Projective Gauge Theory and Thomas-Whitehead Gravity

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What Else can String Symmetries Tell us About 4D Gravity?

- Kac-Moody algebra are related to Yang-Mills theories in higher dimensions
- What was the analog for the Virasoro algebra?
- Would these higher dimensional contributions correspond to Dark Matter/Dark Matter?
- Is there a mathematically natural origin of the inflaton?

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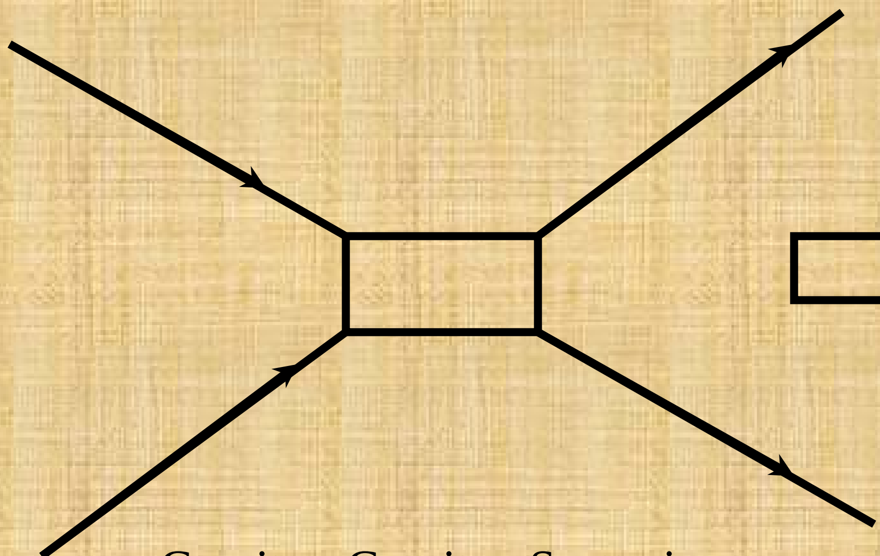
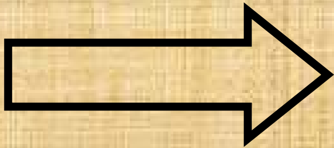
String Theory can “regulate” gravitational diagrams

Despite classical success GR fails to explain quantum gravitation despite its weak couplings. Feynman diagrams should have worked.



Graviton-Graviton Scattering



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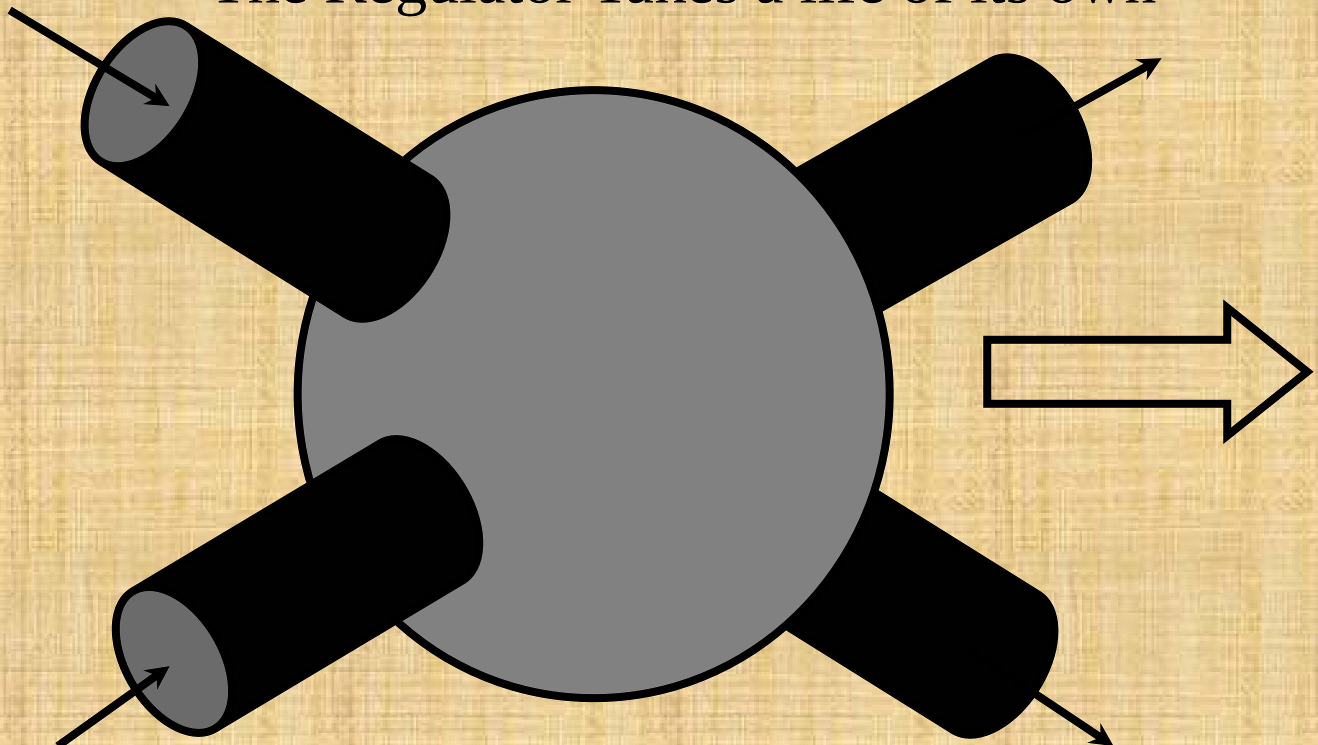
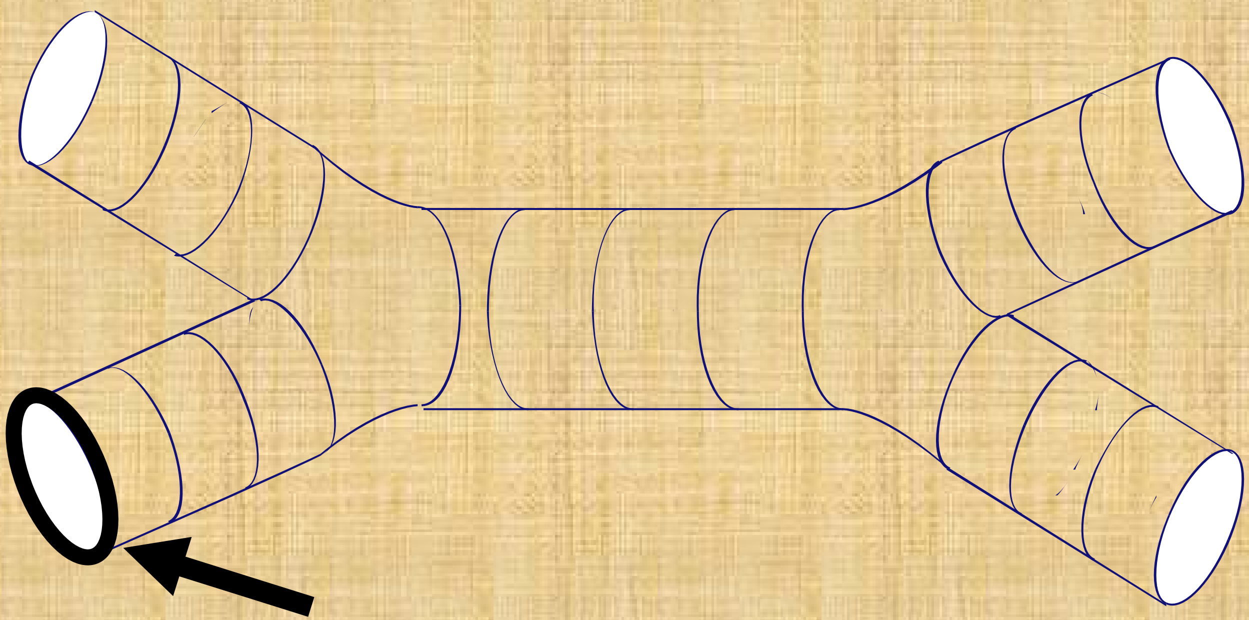
String Theory "Regulates" Feynman Diagrams

