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Beam Dynamics with Synchrotron Radiation

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Contents

Part 1: Beam dynamics with synchrotron radiation in electron storage rings.

- Introduction: the need for low emittance lattices.
- The physics of radiation damping:
  - damping of synchrotron oscillations;
  - damping of betatron oscillations.
- The physics of quantum excitation:
  - excitation of synchrotron oscillations;
  - excitation of betatron oscillations.
- Formulae for equilibrium emittances.

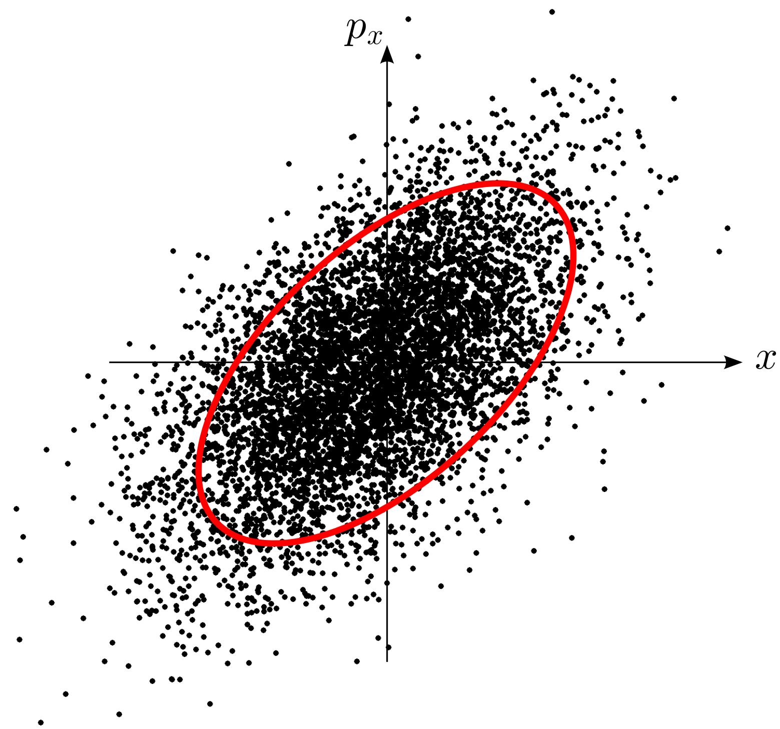
Part 2: Lattice design for low-emittance electron storage rings.

Emittance

The emittance of an electron beam is a measure of the area occupied by the beam in phase space.

In the absence of coupling and dispersion, the horizontal emittance is given by:

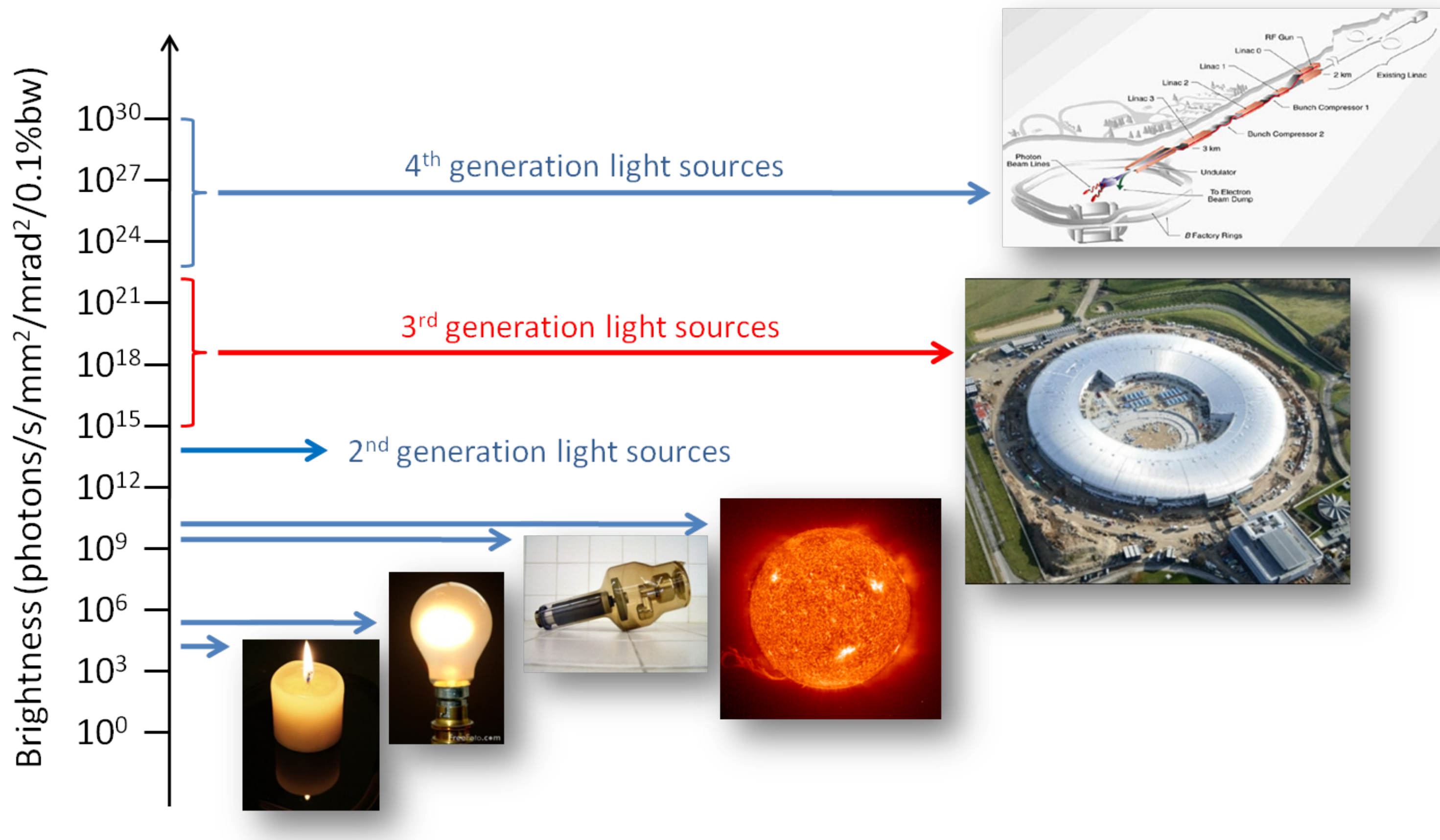
$$\varepsilon_x = \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2}. \qquad (1)$$



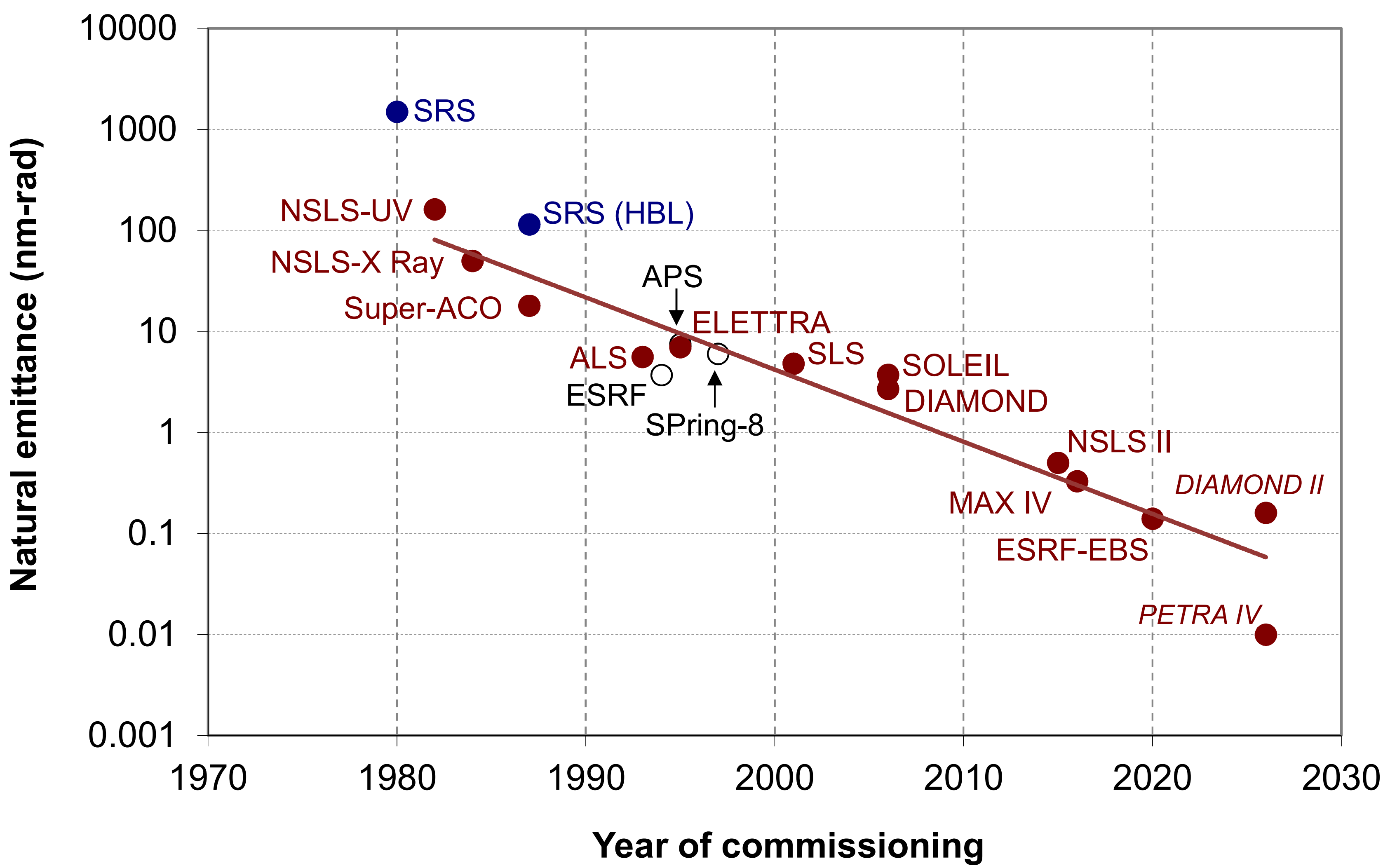
With some approximations, the emittance of a beam remains constant as the beam moves around a storage ring.

