

# Lattice Design

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The analysis of the cell stability and betatron functions can be done via an algorithmic approach using the method presented yesterday <sup>1</sup>:

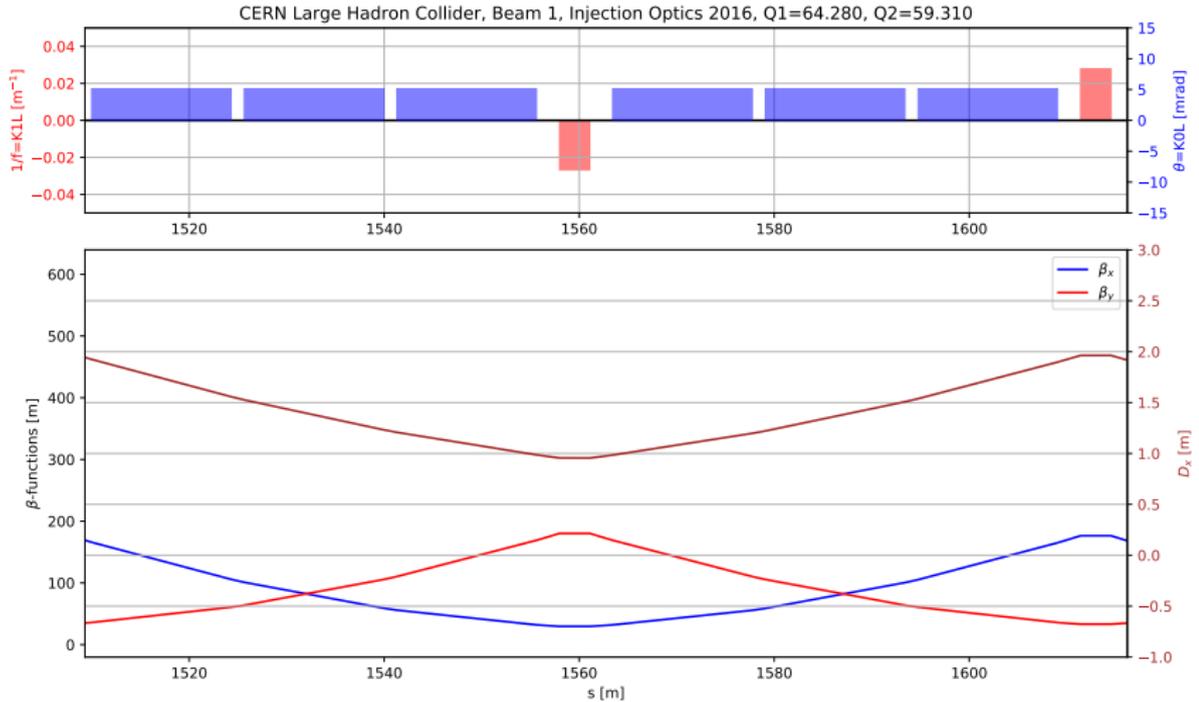
- 1 compute *symbolically* the  $M_{OTM}$ ,
- 2 diagonalize it  $M_{OTM} = PDP^{-1}$ , with  $\det(P) = -i$  and  $P_{11} = P_{12}$ ,
- 3 impose that all the eigenvalues amplitude is 1 to get the stability condition,
- 4 study P to get the periodic solution for  $\beta$  and  $\alpha$  at the start of the cell,
- 5 propagate the solution from the start of the cell along the different lattice element.

We will start considering a **FODO cell**.

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<sup>1</sup>[LatticeCellStudies.ipynb](#)

# The CERN Large Hadron Collider FODO cell



# The FODO cell description

Let's consider a FODO cell of length  $L_{cell}$  in **thin lens approximation**, where

- 1 the space of the focusing (F) and defocusing (D) quadrupoles is equal to  $L_{cell}/2$  and
- 2 the focal length of the F and D quadrupoles equal in module, that is  $f_D = -f_F$  with  $f_F > 0$ .

For convenience, we will start and end the FODO cell with half of an F quadrupole (i.e., with focal length  $2 \times f_F$ ) and we will consider, as first step, the horizontal plane.

# The FODO $M_{OTM}$ diagonalization

Using symbolic tools (e.g., *sympy*) one can compute

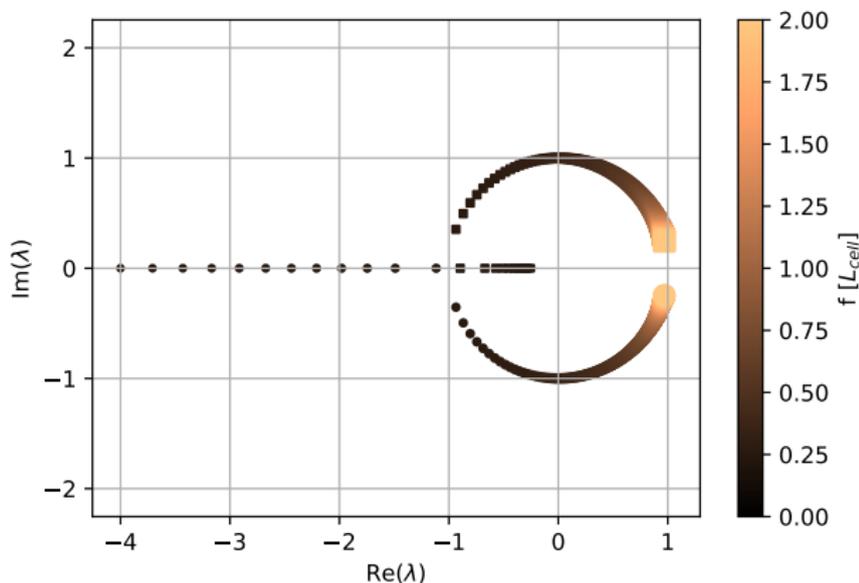
$$M_{OTM} = \begin{bmatrix} -\frac{L_{cell}^2}{8f^2} + 1 & \frac{L_{cell}^2}{4f} + L_{cell} \\ \frac{L_{cell}(L_{cell}-4f)}{16f^3} & -\frac{L_{cell}^2}{8f^2} + 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{-L_{cell}^2 + L_{cell}\sqrt{L_{cell}^2 - 16f^2} + 8f^2}{8f^2} & 0 \\ 0 & \frac{-L_{cell}^2 - L_{cell}\sqrt{L_{cell}^2 - 16f^2} + 8f^2}{8f^2} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{\sqrt{f}}{\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}(-L_{cell}+4f)}} & \frac{\sqrt{f}}{\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}(-L_{cell}+4f)}} \\ \frac{1}{2\sqrt{f}\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}\sqrt{L_{cell}^2-16f^2}}} & \frac{1}{2\sqrt{f}\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}\sqrt{L_{cell}^2-16f^2}}} \end{bmatrix}$$

# The FODO stability I

The stability on the horizontal plane is achieved if  $\lambda_1$  and  $\lambda_2$  have unitary module.



## The FODO stability II

This implies  $-1 < \frac{\lambda_1 + \lambda_2}{2} = \cos \mu < 1$ , that is

$$\boxed{\frac{L_{cell}}{4} < f}.$$

The stability condition in the vertical plane is exactly equivalent, since  $D(f) = D(-f)$ .

The stability condition of a FODO lattice (thin lens approximation and no dipoles) imposes an F quadrupole with  $f$  larger than  $L_{cell}/4$ .

# The FODO phase advance

Remembering that

$$\mu = \arccos \frac{\lambda_1 + \lambda_2}{2},$$

one gets

$$\mu = \arccos \left( 1 - \frac{L_{cell}^2}{8f^2} \right),$$

or, equivalently, from<sup>2</sup>

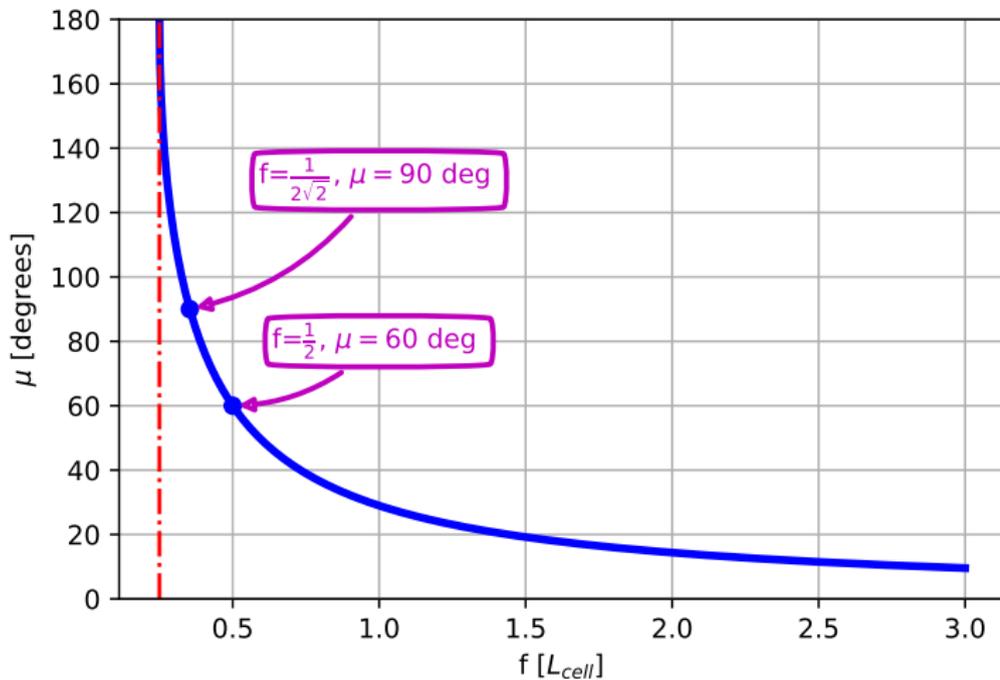
$$\sin \left( \frac{\arccos(1 - x)}{2} \right) = \sqrt{\frac{x}{2}}$$

