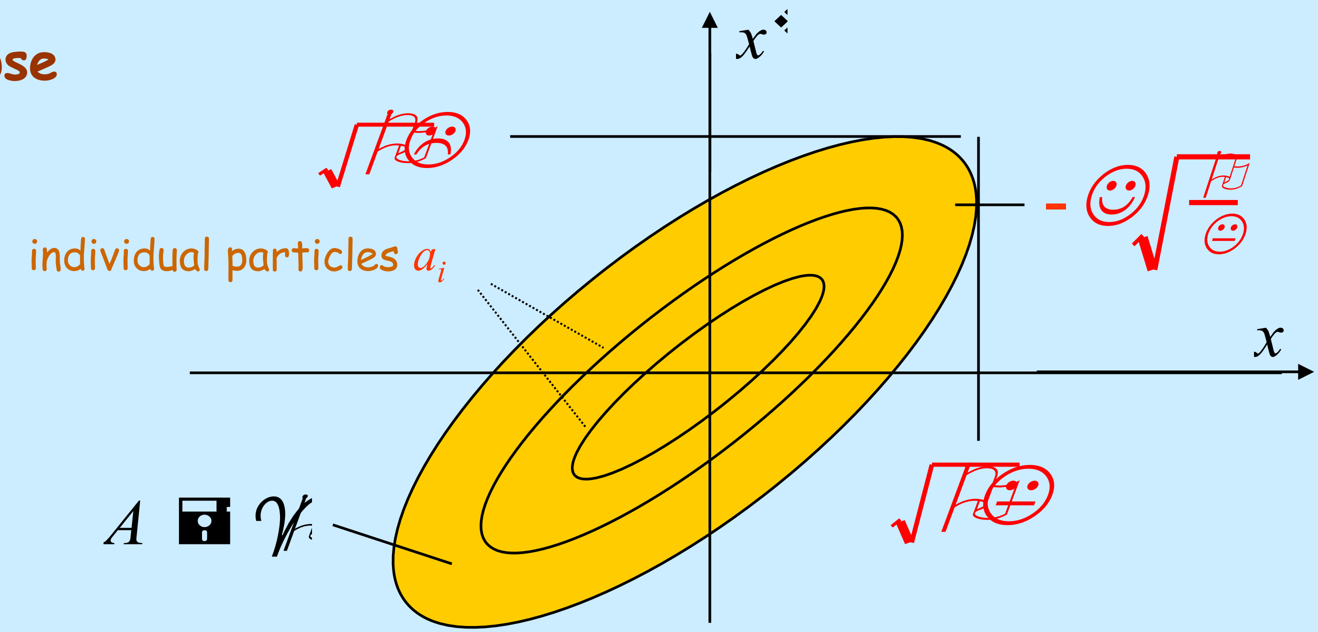
during betatron oscillations,

each particle i moves on its own ellipse with area a_i

in phase space

motion in phase space

phase space ellipse



Phase Space Dynamics

many particles

we may follow single particles through arbitrary beam transport line

follow 10^9 ✍ 10^{10} or more particles with different initial conditions and you have a beam, showing beam envelope and dispersion ! ?

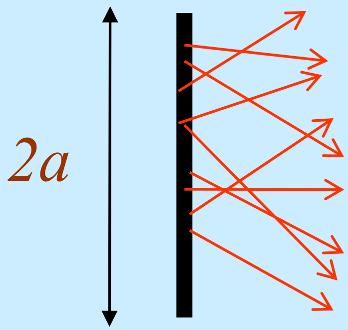
that is too cumbersome !

there must be a better way

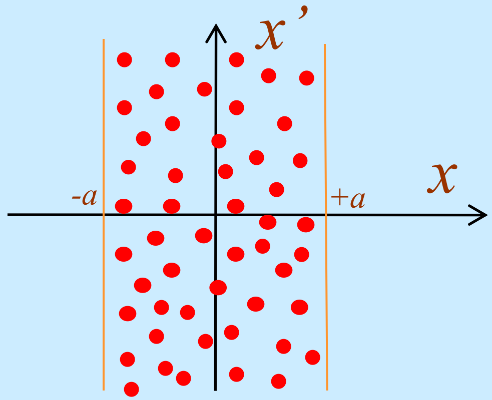
and there is: **observe beam in phase space**