In a lattice with given focusing strength (i.e. fixed optics), a smaller emittance leads to a smaller beam size.

$$\text{Brightness} = \frac{\text{radiation power per unit bandwidth}}{\text{phase space area of source}}. \qquad (2)$$

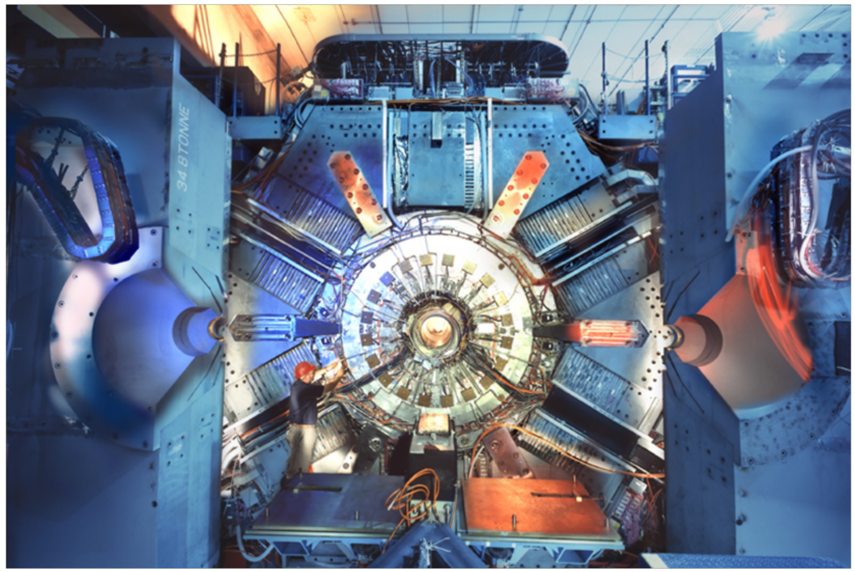


High photon brightness needs low electron beam emittance



Low emittance is important for colliders

Luminosity is a key figure of merit for colliders. The luminosity depends directly on the horizontal and vertical emittances.



Dynamical effects associated with the collisions mean that it is sometimes helpful to *increase* the horizontal emittance; but generally, reducing the vertical emittance as far as possible helps to increase the luminosity.

Lecture 1 objectives: linear dynamics with synchrotron radiation

In this lecture, we shall:

- describe the damping of synchrotron and betatron oscillations by the emission of electromagnetic radiation;
- discuss how quantum excitation leads to equilibrium values for the longitudinal and transverse beam emittances;
- give expressions for the damping times and equilibrium emittances in terms of the *synchrotron radiation integrals*.

Our first goal is to understand how synchrotron radiation leads to the damping of synchrotron oscillations.

We shall proceed as follows:

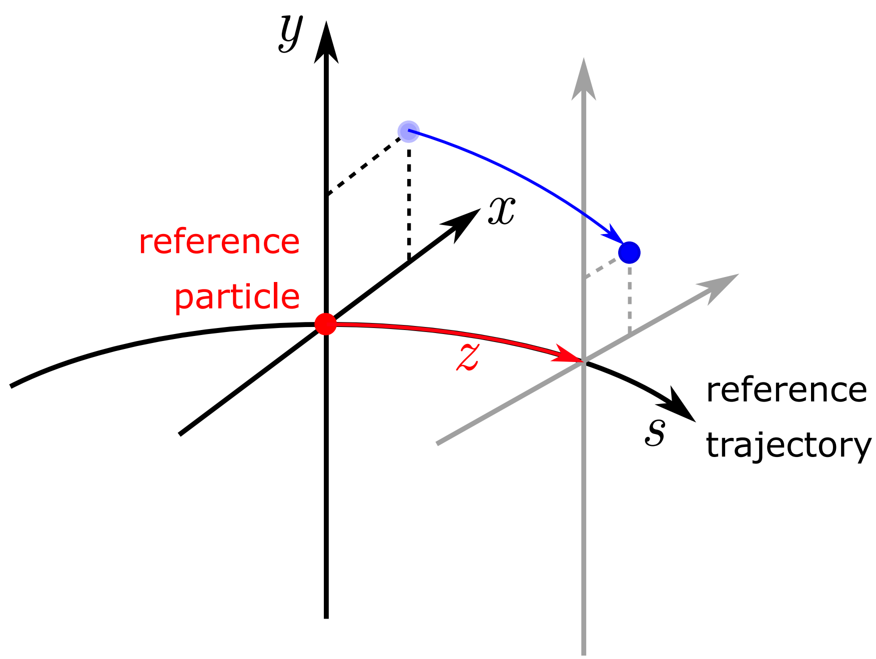
- We write down the equations of motion for a particle performing synchrotron motion, including the radiation energy loss.

- We express the energy loss per turn as a function of the energy of the particle: this leads to a "damping term" into the equations of motion.

- Solving the equations of motion gives synchrotron oscillations with amplitude that decays exponentially.

We describe the longitudinal motion of a particle in a storage ring in terms of the variables z and δ.

The co-ordinate z is the longitudinal position of a particle with respect to a *reference particle*.

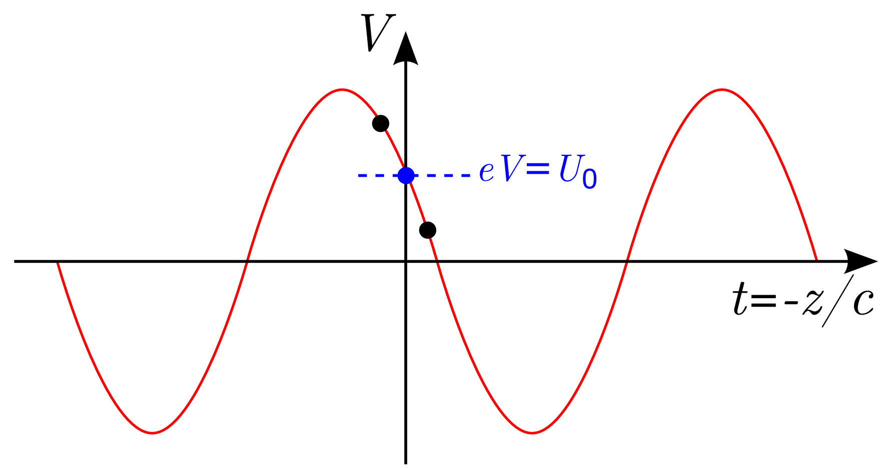


The reference particle is moving round the ring on the reference trajectory, with the reference energy E_0.

δ is the energy deviation of a particle with energy E:

$$\delta = \frac{E - E_0}{E_0}. \tag{3}$$

A particle moving around a storage ring gains energy from the RF cavities, and loses energy by synchrotron radiation.



Averaged over one turn, the change in energy deviation δ is:

$$\Delta\delta = \frac{eV_{\mathrm{rf}}}{E_0}\sin\left(\phi_s - \frac{\omega_{\mathrm{rf}} z}{c}\right) - \frac{U}{E_0}, \tag{4}$$

where V_{rf} is the RF voltage, ω_{rf} the RF frequency, and U is the energy lost by the particle through synchrotron radiation.

