



Instabilities Part II: Longitudinal wake fields – impact on machine elements and beam dynamics

Giovanni Rumolo and Kevin Li

- We have learned about the concept of **particle distributions** as well as some **peculiarities of multiparticle dynamics** in accelerators, decoherence, filamentation.
- We will learn the basic **concept of wake fields** and how these can be characterized as a **collective effect** in that they depend on the particle distribution.
- We will have a basic understanding of multiparticle systems and wakefields and are now ready to look at the **impact of these** in the longitudinal and transverse planes.

Part 2: Multiparticle dynamics with wake fields – impact on machine elements and longitudinal beam dynamics

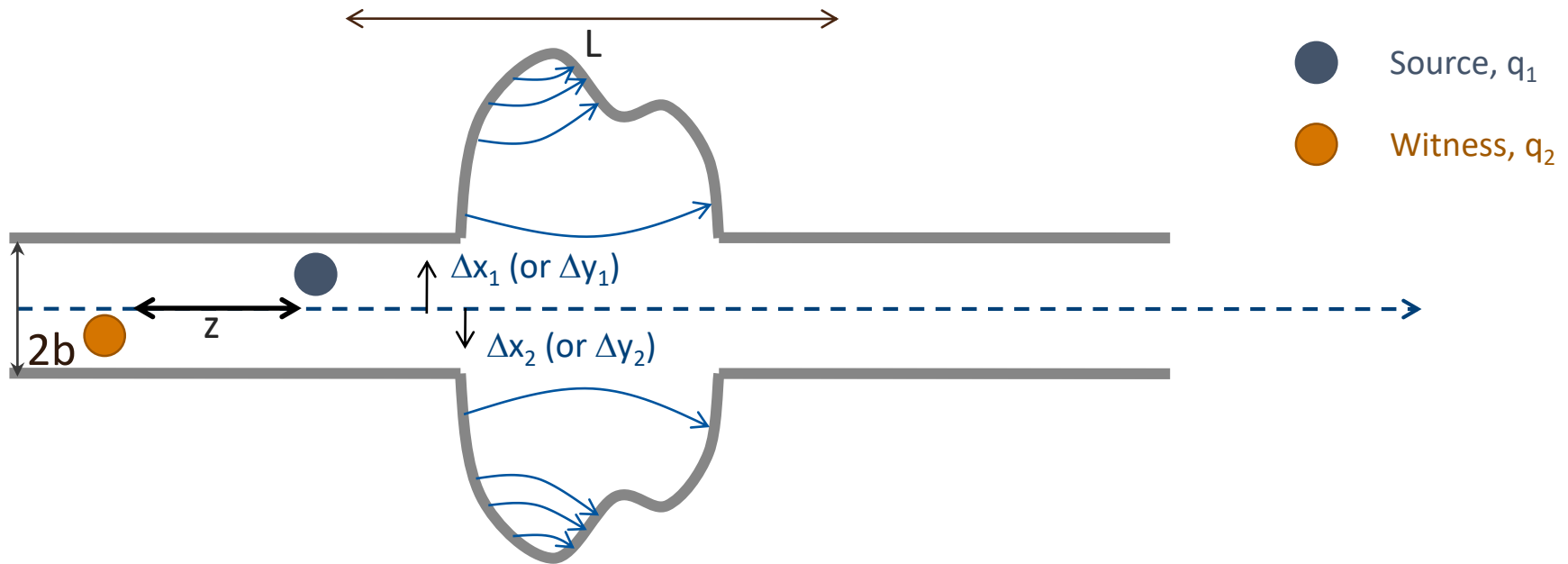
- General introduction to wake fields
- Longitudinal wake fields and the longitudinal wake function
- Energy loss – beam induced heating and stable phase shift
- Potential well distortion, bunch lengthening and microwave instability

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Wake functions: general definition

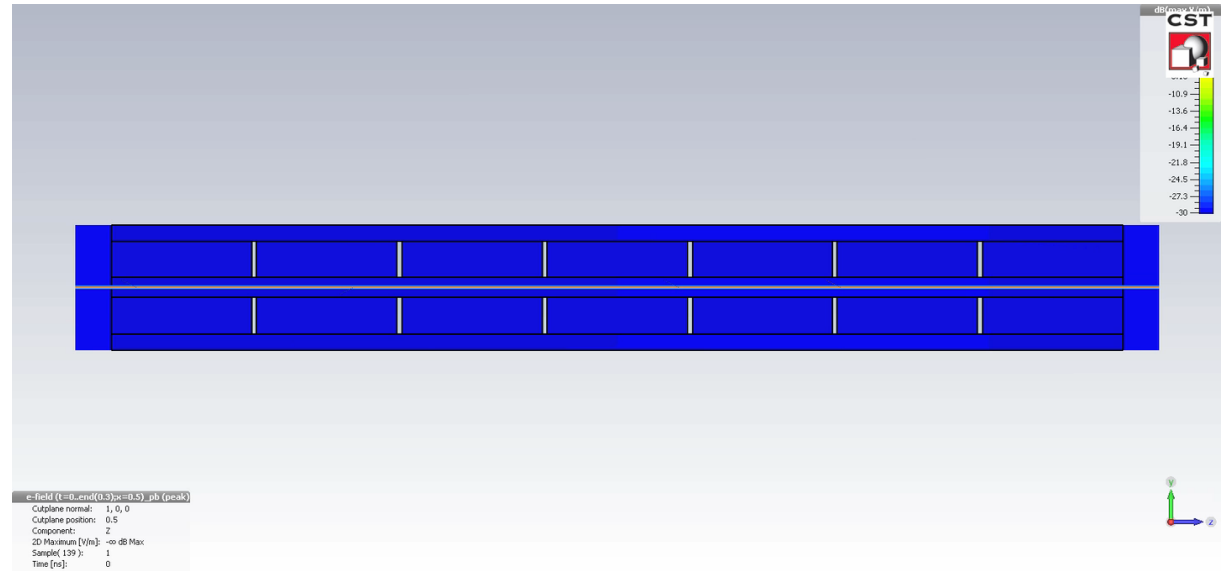
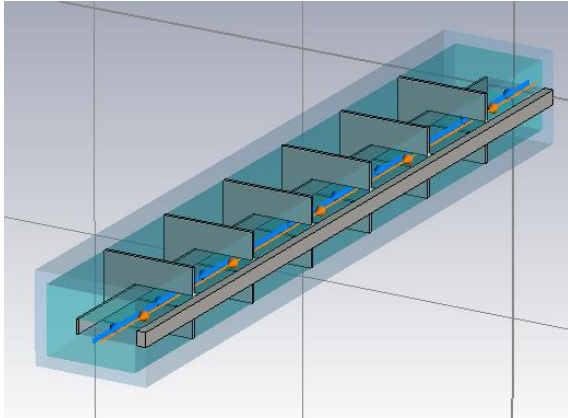