- Adds tiny Dimension to point particle
- Size of fattening is on the order of Planck length
- The Regulator Takes a life of its own



Finite and familiar potential

One Dimension: Examine Primitive Strings Properties To Understand More About Gravity



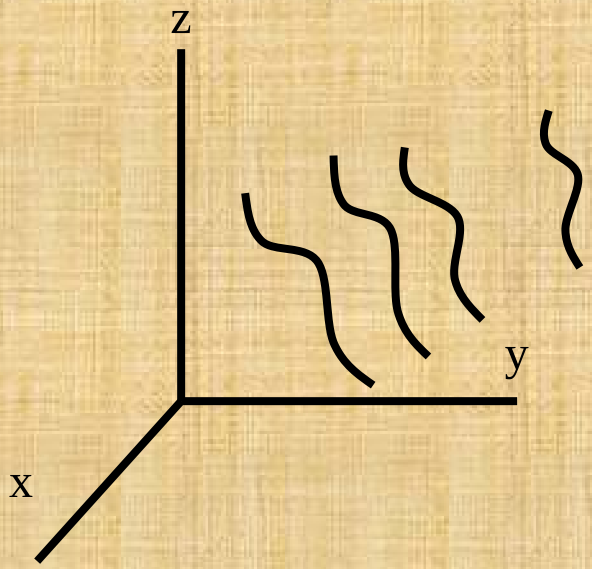
Mathematically Examine the Group properties of this tiny correction:

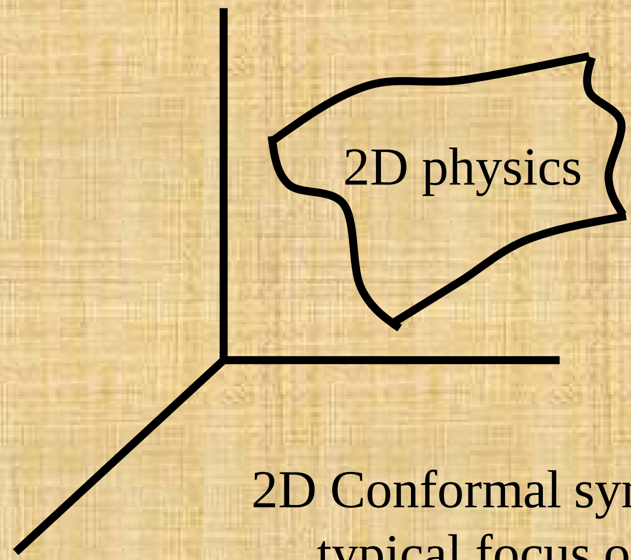
Virasoro Group and Algebra

Strings add new symmetry beyond point particle symmetries in Feynman diagrams

$$S_L = \frac{1}{\alpha'} \int \frac{d^2\sigma}{2\pi} \sqrt{-g} g^{mn} \tfrac{1}{2} (\partial_m X^a) \cdot (\partial_n X^b) \eta_{ab}$$

Open String sweeps out a 2D world sheet as it propagates.