we get

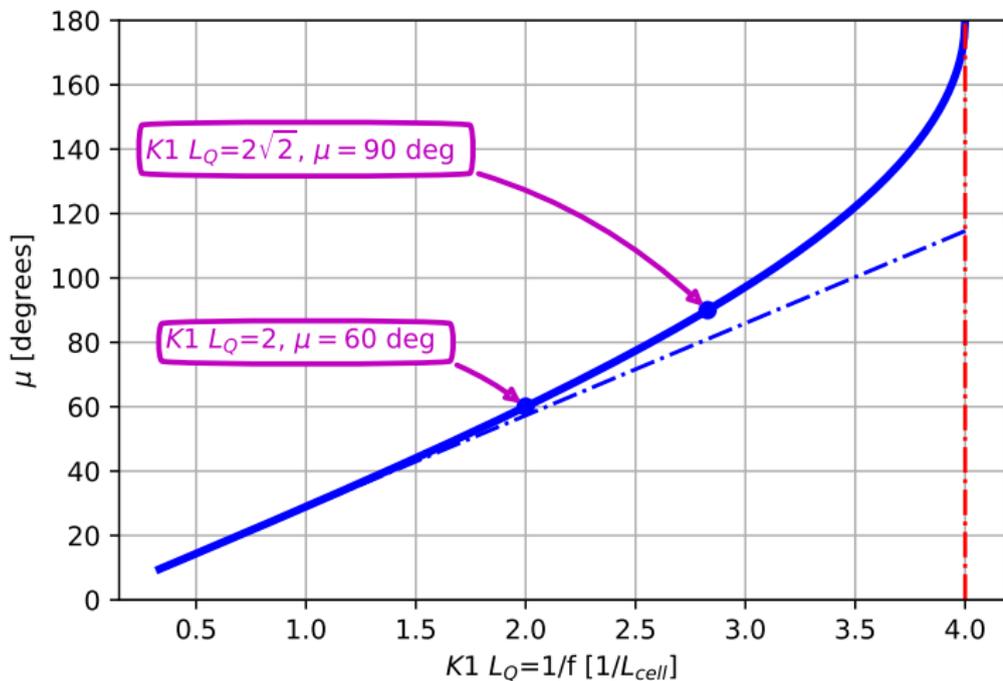
$$\sin \left( \frac{\mu}{2} \right) = \frac{L_{cell}}{4f}.$$

<sup>2</sup>[LatticeCellStudies.ipynb](http://LatticeCellStudies.ipynb)

# $\mu$ vs $f$ and $1/f$



# $\mu$ vs $f$ and $1/f$



Remembering that

$$P = \begin{pmatrix} \sqrt{\frac{\beta}{2}} & \sqrt{\frac{\beta}{2}} \\ \frac{-\alpha + i}{\sqrt{2\beta}} & \frac{-\alpha - i}{\sqrt{2\beta}} \end{pmatrix}$$

we have

$$\boxed{\beta(0) = 2 P_{11}^2} \text{ and } \boxed{\alpha(0) = -P_{11}(P_{21} + P_{22})}.$$

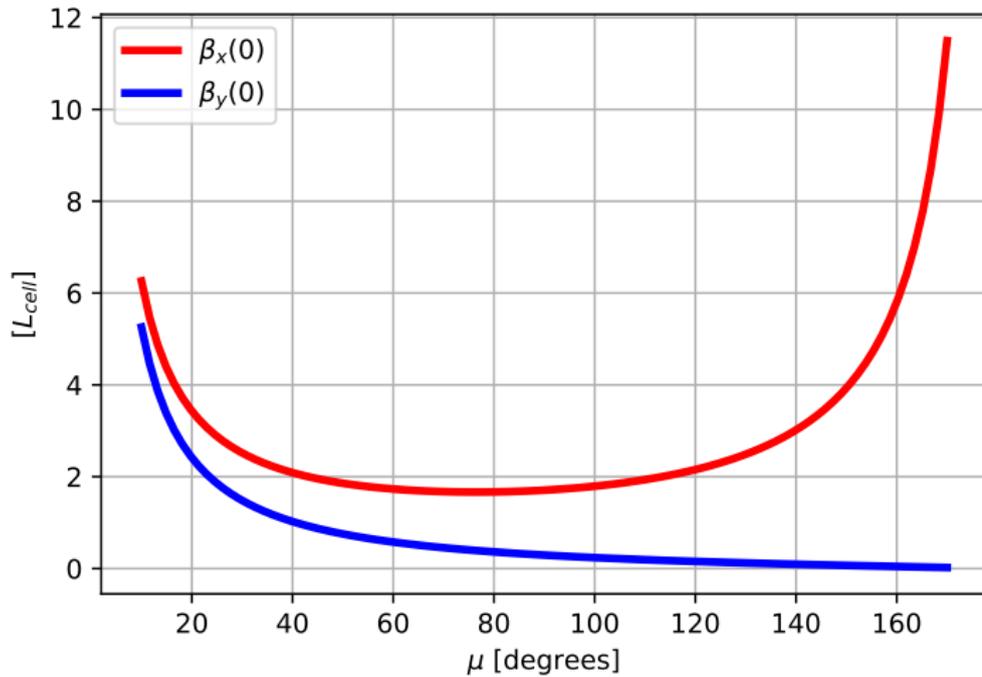
This yields

$$\beta_x(0) = \frac{2f\sqrt{4f + L_{cell}}}{\sqrt{4f - L_{cell}}} = L_{cell} \frac{1 + \sin(\mu/2)}{\sin(\mu)}$$
$$\alpha_x(0) = 0.$$

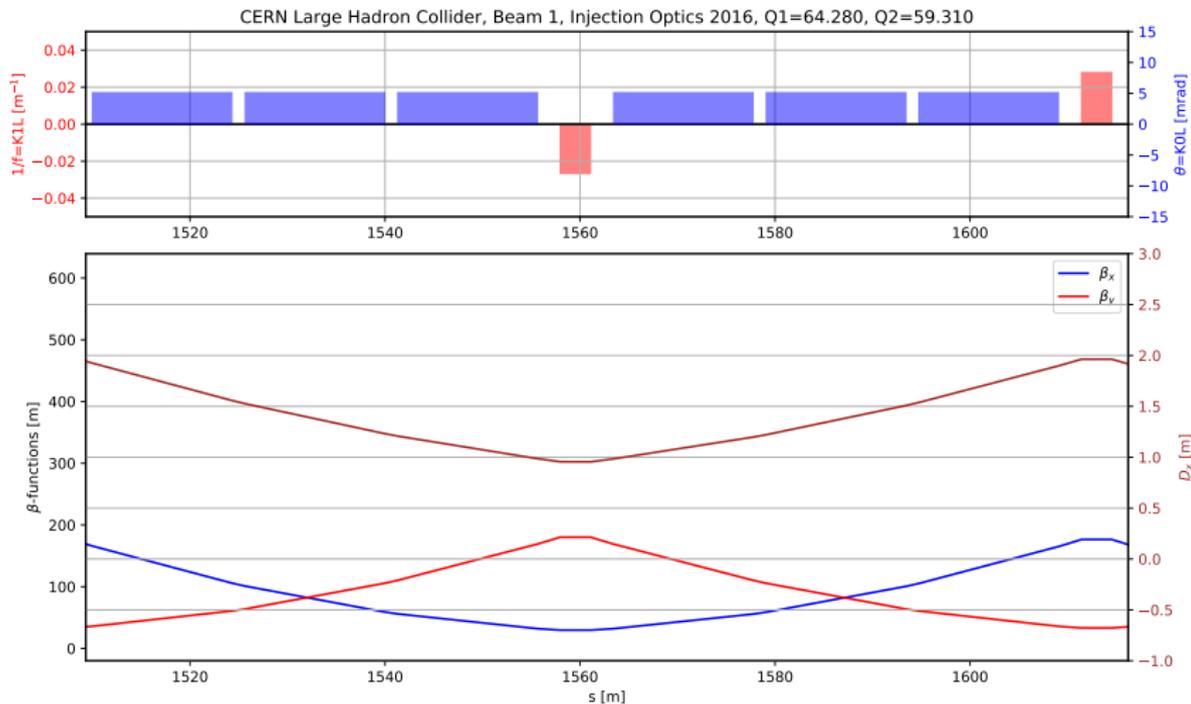
With a similar approach, we can compute the  $y$ -plane optical functions by considering  $P(-f)$ , getting

$$\beta_y(0) = \frac{2f\sqrt{4f - L_{cell}}}{\sqrt{4f + L_{cell}}} = L_{cell} \frac{1 - \sin(\mu/2)}{\sin(\mu)}$$
$$\alpha_y(0) = 0.$$

