DATA ANALYSIS

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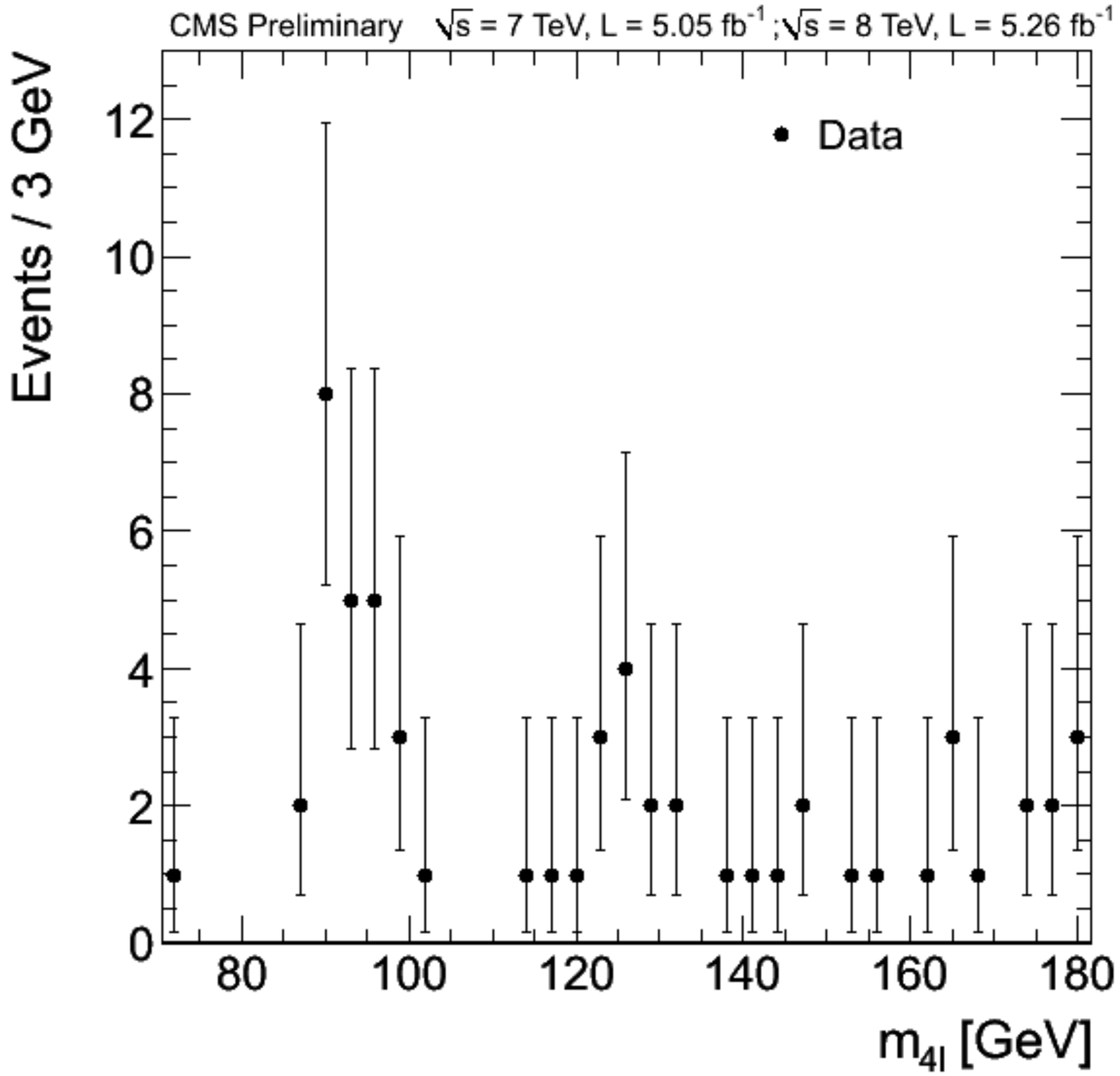
LECTURES OUTLINE

1) Introduction to Data Analysis
2) Probability density functions and Monte Carlo methods
3) Parameter estimation and Confidence intervals
4) Hypothesis testing and p-value

INTRODUCTION TO DATA ANALYSIS

EVENT DISTRIBUTION



- Does the observed data agree with our expectations from the Standard Model?