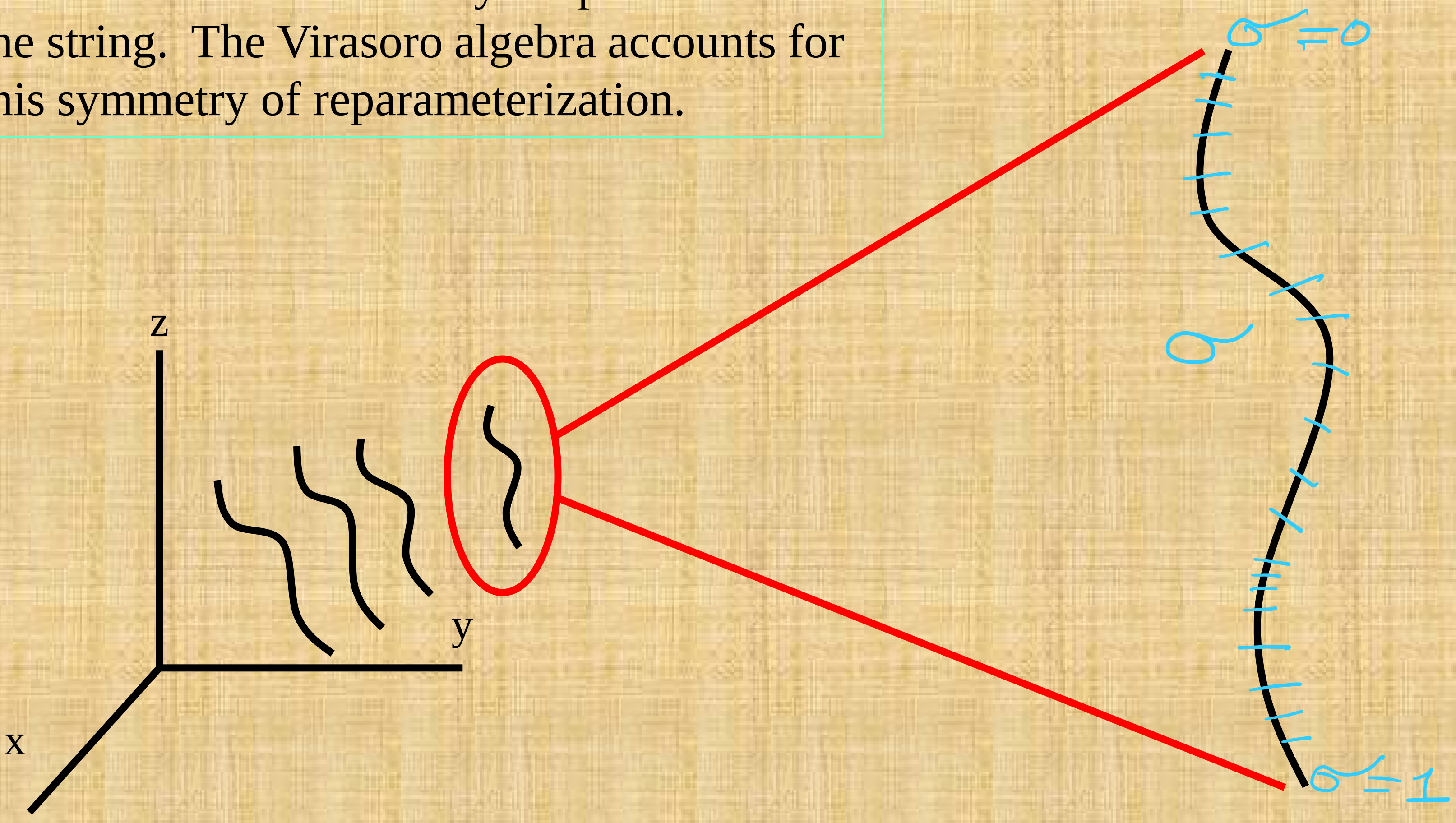




2D Conformal symmetry is the typical focus of attention

The Virasoro Algebra: A one dimensional symmetry, we will start in one dimension.

There are uncountable ways to parameterize the string. The Virasoro algebra accounts for this symmetry of reparameterization.



By marrying the Virasoro Algebra and Kac- Moody Algebra (Affine Lie Algebra), we can track how to make contact with higher dimensions as Yang-Mills is related to Kac-Moody algebra.

Algebra of String Symmetries:

- One Dimensional Coordinate Transformations:

$$[L_N, L_M] = (N-M)\, L_{N+M} + (cN^3 + hN)\, \delta_{N+M,0}$$

- One Dimensional Gauge Transformations:

$$\left[J_N^\alpha, J_M^\beta\right] = i\, f^{\alpha\beta\gamma}\, J_{N+M}^\beta + N\, k\, \delta_{N+M,0}\, \delta^{\alpha\beta}$$

- Semi-Direct Product:

$$[L_N, J_M^\alpha] = -M\; J_{N+M}^\alpha$$

Virasoro algebra as centrally extended Lie algebra of 1-D vector fields: ξ, η, while "a" and "b" are central elements.

$$[(\xi, a), (\eta, b)] = (\xi \circ \eta, ((\xi, \eta))_0)$$

Lie Algebra of vector fields

Central extension via a Gelfand-Fuchs *2-cocycle*

$$((\xi, \eta))_0 = \frac{c}{2\pi} \int (\xi \eta''') d\theta$$

Gelfand-Fuchs 2-cocycle.

$$\eta \partial_\theta \rightarrow \eta''' \, d\theta^2 \longrightarrow (\tilde{\nabla}_\alpha G^{\rho\nu} \tilde{\nabla}_\rho \tilde{\nabla}_\nu \eta^\beta G_{\beta\mu}) \, d\theta^\alpha d\theta^\mu$$