# $\beta$ -function vs $\mu$



# $\beta$ -function vs $\mu$



# Chromaticity of a FODO I

The definition of the linear chromaticity is

$$\xi = \frac{\Delta Q}{\frac{\Delta p}{p_0}} = \frac{1}{2\pi} \frac{\Delta \mu}{\frac{\Delta p}{p_0}}. \quad (1)$$

From the relation

$$f \left( \frac{\Delta p}{p_0} \right) = f \times \left( 1 + \frac{\Delta p}{p_0} \right) \quad (2)$$

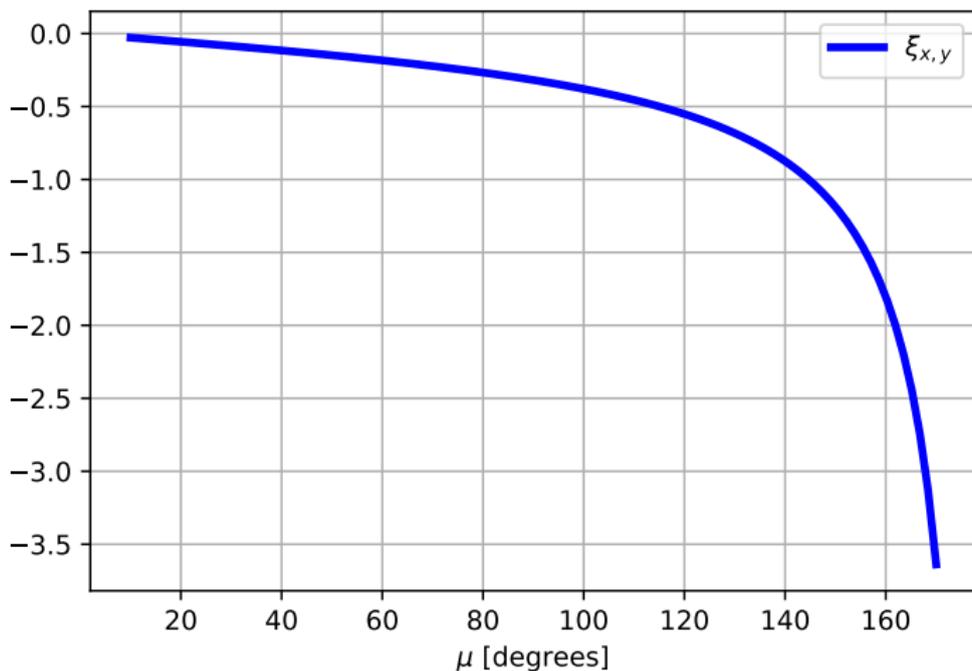
and from

$$\sin \left( \frac{\mu}{2} \right) = \frac{L_{cell}}{4f}, \quad (3)$$

one can compute the FODO lattice chromaticity

$$\xi = -\frac{1}{4\pi} \frac{L_{cell}}{f} \frac{1}{\cos(\mu/2)} = \boxed{-\frac{1}{\pi} \tan \left( \frac{\mu}{2} \right)} \quad (4)$$

# Chromaticity of a FODO II



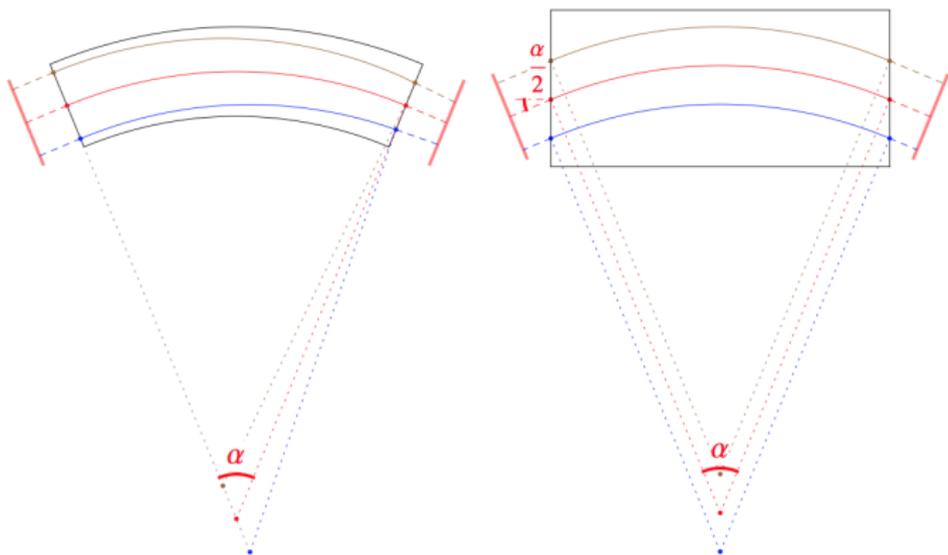
From the FODO lattice we can define at least two additional “flavours”:

- ① different focal length in the F and D quadrupoles,
- ② uneven distance between quadrupoles.

The stability of the two cases is discussed in [LatticeCellStudies.ipynb](https://github.com/LatticeCellStudies.ipynb)

## FODO flavours II

In addition an example on the effect of the dipoles (sector and rectangular bends) and thick quadrupoles can be computed using MAD-X.

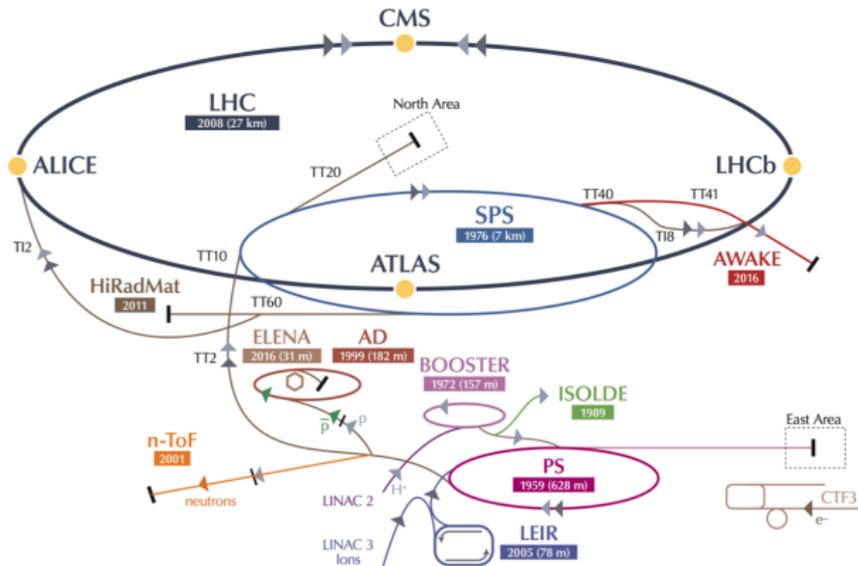


Starting from the FODO we can consider other lattice cells. As an example, by putting back-to-back two OFOD's, we have a triplet cell (OFODDOFO).

An example of triplet lattice analysis is presented in [LatticeCellStudies.ipynb](#), where the stability condition is discussed.

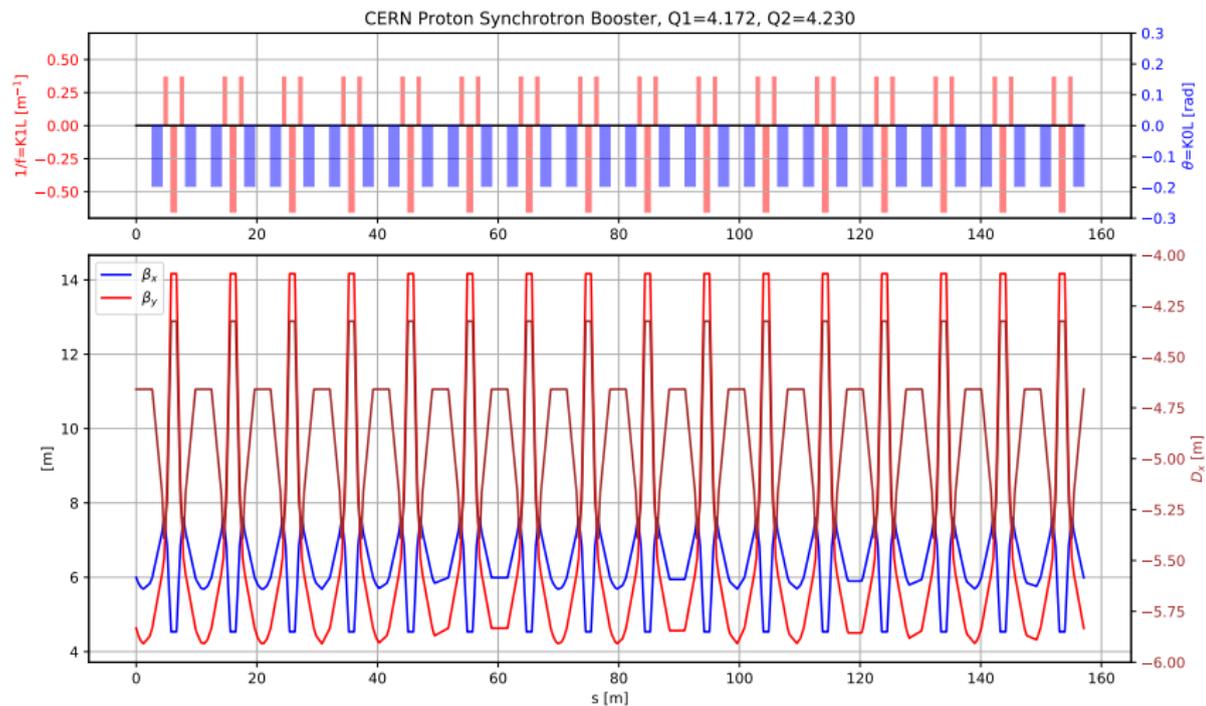
# An stroll along CERN Accelerator Complex

In the following we present few of the CERN Accelerator Complex optics ([acc-models.web.cern.ch](http://acc-models.web.cern.ch))<sup>3</sup>.