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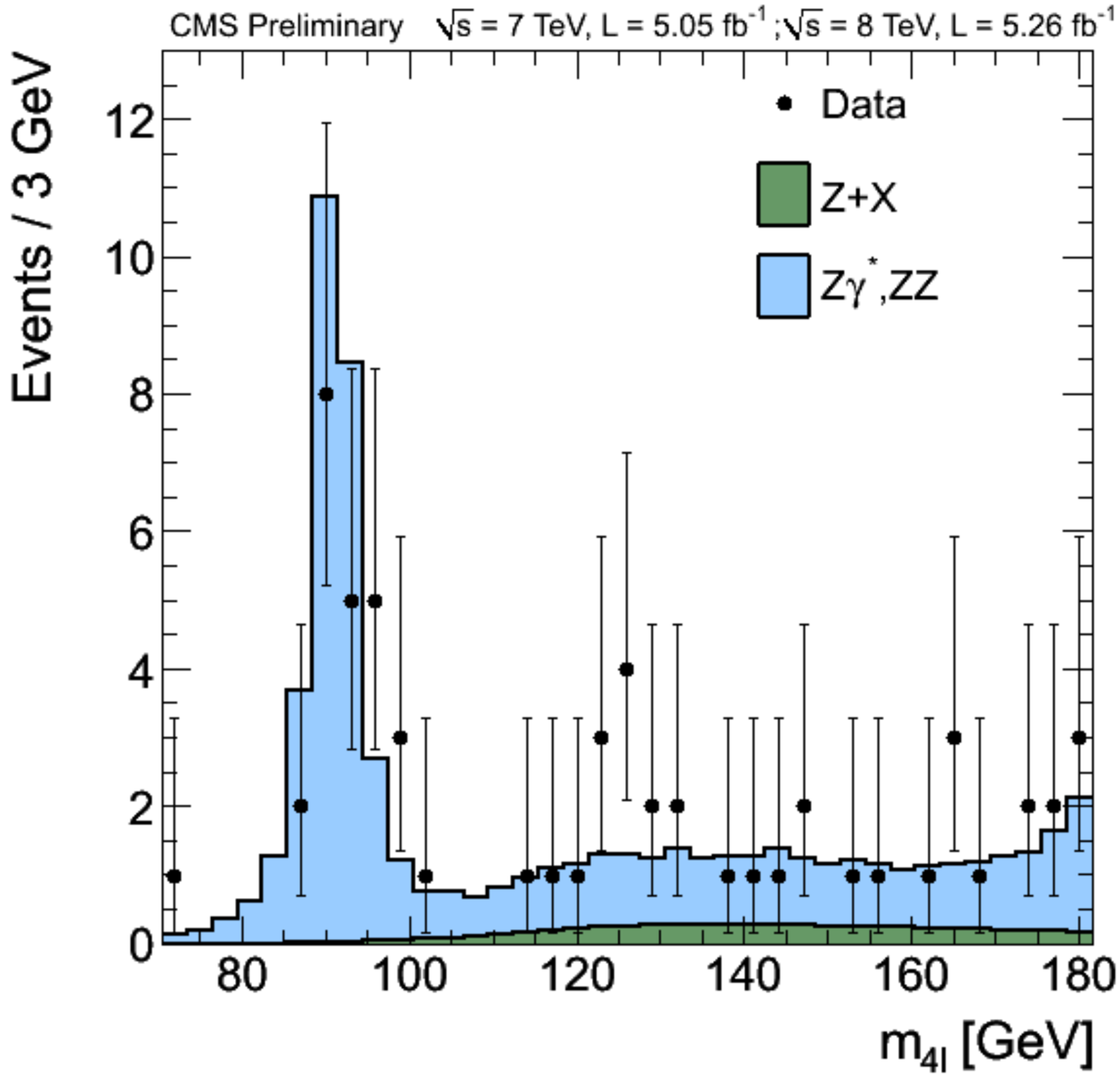

- We can not tell until we can compare to the expected distribution
- Is there any place on where data does not agree with the expectation? Where? How significant?