to describe particle beams in phase space we use
2 conjugate variables, like

$$(x, p_x)$$
$$(y, p_y)$$
$$(z, p_z)$$

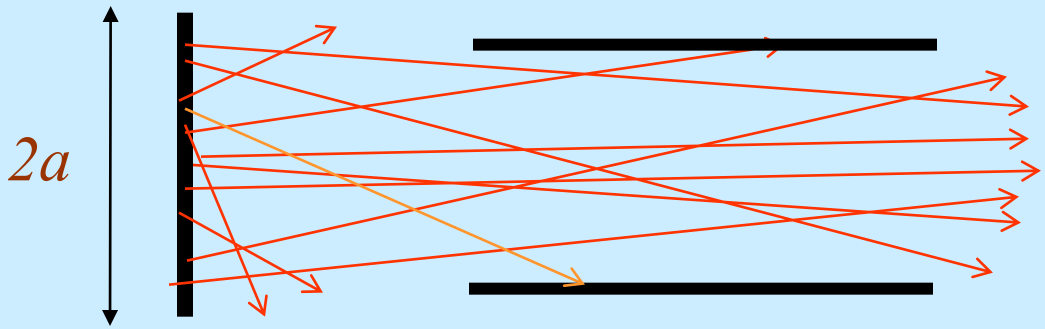


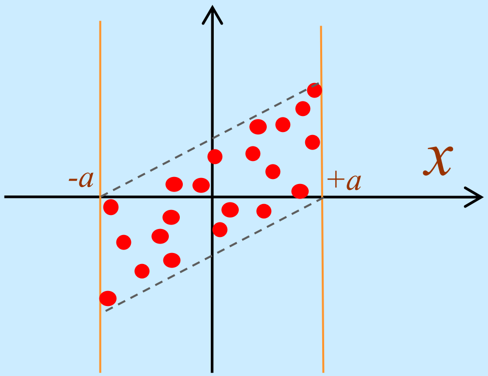
diffuse particle source



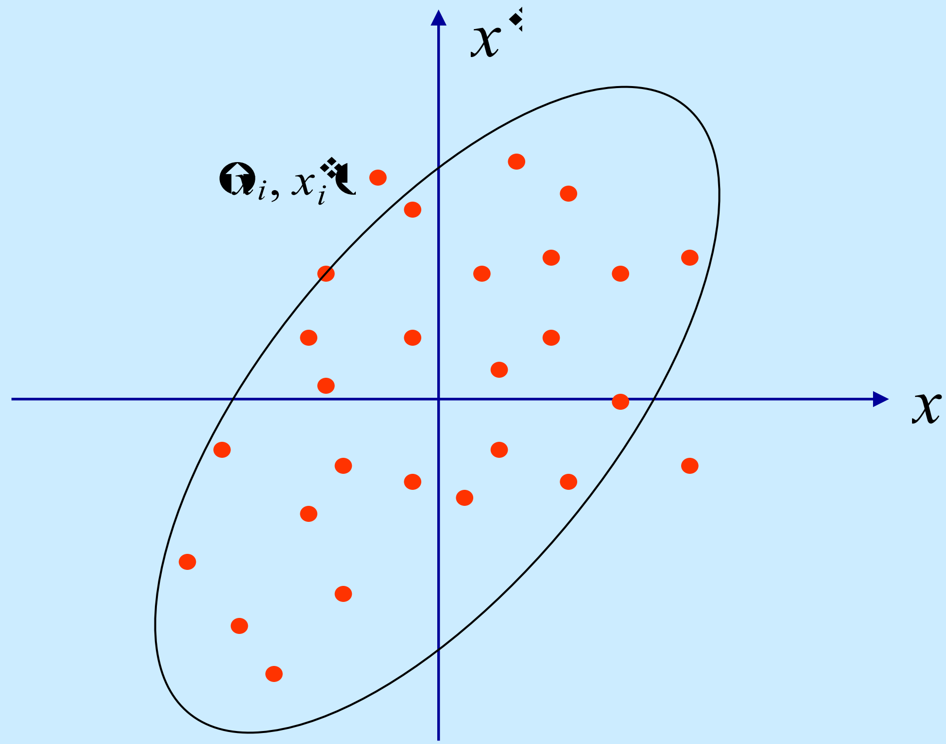
phase space

insert tube





many particles



randomly distributed particles can be enclosed by an ellipse, the **phase ellipse**

What is the significance of the phase ellipse?