Wake function is the **integrated force** felt by a witness charge following a source charge, thus associated to an ‘energy kick’:

- In general, for two point-like particles, we have

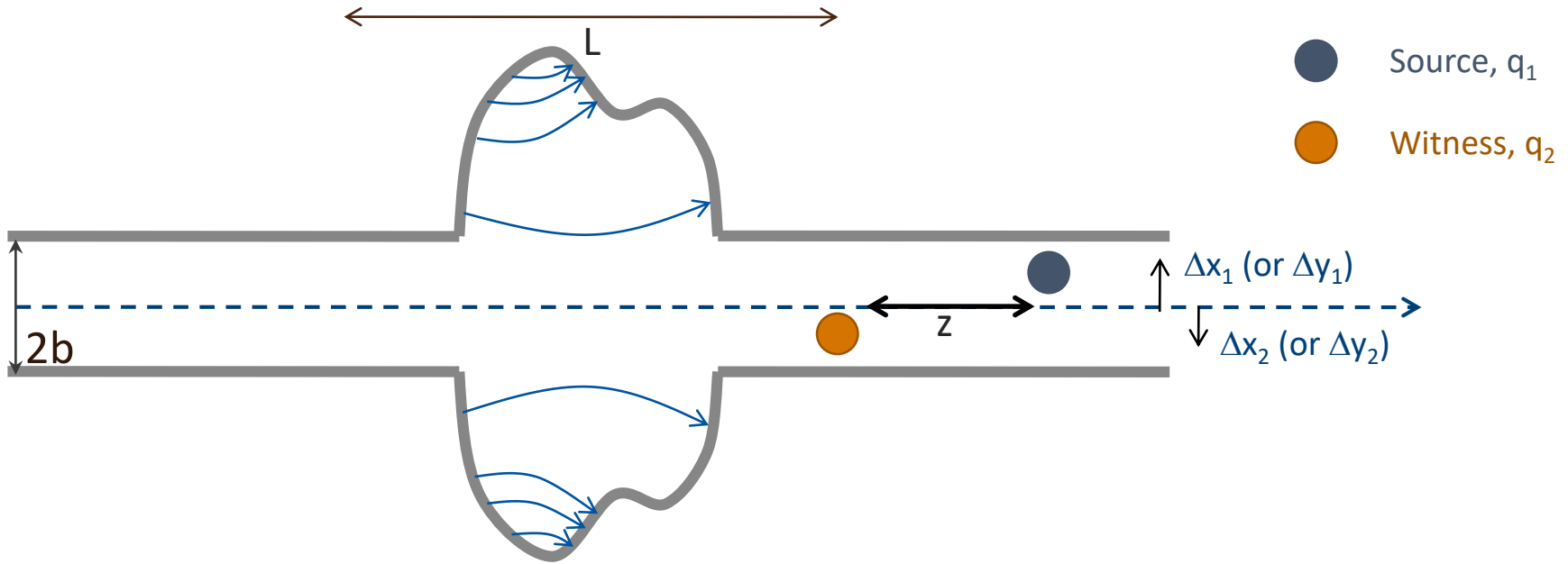
$$\Delta E_2 = \int F(x_1, x_2, z, s) ds = -q_1 q_2 w(x_1, x_2, z)$$

w is typically expanded in the transverse offsets of source and witness particles. This yields the different types of wake fields (dipole, quadrupole, coupling wakes)



- The **wake function** is a type of **electromagnetic response** of a device to a charge pulse. It is an intrinsic property of this device and depends on
 - The device's **geometry** (transitions, cavities, etc.)
 - The **electromagnetic properties** of the materials exposed to the beam (e.g. PEC, finite conductivity, lossy materials, metamaterials, etc.)
- The wake function describes the **electromagnetic coupling between two point charges** as a function of the distance between them.

Longitudinal wake function



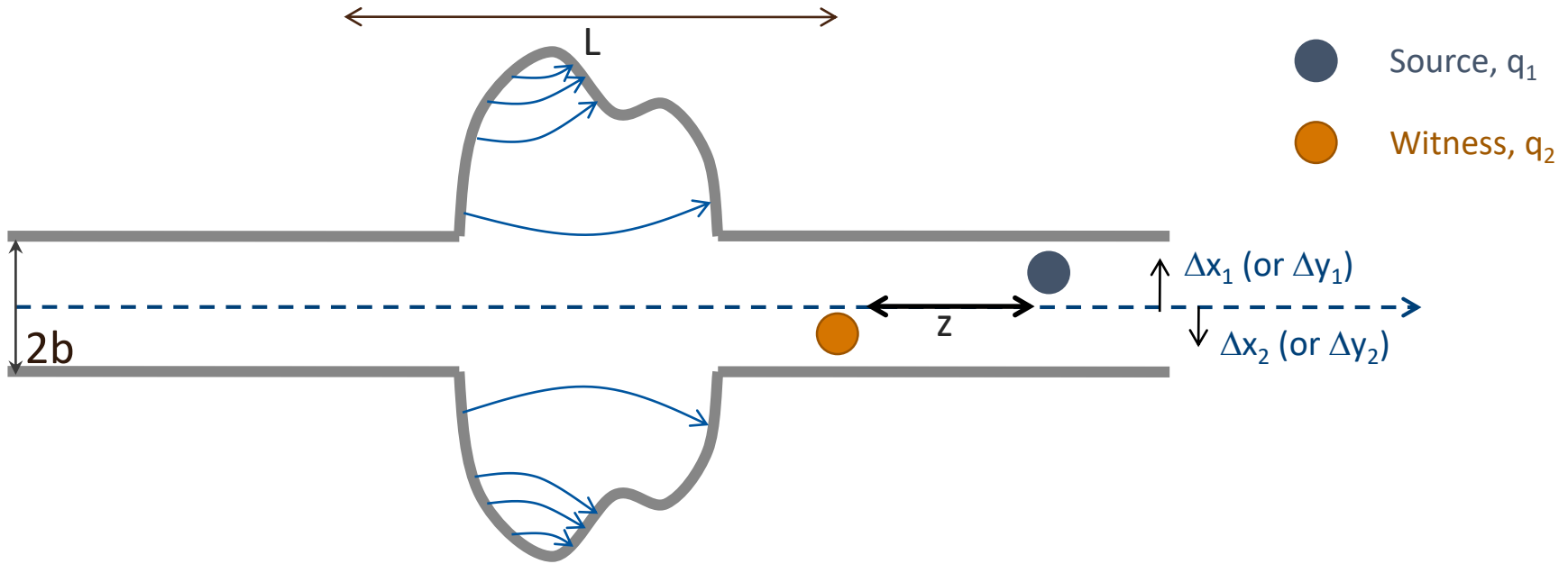
- Longitudinal wake fields

$$\int F_z(x_1, x_2, z, s) ds = -q_1 q_2 \left(\boxed{W_{\parallel}(z)} + \boxed{O(\Delta x_1) + O(\Delta x_2)} \right)$$