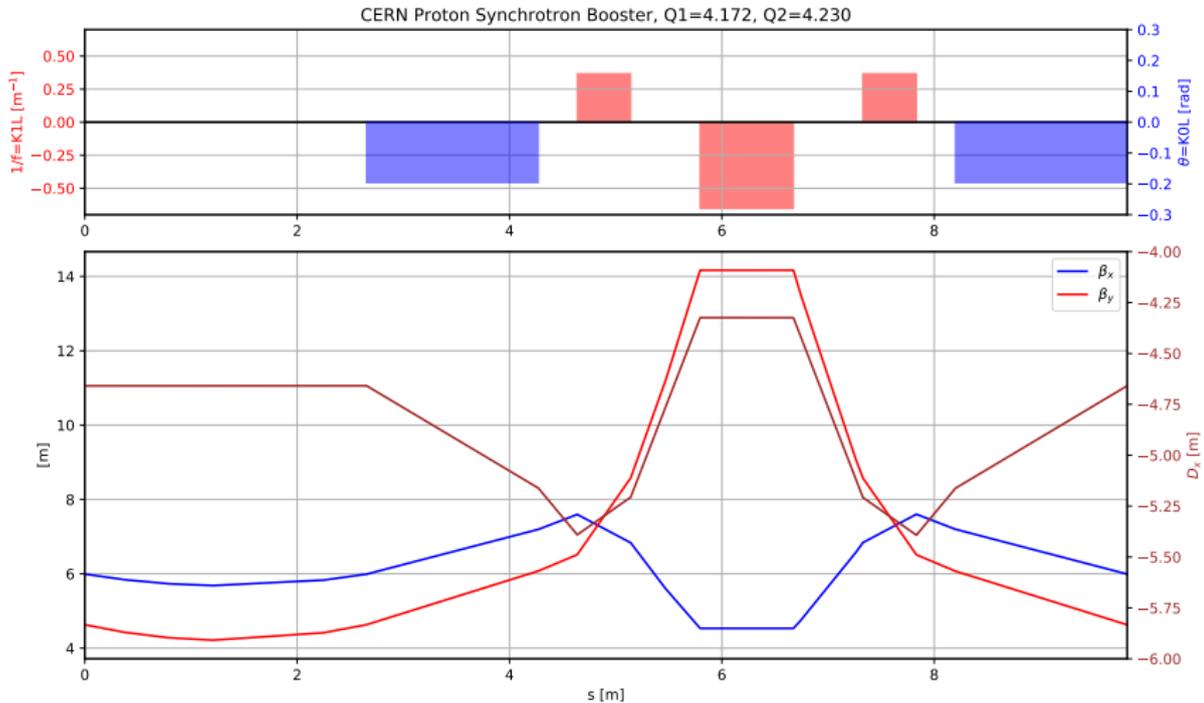


<sup>3</sup>[Lattices.ipynb](http://Lattices.ipynb)