Quadratic differential

Quadratic differentials are *dual* to the Virasoro algebra

One-forms are *dual* to the Kac-Moody Algebra

$$\tilde{L}_N = e^N_{ab}\, dx^a\, dx^b = -ie^{-iN\theta}\, d\theta^2$$

Modes of a quadratic differentials in Virasoro algebra

$$\tilde{J}^\alpha_N = A^{N,\alpha}_a\, dx^a = \tau^\alpha e^{-iN\theta}\,,$$

Modes of a one-forms which are the duals of the affine Lie algebras

Kac-Moody Algebra and Dual

$$< \Lambda | A >= tr \int \Lambda A_j dx^j$$

$$\Lambda^I_J = \sum_{\alpha=1}^{q} \sum_{N=-\infty}^{\infty} \Lambda^N_\alpha (J^\alpha_N)^I_J$$

Non-Abelian Gauge Parameters

$$A = \sum_{\alpha=1}^{M} \sum_{n=-\infty}^{\infty} \Lambda^n_\alpha \tilde{J}^\alpha_n$$

Non-Abelian Gauge Connections
(Yang-Mills Fields)

Virasoro Algebra and Dual

$$< \xi | \mathcal{D} >= \int \xi^i \mathcal{D}_{ij} dx^j$$

$$\xi^j = \sum_{N=-\infty}^{\infty} e^j_N \xi^N$$

One Dimensional Vector Fields

$$\mathcal{D} = \sum_{n=-\infty}^{\infty} \mathcal{D}^n \tilde{L}_n$$

Projective Connections
(as we shall see)

The Duals of the Virasoro Algebra are the Coadjoint Elements:

$$\langle(\xi, a)|(B, c)\rangle \equiv \int (\xi B) d\theta + ac$$

1) Kirillov recognizes that the Coadjoint elements genralize Gelfand-Fuchs
2) The coadjoint elements transform as:

$$ad^*_{(\eta,d)}(B, c) = (\eta B' + 2\eta' B - c\eta''', 0)$$

3) The Coadjoint Orbits have a Natural Symplectic two-form Ω that when integrated leads to 2D Polyakov Gravity and Wess-Zumino Witten Models.
4) The Coadjoint elements are in one-to-one correspondence with Sturm-Liouville potentials which are directly related to projective structure.
5) The projective structure is the desendent from the higher dimensional projective geometry of Thomas and Whitehead.

Coadjoint Orbits admit a natural symplectic structure which interprets the coadjoint elements as fields and coupling constants.

- Kac-Moody and Virasoro correspond to Wess-Zumino-Witten and Polyakov quantum 2D gravity with Background Fields A and D

$$S = \int \Omega$$

- This innocent expression becomes

THE 2D STRING ACTION FOR 2D GRAVITY:

Basically, “Bosonized Fermions” coupling to Background Fields

$$\Psi^{Ia} : h_{\mu\nu} = \bar{\Psi}^I \gamma_\mu \gamma_\nu \Psi_I \simeq \frac{\partial_t s}{\partial_x s}$$

$$S = \int d^2x\, D(x,t) \frac{\partial_t s}{\partial_x s} + \int d^2x\, A(x,t)\, g^{-1}\partial_t g$$

$$+ \frac{c\mu}{48\pi} \int \left[\frac{\partial_x^2 s}{\partial_x s} \partial_t \partial_x s - \frac{(\partial_x^2 s)^2 (\partial_t s)}{(\partial_x s)^3} \right] d^2x$$

$$- k\mu \int \mathrm{Tr}(g^{-1}\partial_x g)(g^{-1}\partial_t g)\, d^2x$$

$$- k\mu \int \mathrm{Tr}(g^{-1}\partial_x g)[(g^{-1}\partial_t g),(g^{-1}\partial_z g)]\, d^3x$$

$$\Psi^{Ia} : (J^\mu)^I_J = \bar{\Psi}^I \gamma^\mu \Psi_J \simeq (g^{-1}\partial_t g)^I_J$$

THE 2D STRING ACTION FOR 2D GRAVITY:
This Would Be Related to Pure Gauge in Higher Diemsnions

Polyakov 2D Gravitational Theory

$$S = \int d^2x\, D(x,t)\frac{\partial_t s}{\partial_x s} - \int d^2x\, A(x,t)\, g^{-1}\partial_t g$$

$$+\frac{c\mu}{48\pi}\int\left[\frac{\partial_x^2 s}{\partial_x s}\partial_t\partial_x s - \frac{(\partial_x^2 s)^2(\partial_t s)}{(\partial_x s)^3}\right]d^2x$$

$$-\,k\mu\int \mathrm{Tr}(g^{-1}\partial_x g)(g^{-1}\partial_t g)\,d^2x$$

$$-\,k\mu\int \mathrm{Tr}(g^{-1}\partial_x g)[(g^{-1}\partial_t g),(g^{-1}\partial_z g)]\,d^3x$$

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Wess Zumino Witten: gauge theory

Background fields couple to the bosonized fermions.

$$S = \int d^2x \, D(x,t)\frac{\partial_t s}{\partial_x s} - \int d^2x \, A(x,t) g^{-1}\partial_t g$$

$$+\frac{c\mu}{48\pi}\int\left[\frac{\partial_x^2 s}{\partial_x s}\partial_t\partial_x s - \frac{(\partial_x^2 s)^2(\partial_t s)}{(\partial_x s)^3}\right]d^2x$$

$$-k\mu\int \mathrm{Tr}(g^{-1}\partial_x g)(g^{-1}\partial_t g)d^2x$$

$$-k\mu\int \mathrm{Tr}(g^{-1}\partial_x g)[(g^{-1}\partial_t g),(g^{-1}\partial_z g)]d^3x$$

B. Rai and V.G.J.R

Nucl. Phys B341 119-133 (1990)