Zeroth order with source and test centred
usually dominant

Higher order terms
Usually negligible for small offsets

Longitudinal wake function



- Longitudinal wake fields

$$\Delta E_2 = \int F_z(z, s) ds = -q_1 q_2 W_{\parallel}(z)$$

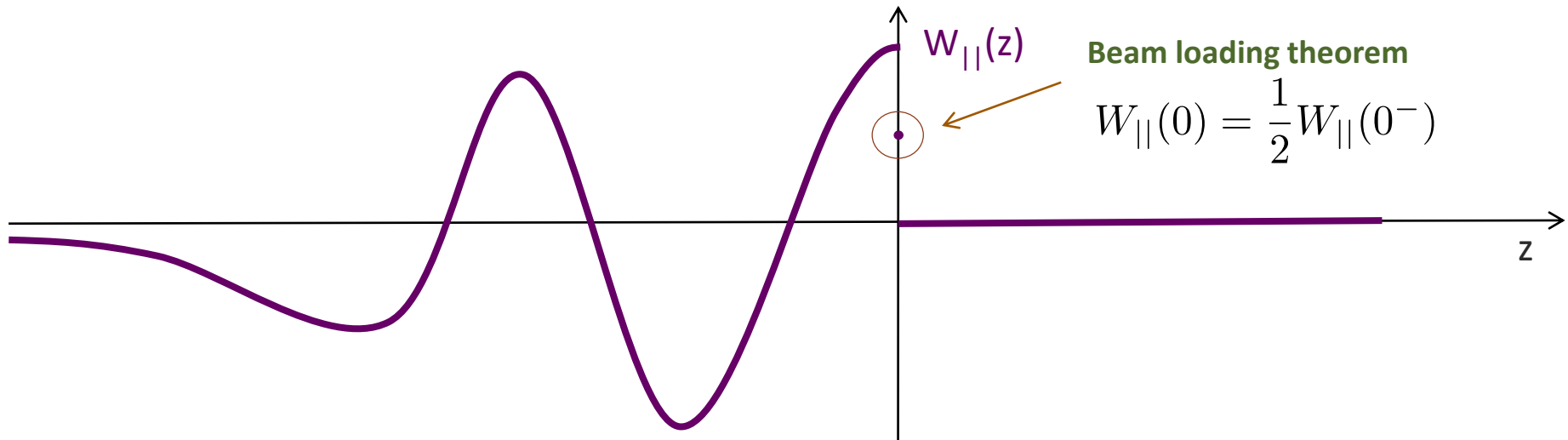
$$\rightarrow \frac{\Delta E_2}{E_0} = \left(\frac{\gamma^2 - 1}{\gamma} \right) \frac{\Delta p_2}{p_0}$$

Energy kick of the witness particle from longitudinal wakes

Longitudinal wake function

$$W_{\parallel}(z) = -\frac{\Delta E_2}{q_1 q_2} \xrightarrow[q_2 \rightarrow q_1]{z \rightarrow 0} W_{\parallel}(0) = -\frac{\Delta E_1}{q_1^2}$$

- The value of the wake function in $z=0$ is related to the **energy lost by the source particle** in the creation of the wake
- $W_{\parallel}(0) > 0$ since $\Delta E_1 < 0$
- $W_{\parallel}(z)$ is discontinuous in $z=0$ and it vanishes for all $z > 0$ because of the ultra-relativistic approximation



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- The **wake function** of an accelerator component is basically its **Green function in time domain** (i.e., its response to a pulse excitation)
 - Very useful for macroparticle models and simulations, because it can be used to describe the driving terms in the single particle equations of motion!
- We can also describe it as a **transfer function in frequency domain**
 - This is the definition of **longitudinal beam coupling impedance** of the element under study

$$\boxed{Z_{\parallel}(\omega)} = \int_{-\infty}^{\infty} \boxed{W_{\parallel}(z)} \exp\left(-\frac{i\omega z}{c}\right) \frac{dz}{c}$$

\downarrow \downarrow

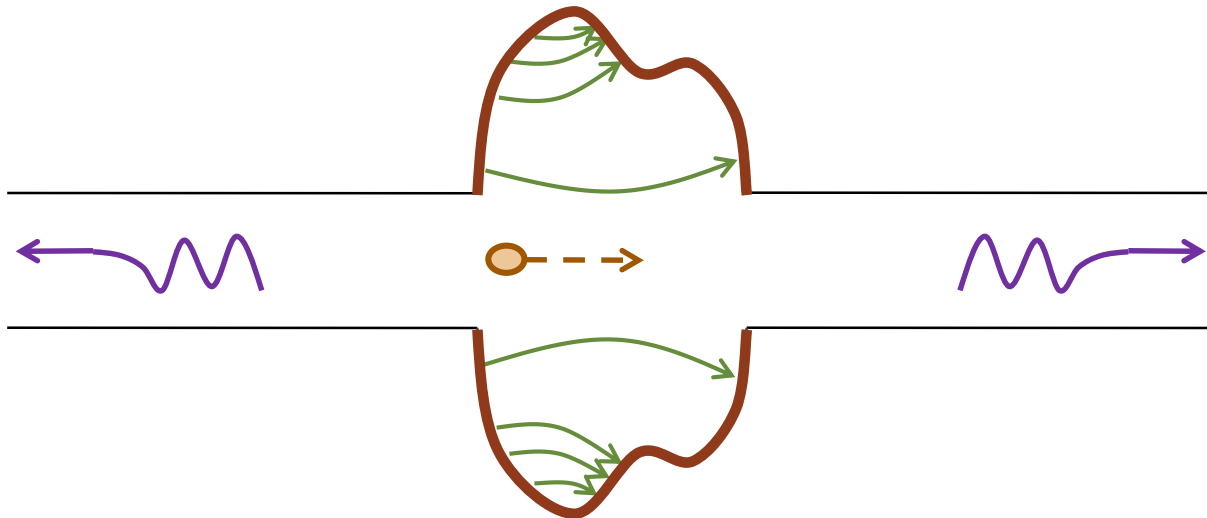
$[\Omega]$ $[\Omega/s]$

The energy balance

$$W_{\parallel}(0) = \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} (Z_{\parallel}(\omega)) d\omega = -\frac{\Delta E_1}{q_1^2}$$

What happens to the energy lost by the source?

- In the global energy balance, the energy lost by the source splits into:
 - Electromagnetic energy of the **modes that remain trapped** in the object
 - Partly dissipated on **lossy walls** or into purposely designed inserts or HOM absorbers
 - Partly transferred to **following particles** (or the same particle over successive turns), possibly feeding into an instability!
 - Electromagnetic energy of **modes that propagate** down the beam chamber (above cut-off), eventually lost on surrounding lossy materials



The energy balance

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What happens to the energy lost by the source?