Liouville's Theorem:

Phase space density is constant

no particle can escape the phase ellipse

any particle within the ellipse will stay within the ellipse

any particle outside the ellipse will stay outside

particle with maximum amplitude $\hat{a}$
defines whole beam with
beam emittance

$$\epsilon_{x,y} = \hat{a}^2$$

because of
Liouville's theorem

for bell shaped or Gaussian distribution
we define the beam emittance by

$$\epsilon_u = \frac{\langle u^2 \rangle}{\beta_u} = \frac{1}{2}\langle a_i^2 \rangle$$

$$u = x \text{ or } y$$

general definition of emittance for any distribution of particle

$$\epsilon = \sqrt{\langle u^2 \rangle \langle u'^2 \rangle - \langle u u' \rangle^2}$$

transformation of betatron functions

$$\begin{pmatrix} \beta(z) \\ \alpha(z) \\ \gamma(z) \end{pmatrix} = \begin{pmatrix} a^2 & -2ad & b^2 \\ -ac & 2(ad+bc) & -bd \\ c^2 & -2cd & d^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

where

$$\mathrm{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is transformation matrix for trajectories

betatron function in a drift space

example: drift space of length s:

$$\mathcal{M}_{\text{drift}} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \longrightarrow \mathcal{M}_{\beta} = \begin{pmatrix} 1 & -2s & s^2 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix}$$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

$$\alpha(s) = \alpha_0 - \gamma_0 s$$

$$\gamma(s) = \gamma_0$$

beam waist: $\alpha_w = 0$

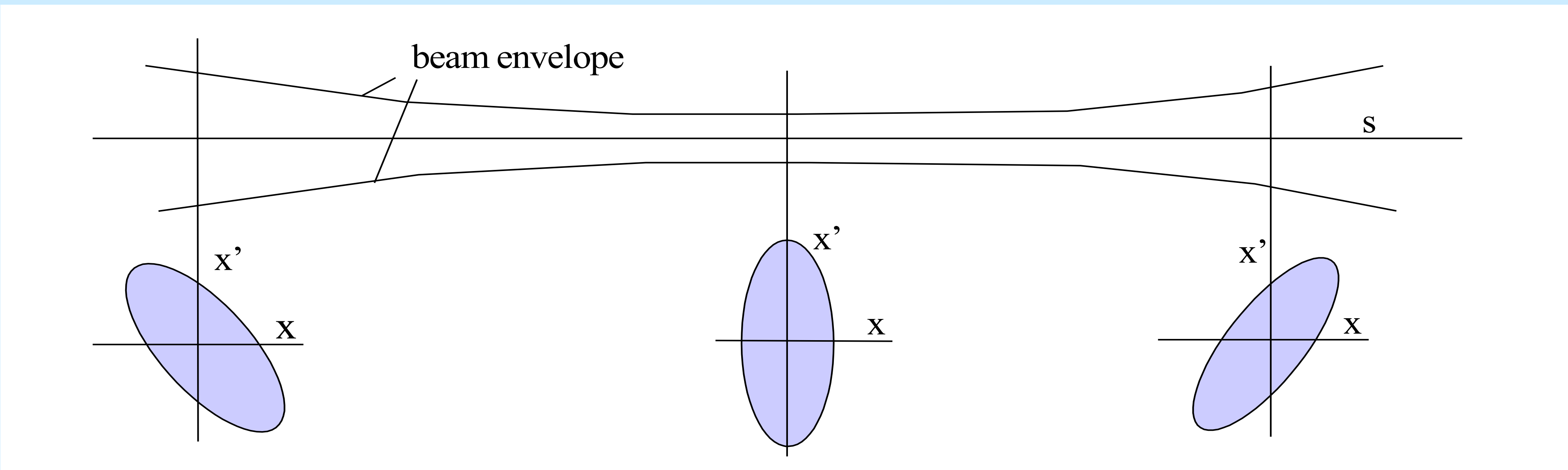
$$\beta(s) = \beta_w + \frac{s^2}{\beta_w}$$

or in terms of beam sizes: $\sigma^2(s) = \sigma_w^2 + \frac{\epsilon^2 s^2}{\sigma_w^2} = \sigma_w^2\left(1 + \frac{\epsilon^2 s^2}{\sigma_w^4}\right)$

phase ellipse along a drift space

$\sigma_u = \sqrt{\beta_u \epsilon_u}$ where $u = x, y$ $\qquad \beta_x = \frac{\sigma_x^2}{\epsilon_x(s)} = \frac{\sigma_x^2}{\epsilon_x(s)}$