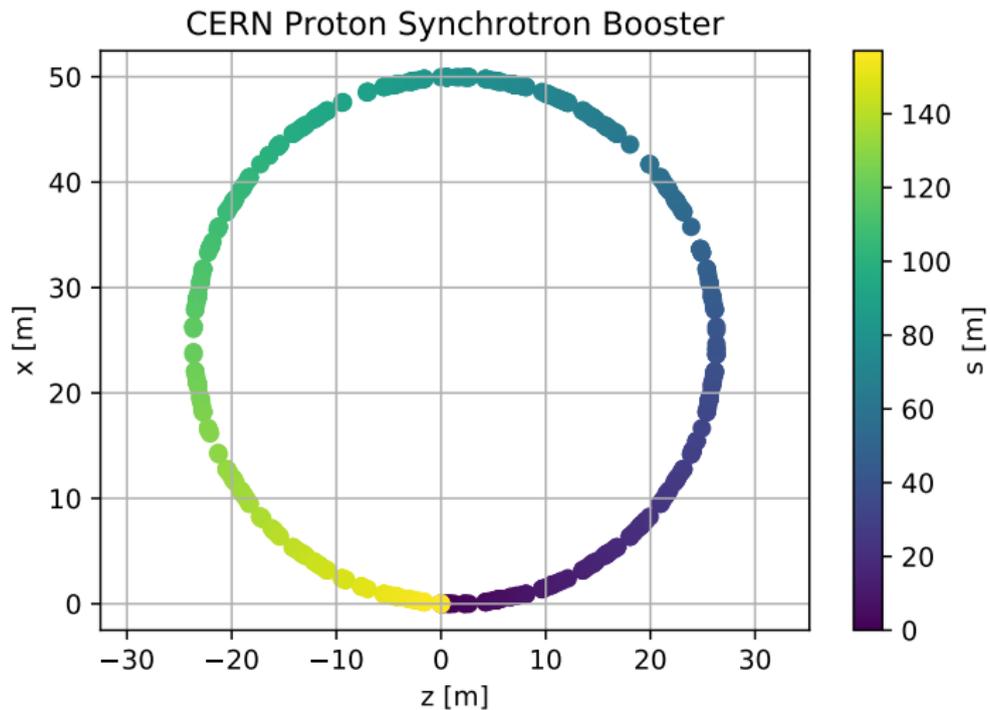
# CERN Proton Synchrotron Booster



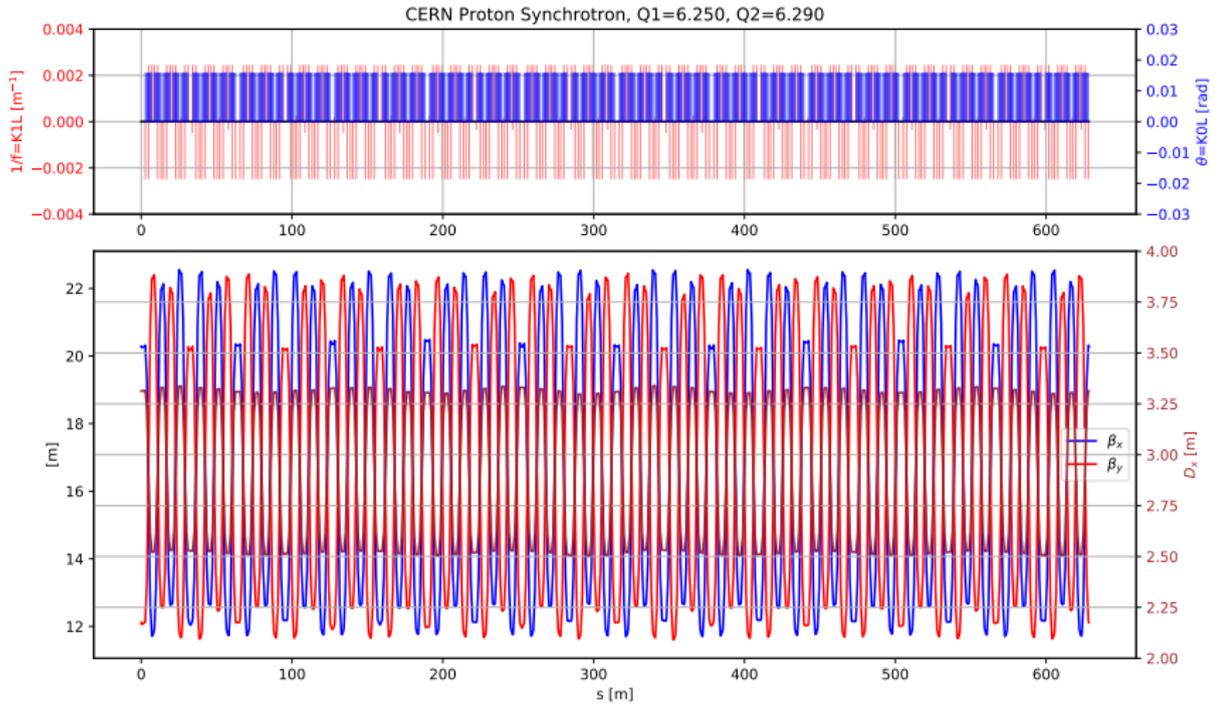
# CERN Proton Synchrotron Booster



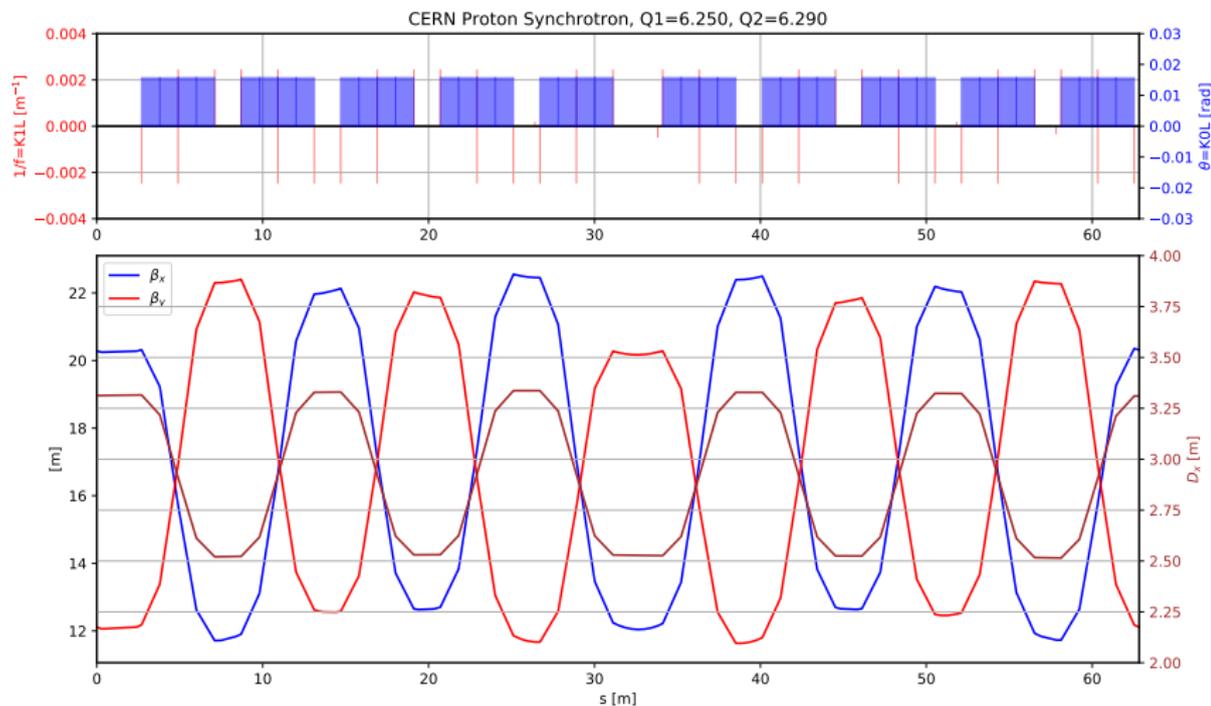
# CERN Proton Synchrotron Booster



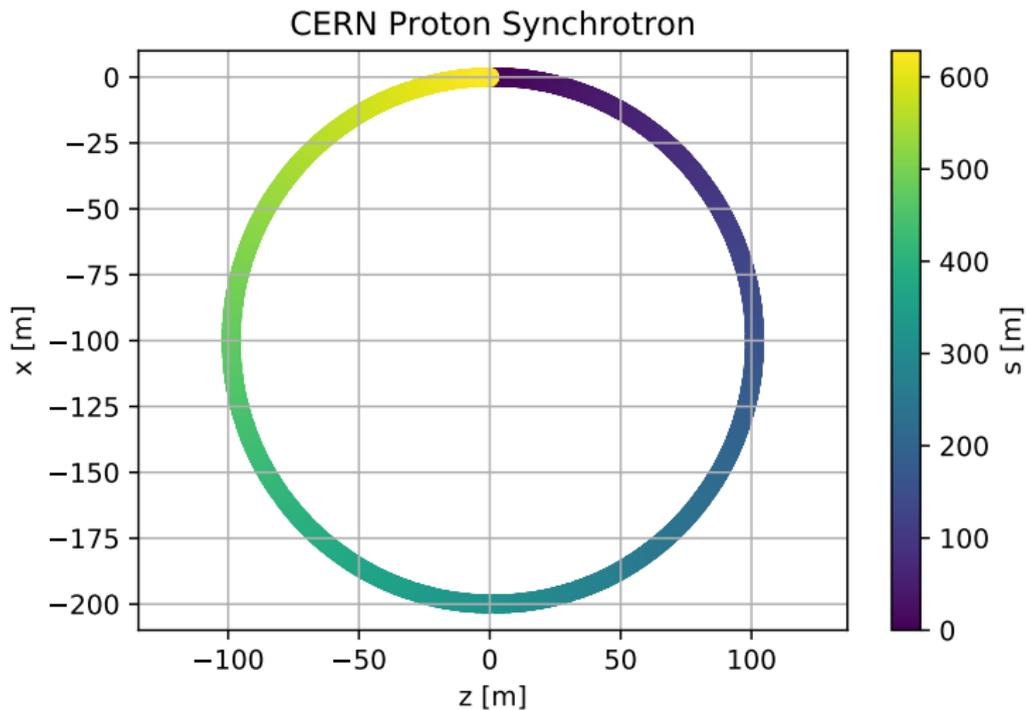