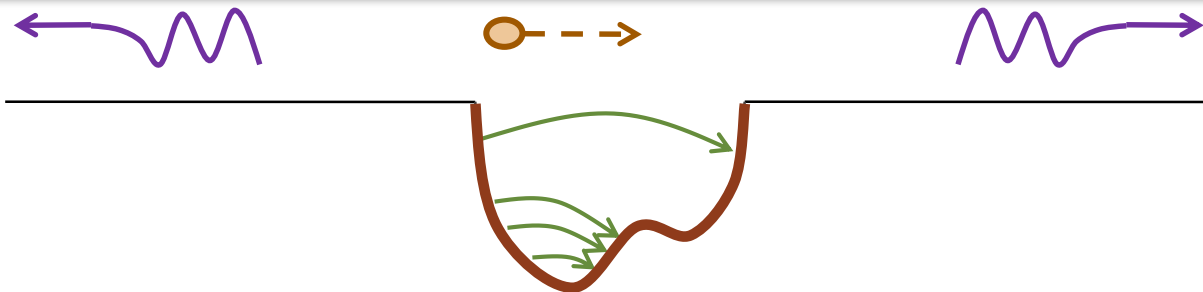
- In the global energy balance, the energy lost by the source splits into

-

The energy loss of a particle bunch

-

- ⇒ causes **beam induced heating** of the machine elements (damage, outgassing) or **sparking** due to high field
- ⇒ feeds into both **longitudinal and transverse instabilities** through the associated EM fields
- ⇒ is compensated by the RF system determining a **stable phase shift**

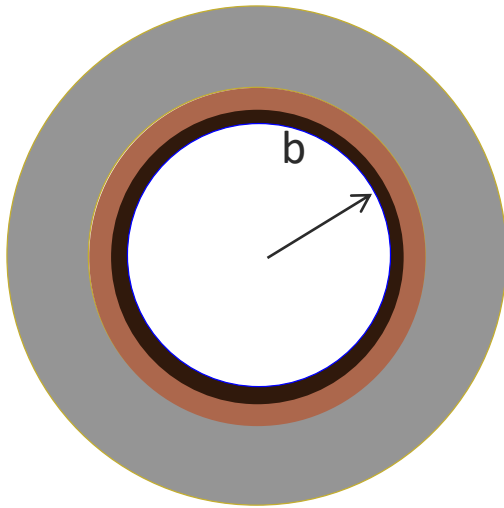


ct
absorbers
ve turns),
chamber

How are wakes and impedances computed?

- **Analytical or semi-analytical** approach, when geometry is simple (or simplified)
 - Solve Maxwell's equations with the correct source terms, geometries and boundary conditions up to an advanced stage
 - Find closed expressions or execute the last steps numerically to derive wakes and impedances

→ An example: **axisymmetric beam chamber** with several layers with different EM properties



$$\nabla \times \vec{E} = -i\omega \vec{B}$$

$$\nabla \cdot \vec{E} = \frac{\tilde{\rho}}{\epsilon_0 \epsilon_1(\omega)}$$

$$\nabla \times \vec{B} = \mu_0 \mu_1(\omega) \vec{J} + i\omega \frac{\mu_1(\omega) \epsilon_1(\omega)}{c^2} \vec{E}$$

$$\nabla \cdot \vec{B} = 0$$

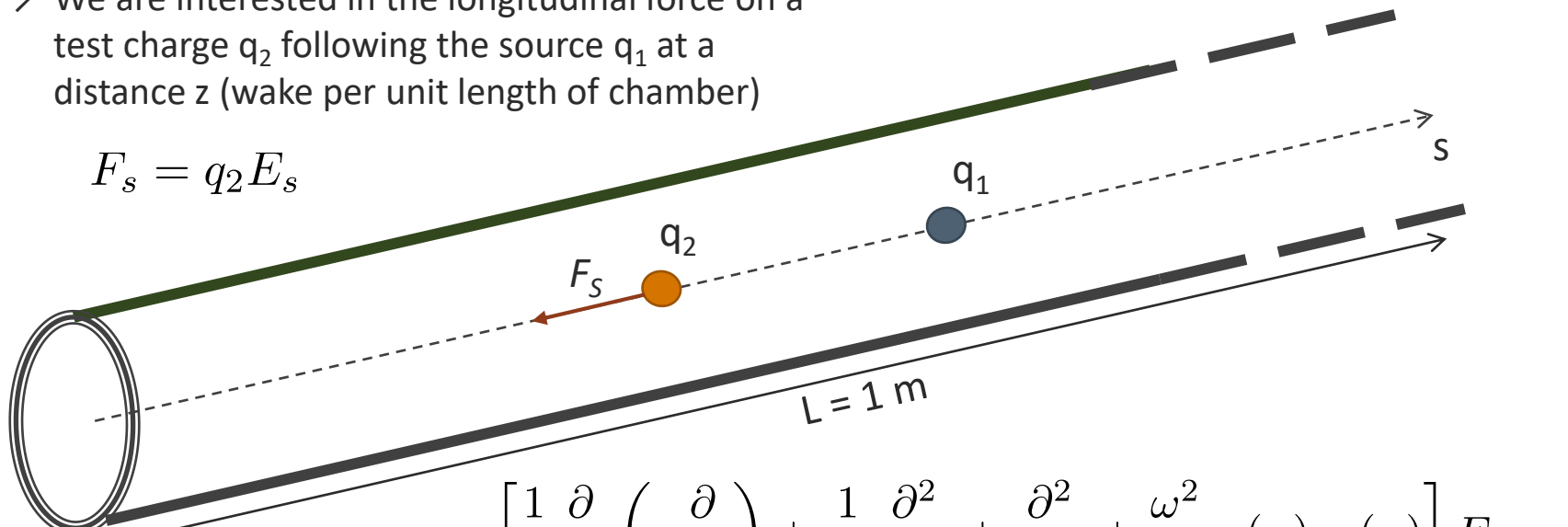
+ Boundary conditions

$$\tilde{\rho}(r, \theta, s, \omega) = \frac{q_1}{r_1 v} \delta(r - r_1) \delta_P(\theta) \exp\left(-\frac{i\omega s}{v}\right)$$

$$\vec{J}(r, \theta, s, \omega) = \tilde{\rho}(r, \theta, s, \omega) \vec{v}$$

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- We are interested in the longitudinal force on a test charge q_2 following the source q_1 at a distance z (wake per unit length of chamber)



$F_s = q_2 E_s$

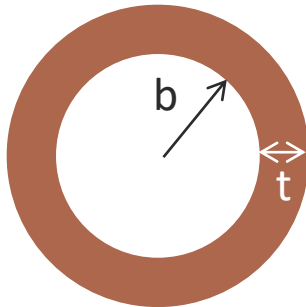
$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial s^2} + \frac{\omega^2}{c^2} \epsilon_1(\omega) \mu_1(\omega) \right] E_s =$$

$$= \frac{1}{\epsilon_0 \epsilon_1(\omega)} \frac{\partial \tilde{\rho}}{\partial s} + i \omega \mu_0 \mu_1(\omega) \tilde{\rho} v$$