$\sigma_{u'} = \sqrt{\gamma_u \epsilon_u}$



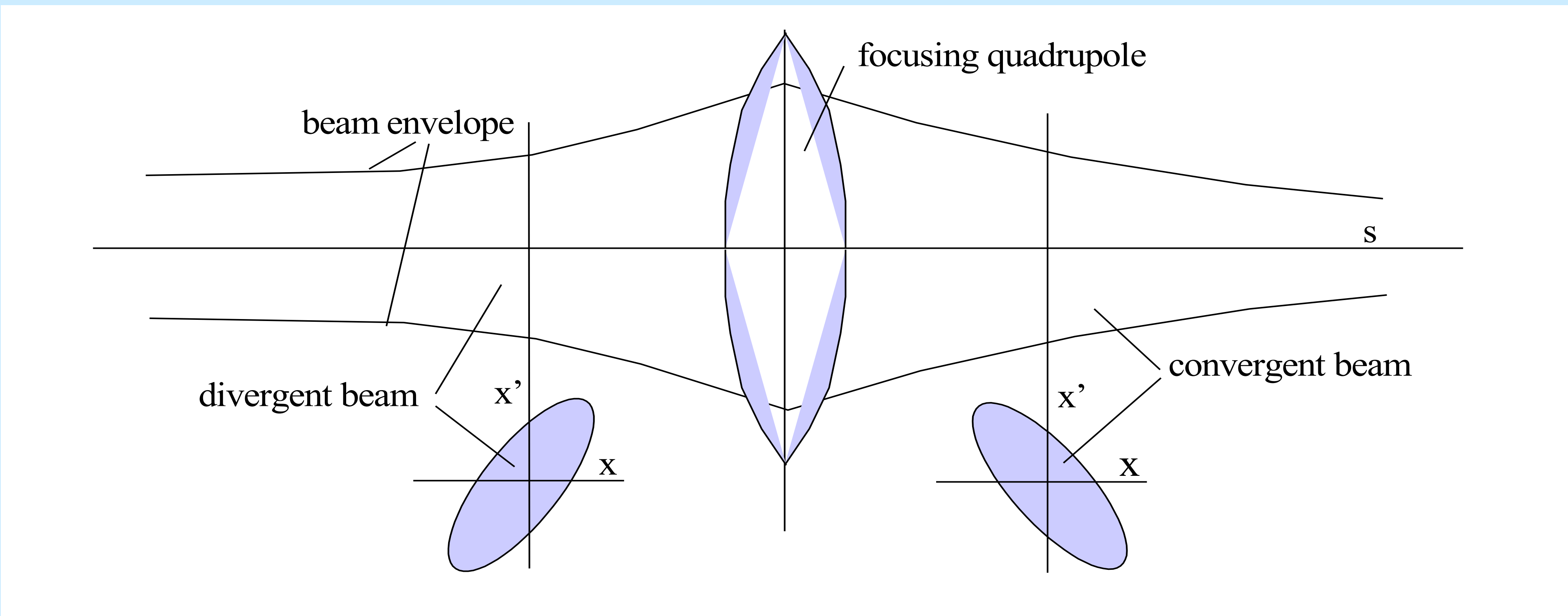
phase ellipse in a thin quadrupole

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ -\frac{1}{f^2} & 2\frac{1}{f} & 1 \end{pmatrix} \rightarrow$$

[illegible] = [illegible]

[illegible] = $\frac{1}{f}$ [illegible] + [illegible]