# CERN Proton Synchrotron



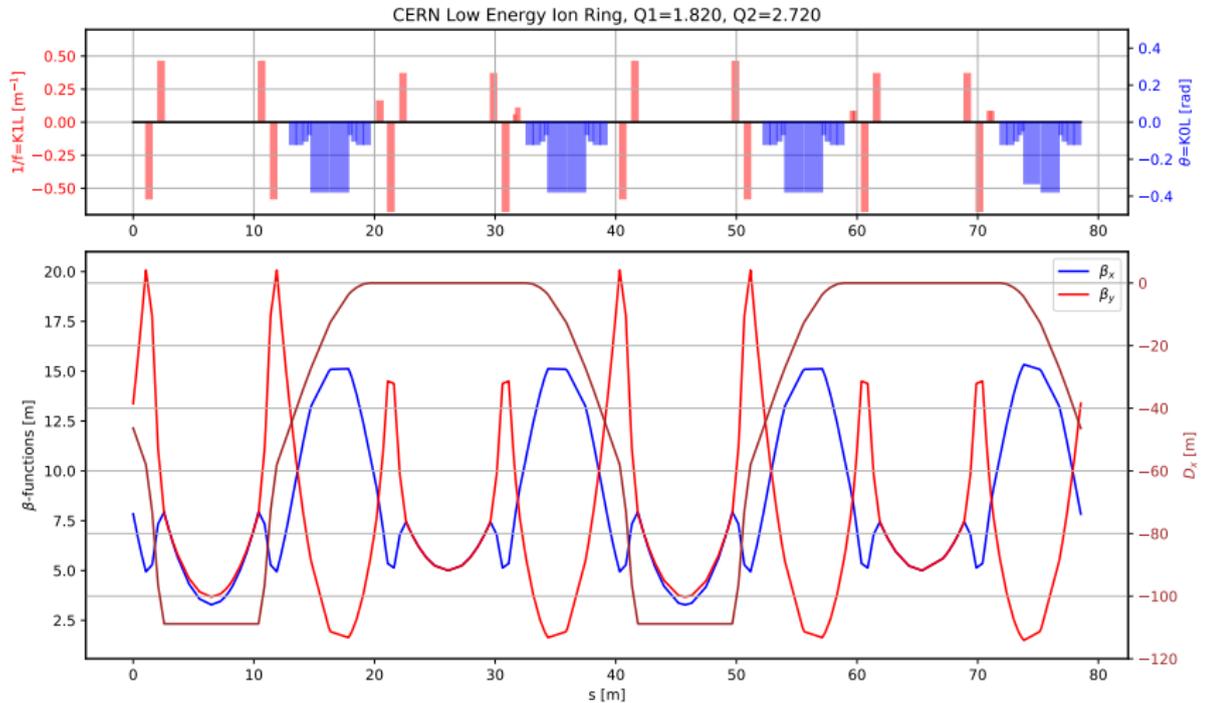
# CERN Proton Synchrotron



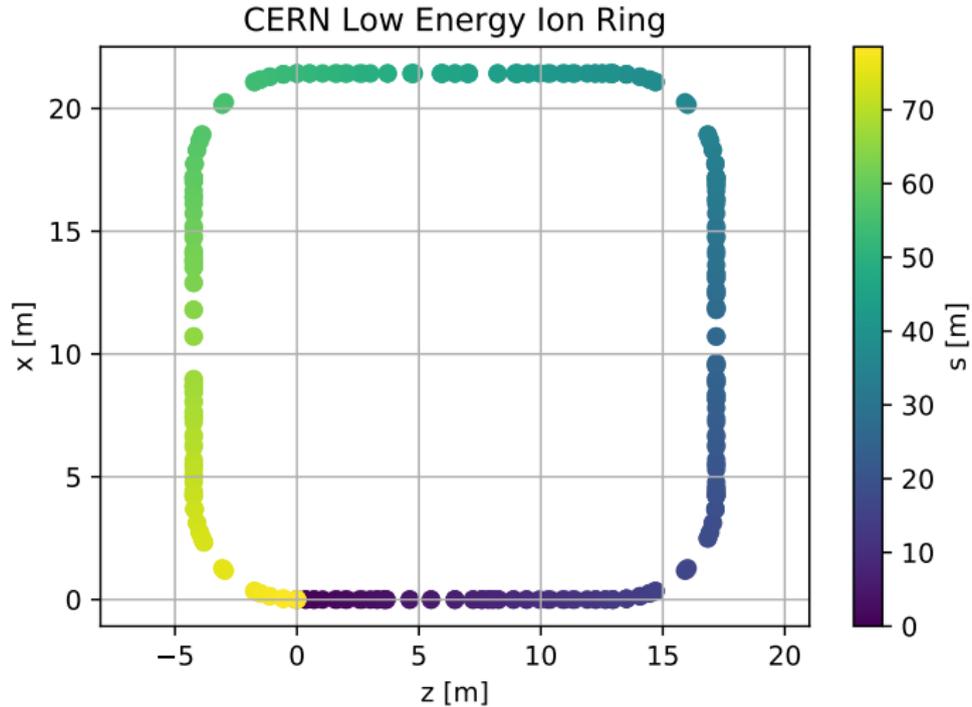
# CERN Proton Synchrotron



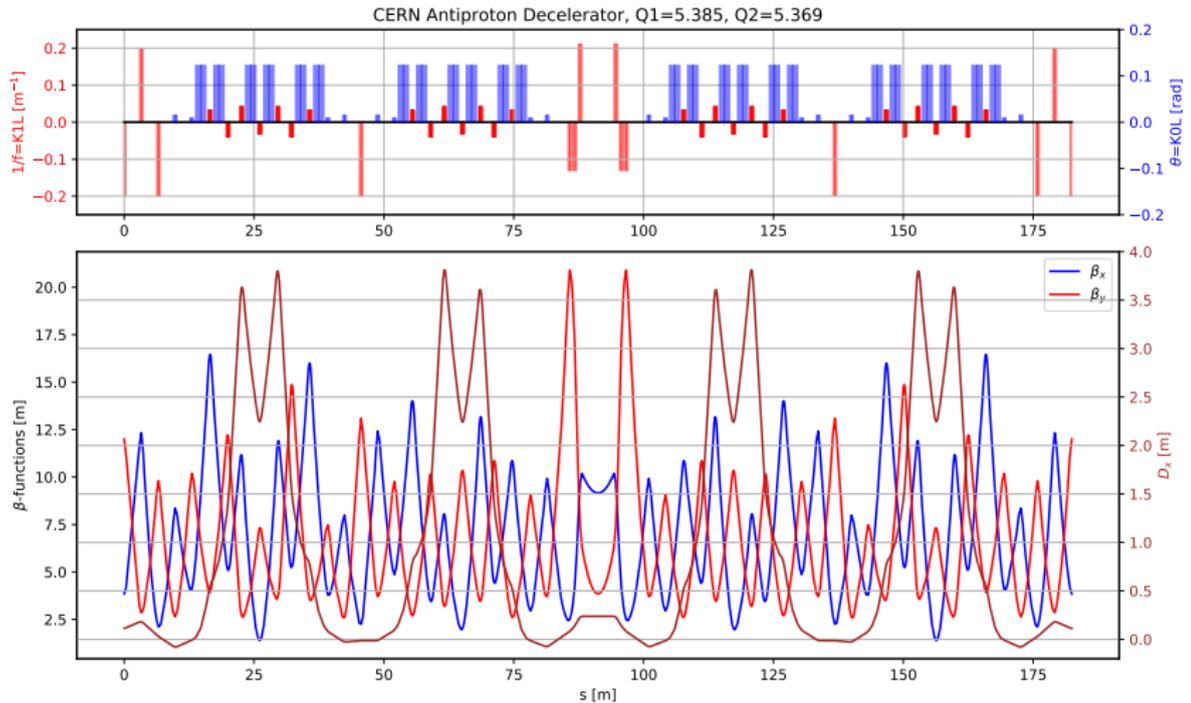
# CERN Low Energy Ion Ring



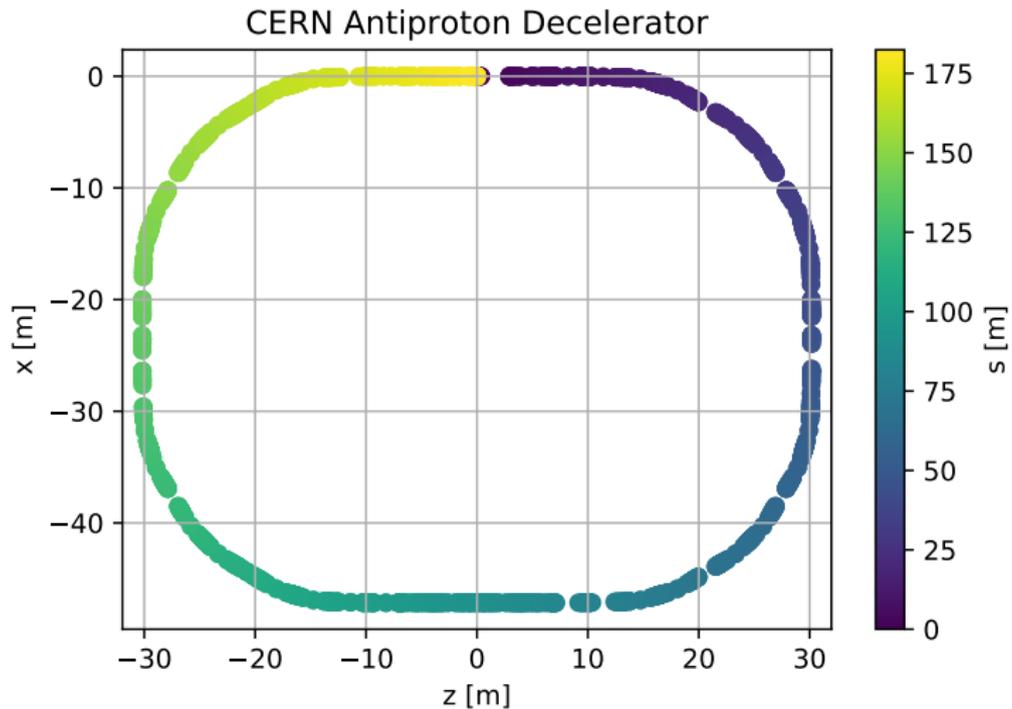
# CERN Low Energy Ion Ring



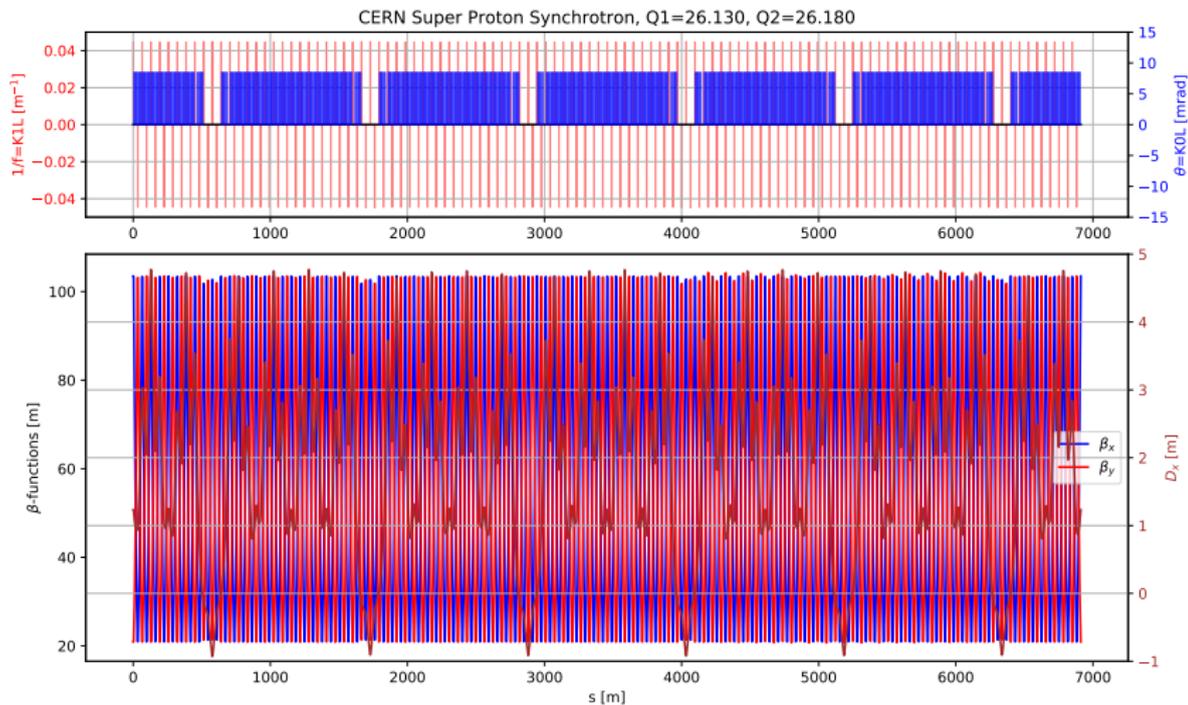