The "synchronous phase" ϕ_s is defined by the condition:

$$\Delta\delta = 0 \text{ when } z = 0 \text{ and } \delta = 0, \qquad \text{i.e. } \sin(\phi_s) = \frac{U_0}{eV_{\mathrm{rf}}}. \tag{5}$$

The change in the longitudinal co-ordinate z as a particle makes one turn around a storage ring (circumference C_0) is given by the *momentum compaction factor* α_p:

$$\Delta z = -\alpha_p C_0 \delta. \tag{6}$$

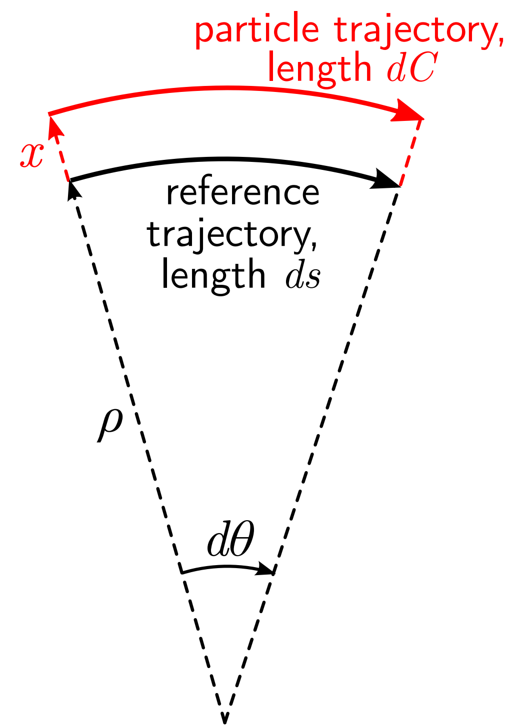
The momentum compaction factor can be written:

$$\alpha_p = \frac{I_1}{C_0}, \tag{7}$$

where the *first synchrotron radiation integral* I_1 is:

$$I_1 = \left. \frac{dC}{d\delta} \right|_{\delta=0} = \oint \frac{\eta_x}{\rho} \, ds. \tag{8}$$

Here, η_x is the dispersion, and ρ is the radius of curvature at a given point along the beam trajectory.



Let us assume that over a single revolution, the change in the energy deviation $\Delta\delta$ and the change in the longitudinal co-ordinate Δz are both small.

In that case, we can write the longitudinal equations of motion for the particle:

$$\frac{d\delta}{dt} = \frac{eV_{\mathsf{rf}}}{E_0 T_0} \sin\left(\phi_s - \frac{\omega_{\mathsf{rf}} z}{c}\right) - \frac{U}{E_0 T_0}, \tag{9}$$

$$\frac{dz}{dt} = -\alpha_p c \delta, \tag{10}$$

where $T_0 = C_0/c$ is the revolution period.

In solving these equations, we need to take into account the fact that the energy loss per turn U depends on the energy deviation δ...

The energy loss per turn U depends on the energy of the particle: particles with higher energy radiate more synchrotron radiation power.

Assuming that $|\delta| \ll 1$, we work to first order in δ, so that:

$$U = U_0 + \Delta E \left.\frac{dU}{dE}\right|_{E=E_0} = U_0 + E_0\delta \left.\frac{dU}{dE}\right|_{E=E_0}. \tag{11}$$

Also, we assume that the particle arrives at each RF cavity at a phase close to the synchronous phase, so that:

$$\sin\left(\phi_s - \frac{\omega_{\mathsf{rf}} z}{c}\right) \approx \sin(\phi_s) - \cos(\phi_s)\frac{\omega_{\mathsf{rf}} z}{c}. \tag{12}$$

With these assumptions, and combining the equations of motion (9) and (10) we find the equation of motion for the energy deviation:

$$\frac{d^2\delta}{dt^2} + 2\alpha_E \frac{d\delta}{dt} + \omega_s^2 \delta = 0, \tag{13}$$

where the synchrotron oscillation frequency ω_s is given by*:

$$\omega_s^2 = -\frac{eV_{\mathsf{rf}}}{E_0} \cos(\phi_s) \frac{\omega_{\mathsf{rf}}}{T_0} \alpha_p, \tag{14}$$

and the damping constant α_E is:

$$\alpha_E = \frac{1}{2T_0} \left. \frac{dU}{dE} \right|_{E=E_0}. \tag{15}$$

Equation (13) is the equation of motion for a damped harmonic oscillator, with frequency ω_s and damping constant α_E.

*Note that for stable oscillations, we require $\cos(\phi_s) < 0$.

If $\alpha_E \ll \omega_s$, the energy deviation and longitudinal co-ordinate vary as:

$$\delta(t) = \delta_0 \, e^{-\alpha_E t} \sin(\omega_s t - \theta_0), \tag{16}$$

$$z(t) = \frac{\alpha_p c}{\omega_s} \delta_0 \, e^{-\alpha_E t} \cos(\omega_s t - \theta_0), \tag{17}$$

where δ_0 and θ_0 are constants (respectively, the amplitude and phase of the oscillation at $t = 0$).

To evaluate the damping constant α_E, we need to know how the energy loss per turn U depends on the particle energy E...

From classical electromagnetic theory, an ultrarelativistic particle ($\beta = v/c \approx 1$) with energy E in a magnetic field B emits electromagnetic radiation with power:

$$P_\gamma \approx \frac{C_\gamma}{2\pi} e^2 c^3 E^2 B^2, \qquad \text{where} \qquad C_\gamma = \frac{e^2}{3\epsilon_0 (mc^2)^4}. \tag{18}$$

For electrons, $C_\gamma \approx 8.846 \times 10^{-5}\,\mathrm{m/GeV^3}$.

To find the energy loss per turn, we integrate P_γ over one revolution period. For a particle on the reference trajectory, with $\delta = 0$, we find (see Appendix B):

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2, \tag{19}$$

where the second synchrotron radiation integral is:

$$I_2 = \oint \frac{1}{\rho^2}\, ds. \tag{20}$$

To find the damping constant α_E (15) we need (again from Appendix B):

$$\left.\frac{dU}{dE}\right|_{E=E_0} = j_z \frac{U_0}{E_0}, \tag{21}$$

where j_z is the *longitudinal damping partition number*:

$$j_z = 2 + \frac{I_4}{I_2}. \tag{22}$$

The fourth synchrotron radiation integral I_4 accounts for dispersion and any field gradient in the dipoles:

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \quad \text{where } k_1 = \frac{ec}{E_0}\frac{\partial B_y}{\partial x}. \tag{23}$$

If the dipoles have no field gradient, then usually $I_4 \ll I_2$, in which case $j_z \approx 2$.

The longitudinal damping time τ_z is defined by:

$$\tau_z = \frac{1}{\alpha_E} = \frac{2}{j_z}\frac{E_0}{U_0}T_0. \tag{24}$$

The longitudinal emittance can be defined as:

$$\varepsilon_z = \sqrt{\langle z^2\rangle\langle\delta^2\rangle - \langle z\delta\rangle^2}. \tag{25}$$

Since the amplitudes of the synchrotron oscillations decay with time constant τ_z, the damping of the longitudinal emittance can be written:

$$\varepsilon_z(t) = \varepsilon_z(0)\exp\left(-2\frac{t}{\tau_z}\right), \tag{26}$$

where $\varepsilon_z(0)$ is the longitudinal emittance at $t = 0$.

Let us now consider the effect of synchrotron radiation on betatron oscillations.

In the case of synchrotron oscillations, we assumed that the changes in the longitudinal variables were small over a single turn around the ring.

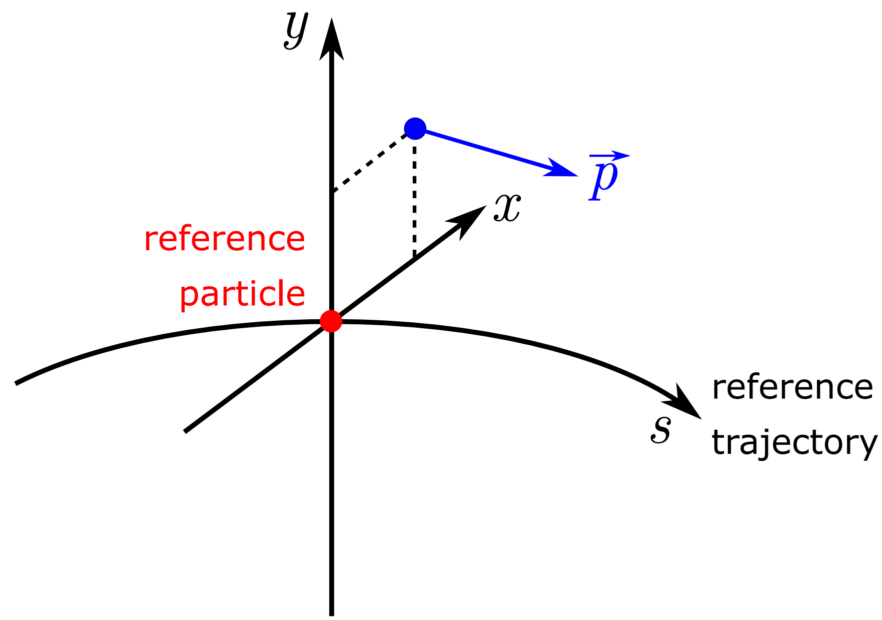
In other words, we assumed that the synchrotron frequency was small compared to the revolution frequency.