[illegible] = $-\frac{1}{f^2}$ [illegible] + $\frac{2}{f}$ [illegible] + [illegible]



particle trajectories and betatron functions

general solution $x(z) = a\sqrt{\beta(z)}\cos[\psi(z)] + b\sqrt{\beta(z)}\sin[\psi(z)]$

with derivative

$$x'(z) = -a\left\{\frac{\alpha(z)}{\sqrt{\beta(z)}}\cos[\psi(z)] + \frac{1}{\sqrt{\beta(z)}}\sin[\psi(z)]\right\} + b\left\{-\frac{\alpha(z)}{\sqrt{\beta(z)}}\sin[\psi(z)] + \frac{1}{\sqrt{\beta(z)}}\cos[\psi(z)]\right\}$$

determine constants a and b:

for $z=0$

$$x_0 = a\sqrt{\beta_0} \qquad a = \frac{x_0}{\sqrt{\beta_0}}$$

$$x_0' = -a\frac{\alpha_0}{\sqrt{\beta_0}} + b\frac{1}{\sqrt{\beta_0}} \qquad b = \left(\frac{x_0}{\sqrt{\beta_0}}\alpha_0 + \sqrt{\beta_0}\,x_0'\right)$$

in matrix formulation

$$\begin{pmatrix} C(z) & S(z) \\ C'(z) & S'(z) \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}\left(\cos\psi + \alpha_0 \sin\psi\right) & \sqrt{\beta\beta_0}\,\sin\psi \\ \frac{1}{\sqrt{\beta_0\beta}}\left[(\alpha_0 - \alpha)\cos\psi - (1 + \alpha\alpha_0)\sin\psi\right] & \sqrt{\frac{\beta_0}{\beta}}\left(\cos\psi - \alpha\sin\psi\right) \end{pmatrix}$$

where $\beta = \beta(z),\ \alpha = \alpha(z),\ \psi = \psi(z)$

$$x = C_x(z)\,x_0 + S_x(z)\,x'_0 \qquad \text{horizontal}$$

$$y = C_y(z)\,y_0 + S_y(z)\,y'_0 \qquad \text{vertical}$$

trajectory

use this formalism to manipulate particle trajectories

- a kick of angle θ displaces trajectory by

$$x(z) = \left(\sqrt{\beta_0\beta_1}\,\sin\Delta\psi_1\right)\theta_1$$

- several kicks add up linearly

$$x(z) = \left(\sqrt{\beta_0\beta_1}\,\sin\Delta\psi_1\right)\theta_1 + \left(\sqrt{\beta_0\beta_2}\,\sin\Delta\psi_2\right)\theta_2 + \ldots\ldots$$

a kick is most effective where β is large

Dispersion

energy errors

- not all particles are ideal……….have the ideal energy
- we must consider chromatic effects
- chromatic focusing is a second order effect and can be neglected…for now
- bending, however, depends linearly on the particle energy
- for off-momentum particles the bending angle is different from the ideal value

Dispersion Function

a particle with a somewhat higher energy than the ideal energy will not get bent as much.

How do they get bent by exactly 360 degrees per turn?

The particle with higher energy travels outside the ideal orbit and catches the missing bending angle in the quadrupoles which exhibit predominantly focusing properties.

A similar focusing effect occurs for lower energy particles being bent towards the ideal orbit by quadrupoles.

trajectory for off-momentum particles is determined by
Dispersion Function

dispersion in transport line

$$D(z) = \int_0^z \frac{1}{\rho_0(\tilde z)}\left[S(z)C(\tilde z) - C(z)S(\tilde z)\right] d\tilde z$$

transformation of off-momentum particle ($\delta = \Delta p/p_0$)

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \begin{pmatrix} C(z) & S(z) & D(z) \\ C'(z) & S'(z) & D'(z) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ \delta \end{pmatrix}$$