Can these Background have meaning in any dimension? If so, D becomes a fundamental field in a theory of gravitation just as A is fundamental as a gauge field (W, Z, photons). D_{ab} is called the *diffeomorphism* field.

$$S = \int d^2x\, D(x,t)\frac{\partial_t s}{\partial_x s} - \int d^2x\, A(x,t)\, g^{-1}\partial_t g$$

$$+ \ldots.$$

$$S = \int d^2x \sqrt{h}\, D_{ab}\, h^{ab} - \int d^2x \sqrt{h}\, A_a J^a$$

$$+ \ldots.$$

Our usual Gauge Fields

h^{ab} and J^a *are the bos*onized fermions

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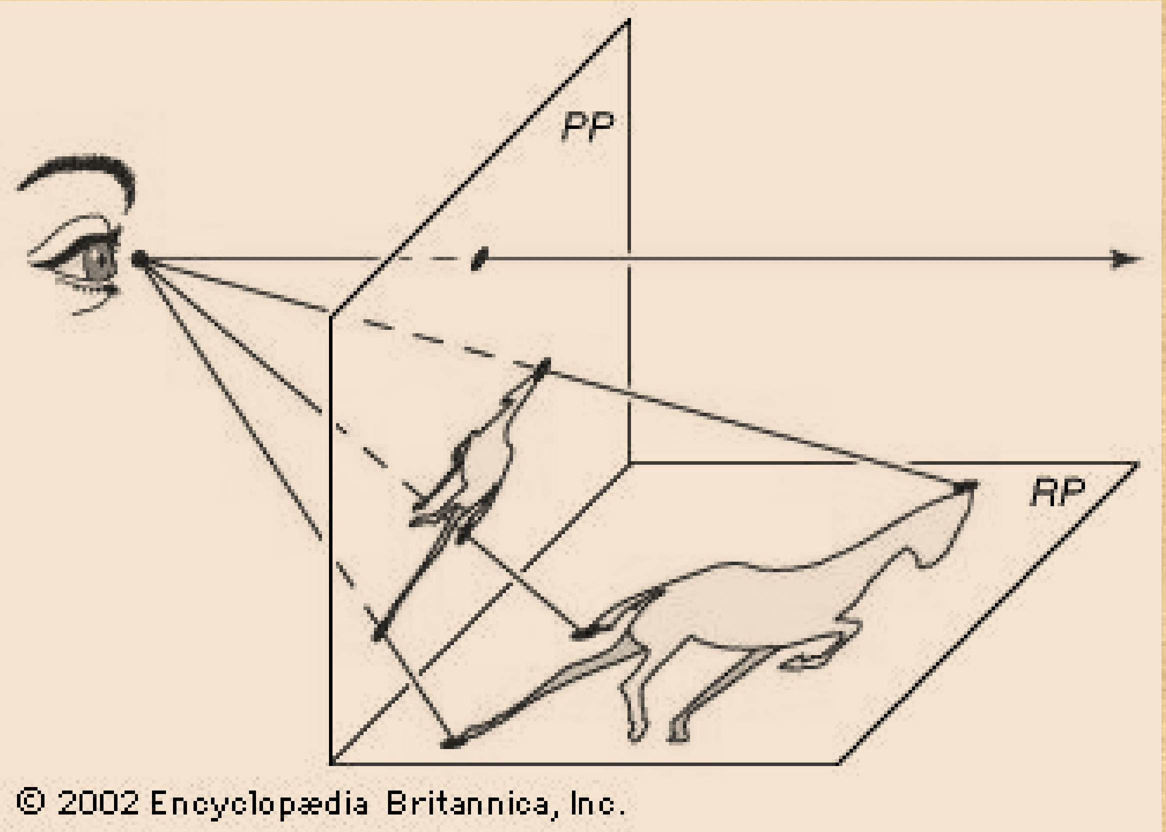
V.G.J.R, Phys. Letters B336 (1994) 343-346

Answer: Projective (Structures) Geometry is the requisite structure.



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Projective Symmetry in Geodesics and Geodetics:

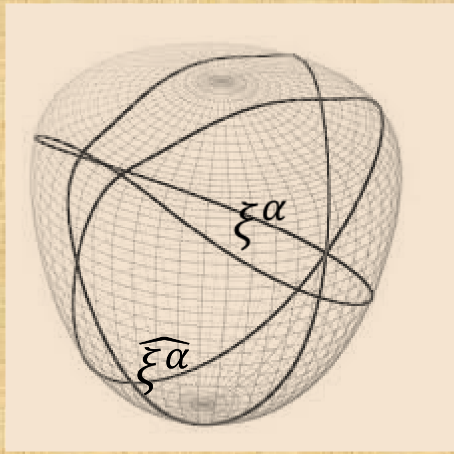
Changing the Connection by a one-form v_b yields same geodesics

$$\hat{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + \delta^a{}_b v_c + \delta^a{}_c v_b$$

projective transformation

$$\xi^d \hat{\nabla}_d \xi^a = \frac{d^2 x^a}{d\tau^2} + \hat{\Gamma}^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$$