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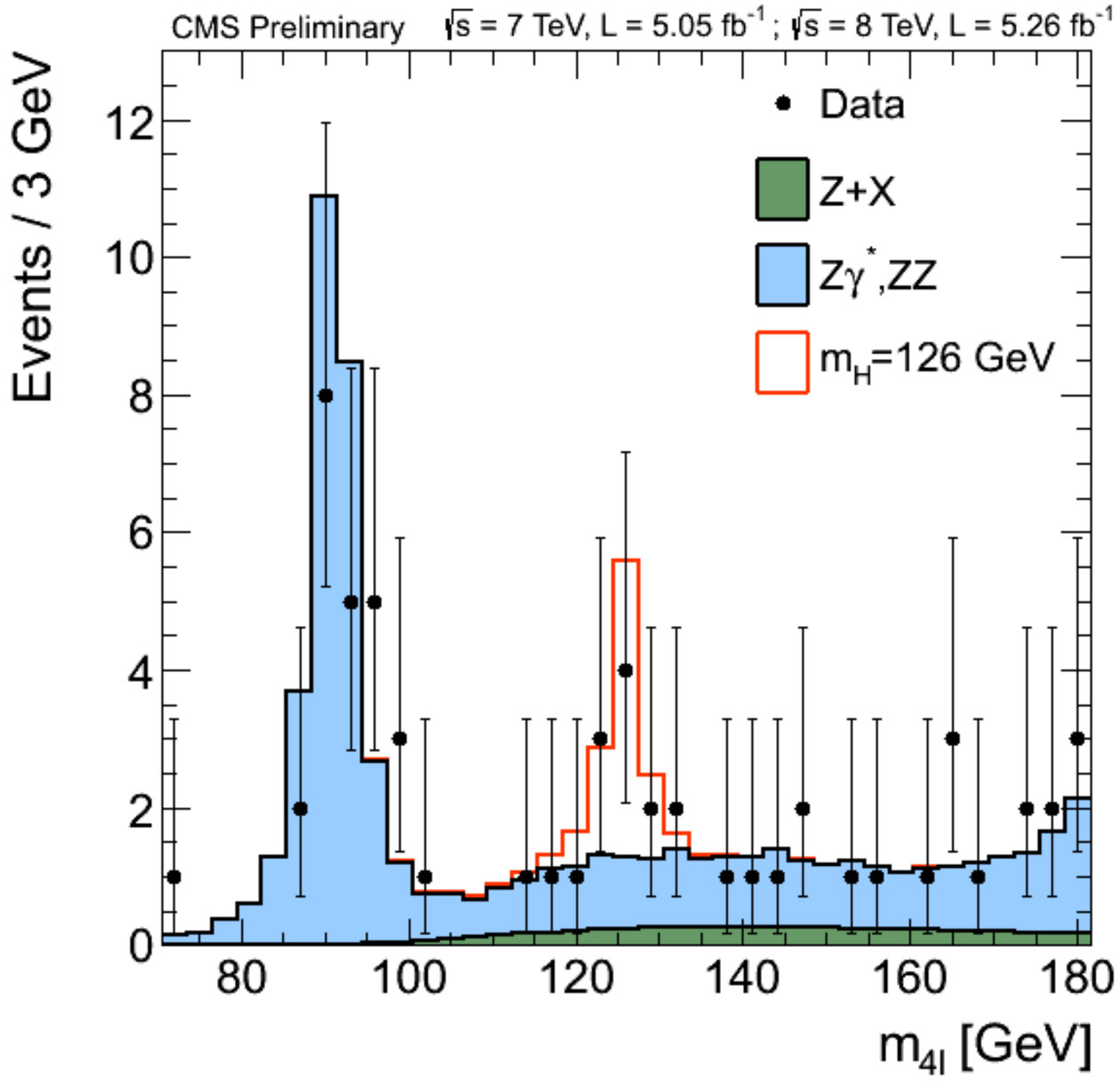


- When can we tell that we have discovered something new?
- Can we ever be 100% sure?

WHAT IS DATA ANALYSIS?