This is not a valid assumption in the case of betatron oscillations, so we shall have to take a different approach to the analysis.

We shall first consider vertical betatron oscillations: this turns out to be a simpler case than horizontal betatron oscillations.

We describe the vertical motion of a particle in a storage ring in terms of the variables y and p_y.

The co-ordinate y is the vertical position of a particle with respect to the *reference trajectory*.



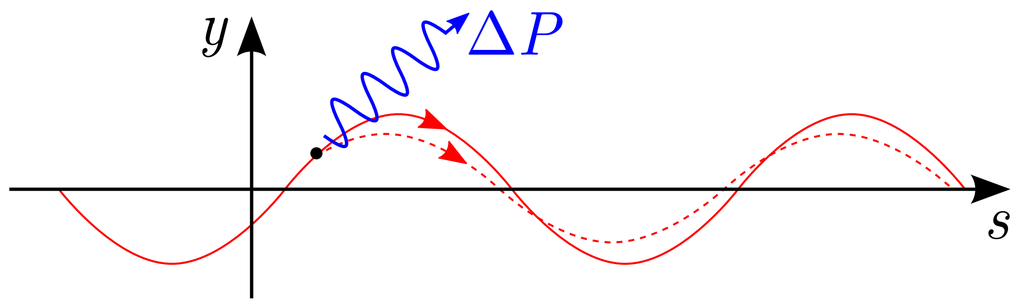
The conjugate momentum p_y is the vertical momentum of a particle scaled by the *reference momentum* P_0:

$$p_y = \frac{\gamma m v_y}{P_0}, \tag{27}$$

where γ is the relativistic factor, and v_y is the vertical velocity of the particle.

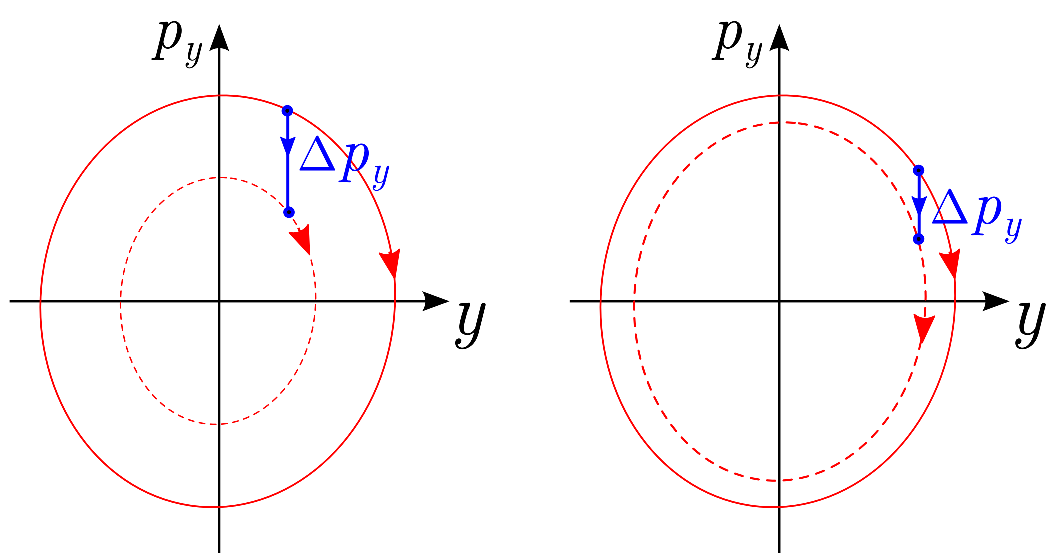
Radiation damping of betatron oscillations is a result of particles losing momentum by emitting synchrotron radiation.

Synchrotron radiation is emitted in a narrow cone (opening angle $1/\gamma$) around the direction of motion of the particle.



If a particle emits radiation with momentum ΔP, the change in the vertical conjugate momentum of the particle is:

$$\Delta p_y = -p_y \frac{\Delta P}{P_0}. \qquad (28)$$



The amplitude of the betatron oscillations of a given particle can be characterised by the *betatron action*:

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2. \tag{29}$$

The vertical emittance of a beam is the average of the vertical betatron action of all particles in the beam:

$$\varepsilon_y = \langle J_y \rangle. \tag{30}$$

If all particles (at random betatron phases) lose an equal amount of momentum ΔP, the change in vertical emittance is found to be (an exercise for the student!†):

$$\Delta\varepsilon_y = \langle \Delta J_y \rangle = -\varepsilon_y \frac{\Delta P}{P_0}. \tag{31}$$

†The results in Appendix A may be useful.

In the absence of radiation effects, the action J_y of each particle remains constant as the particles move round a storage ring.

Then, assuming that the rate of change of emittance (from emission of radiation) is slow compared to the revolution frequency, the rate of change of the emittance can be found by averaging the momentum loss around the ring:

$$\frac{d\varepsilon_y}{dt} = -\frac{\varepsilon_y}{T_0}\oint \frac{dP}{P_0} \approx -\frac{U_0}{E_0 T_0}\varepsilon_y = -\frac{2}{\tau_y}\varepsilon_y. \tag{32}$$

Here, T_0 is the revolution period, E_0 is the reference energy, and U_0 is the energy loss in one turn.

The approximation in the above formulae is valid for an ultra-relativistic particle, which has $E \approx Pc$.

The evolution of the vertical emittance is given by:

$$\varepsilon_y(t) = \varepsilon_y(0) \exp\left(-2\frac{t}{\tau_y}\right), \tag{33}$$

where the vertical damping time τ_y is:

$$\tau_y = 2\frac{E_0}{U_0}T_0. \tag{34}$$

Note the similarity with the formula for the evolution of the longitudinal emittance (26):

$$\varepsilon_z(t) = \varepsilon_z(0) \exp\left(-2\frac{t}{\tau_z}\right). \tag{35}$$

where the longitudinal damping time is:

$$\tau_z = \frac{2}{j_z}\frac{E_0}{U_0}T_0. \tag{36}$$

Since (in many cases) $j_z \approx 2$, the vertical damping time is often about twice the longitudinal damping time.

Typically, in an electron storage ring, the damping time is of order several tens of milliseconds, while the revolution period is of order of a microsecond.

Therefore, radiation effects are indeed “slow” compared to the revolution frequency.

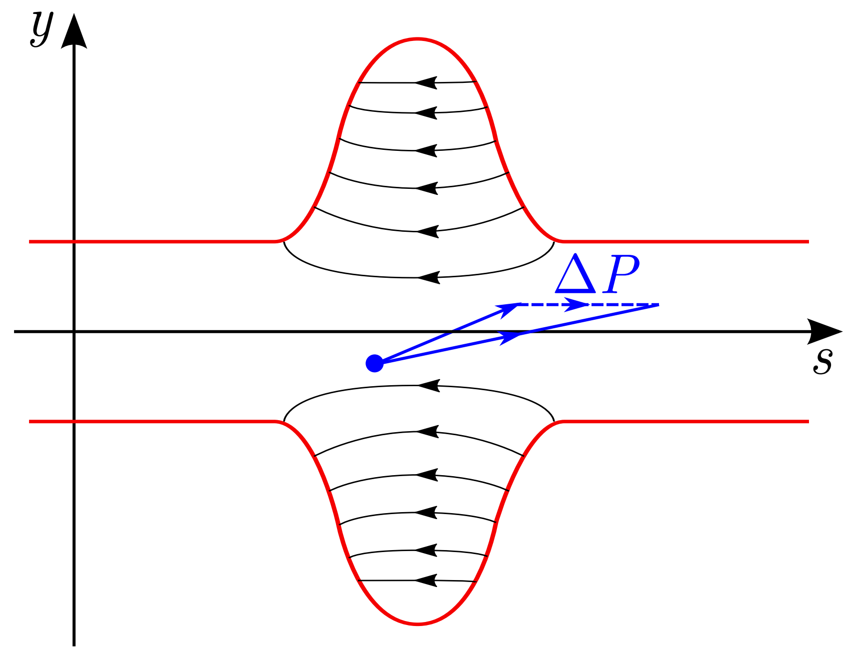
But note that we made the assumption that the momentum of the particle was close to the reference momentum, i.e. $P \approx P_0$.