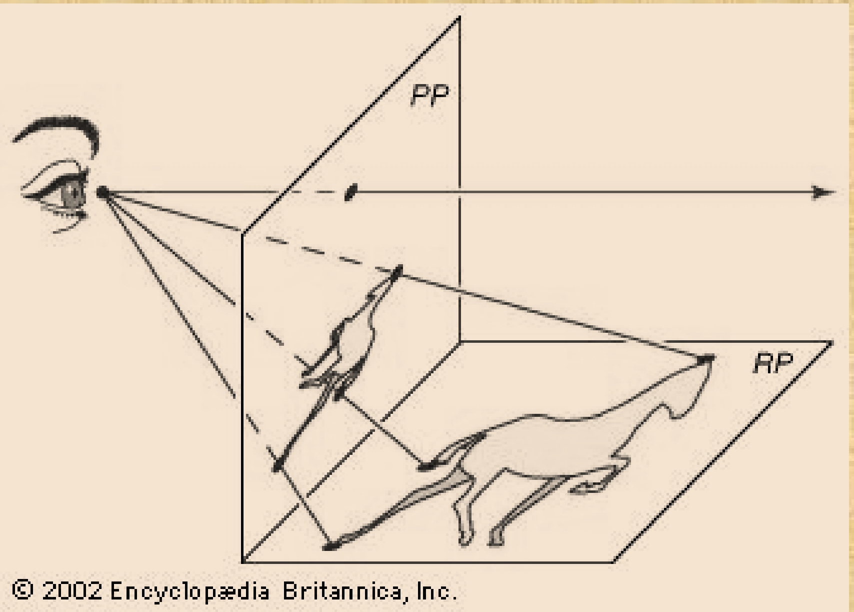
$$\xi^d \nabla_d \xi^a = \frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = f \frac{dx^a}{d\tau}$$



$$f(\tau) = -2 v_b \frac{dx^b}{d\tau}$$

Projective Connection: $\widetilde{\nabla}_c$

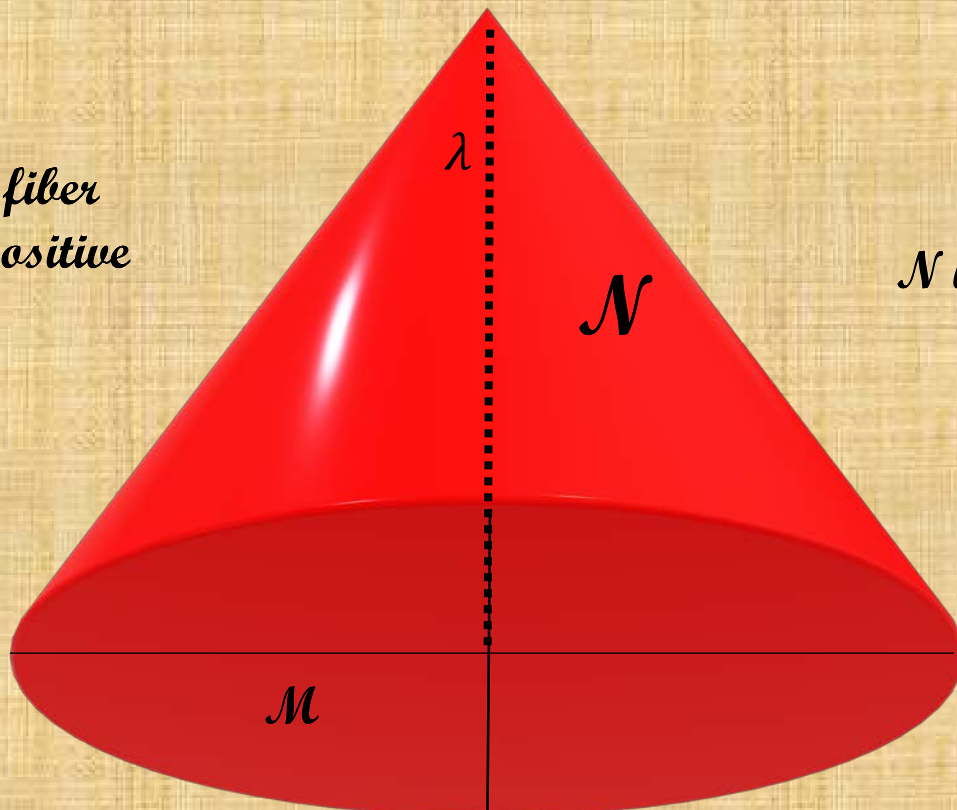
- Work of E. Cartan 1924
- T.Y. Thomas 1925
- J.H.C. Whitehead 1931



- Write a connection where projective transformations are a gauge symmetry
- Different parametrizations are just different section of the fiber bundle.

The Thomas Cone, $\mathcal{N}$: the fiber bundle for the projective connection $\widetilde{\nabla}_c$

λ is the fiber and is positive definite



$\mathcal{N}$ is the Thomas Cone

$\mathcal{M}$ is the 4D manifold

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Coordinate transformations on $\mathcal{N}$

The Volume fiber

$$(\lambda, x^0, \ldots, x^{m-1}) \to (\lambda|J|^{-\frac{1}{m+1}}, y^0, \ldots, y^{m-1})$$

$$J^a{}_b = \left[\frac{\partial y^a}{\partial x^b}\right]$$

1) $\widetilde{\nabla}_c$ transforms covariantly on the Thomas Cone

2) There is a one-to-one correspondence with general coordinate transformations on the underlying manifold and the coordinate transformations of the Thomas Cone.

3) The λ coordinate acts as the "volume" and scaled like the volume.

Explicit Realization $\widetilde{\nabla}_c$:

$$
\tilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \begin{cases}
\tilde{\Gamma}^{\lambda}{}_{\lambda a} = \tilde{\Gamma}^{\lambda}{}_{a\lambda} = 0 \\
\tilde{\Gamma}^{\alpha}{}_{\lambda\lambda} = 0 \\
\tilde{\Gamma}^{a}{}_{\lambda b} = \tilde{\Gamma}^{a}{}_{b\lambda} = \omega_{\lambda}\, \delta^{a}_{b} \\
\tilde{\Gamma}^{a}{}_{bc} = \Pi^{a}{}_{bc} \\
\tilde{\Gamma}^{\lambda}{}_{ab} = \Upsilon^{\lambda} \mathcal{D}_{ab}
\end{cases}
$$