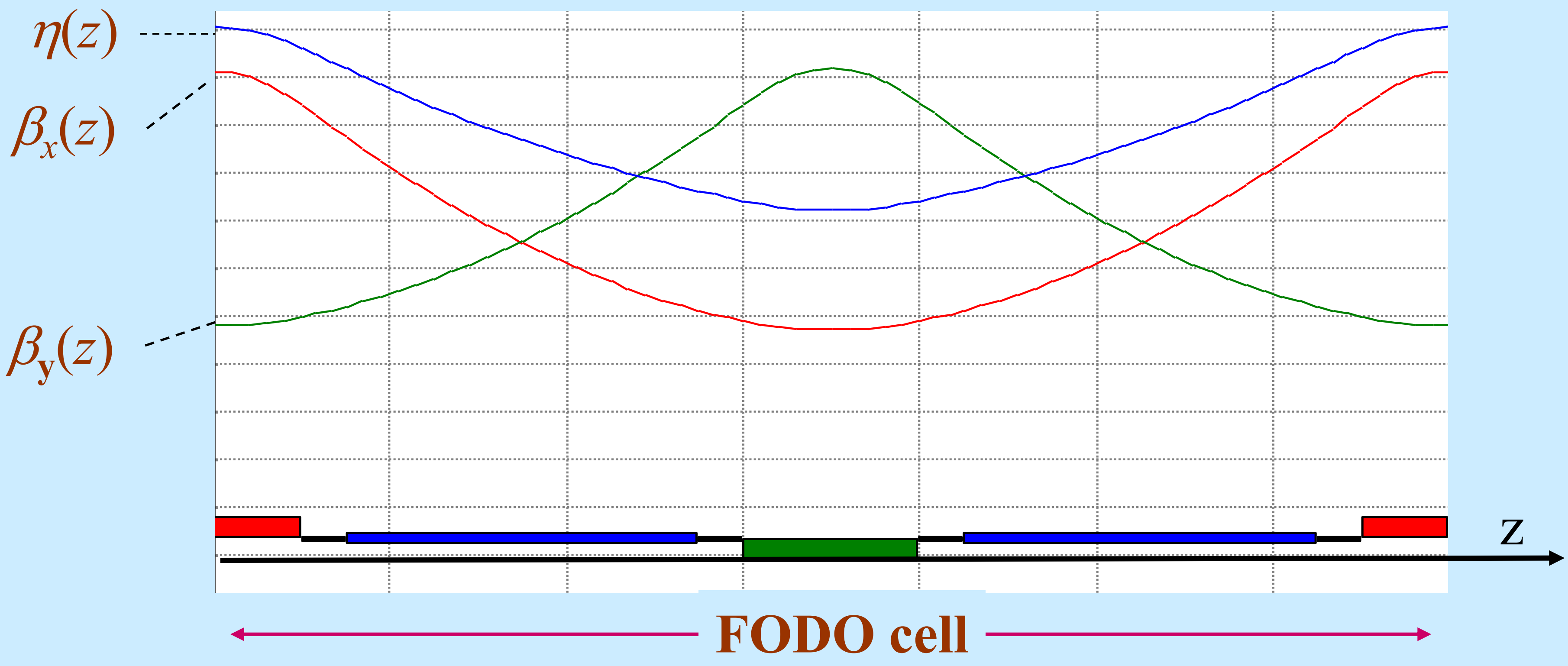
charge center of a particle beam transforms like a single particle the periodic η-function in a circular accelerator

$$\eta(z) = \frac{\sqrt{\beta(z)}}{2\sin \pi\nu} \int_z^{z+L_\mathrm{p}} \frac{\sqrt{\beta(\zeta)}}{\rho(\zeta)} \cos\left[\nu\left(\varphi(z) - \varphi(\zeta) + \pi\right)\right] d\zeta$$

Note: the integration must start at z and go around the ring of length L_p to location z again!

Any particle with an energy deviation δ will perform betatron oscillations about the reference orbit

$$x = x_\beta + x_\delta = C(z)\,x_0 + S(z)\,x_0' + \eta(z)\,\delta$$


η(z)
βx(z)
βy(z)
z
FODO cell

Momentum Compaction Factor

the path length along a beam line or the reference orbit is in general

$$\mathrm{d}L = \left(1 + \frac{\eta\delta}{\rho}\right)\mathrm{d}z$$

or after integration $L = L_0 + \delta \int \frac{\eta}{\rho}\,\mathrm{d}z$

The variation of the path length with energy is then

$$\frac{\Delta L}{L_0} = \delta \frac{1}{L_0}\int \frac{\eta}{\rho}\,\mathrm{d}z = \alpha_c \delta$$

where α_c is the momentum compaction factor

$$\alpha_c = \frac{\Delta L/L_0}{\Delta p/p_0}$$

Measurement of the η-function

A change Δf_{Rf} in the Rf-frequency alters the path of the particles by

$$\Delta x = -\eta \frac{1}{\alpha_c} \frac{\Delta f_{Rf}}{f_{Rf}}$$

Measurement of the beam position before and after change of the Rf-frequency gives the value of the η-function at the location of the position monitors.