# CERN Antiproton Deceleration



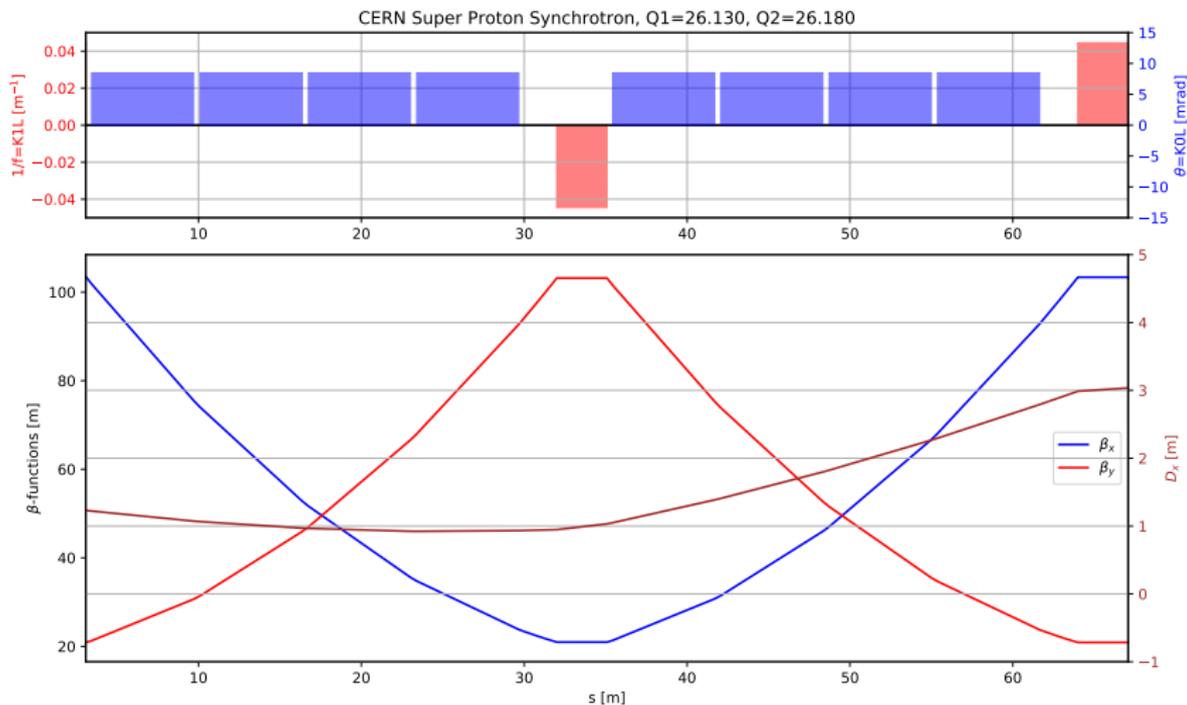
# CERN Antiproton Deceleration



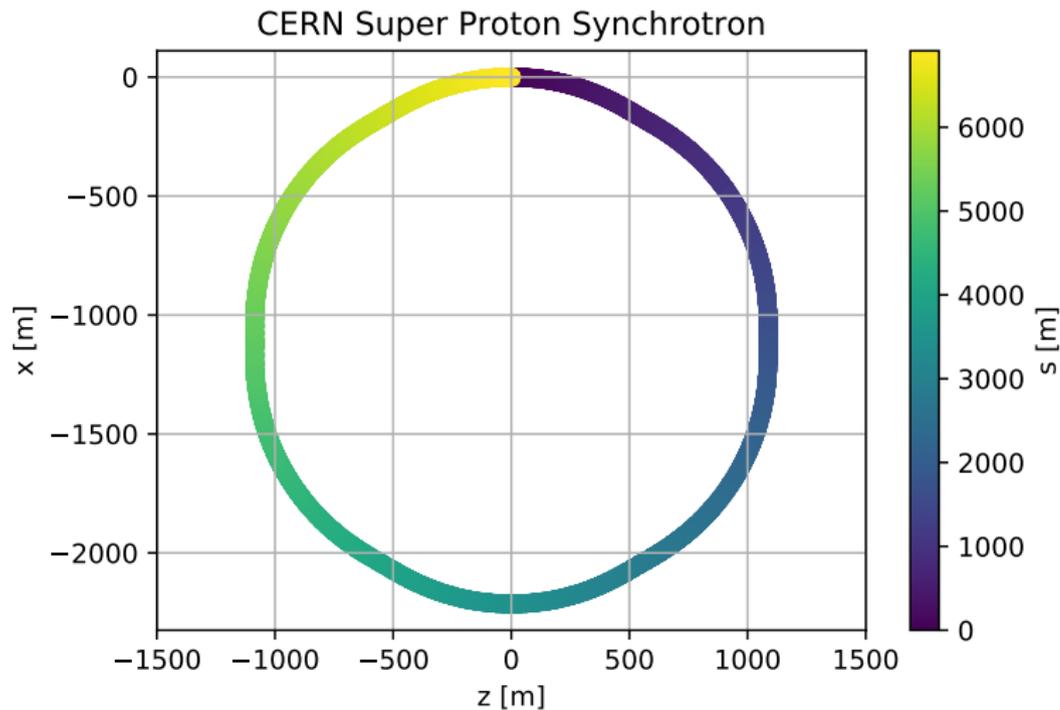
# CERN Super Proton Synchrotron



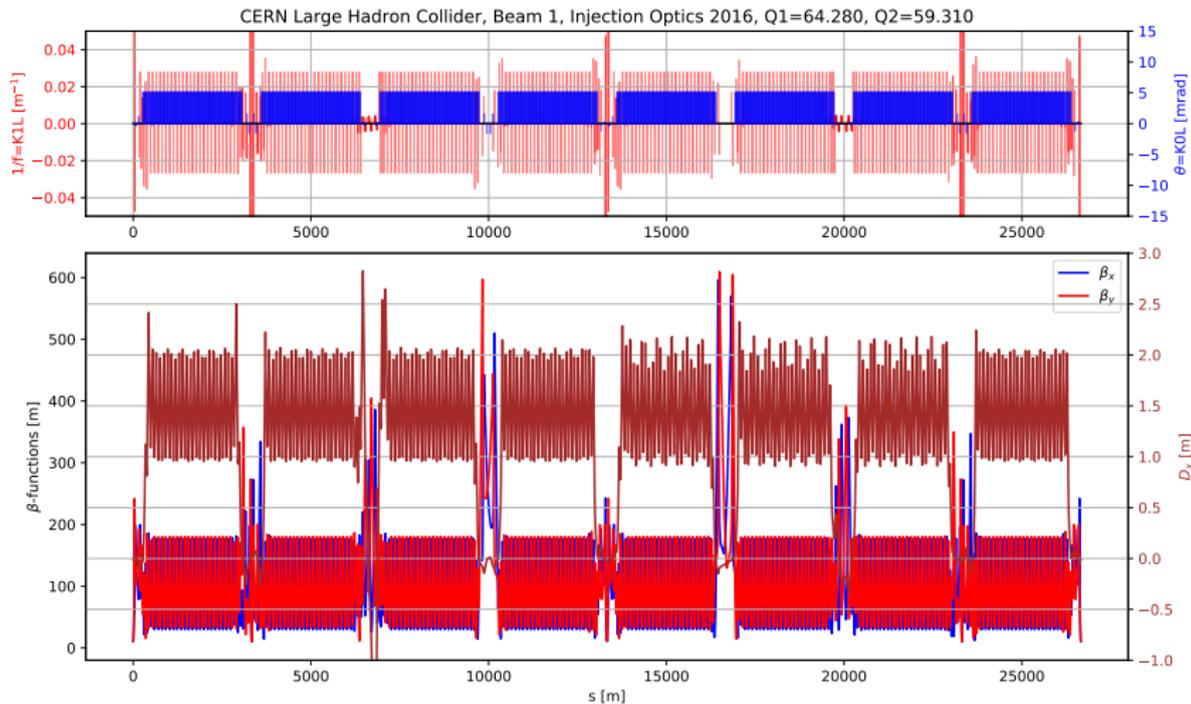
# CERN Super Proton Synchrotron



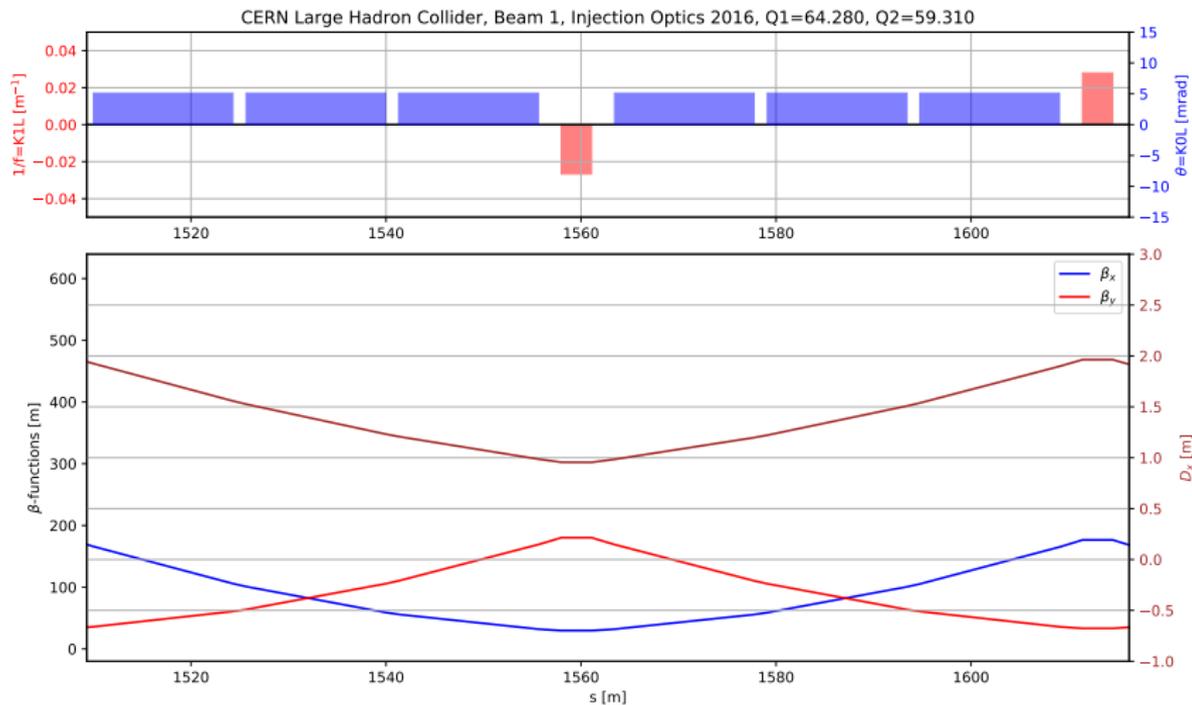