*"Data analysis is a process for obtaining **raw data** and converting it into information useful for decision-making by users. Data are collected and analyzed to answer questions, test hypotheses or disprove theories."*

- Data analysis uses statistics for presentation and interpretation (explanation) of data
- A mathematical foundation for statistics is the probability theory

DATA ANALYSIS IN THE INDUSTRY

RAW DATA

$(\text{search string}_1, \text{location}_1)^{\text{user 1}}$
$(\text{search string}_2, \text{location}_2)^{\text{user 1}}$
...
$(\text{search string}_n, \text{location}_n)^{\text{user 1}}$
$(\text{search string}_1, \text{location}_1)^{\text{user 2}}$
...
$(\text{search string}_m, \text{location}_m)^{\text{user 2}}$
$(\text{search string}_1, \text{location}_1)^{\text{user 3}}$
...
$(\text{search string}_1, \text{location}_1)^{\text{user k}}$
...

DATA ANALYSIS

- Maximum Likelihood fit
- Significance
- Hypothesis testing
- P-value
- Neural Networks

USABLE INFORMATION

DATA ANALYSIS IN HEP

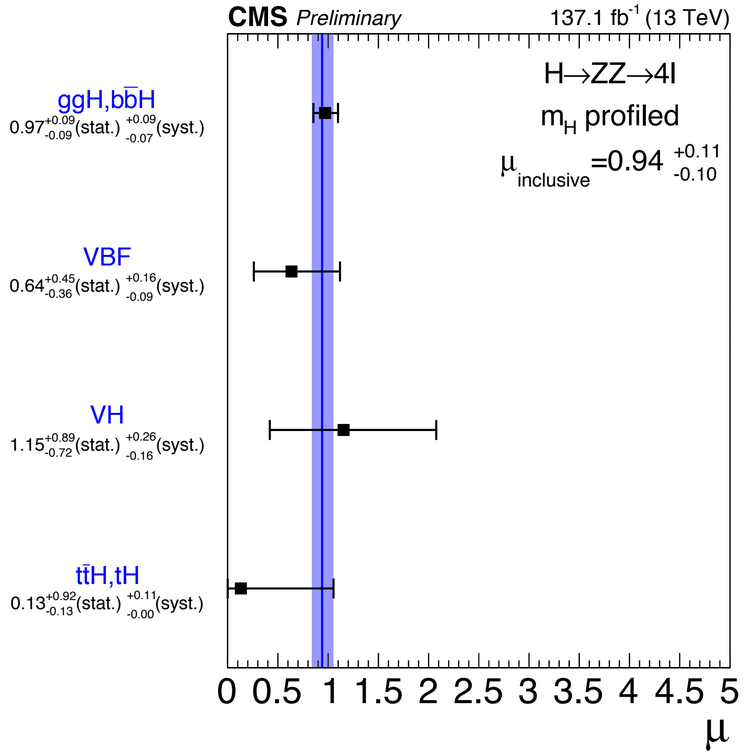
RAW DATA

$(p_{x1}, p_{y1}, p_{z1}, E_1)^{\text{event 1}}$
$(p_{x2}, p_{y2}, p_{z2}, E_2)^{\text{event 1}}$
...
$(p_{xn}, p_{yn}, p_{zn}, E_n)^{\text{event 1}}$
$(p_{x1}, p_{y1}, p_{z1}, E_1)^{\text{event 2}}$
...
$(p_{xm}, p_{ym}, p_{zm}, E_m)^{\text{event 2}}$
$(p_{x1}, p_{y1}, p_{z1}, E_1)^{\text{event 3}}$
...
$(p_{x1}, p_{y1}, p_{z1}, E_1)^{\text{event k}}$
...

DATA ANALYSIS

Maximum Likelihood fit
Significance
Hypothesis testing
P-value
Neural Networks

USABLE INFORMATION



DATA ANALYSIS IN HEP

- **Main goals** are:
  - estimate (measure) the parameters
  - quantify the uncertainty of the parameter estimates
  - test the extent to which the predictions of a theory are in agreement with the data
- Use of **statistics** for presentation and interpretation (explanation) of data
- A mathematical foundation for statistics is **the probability theory**
- Why is statistics even needed?
  - theory predictions in quantum mechanics are not deterministic
  - finite size of data sample
  - imperfection of the measurement

DATA ANALYSIS GENERAL PICTURE

Sampling reality

2 EXPERIMENT

1

$$ie(W_\mu^- W_\nu^+ - W_\mu^+ W_\nu^-)|^2 -$$
$$- W_\nu^+ A_\mu) + ig's_w (W_\mu^+ Z_\nu -$$
$$- \partial_\nu Z_\mu + ig' c_w (W_\mu^- W_\nu^+ - W$$

Physical phenomena

Described by a theory

Described by PDFs,
depending on unknown parameters
with true values
$\theta^{true}=(m_H^{true},\Gamma_H^{true},\ldots,\sigma^{true})$

4 DATA ANALYSIS

3

Data sample

$x = (x_1,x_2,\ldots,x_N)$

x is a multivariate random variable

5 **Results**

- **parameter estimates**
- **confidence limits**
- **hypothesis tests**

PROBABILITY DEFINITION

What is probability anyway?