If the particle continues to radiate without any restoration of energy, eventually this assumption will no longer be valid...

However, electron storage rings contain RF cavities to restore the energy lost through synchrotron radiation. But then, we should consider the change in momentum of a particle as it moves through an RF cavity.

RF cavities are usually designed to provide a longitudinal electric field.

There is then no change in the transverse momentum when a particle passes through the cavity.



Therefore, we do not have to consider explicitly the effects of RF cavities on the emittance of the beam.

Damping of horizontal emittance

Analysis of radiation effects on the vertical emittance was relatively straightforward. When we consider the horizontal emittance, there are three complications that we need to address:

- The horizontal motion of a particle is often strongly coupled to the longitudinal motion (by dispersion): when a particle emits radiation, its horizontal co-ordinate *with respect to the closed orbit* will change.

- Where the reference trajectory is curved (usually, in dipoles), the path length taken by a particle depends on the horizontal co-ordinate with respect to the reference trajectory.

- Dipole magnets are sometimes built with a gradient, so that the vertical field seen by a particle in a dipole depends on the horizontal co-ordinate of the particle.

Taking all the above effects into account, we can proceed along the same lines as for the analysis of the vertical emittance:

- Write down the changes in co-ordinate x and momentum p_x resulting from an emission of radiation with momentum dp (taking into account the additional effects of dispersion).
- Substitute expressions for the new co-ordinate and momentum into the expression for the horizontal betatron action, to find the change in the action resulting from the radiation emission.
- Average over all particles in the beam, to find the change in the emittance resulting from radiation emission from each particle.
- Integrate around the ring (taking account of changes in path length and field strength with x in the bends) to find the change in emittance over one turn.

The algebra gets somewhat cumbersome, and is not especially enlightening. See Appendix C for more details. Here, we just quote the result...

The horizontal emittance decays exponentially:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x, \tag{37}$$

where the horizontal damping time is given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0. \tag{38}$$

The horizontal damping partition number j_x is:

$$j_x = 1 - \frac{I_4}{I_2}, \tag{39}$$

where the fourth synchrotron radiation integral is given by (23):

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}. \tag{40}$$

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2, \qquad C_\gamma \approx 8.846 \times 10^{-5} \text{m/GeV}^3. \tag{41}$$

The emittances damp exponentially:

$$\varepsilon_x(t) = \varepsilon_{x0} \exp\left(-2\frac{t}{\tau_x}\right), \tag{42}$$

(and similarly for ε_y and ε_z). The radiation damping times are:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0, \quad \tau_y = \frac{2}{j_y}\frac{E_0}{U_0}T_0, \quad \tau_z = \frac{2}{j_z}\frac{E_0}{U_0}T_0. \tag{43}$$

The damping partition numbers are:

$$j_x = 1 - \frac{I_4}{I_2}, \quad j_y = 1, \quad j_z = 2 + \frac{I_4}{I_2}. \tag{44}$$

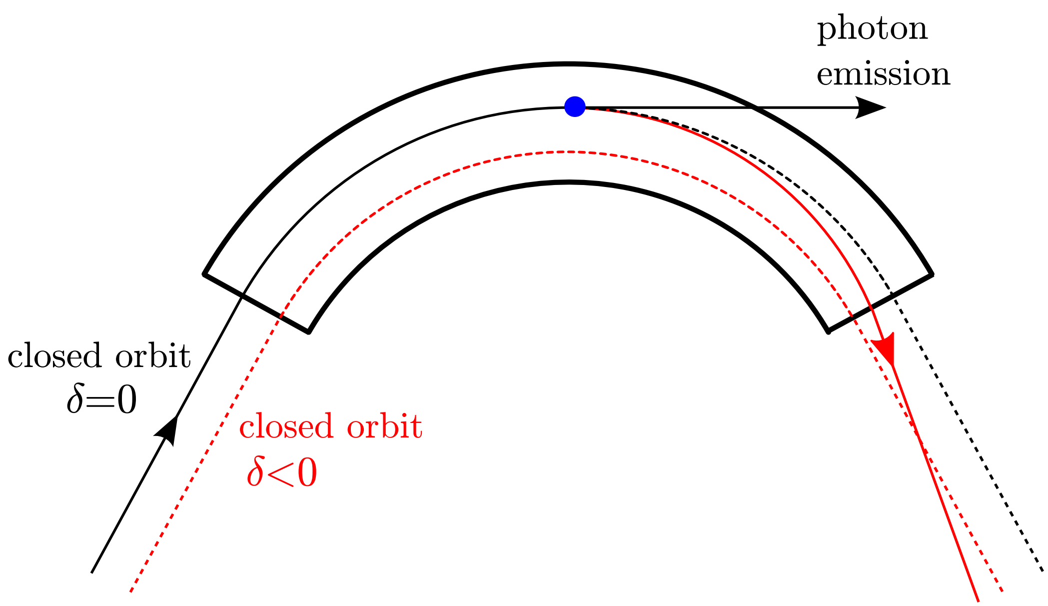
The second and fourth synchrotron radiation integrals are:

$$I_2 = \oint \frac{1}{\rho^2}\, ds, \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds. \tag{45}$$

If radiation were a purely classical process, the emittances would damp to (nearly) zero.

However, radiation is emitted in discrete quanta (photons).

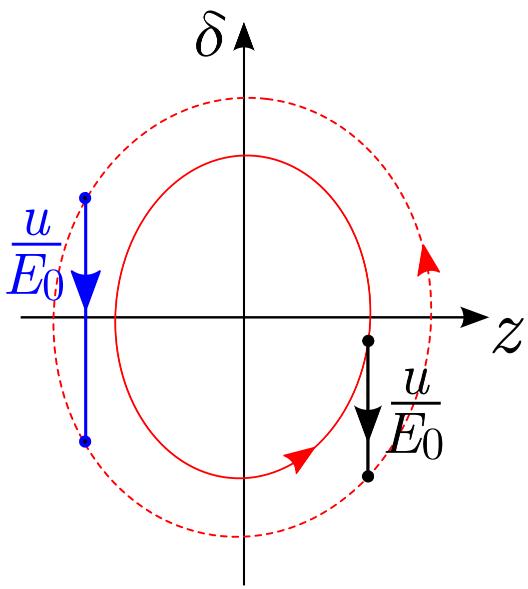
Because of dispersion, the horizontal betatron amplitude of a particle can *increase* when it emits a photon.



The beam eventually reaches an equilibrium distribution determined by a balance between the radiation damping and the quantum excitation.

First, consider quantum excitation of longitudinal emittance.

The longitudinal emittance is a measure of the average amplitude of synchrotron oscillations of particles in a bunch.



The change in the synchrotron amplitude of a particle when it emits a photon depends on the energy of the photon and on the synchrotron phase of the particle at the point of emission.

Since photon emission does not change the longitudinal co-ordinate, let us begin by considering the change in energy spread of a bunch, resulting from photon emission...

Suppose that the particles in a bunch emit photons with energy distribution $N(u)$, so that the mean square photon energy is:

$$\langle u^2 \rangle = \int_0^\infty N(u) u^2 \, du. \tag{46}$$

Taking into account radiation damping, the rate of change of the mean square energy deviation of the particles is (see Appendix D):

$$\frac{d\sigma_\delta^2}{dt} = \frac{1}{2E_0^2} \int_0^\infty \dot{N}(u) u^2 \, du - \frac{2}{\tau_z} \sigma_\delta^2, \tag{47}$$

where $\dot{N}(u)\, du$ is the number of photons with energy between u and $u + du$ emitted per unit time.

We assume that we can average the photon emission around the circumference of the ring.

Then, using the results in Appendix E for the radiation spectrum, we find that the rate of change of the mean square energy deviation is:

$$\frac{d\sigma_\delta^2}{dt} = C_q \gamma^2 \frac{2}{j_z \tau_z} \frac{I_3}{I_2} - \frac{2}{\tau_z} \sigma_\delta^2, \tag{48}$$

where the *third synchrotron radiation integral* I_3 is defined:

$$I_3 = \oint \frac{1}{|\rho^3|} \, ds, \tag{49}$$

and the "quantum radiation constant" C_q is given by:

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \quad (\approx 3.832 \times 10^{-13} \text{ for electrons}). \tag{50}$$

Quantum excitation gives a steady increase in the mean square energy spread, while damping gives an exponential decay.

It follows that there is an equilibrium energy spread for which the quantum excitation is exactly balanced by the damping. The equilibrium can be found from $d\sigma_\delta^2/dt = 0$:

$$\sigma_{\delta 0}^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}. \qquad (51)$$

This is often referred to as the "natural" energy spread, since collective effects can often lead to an increase in the energy spread with increasing bunch charge.