# CERN Super Proton Synchrotron



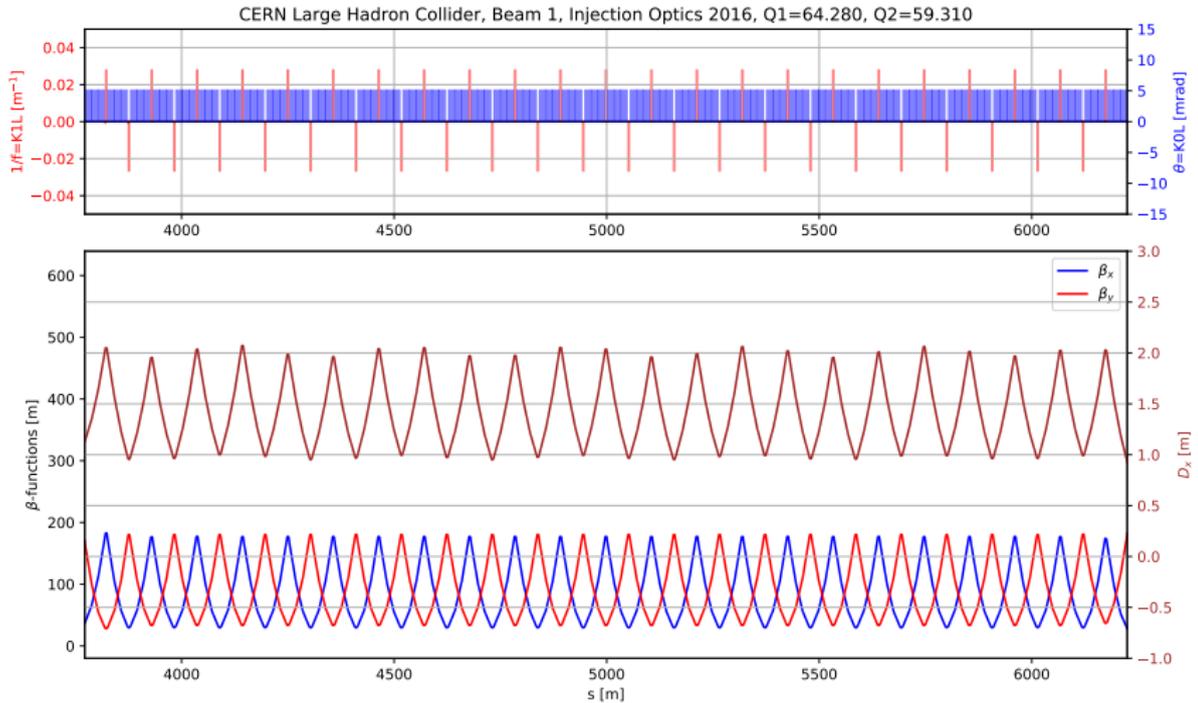
# CERN Large Hadron Collider Beam 1



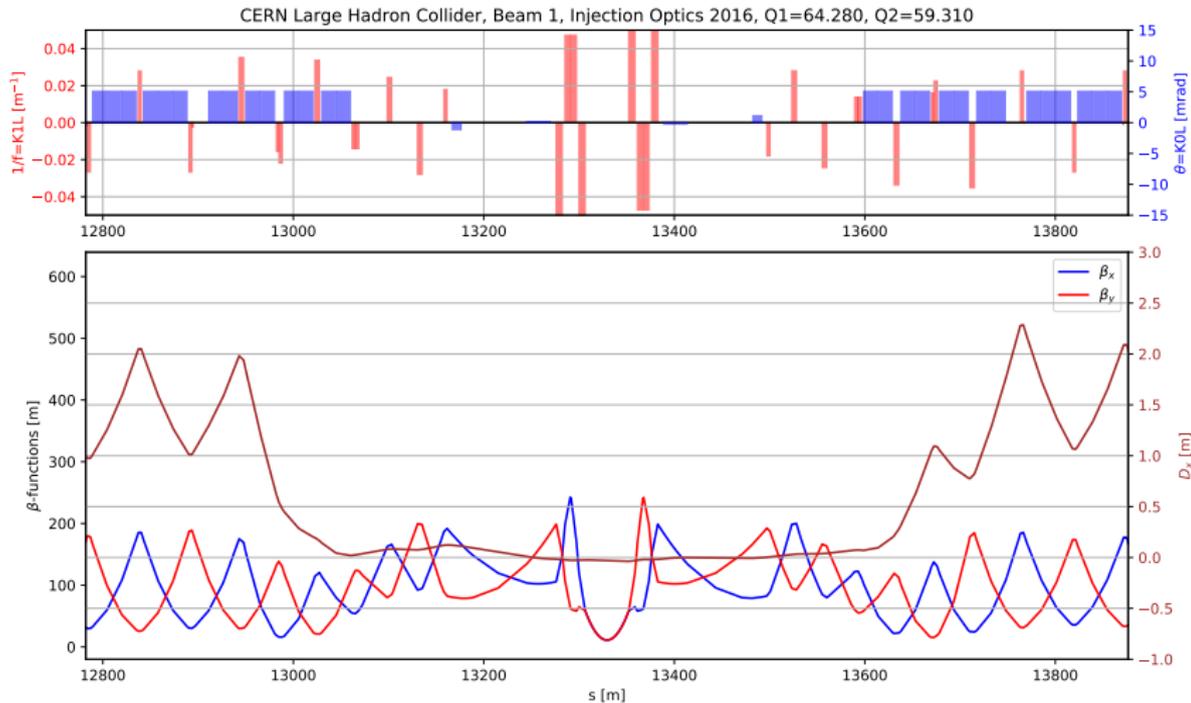
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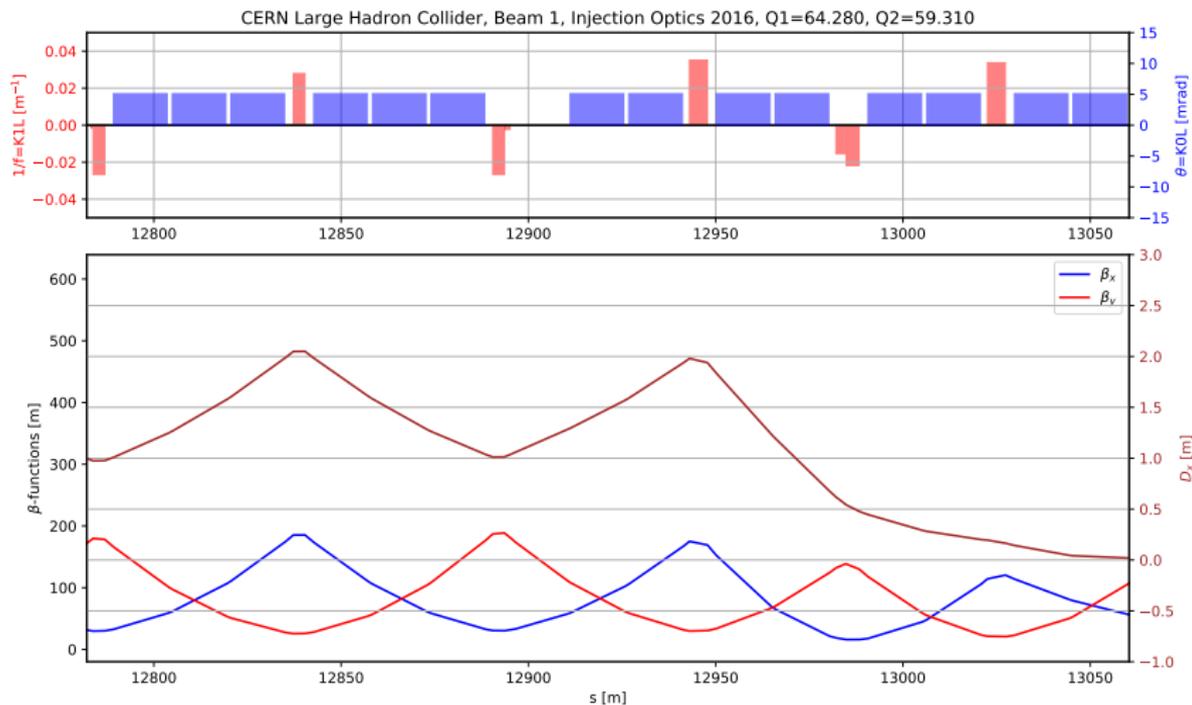
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