So far, we considered perfect beam lines and accelerators:

The bad news is:

no need for diagnostics

The good news is:

we are not perfect and

we desperately need diagnostics

Diagnostics help us to find imperfections

Dipole Field Errors

in

Circular Accelerators

to get the dispersion function, we solved the differential equation

$$x'' + kx = \frac{1}{\rho}\delta$$

which gives the trajectory for energy deviating particles.

One might consider another viewpoint:

the particle energy is not wrong,

what's wrong are the magnetic fields!

and we should be solving

$$x'' + kx = \frac{1}{\rho}$$

This differential equation has the same form as the one for the dispersion function.

We look for the periodic solution, since these dipole errors define a perturbed closed orbit. The solution for the perturbed closed orbit is

$$\Delta x(z) = \frac{\sqrt{\beta(z)}}{2\sin\pi\nu} \oint_z^{z+L} \sqrt{\beta(\zeta)}\, \frac{1}{\rho} \cos\left[\nu\left(\varphi(z) - \varphi(\zeta) + \pi\right)\right] d\zeta$$

Of course, we do not know, where the dipole field errors are located, but we can measure their effects on the orbit in beam position monitors (BPM) around the ring.

Orbit Distortions

- we don't know the location of the errors
- or the magnitude of these errors

- all we know are the orbit distortions at BPMs

assuming we have a set of beam position monitors (BPM) distributed along the orbit at positions j, which are used to measure the orbit distortion.

$$\Delta x_j = \frac{\sqrt{\beta_j}}{2\sin\pi\nu}\int_z^{z+L}\sqrt{\beta(z)}\,\Delta\frac{1}{\rho}\cos\left[\nu\left(\varphi_j-\varphi(z)+\pi\right)\right]dz$$

dipole errors can come from many sources, like

- dipole errors
- fields from vacuum pump magnets
- earth magnetic field
- misaligned quadrupole, sextupole
- rotation of bending magnet
- etc.

Quadrupole/multipole field: $B_y = a_n x^n$

horizontal misalignment by δx : $B_y = g(x + \delta x)$

field error $\delta B_y = g\,\delta x$ → kick angle $\delta\theta_x = k\ell\,\delta x$

field from a misaligned sextupole:

$$B_y = \tfrac{1}{2} B_y''(x + \delta x)^2 = \tfrac{1}{2} B_y'' x^2 + B_y''\,\delta x\, x + \tfrac{1}{2} B_y''\,\delta x^2$$

consists of original sextupole field, a quadrupole term and a dipole term.

Generation of lower order fields is called: “feed down”

Orbit Correction

To correct these errors, we install small dipole magnets as "steering magnets" along the orbit similar to the BPMs. Since we know the position of these steering magnets, we can use them to generate a **known orbit distortion** which just "cancels" the orbit distortion from unknown errors at the BPMs.

Mathematically, we write for the orbit distortion Δx_j generated at BPM-j by steering magnets-i

$$\Delta x_j = \frac{\sqrt{\beta_j}}{2\sin\pi\nu}\sum_i \sqrt{\beta_i}\,\theta_i\cos[\nu(\varphi_j - \varphi_i + \pi)]$$

where the θ_i are the beam deflection angles by the steerers.

To correct orbit, use steerers to produce orbit distortion:

$$\Delta x_{j,\mathrm{steerers}} = -\Delta x_{j,\mathrm{meas}}$$

for one pair of BPM and steering magnet

$$\Delta x_j = \frac{\sqrt{\beta_j}}{2\sin\pi\nu}\sqrt{\beta_i}\cos\left(\nu|\varphi_j-\varphi_i|-\nu\pi\right)\theta_i = R_{ji}\,\theta_i$$

the coefficients R_{ji} are called the **Response Matrix elements**

and matrix made-up of these elements is called

Response Matrix

the number of rows are equal to the number of BPMs

the number of columns are equal to the number of steerers

Response Matrix for a circular accelerator

horizontal $$R_{ji,x} = \frac{1}{2\sin\pi\nu_x}\sqrt{\beta_{xj}\beta_{xi}}\cos\left[\nu_x\left(\varphi_{xj}-\varphi_{xi}+\pi\right)\right]+\frac{\eta_{xj}\eta_{xi}}{\alpha_c L_0}$$

vertical $$R_{ji,y} = \frac{1}{2\sin\pi\nu_y}\sqrt{\beta_{yj}\beta_{yi}}\cos\left[\nu_y\left(\varphi_{yj}-\varphi_{yi}+\pi\right)\right]$$