The natural energy spread is determined by the beam energy and by the bending radii of the dipoles: note that it does not depend on the RF parameters (voltage or frequency).

The bunch length σ_z in a *matched distribution*[‡] with energy spread σ_δ is:

$$\sigma_z = \frac{\alpha_p c}{\omega_s}\sigma_\delta. \tag{52}$$

For a given energy spread, we can reduce the bunch length, either:

- by increasing the RF voltage, or
- by increasing the RF frequency.

An increase in RF voltage or frequency increases the synchrotron frequency ω_s, but does not change the energy spread.

[‡]Note: a *matched distribution* in phase space has the same shape as the path mapped out by a single particle when observed on successive turns. Neglecting radiation effects, a matched distribution stays the same on successive turns of the bunch around the ring.

Let us now consider the quantum excitation of the horizontal emittance.

From the change in the co-ordinate and momentum when a particle emits radiation carrying momentum dp, we find that the betatron action changes as:

$$\frac{dJ_x}{dt} = -\frac{w_1}{P_0}\frac{dp}{dt} + \frac{w_2}{P_0^2}\frac{(dp)^2}{dt}, \tag{53}$$

where w_1 and w_2 are functions of the Courant–Snyder parameters, the dispersion, the co-ordinate x and the momentum p_x (see Appendix C).

In the classical approximation, we can take $dp \to 0$ in the limit of small time interval, $dt \to 0$.

In this approximation, the second term on the right hand side in the above equation vanishes, and we are left only with damping.

But since radiation is quantized, the limit $dp \to 0$ is not "physical".

To take account of the quantization of synchrotron radiation, we write:

$$\frac{dp}{dt} = \frac{1}{c} \int_0^\infty \dot{N}(u)\, u\, du, \tag{54}$$

and:

$$\frac{(dp)^2}{dt} = \frac{1}{c^2} \int_0^\infty \dot{N}(u)\, u^2\, du. \tag{55}$$

Here (as before) $\dot{N}(u)\, du$ is the number of photons emitted per unit time with energy between u and $u + du$.

In Appendix E, we show that with (53), these relations lead to the equation for the evolution of the emittance:

$$\frac{d\varepsilon_x}{dt} = \frac{2}{j_x \tau_x} C_q \gamma^2 \frac{I_5}{I_2} - \frac{2}{\tau_x} \varepsilon_x. \tag{56}$$

The *fifth synchrotron radiation integral* I_5 is given by:

$$I_5 = \oint \frac{\mathcal{H}}{|\rho^3|} \, ds, \tag{57}$$

where the "curly-H" function $\mathcal{H}$ is defined:

$$\mathcal{H} = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2. \tag{58}$$

C_q is the quantum radiation constant that we saw earlier (50).

Using Eq. (56) we see that there is an equilibrium horizontal emittance ε_0, for which the damping and excitation rates are equal:

$$\frac{d\varepsilon_x}{dt} = 0 \quad \text{when} \quad \varepsilon_x = \varepsilon_0 = C_q \frac{\gamma^2}{j_x} \frac{I_5}{I_2}. \tag{59}$$

Note that ε_0 is determined by the beam energy, the lattice functions (Courant–Snyder parameters and dispersion) in the dipoles, and the bending radius in the dipoles.

ε_0 is sometimes called the "natural emittance" of the lattice, since it includes only the most fundamental effects that contribute to the emittance: radiation damping and quantum excitation.

Typically, third generation synchrotron light sources have natural emittances of order a few nanometres. With beta functions of a few metres, this implies horizontal beam sizes of tens of microns (in the absence of dispersion).

As the current is increased, interactions between particles in a bunch can increase the emittance above the natural emittance.

Finally, let us consider the quantum excitation of the vertical emittance.

In principle, we can apply the formulae that we derived for the quantum excitation of the horizontal emittance, making appropriate substitutions of vertical quantities for horizontal ones.

In many storage rings, the vertical dispersion in the absence of alignment, steering and coupling errors is zero, so $\mathcal{H}_y = 0$.

However, the equilibrium vertical emittance is larger than zero, because the vertical opening angle of the radiation excites some vertical betatron oscillations.

The fundamental lower limit on the vertical emittance, from the opening angle of the synchrotron radiation, is given by[§]:

$$\varepsilon_y = \frac{13}{55} \frac{C_q}{j_y I_2} \oint \frac{\beta_y}{|\rho^3|} \, ds. \tag{60}$$

In most storage rings, this is an extremely small value, typically four orders of magnitude smaller than the natural (horizontal) emittance.

In practice, the vertical emittance is dominated by magnet alignment errors. Storage rings typically operate with a vertical emittance that is of order 1% of the horizontal emittance, but many can achieve emittance ratios somewhat smaller than this.

[§] T. Raubenheimer, SLAC Report 387, p.19 (1991).

Including the effects of radiation damping and quantum excitation, the emittances vary as:

$$\varepsilon(t) = \varepsilon(0)\,\mathrm{exp}\left(-2\frac{t}{\tau}\right) + \varepsilon(\infty)\left[1 - \mathrm{exp}\left(-2\frac{t}{\tau}\right)\right]. \tag{61}$$

The damping times are given by:

$$j_x\tau_x = j_y\tau_y = j_z\tau_z = 2\frac{E_0}{U_0}T_0. \tag{62}$$

The damping partition numbers are given by:

$$j_x = 1 - \frac{I_4}{I_2}, \qquad j_y = 1, \qquad j_z = 2 + \frac{I_4}{I_2}. \tag{63}$$

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi}E_0^4 I_2, \qquad C_\gamma = 9.846 \times 10^{-5}\ \mathrm{m/GeV}^3. \tag{64}$$

The natural emittance is:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}, \qquad C_q = 3.832 \times 10^{-13} \text{ m.} \tag{65}$$

The natural energy spread and bunch length are given by:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}, \qquad \sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta. \tag{66}$$

The momentum compaction factor is:

$$\alpha_p = \frac{I_1}{C_0}. \tag{67}$$

The synchrotron frequency and synchronous phase are given by:

$$\omega_s^2 = -\frac{eV_{\text{rf}}}{E_0} \frac{\omega_{\text{rf}}}{T_0} \alpha_p \cos(\phi_s), \qquad \sin(\phi_s) = \frac{U_0}{eV_{\text{rf}}}. \tag{68}$$

The synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho}\, ds, \tag{69}$$

$$I_2 = \oint \frac{1}{\rho^2}\, ds, \tag{70}$$

$$I_3 = \oint \frac{1}{|\rho|^3}\, ds, \tag{71}$$

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}, \tag{72}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3}\, ds, \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2. \tag{73}$$

Appendices

The horizontal action J_x and phase ϕ_x are defined:

$$2J_x = \gamma_x x^2 + 2\alpha_x x p_x, + \beta_x p_x^2, \tag{74}$$

$$\tan(\phi_x) = -\beta_x \frac{p_x}{x} - \alpha_x, \tag{75}$$

where x is the transverse horizontal co-ordinate with respect to the reference trajectory, and p_x is the conjugate momentum:

$$p_x = \frac{\gamma m v_x}{P_0}, \tag{76}$$

where γ is the relativistic factor for the particle (which has mass m), v_x is the horizontal velocity, and P_0 is the reference momentum.

The quantities α_x, β_x and γ_x are the Courant–Snyder parameters (sometimes called the Twiss parameters), and satisfy:

$$\beta_x \gamma_x - \alpha_x^2 = 1. \tag{77}$$

Corresponding definitions apply for the vertical action-angle variables (J_y, ϕ_y) in terms of y and p_y, and for the longitudinal action-angle variables (J_z, ϕ_z) in terms of z and δ.

Inverting (74) and (75) gives:

$$x = \sqrt{2\beta_x J_x}\,\cos(\phi_x), \tag{78}$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}\,(\sin(\phi_x) + \alpha_x \cos(\phi_x))\,. \tag{79}$$

If the angles ϕ_x of the particles in a bunch are uncorrelated, so that:

$$\langle \sin(\phi_x) \rangle = \langle \cos(\phi_x) \rangle = 0, \tag{80}$$

where the brackets $\langle \cdot \rangle$ represent an average over all particles in a bunch, then:

$$\langle x \rangle = \langle p_x \rangle = 0, \tag{81}$$

i.e. the centroid of the bunch has no horizontal offset (or momentum) with respect to the reference trajectory.