“Unfortunately, statisticians do not agree on basic principles.”
- Fred James

Mathematical (axiomatic) definition

Classical definition

Frequentist definition

Bayesian (subjective) definition

MATHEMATICAL DEFINITION

- Developed in 1933 by Kolmogovor in his “Foundations of the Theory of Probability”

- Define an exclusive set of all possible elementary events x_i
  - Exclusive means the occurrence of one of them implies that none of the others occurs

- For every event x_i, there is a probability $P(x_i)$ which is a real number satisfying the Kolmogorov Axioms of Probability:
  I) $P(x_i) \geq 0$
  II) $P(x_i \text{ or } x_j) = P(x_i) + P(x_j)$
  III) $\sum P(x_i) = 1$

- From these properties more complex probability expressions can be deduced
  - For non-elementary events, i.e. set of elementary events
  - For non-exclusive events, i.e. overlapping sets of elementary events

- Entirely free of meaning, does not tell what probability is about

CLASSICAL DEFINITION

"The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence, and in determining the number of cases favourable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability, which is thus simply a fraction whose numerator is the number of favourable cases and whose denominator is the number of all the cases possible."

- Pierre-Simon Laplace,

A Philosophical Essay on Probabilities

FREQUENTIST DEFINITION

- Experiment performed N times, outcome x occurs N(x) times
- Define probability: $$P(x) = \lim_{N \to \infty} \frac{N(x)}{N}$$
- Such a probability has big restrictions:
  - depends on the sample, not just a property of the event
  - experiment must be repeatable under identical conditions
  - For example one can’t define a probability that it’ll snow tomorrow
- Probably the one you’re implicitly using in everyday life
- Frequentist statistics is often associated with the names of *Jerzy Neyman* and *Egon Pearson*

BAYESIAN DEFINITION

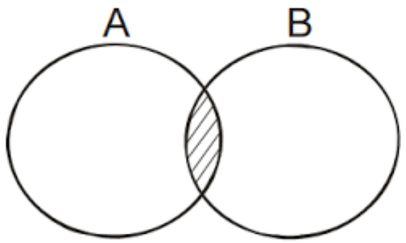
- Define probability: P(x) = **degree of belief** that x is true

- It can be quantified with betting odds:
  - What's amount of money one's willing to bet based on their belief on the future occurrence of the event

- In particle physics frequency interpretation often most useful, but Bayesian probability can provide more natural treatment of non-repeatable phenomena

BAYES' THEOREM

- Define conditional probability: P(A|B) = P(A∩B)/P(B)
  - probability of A happening given B happened
  - for independent events P(A|B) = P(A), hence P(A∩B)=P(A)P(B)



- From the definition of conditional probability Bayes' theorem states:

$$P(T|D) = \frac{P(D|T)P(T)}{P(D)}$$

- T is a **theory** and D is the **data**
- P(T) is the **prior probability** of T: the probability that T is correct before the data D was seen
- P(D|T) is the **conditional probability** of seeing the data D given that the theory T is true.
  - P(D|T) is called the likelihood.
- P(D) is the **marginal probability** of D.
  - P(D) is the prior probability of witnessing the data D under all possible theories
- P(T|D) is the **posterior probability**: the probability that the theory is true, given the data and the previous state of belief about the theory