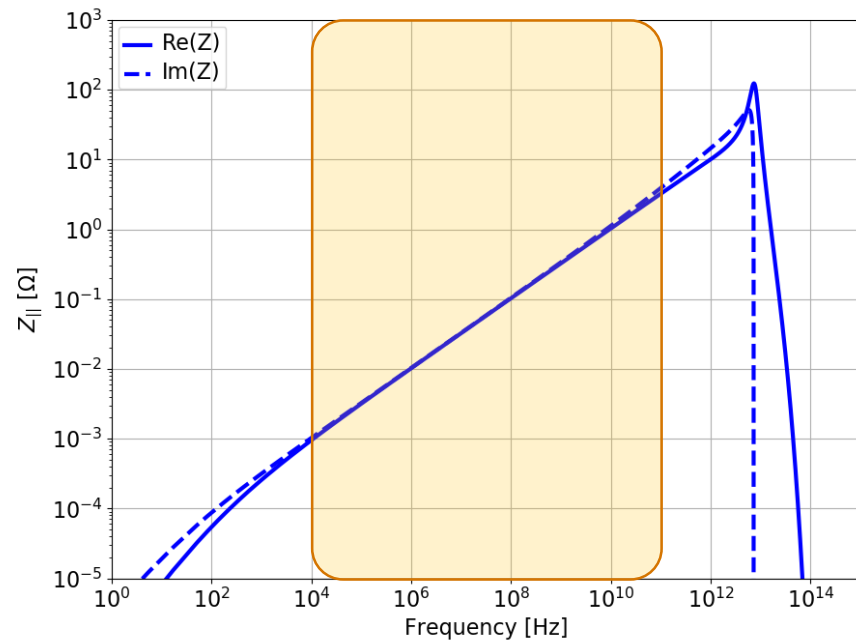
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→ An example: a 1 m long Cu pipe with radius $b=2$ cm and thickness $t = 4$ mm in vacuum



- Highlighted region shows the typical $\omega^{1/2}$ scaling
- Scaling is with respect to b :
 - Longitudinal impedance $\sim b^{-1}$

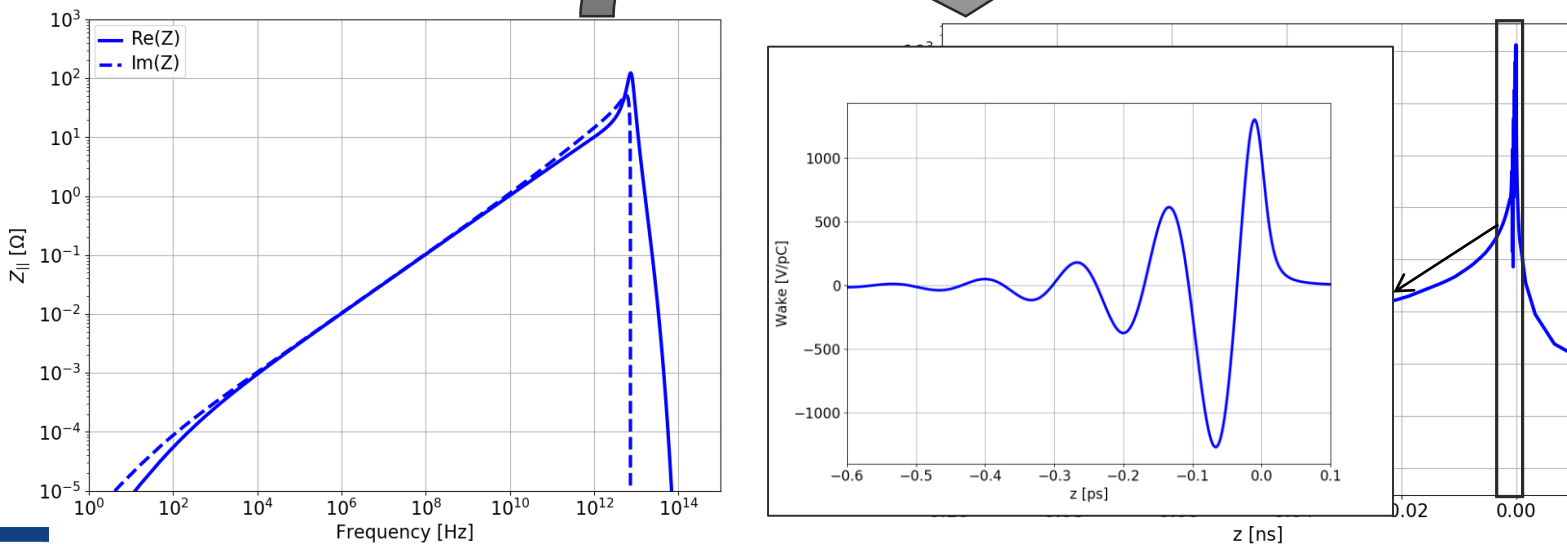


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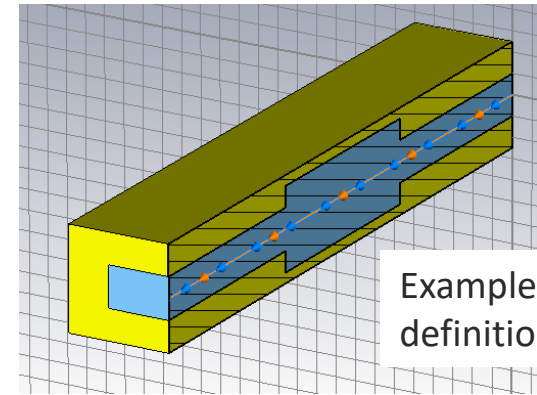
\mathcal{F}^{-1}



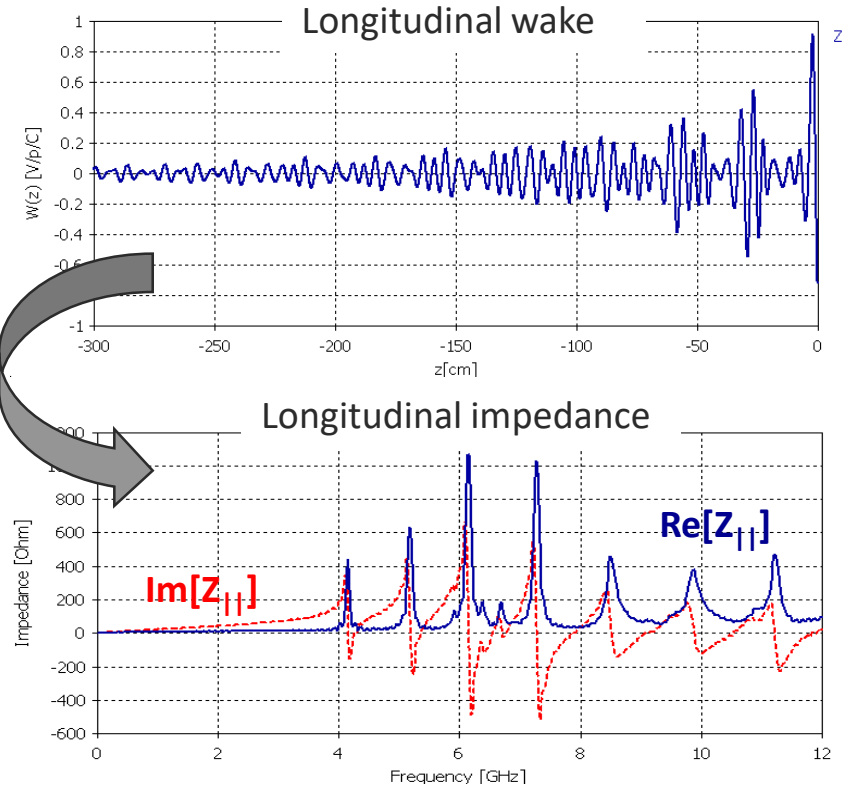
How are wakes and impedances computed?