In that case, the second order moments of the beam distribution are:

$$\langle x^2 \rangle = \beta_x \varepsilon_x, \quad \langle x p_x \rangle = -\alpha_x \varepsilon_x, \quad \langle p_x^2 \rangle = \gamma_x \varepsilon_x, \tag{82}$$

where the horizontal emittance ε_x is given by:

$$\varepsilon_x = \langle J_x \rangle = \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2}. \tag{83}$$

The energy loss per turn U for a particle is found by integrating the radiation power over one orbit of the ring:

$$U = \frac{1}{c} \oint P_\gamma \left(1 + \frac{\eta_x}{\rho}\delta \right) ds = \frac{C_\gamma}{2\pi} e^2 c^2 E^2 \oint B^2 \left(1 + \frac{\eta_x}{\rho}\delta \right) ds, \qquad (84)$$

where ρ is the radius of curvature of the reference trajectory, and the radiation power P_γ is given by (18).

Using $B\rho \approx E/ec$, we find that for a particle with the reference energy E_0 following the reference trajectory, the energy loss per turn is:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2, \qquad (85)$$

where the second synchrotron radiation integral is:

$$I_2 = \oint \frac{1}{\rho^2} ds. \qquad (86)$$

The longitudinal damping constant α_E (15) is given by the derivative of U with respect to the energy E.

To calculate α_E, we have to take into account the fact that the dipoles in the ring may have a field gradient: if they do, then the field seen by a particle will depend on its trajectory through the dipole, and hence (because of dispersion) on its energy.

If the dipoles have a field gradient:

$$B = B_0 + \frac{E_0}{ec} k_1 \eta_x \delta, \qquad (87)$$

where E_0 is the reference energy, the energy loss per turn (84) becomes:

$$U = \frac{C_\gamma}{2\pi} e^2 c^2 E_0^2 (1+\delta)^2 \oint \left(B_0 + \frac{E_0}{ec} k_1 \eta_x \delta \right)^2 \left(1 + \frac{\eta_x}{\rho} \delta \right) ds. \qquad (88)$$

Then, using $E = (1+\delta)E_0$, we find:

$$\left.\frac{dU}{dE}\right|_{E=E_0} = \frac{1}{E_0} \left.\frac{dU}{d\delta}\right|_{\delta=0} = 2\frac{U_0}{E_0} + \frac{1}{E_0} \frac{C_\gamma}{2\pi} e^2 c^2 E_0^2 \oint \left(\frac{B_0^2 \eta_x}{\rho} + 2\frac{B_0 E_0}{ec} k_1 \eta_x \right) ds. \qquad (89)$$

From (85), we can write:

$$\frac{C_\gamma}{2\pi} e^2 c^2 E_0^2 = \frac{e^2 c^2}{E_0^2} \frac{U_0}{I_2}. \tag{90}$$

Using (90), equation (89) becomes:

$$\left. \frac{dU}{dE} \right|_{E=E_0} = 2\frac{U_0}{E_0} + \frac{U_0}{E_0 I_2} \oint \frac{ec}{E_0} B_0 \eta_x \left(\frac{ec}{E_0} \frac{B_0}{\rho} + 2k_1 \right) ds = \left(2 + \frac{I_4}{I_2} \right) \frac{U_0}{E_0}, \tag{91}$$

where the fourth synchrotron radiation integral I_4 is:

$$I_4 = \oint \frac{ec}{E_0} B_0 \eta_x \left(\frac{ec}{E_0} \frac{B_0}{\rho} + 2k_1 \right) ds = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1 \right) ds, \tag{92}$$

where, in the final step, we have again used $B_0 \rho \approx E_0 / ec$.

Hence, we have finally for the longitudinal damping constant:

$$\alpha_E = \frac{1}{2T_0} \left. \frac{dU}{dE} \right|_{E=E_0} = \frac{j_z}{2T_0} \frac{U_0}{E_0}, \tag{93}$$

where $j_z = 2 + I_4/I_2$ is the longitudinal damping partition number.

Appendix C: Radiation damping of horizontal emittance

In this Appendix, we derive the expression for radiation damping of the horizontal emittance:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x, \tag{94}$$

where:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0, \qquad j_x = 1 - \frac{I_4}{I_2}. \tag{95}$$

To derive these formulae, we proceed as follows:

1. We find an expression for the change of horizontal action of a single particle when emitting radiation with momentum dp.

2. We integrate around the ring to find the change in action per revolution period.

3. We average the action over all the particles in the bunch, to find the change in emittance per revolution period.

To begin, we note that, in the presence of dispersion, the action J_x is written:

$$2J_x = \gamma_x \tilde{x}^2 + 2\alpha_x \tilde{x}\tilde{p}_x + \beta_x \tilde{p}_x^2, \tag{96}$$

where:

$$\tilde{x} = x - \eta_x \delta, \qquad \text{and} \qquad \tilde{p}_x = p_x - \eta_{px}\delta. \tag{97}$$

After emission of radiation carrying momentum dp, the variables change to:

$$\delta' = \delta - \frac{dp}{P_0}, \quad \tilde{x}' = \tilde{x} + \eta_x \frac{dp}{P_0}, \quad \tilde{p}_x' = \tilde{p}_x \left(1 - \frac{dp}{P_0}\right) + \eta_{px}(1-\delta)\frac{dp}{P_0}. \tag{98}$$

We write the resulting change in the action as:

$$J_x' = J_x + dJ_x. \tag{99}$$

Substituting the new values (98) into the expression for the horizontal action (96), we find that the change in the horizontal action is:

$$dJ_x = -\frac{w_1}{P_0}dp + \frac{w_2}{P_0^2}dp^2 \qquad \therefore \qquad \frac{dJ_x}{dt} = -\frac{w_1}{P_0}\frac{dp}{dt} + \frac{w_2}{P_0^2}\frac{dp^2}{dt}, \tag{100}$$

where, in the limit $\delta \to 0$:

$$w_1 = \alpha_x x p_x + \beta_x p_x^2 - \eta_x(\gamma_x x + \alpha_x p_x) - \eta_{px}(\alpha_x x + \beta_x p_x), \tag{101}$$

and:

$$w_2 = \frac{1}{2}\left(\gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2\right) - \left(\alpha_x \eta_x + \beta_x \eta_{px}\right) p_x + \frac{1}{2}\beta_x p_x^2. \tag{102}$$

Treating radiation as a classical phenomenon, we can take the limit $dp \to 0$ in the limit of small time interval, $dt \to 0$.

In this approximation:

$$\frac{dJ_x}{dt} \approx -w_1 \frac{1}{P_0}\frac{dp}{dt} \approx -w_1 \frac{P_\gamma}{P_0 c}, \tag{103}$$

where P_γ is the *rate of energy loss* of the particle through synchrotron radiation.

To find the *average* rate of change of horizontal action, we integrate over one revolution period:

$$\frac{dJ_x}{dt} = -\frac{1}{T_0}\oint w_1 \frac{P_\gamma}{P_0 c}\,dt. \qquad (104)$$

We have to be careful changing the variable of integration where the reference trajectory is curved:

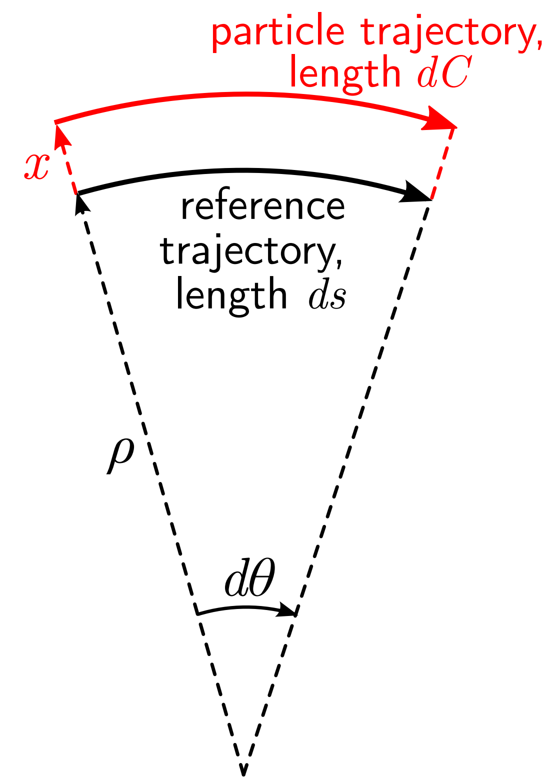
$$dt = \frac{dC}{c} = \left(1 + \frac{x}{\rho}\right)\frac{ds}{c}. \qquad (105)$$