Greek coordinates on $\mathcal{N}$, while Latin restricted to $\mathcal{M}$

Explicit Realization $\widetilde{\nabla}_c$:

$$\tilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \begin{cases} \tilde{\Gamma}^{\lambda}{}_{\lambda a} = \tilde{\Gamma}^{\lambda}{}_{a\lambda} = 0 \\ \tilde{\Gamma}^{\alpha}{}_{\lambda\lambda} = 0 \\ \tilde{\Gamma}^{a}{}_{\lambda b} = \tilde{\Gamma}^{a}{}_{b\lambda} = \omega_{\lambda}\, \delta^{a}_{b} \\ \tilde{\Gamma}^{a}{}_{bc} = \Pi^{a}{}_{bc} \\ \tilde{\Gamma}^{\lambda}{}_{ab} = \Upsilon^{\lambda} \mathcal{D}_{ab} \end{cases}$$

Fundamental Projective Invariant

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Explicit Realization $\widetilde{\nabla}_c$:

$$\tilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \begin{cases} \tilde{\Gamma}^{\lambda}{}_{\lambda a} = \tilde{\Gamma}^{\lambda}{}_{a\lambda} = 0 \\ \tilde{\Gamma}^{\alpha}{}_{\lambda\lambda} = 0 \\ \tilde{\Gamma}^{a}{}_{\lambda b} = \tilde{\Gamma}^{a}{}_{b\lambda} = \omega_{\lambda}\,\delta^{a}_{b} \\ \tilde{\Gamma}^{a}{}_{bc} = \Pi^{a}{}_{bc} \\ \tilde{\Gamma}^{\lambda}{}_{ab} = \Upsilon^{\lambda}\mathcal{D}_{ab} \end{cases}$$

This component is called the Diffeomorphism field that relates back to Virasoro algebra.

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The Projective Geometry:

$$[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta] V^\gamma = \mathcal{K}^\gamma{}_{\rho\alpha\beta} V^\rho$$

$$[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta] V_\gamma = -\mathcal{K}^\rho{}_{\gamma\alpha\beta} V_\rho$$

$$\mathcal{K}^a{}_{bcd} = \mathcal{R}^a{}_{bcd} + \delta^a{}_{[c} \mathcal{D}_{d]b}$$

$$\mathcal{K}^\lambda{}_{cab} = \lambda \partial_{[a} \mathcal{D}_{b]c} + \lambda \Pi^d{}_{c[b} \mathcal{D}_{a]d}$$

The Thomas-Whitehead Gravity: The ingredients

1) Metric tensor: g_{ab}

2) Projective Connection Coefficients:

$$\widetilde{\nabla}_c \longrightarrow \pi^a{}_{bc} \text{ and } D_{ab}$$

D_{ab}, the Diffeomorphism Field, guarantees covariance of $\widetilde{\nabla}_c$ on the Thomas Cone

Brensinger and V.G.J.R, IJMPA, Vol 33, No. 36 (2018) 1850223

1) Diffeomorphism Field’s relation to Coadjoint Elements of the Virasoro Algebra in one dimension

$$\mathcal{D}'_{ab} = \frac{\partial x^m}{\partial x'^a}\frac{\partial x^n}{\partial x'_b}\mathcal{D}_{mn} - \frac{1}{(\mathrm{d}+1)^2}\frac{\partial \log J}{\partial x'^a}\frac{\partial \log J}{\partial x'^b} - \frac{1}{\mathrm{d}+1}\frac{\partial^2 \log J}{\partial x'^a \partial x'^b} + \frac{1}{\mathrm{d}+1}\frac{\partial \log J}{\partial x'^c}\Pi'^c{}_{ab}$$

Projective Connection	**Coadjoint Element**
$\delta D = 2\xi' D + D'\xi + \frac{c\tilde{\mu}}{2\pi}\xi'''$	$ad^*_{(\eta,d)}(B,c) = (\eta B' + 2\eta' B - c\eta''', 0)$

2) Diffeomorphism Field Explains 2D Coupling in Polyakov action

$$S_{PEH} = \int \sqrt{|G|}\; K \; d\lambda \; d^m x \longrightarrow S = \int d^2x\, D(x,t)\frac{\partial_t s}{\partial_x s}$$

Brensinger and V.G.J.R, IJMPA, Vol 33, No. 36 (2018) 1850223

TW Gravity: Projective Einstein-Hilbert and Projective Gauss-Bonnet (Series in Gauss-Bonnet Terms)

$$S = S_{PEH} + S_{PGB}$$

$$S_{PEH} = -\frac{1}{2\tilde{\kappa}_0 \lambda_0} \int d\lambda \, d^{\mathrm{d}}x \sqrt{|G|} \mathcal{K}^{a}{}_{bcd} (\delta^{c}{}_{a} g^{bd})$$

$$S_{PGB} = -\frac{\tilde{J}_0 c}{\lambda_0} \int d\lambda \, d^{\mathrm{d}}x \sqrt{|G|} \, \left(\mathcal{K}^{\alpha}{}_{\beta\gamma\rho} \mathcal{K}_{\alpha}{}^{\beta\gamma\rho} - 4 \mathcal{K}_{\alpha\beta} \mathcal{K}^{\alpha\beta} + \mathcal{K}^2 \right)$$

- Now we add dynamics to the action for the diffeomorphism field.
- Diff field transforms non-tensorially on $\mathcal{M}$.
- We use Gauss-Bonnet to avoid higher derivative theories
- TW Gravity collapse to Einstein-Hilbert when Diff field vanishes

S. Brensinger, K. Heitritter, V. G. J. Rodgers and K. Stiffler, *General structure of Thomas–Whitehead gravity*, *Phys. Rev. D* **103** (2021) 044060 [2009.06730].