• Numerical approach

- Different codes have been developed over the years to solve numerically Maxwell's equations in arbitrarily complicated structures
- Examples are CST Studio Suite (Particle Studio, Microwave Studio), ABCI, GdFidL, HFSS, ECHO2(3)D. Exhaustive list can be found from the program of the [ICFA mini-Workshop on "Electromagnetic wake fields and impedances in particle accelerators"](#), Erice, Sicily, 23-28 April, 2014
- Computations can become very **challenging** if high frequency resolution (long wake) or knowledge of impedance spectrum at high frequency (short excitation) are required, especially for large/complicated geometries



Example of a cavity definition in CST

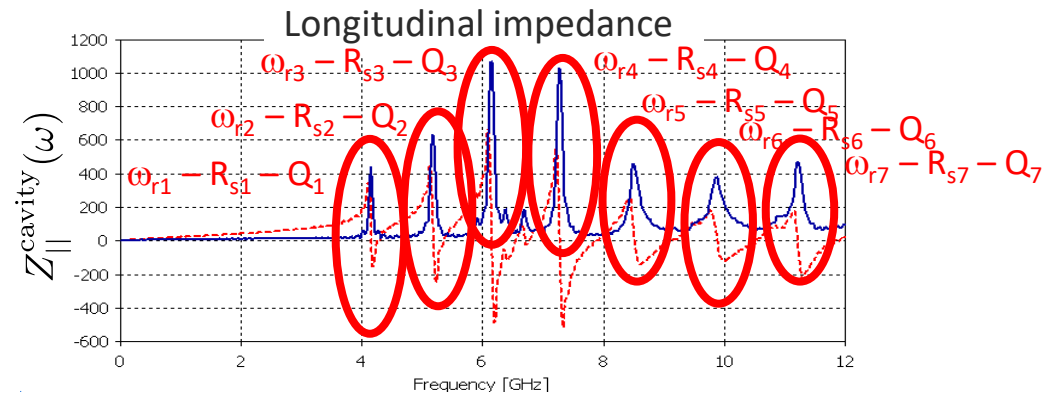


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How are wakes and impedances computed?

Numerical approach

- To limit numerical noise, in cases with many resonances, the resonances are first characterized through their frequencies (ω_{r_i}), shunt impedances (R_{s_i}) and quality factors (Q_i)



- Then analytical formulae for resonators are used in computations

$$Z_{||}^{\text{Res}}(\omega) = \frac{R_{s||}}{1 + iQ \left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega} \right)}$$

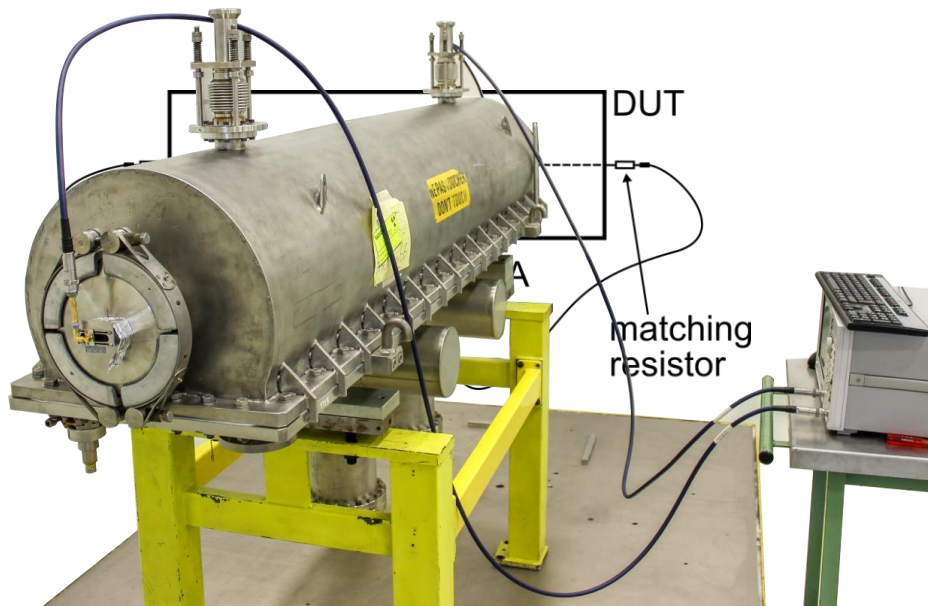
$$W_{||}^{\text{Res}}(z) = \begin{cases} 2\alpha_z R_{s||} \exp\left(\frac{\alpha_z z}{c}\right) \left[\cos\left(\frac{\bar{\omega} z}{c}\right) + \frac{\alpha_z}{\bar{\omega}} \sin\left(\frac{\bar{\omega} z}{c}\right) \right] & \text{if } z < 0 \\ \alpha_z R_{s||} & \text{if } z = 0 \\ 0 & \text{if } z > 0 \end{cases}$$

$$\alpha_z = \frac{\omega_r}{2Q} \quad \bar{\omega} = \sqrt{\omega_r^2 - \alpha_z^2}$$

$$Z_{||}^{\text{cavity}}(\omega) = \sum Z_{||i}^{\text{Res}}(\omega)$$

How are wakes and impedances computed?