EXAMPLE

- You meet an old friend in a pub. He proposes that the next round should be payed by whoever of the two extracts the card of lower value from a pack of cards
- This situation happens many times in the following days. What is the probability that your friend cheats if you end up paying *wins* consecutive times?*
- You assume:
  - P(*cheat*) = 5% and P(*honest*) = 95% (surely an old friend is an unlikely cheater…)
  - P(*wins* | *cheat*) = 1 and P(*wins* | *honest*) = 2^{-wins}
- Bayesian solution:

$$P(cheat\,|\,wins) = \frac{P(wins\,|\,cheat)P(cheat)}{P(wins\,|\,cheat)P(cheat) + P(wins\,|\,honest)P(honest)}$$

$$P(cheat\,|\,0) = \frac{1 \cdot P(cheat)}{1 \cdot P(cheat) + 2^{-0}P(honest)} = \frac{0.05}{0.05 + 0.95} = 5\,\%$$

$$P(cheat\,|\,5) = \frac{1 \cdot P(cheat)}{1 \cdot P(cheat) + 2^{-5}P(honest)} = \frac{0.05}{0.05 + 0.03} = 63\,\%$$

*taken from G.D'Agostini, Bayesian Reasoning in HEP, Principles and Applications, CERN-99-03, 1999

LEARNING BY EXPERIENCE

- The process of updating the probability when new experimental data becomes available can be follow easily if we insert
  - P(cheat) = P(cheat | wins-1) and P(honest) = P(honest | wins -1), where wins - 1 indicates the probability assigned after the previous win
  - P(wins=1 | cheat) = P(win | cheat) = 1 and P(wins = 1 | honest) = 0.5
- Iterative application of the Bayes' formula:

$$P(cheat \mid wins) = \frac{P(win \mid cheat)P(cheat \mid wins-1)}{P(win \mid cheat)P(cheat \mid wins-1) + P(win \mid honest)P(honest \mid wins-1)}$$

$$P(cheat \mid wins) = \frac{P(cheat \mid wins-1)}{P(cheat \mid wins-1) + 0.5 * P(honest \mid wins-1)}$$

P(cheat)	**P(cheat | wins)**		
%	wins=5	wins=10	wins=15
1	24%	91%	99.7%
5	63%	98%	99.94%
50	97%	99.9%	99.99%

When you learn from the experience, your conclusion does not longer depend on the initial assumptions!

RANDOM VARIABLES

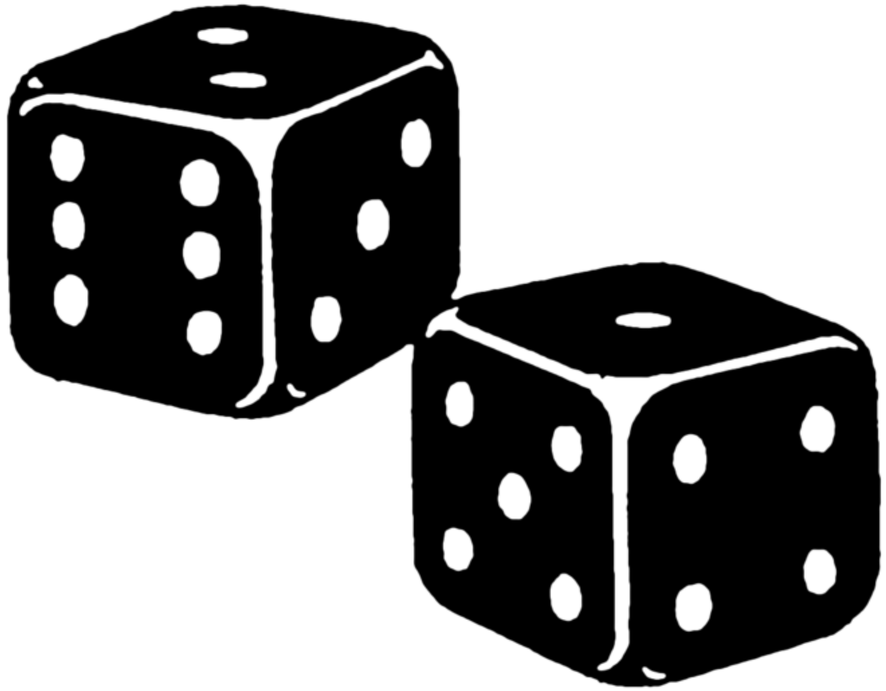
- **Random event** is an event having more than one possible outcome
  - Each outcome may have associated probability
  - Outcome not predictable, only the probabilities known

- Different possible outcomes may take different possible numerical values x_1, x_2, ...
- The corresponding probabilities $P(x_1)$, $P(x_2)$, ... form a **probability distribution**