TW Gravity: Reduces to a Four-Dimensional Theory

$$
\begin{aligned}
S = \left(\int \frac{1}{\lambda}\, d\lambda\right) \Bigg[&- \tfrac{1}{2\tilde{\kappa}_0} \int d^{\mathrm{d}}x \sqrt{|g|}\, \mathcal{K} \\
&- \tilde{J}_0 c \int d^{\mathrm{d}}x \sqrt{|g|} \left(\mathcal{K}^a{}_{bcd}\mathcal{K}_a{}^{bcd} - 4\mathcal{K}_{ab}\mathcal{K}^{ab} + \mathcal{K}^2\right) \\
&+ \tilde{J}_0 c \lambda_0{}^2 \int d^{\mathrm{d}}x \sqrt{|g|} \underbrace{\left(g_a \mathcal{K}^a{}_{bcd} + \breve{\mathcal{K}}_{bcd}\right)}_{\text{tensor}} \underbrace{\left(g_e \mathcal{K}^e{}_{fgh} + \breve{\mathcal{K}}_{fgh}\right)}_{\text{tensor}} g^{bf} g^{cg} g^{dh} \Bigg] \,.
\end{aligned}
$$

Dressed Coupling Constants: suggests "built-in" renormalization

$$
\frac{1}{\tilde{\kappa}_0} \int_{\ell_i}^{\ell_f} d\ell\, \frac{1}{\ell} = \frac{\log(\ell_f/\ell_i)}{\tilde{\kappa}_0} \qquad \Rightarrow \kappa_0 \equiv \frac{\tilde{\kappa}_0}{\log(\ell_f/\ell_i)}
$$

$$
\tilde{J}_0 \int_{\ell_i}^{\ell_f} d\ell\, \frac{1}{\ell} = \tilde{J}_0 \log(\ell_f/\ell_i) \qquad \Rightarrow J_0 \equiv \tilde{J}_0 \log(\ell_f/\ell_i)
$$

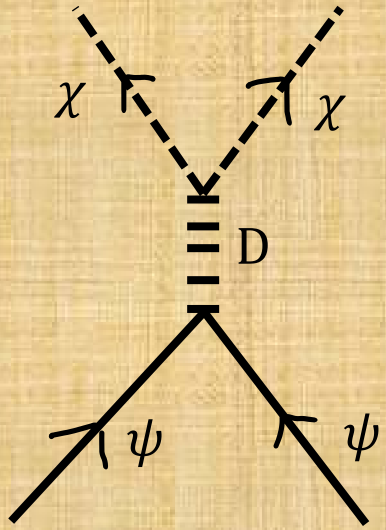
TW Dirac Action Reduces to Four-Dimensional Action

$$\mathcal{L}_{\mathrm{TWD}} = \sqrt{|g|}\left(i\bar{\psi}\nabla\!\!\!/\,\psi + \left(iB_m\bar{\psi}\gamma^m\psi + \Xi\bar{\psi}\gamma^5\psi\right) + M\bar{\psi}\psi\right)$$

Couples to all fermions whether in Standard model or Dark Matter sectors.

$$\Xi \equiv \lambda_0\, \mathcal{D}_{rm} g^{mr} + \text{other terms}$$

Pseudo scalar coupling leads to dark matter as well as dark matter portal.

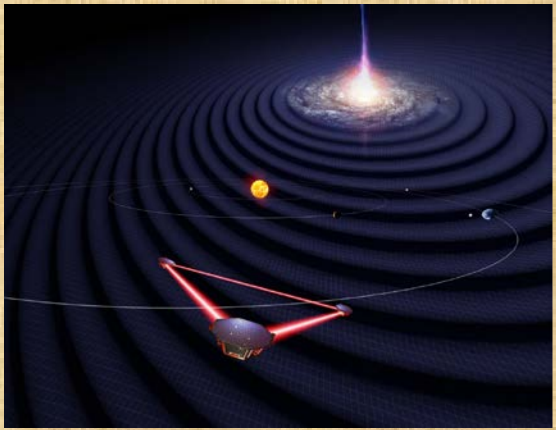


Geodesic Deviations and Radiation:

$$\frac{D^2 X^\alpha}{d\tau^2} = K^\alpha{}_{\beta\sigma\rho}\frac{dx^\beta}{d\tau}\frac{dx^\sigma}{d\tau}X^\rho$$

$$\frac{D^2 X^a}{du} - \left(\mathcal{R}^a{}_{bcd} + \delta_{[c}{}^a\mathcal{D}_{d]b}\right)\frac{dx^b}{du}\frac{dx^c}{du}X^d = \left(\frac{2}{\lambda}\frac{d\lambda}{du}\right)\frac{DX^a}{du}$$

- Diff Fields acts as source of gravitational radiation



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Conclusion

Symmetry	String Theory		Field Theory	
Reparameterization Invariance	Algebra: Virasoro	Coadjoint Elements: (B, q)	Connection: Projective	$\tilde{\nabla}_\alpha(\mathcal{D}_{bc}, \Pi^a{}_{bc})$
Gauge Invariance	Algebra: Affine Lie	Coadjoint Elements: $(A,\ \alpha)$	Connection: Yang-Mills	$D_a(A_b)$

TABLE I: *Correspondence of Symmetries in String Theories to Connections in Field Theories*
Coadjoint elements of the Virasoro algebra, (B, q), consists of a quadratic differential B and a central element q. They are in correspondence with the projective connection components $\mathcal{D}_{ab}$ that appear in the projective covariant derivative $\tilde{\nabla}_\alpha$. Analogously, the coadjoint elements of the affine Lie algebra (Kac-Moody algebra), (A, α), consisting of a one form A and a central element α, and are in correspondence with the Yang-Mills connection, A_a that appears in the gauge covariant derivative D_a.