- **Bench measurements** based on transmission/reflection measurements with stretched wires
 - Seldom used independently to assess impedances due to the perturbation introduced by the measurement set up (flanging, presence of wire)
 - Usefulness mainly lies in that they can be used for validating 3D EM models for simulations



- A **wire** is stretched in the middle of the device to simulate the beam
- **Reflection and transmission coefficients** are measured via a VNA
- The impedance can be calculated by plugging the measured scattering parameters into the **LOG formula**

$$Z_{||} = 2Z_L \ln(S_{21})$$

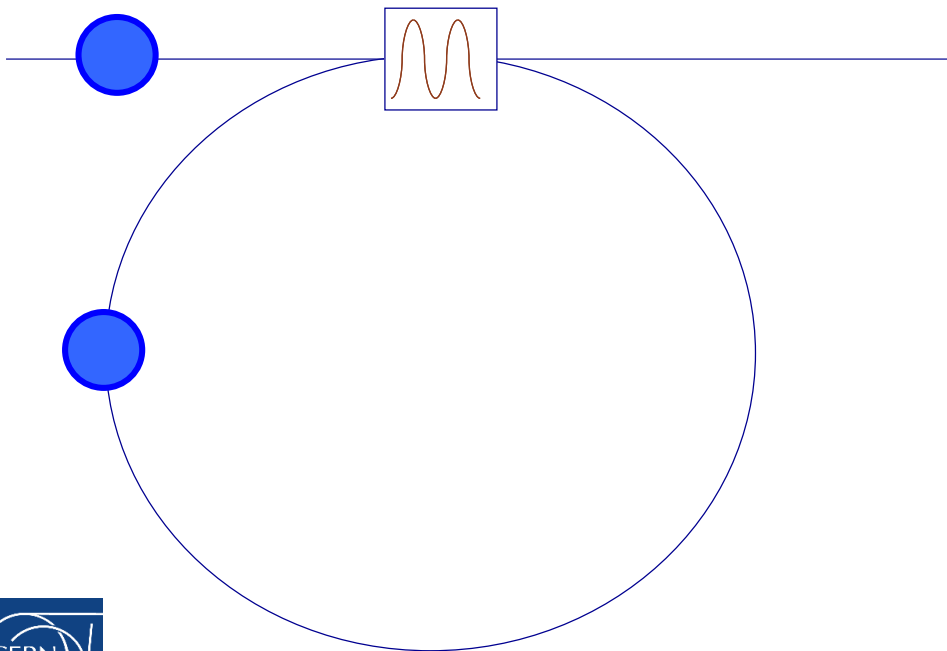
- We have learnt what is a wake function and how it is defined in the **longitudinal plane**. We have introduced the **longitudinal impedance**
- We have seen how longitudinal wake functions are related to the **energy loss** of the source particles.
- We have discussed the **energy balance** which contains all the **fundamental underlying mechanisms** for collective effects related to wake fields and impedances.
- We have shown how wake functions and impedances **can be computed**

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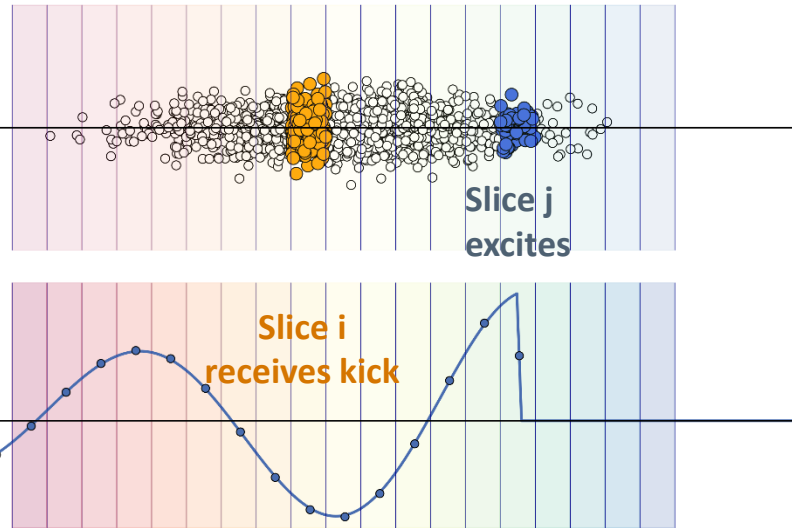
Bunch energy loss per turn

- Single traversal of a bunch through an impedance source
 - We assume a single bunch of particles that goes only once through a known (characterized) wake/impedance source, representing both
 - Single passage (e.g. in a line)
 - Energy loss per turn if the bunch passes every turn but the wake fully decays between subsequent turns
 - Our goal is to calculate how much energy the bunch loses in this passage due to the electromagnetic interaction



Bunch energy loss per turn

- Single traversal of a bunch through an impedance source



$$\Delta E_{ij} = -e^2 W_{||}(z_{ij})$$

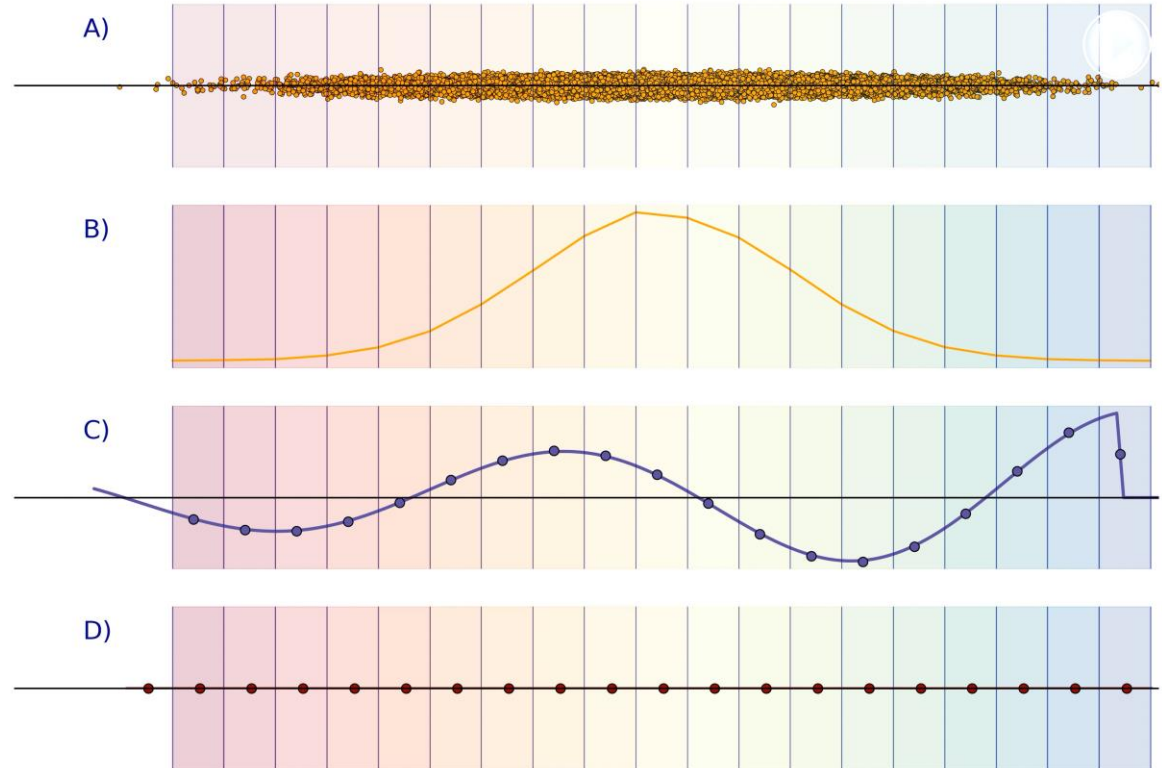
$$\Delta E_{bunch} = -e^2 \sum_{j=1}^{N_b} \sum_{i=1}^{N_b} W_{||}(z_{ij})$$