Response Matrix for a beam line

$$R_{ji} = \sqrt{\beta_j\beta_i}\cos\left(\psi_j-\psi_i\right)$$

$\nu_{x,y}$	tune of circular accelerator	j	location of BPMs
$\psi_{x,y} = \nu_{x,y}\varphi_{x,y}$	phase of betatron oscillations	i	location of steerers
	always use $\psi_j > \psi_i$		always use $\varphi_j > \varphi_i$

$$\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_m \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & \cdots & & R_{1n} \\ R_{21} & R_{22} & & & \\ R_{31} & & \ddots & & \\ \vdots & & & \ddots & \\ R_{m1} & & & & R_{mn} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix}$$

beam position monitors, BPM | Response Matrix | steerers

$$\Delta \boldsymbol{x} = \mathbf{R}\,\theta$$

to correct orbit

create intentional orbit distortion with steerers

such that this distortion just cancels the orbit

distortion created by un-intentional errors

$$\Delta x_{j,\mathrm{steerers}} = -\Delta x_{j,\mathrm{meas}}$$

$$\Delta \boldsymbol{x}_{\mathrm{steerers}} = \mathrm{R}\,\boldsymbol{\theta} = -\Delta \boldsymbol{x}_{\mathrm{BPM}}$$

steerers need to be set to

$$\boldsymbol{\theta} = -\mathrm{R}^{-1}\,\Delta \boldsymbol{x}_{\mathrm{BPM}}$$

this correction 💧🖫✍R✍1✌ x_{BPM}

requires matrix inversion with SVD

matrix has thousands of elements

quite demanding on diagnostics:

- read some 60 - 100 BPMs
- perform mathematical calculations
- set 60 – 100 steerers
- about 10 – 100 times a second
- at very high precision

} **to keep orbit within less than 1μm** we can do more with closed orbit and Response Matrix

the Response Matrix includes information on

all dipole fields - all field gradients

whether intentional by design or unintentional from errors

dipoles and dipole errors

$$R_{ji} = \frac{1}{2\sin\pi\nu}\sqrt{\beta_j\beta_i}\,\cos\left(|\varphi_j-\varphi_i|-\pi\nu\right) - \frac{\eta_j\eta_i}{\alpha_c L_0}$$

quadrupole and gradient field errors

in a real beam line or accelerator

- we start out with a model of the beam line or accelerator
- real machine will be different
- we need to get a realistic model for computer simulations
- measure Response Matrix
- calculate model Response Matrix and compare
- try to close difference by fitting potential errors
- now you have an error loaded but realistic model
- correct errors if possible (calibration, alignment…)
- retune accelerator on computer to get desired parameters

this procedure can be performed with the program

LOCO

which was developed and written by J. Safranek

now also embedded in AT-MATLAB

see also:

1. Experimental Determination of Storage Ring Optics Using Orbit Response Measurements, J. Safranek, Nucl. Instr. and Meth. A388, (1997) pg. 27.
2. Beam Based Diagnostics, Christoph Steier, James Safranek and Greg Portmann, USPAS January 2006, USPAS.fnal.gov, Course Materials
3. MATLAB Based LOCO, www.**slac**.stanford.edu/cgi-wrap/getdoc/**slac**-pub-9464.pdf
4. Accelerator Toolbox for MATLAB, A.Terebilo, SLAC-PUB-8732

another more direct method to determine phase advances

made possible by advances in beam diagnostics:

turn by turn BPMs

- measure beam position at each BPM over 1024 turns
- perform Fourier analysis to

  determine betatron phase at every BPM
- calculate value of betatron function

P. Castro, PAC1993, p2103

- the quest for a "perfect accelerator" requires
- perfect diagnostics - fast and accurate
- anytime diagnostics makes a step forward
- accelerator gain

  new performance characteristics

Thank you