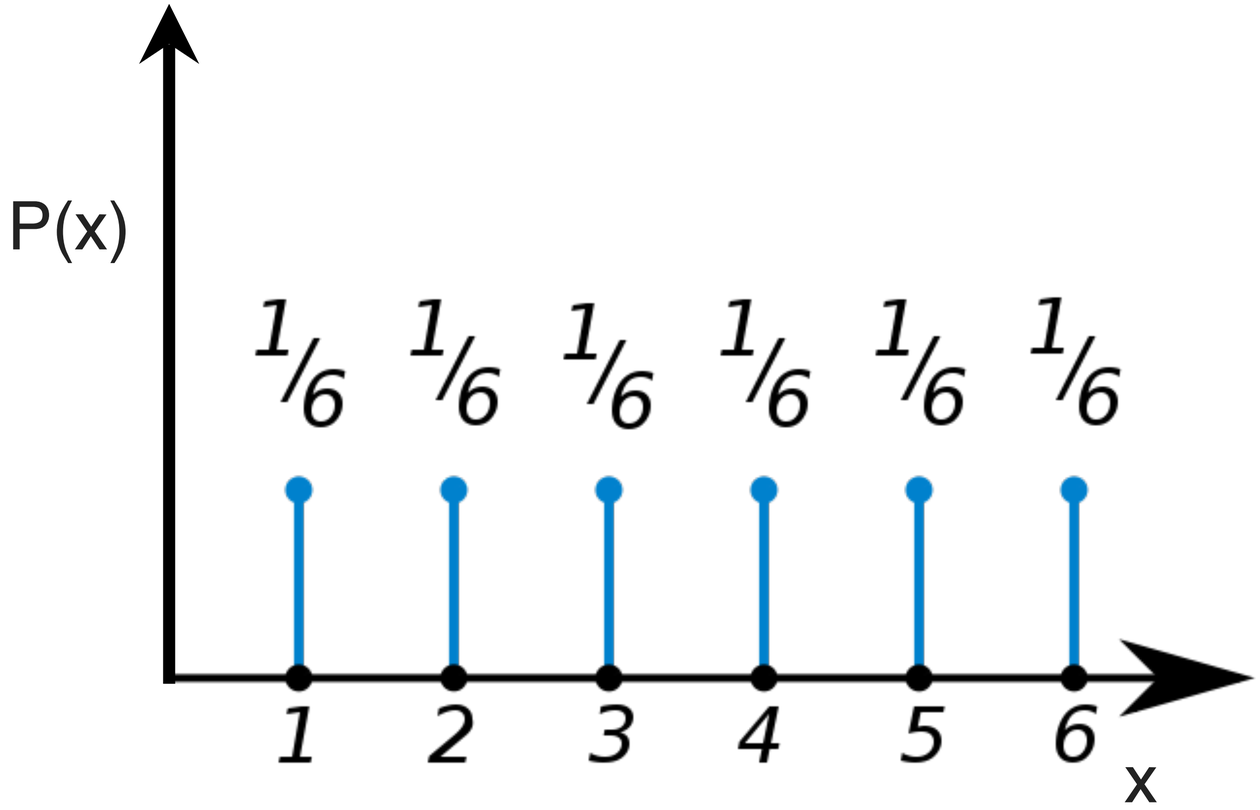
- If observations are independent the distribution of each random variable is unaffected by knowledge of any other observation
- When an experiment consists of N repeated observations of the same random variable x, this can be considered as the single observation of a random vector **x**, with components x_1, x_2, …, x_N

DISCRETE RANDOM VARIABLES

- Rolling a die:
  - Sample space = {1,2,3,4,5,6}
  - Random variable x is the number rolled



- Discrete probability distribution:



CONTINUOUS RANDOM VARIABLES

- A spinner:
  - Can choose a real number from [0,2n]
  - All values equally likely
  - x = the number spun
  - Probability to select any real number = 0
  - Probability to select any **range** of values > 0
    - Probability to choose a number in [0,n] = 1/2
  - Probability to select a number from any range Δx is Δx/2n
  - Now we say that **probability density** p(x) of x is 1/2n

- More general: $$P(A < x < B) = \int_A^B p(x)dx$$