So:

$$\frac{dJ_x}{dt} = -\frac{1}{T_0 P_0 c^2}\oint w_1 P_\gamma \left(1 + \frac{x}{\rho}\right)ds, \qquad (106)$$

where the rate of energy loss is:

$$P_\gamma = \frac{C_\gamma}{2\pi}c^3 e^2 B^2 E^2. \qquad (107)$$



We have to take into account the fact that the field strength in a dipole can vary with position. To first order in x we can write:

$$B = B_0 + x\frac{\partial B_y}{\partial x}. \tag{108}$$

Substituting Eq. (108) into (107), and with the use of (101), we find (after some algebra!) that, averaging over all particles in the beam:

$$\oint \left\langle w_1 P_\gamma \left(1 + \frac{x}{\rho}\right)\right\rangle ds = cU_0 \left(1 - \frac{I_4}{I_2}\right)\varepsilon_x, \tag{109}$$

where:

$$U_0 = \frac{C_\gamma}{2\pi} c E_0^4 I_2, \qquad I_2 = \oint \frac{1}{\rho^2} ds, \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \tag{110}$$

and k_1 is the normalised quadrupole gradient in the dipole field:

$$k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}. \tag{111}$$

Combining Eqs. (106) and (109) we have:

$$\frac{d\varepsilon_x}{dt} = -\frac{1}{T_0}\frac{U_0}{E_0}\left(1 - \frac{I_4}{I_2}\right)\varepsilon_x. \tag{112}$$

Defining the horizontal damping time τ_x:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0, \qquad j_x = 1 - \frac{I_4}{I_2}, \tag{113}$$

the evolution of the horizontal emittance can be written:

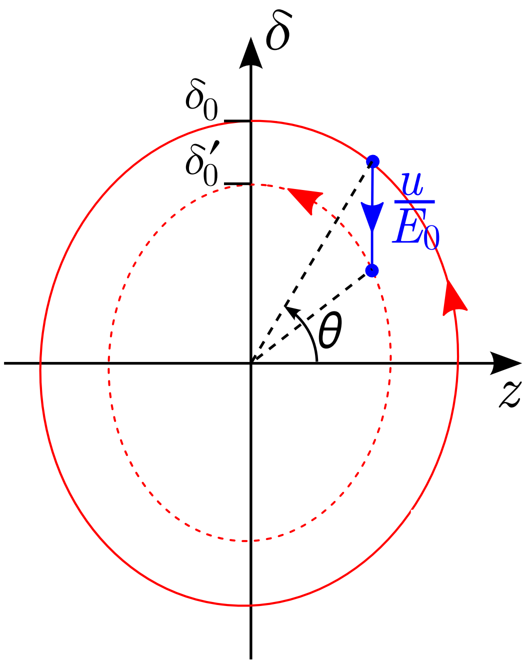
$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x. \tag{114}$$

The quantity j_x is called the *horizontal damping partition number*.

For most synchrotron storage ring lattices, if there is no gradient in the dipoles then j_x is very close to 1.

Consider a particle with longitudinal co-ordinate z and energy deviation δ, which emits a photon of energy u.

In terms of the amplitude δ_0 of the energy oscillation and the synchrotron phase θ, the energy deviation δ' and longitudinal co-ordinate z' after the photon emission are:



$$\delta' = \delta_0' \sin(\theta') = \delta_0 \sin(\theta) - \frac{u}{E_0}. \qquad (115)$$

$$z' = \frac{\alpha_p c}{\omega_s}\delta_0' \cos(\theta') = \frac{\alpha_p c}{\omega_s}\delta_0 \cos(\theta). \qquad (116)$$

$$\therefore\ \delta_0'^2 = \delta_0^2 - 2\delta_0 \frac{u}{E_0}\sin(\theta) + \frac{u^2}{E_0^2}. \qquad (117)$$

Averaging over the bunch gives:

$$\Delta\sigma_\delta^2 = \frac{\langle u^2 \rangle}{2E_0^2} \quad \text{where} \quad \sigma_\delta^2 = \langle \delta^2 \rangle = \frac{1}{2}\langle \delta_0^2 \rangle. \qquad (118)$$

Let us write the number of photons emitted per unit time with energy between u and $u + du$ as $\dot{N}(u)\,du$. Then:

$$\frac{d\langle u^2\rangle}{dt} = \int_0^\infty \dot{N}(u) u^2\,du. \tag{119}$$

Including radiation damping, the energy spread evolves as:

$$\frac{d\sigma_\delta^2}{dt} = \frac{1}{2E_0^2}\left\langle \int_0^\infty \dot{N}(u) u^2\,du \right\rangle_C - \frac{2}{\tau_z}\sigma_\delta^2, \tag{120}$$

where the brackets $\langle\,\rangle_C$ represent an average around the ring.

Using Eq. (129) from Appendix E for $\int \dot{N}(u)u^2\,du$, we find:

$$\frac{d\sigma_\delta^2}{dt} = C_q\gamma^2 \frac{2}{j_z\tau_z}\frac{I_3}{I_2} - \frac{2}{\tau_z}\sigma_\delta^2, \tag{121}$$

where the *third synchrotron radiation integral* I_3 is defined:

$$I_3 = \oint \frac{1}{|\rho^3|}\,ds, \tag{122}$$

and the "quantum radiation constant" is:

$$C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} \quad (\approx 3.832 \times 10^{-13}\text{m for electrons}). \tag{123}$$

In deriving the equation of motion (106) for the action of a particle emitting synchrotron radiation, we made the classical approximation that in a time interval dt, the momentum dp of the radiation emitted goes to zero as dt goes to zero.

In reality, emission of radiation is quantized, so writing "$dp \to 0$" actually makes no sense.

Taking into account the quantization of radiation, the equation of motion for the action (100) should be written:

$$\frac{dJ_x}{dt} = -\frac{w_1}{P_0 c}\int_0^\infty \dot{N}(u)\, u\, du + \frac{w_2}{P_0^2 c^2}\int_0^\infty \dot{N}(u)\, u^2\, du, \tag{124}$$

where $\dot{N}(u)$ is the number of photons emitted per unit time in the energy range from u to $u + du$.

The first term on the right hand side of Eq. (124) just gives the same radiation damping as in the classical approximation.

The second term on the right hand side of Eq. (124) is an excitation term that we previously neglected.

To proceed, we find expressions for the integrals $\int \dot{N}(u)\, u\, du$ and $\int \dot{N}(u)\, u^2\, du$.

The required expressions can be found from the spectral distribution of synchrotron radiation from a dipole magnet. This is given by:

$$\frac{d\mathcal{P}}{d\vartheta} = \frac{9\sqrt{3}}{8\pi} P_\gamma \vartheta \int_\vartheta^\infty K_{5/3}(x)\, dx, \tag{125}$$

where $d\mathcal{P}/d\vartheta$ is the energy radiated per unit time per unit frequency range, and $\vartheta = \omega/\omega_c$ is the radiation frequency ω divided by the critical frequency ω_c:

$$\omega_c = \frac{3}{2}\frac{\gamma^3 c}{\rho}. \tag{126}$$

P_γ is the total energy radiated per unit time, and $K_{5/3}(x)$ is a modified Bessel function.

Since the energy of a photon of frequency ω is $u = \hbar\omega$, it follows that:

$$\dot{N}(u)\, du = \frac{1}{\hbar\omega}\frac{d\mathcal{P}}{d\vartheta}\, d\vartheta. \tag{127}$$

Using (125) and (127), we find:

$$\int_0^\infty \dot{N}(u)\, u\, du = P_\gamma, \tag{128}$$

and:

$$\int_0^\infty \dot{N}(u)\, u^2\, du = 2C_q\gamma^2\frac{E_0}{\rho}P_\gamma. \tag{129}$$

C_q is a constant given by:

$$C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} \approx 3.832 \times 10^{-13}\,\mathrm{m}. \tag{130}$$

The final step is to substitute for the integrals in (124) from (128) and (129), substitute for w_1 and w_2 from (101) and (102), average over the circumference of the ring, and average also over all particles in the beam.

Then, since $\varepsilon_x = \langle J_x \rangle$, we find (for $x \ll \eta_x$ and $p_x \ll \eta_{px}$):

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x + \frac{2}{j_x \tau_x} C_q \gamma^2 \frac{I_5}{I_2} \tag{131}$$

where the fifth synchrotron radiation integral I_5 is given by:

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho^3|}\, ds, \tag{132}$$

The "curly-H" function $\mathcal{H}_x$ is given by:

$$\mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2. \tag{133}$$

The damping time and horizontal damping partition number are given by:

$$j_x \tau_x = 2\frac{E_0}{U_0} T_0, \qquad U_0 = \frac{C_\gamma}{2\pi} c E_0^4 I_2, \tag{134}$$

(U_0 is the energy loss per turn) and:

$$j_x = 1 - \frac{I_4}{I_2}. \tag{135}$$

Note that the excitation term is independent of the emittance.

The quantum excitation does not simply modify the damping time, but leads to a non-zero equilibrium emittance.

The equilibrium emittance ε_0 is determined by the condition:

$$\left.\frac{d\varepsilon_x}{dt}\right|_{\varepsilon_x=\varepsilon_0} = 0. \tag{136}$$

From (131), we see that the equilibrium emittance is given by:

$$\varepsilon_0 = C_q\gamma^2 \frac{I_5}{j_x I_2}. \tag{137}$$