$$\Delta E_{ij} = -e^2 N[j]N[i]W_{||}[(i-j)\Delta z]$$

$$\Delta E_i = -e^2 N[i] \sum_{j=0}^i N[j]W_{||}[(i-j)\Delta z]$$

Bunch energy loss per turn

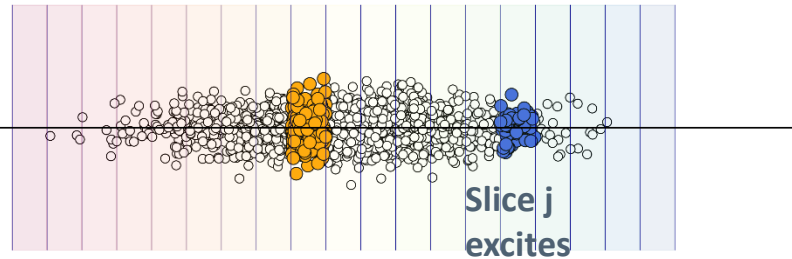
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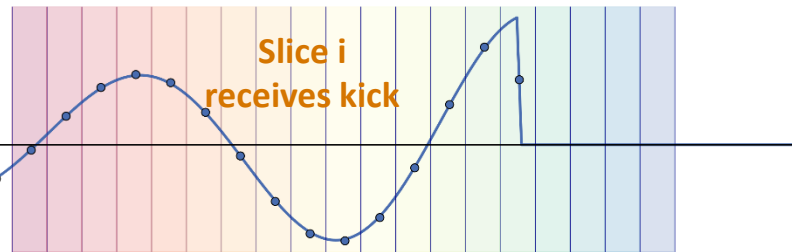
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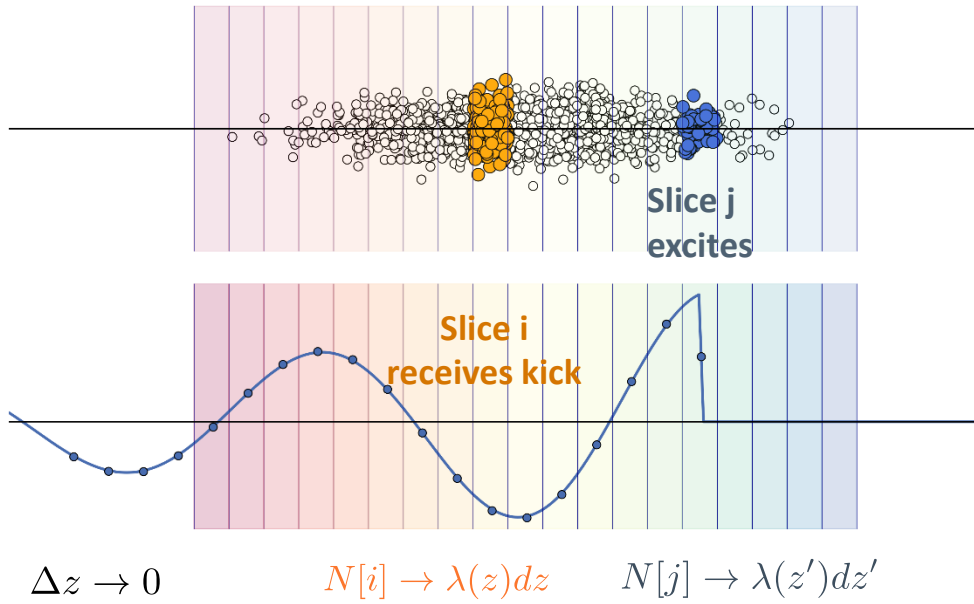
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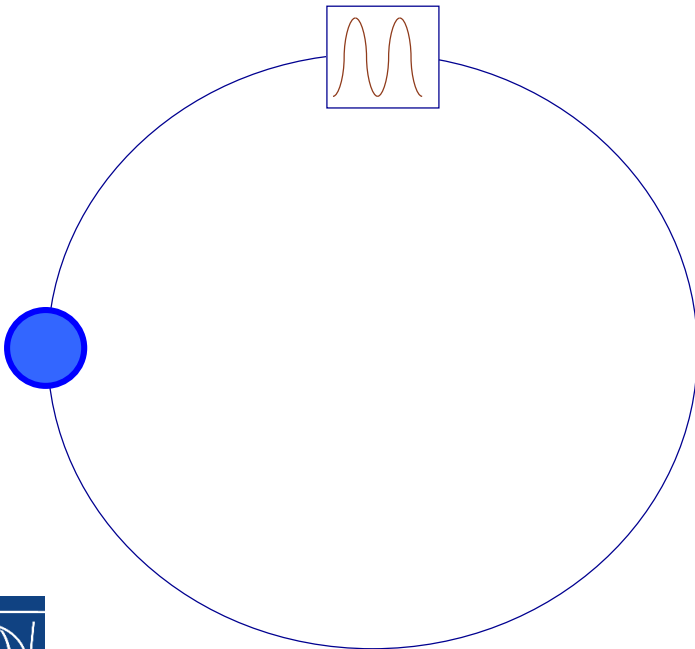
$$\Delta E_i = -e^2 N[i] \sum_{j=0}^i N[j] W_{||}[(i - j)\Delta z]$$

$$\Delta E_{bunch} = -e^2 \int \lambda(z) dz \int \lambda(z') W_{||}(z - z') dz'$$

$$\Delta E_{bunch} = -\frac{e^2}{2\pi} \int \left| \hat{\lambda}(\omega) \right|^2 \text{Re} [Z_{||}(\omega)]$$

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- Multiple traversal of a bunch through an impedance source
 - We assume a single bunch of particles that goes multiple times through a known (characterized) wake/impedance source, representing
 - Energy loss per turn if the bunch passes every turn and the wake fully keeps ringing between subsequent turns
 - Our goal is to calculate how much energy the bunch loses at each passage due to the electromagnetic interaction over several turns



Bunch energy loss per turn

$$\Delta E = -\frac{e^2}{2\pi} \int_{-\infty}^{\infty} \lambda(z) dz \int_{-\infty}^{\infty} dz' \lambda(z') \underbrace{\sum_{k=-\infty}^{\infty} W_{\parallel}(kC + z - z')}_{k=-\infty} dz'$$

$\lambda(z' + kC) = \lambda(z')$, i.e. assuming that the distribution doesn't change from turn to turn

$$\sum_{k=-\infty}^{\infty} W_{\parallel}(kC + z - z') = \frac{c}{C} \sum_{p=-\infty}^{\infty} Z_{\parallel}(p\omega_0) \exp\left[-\frac{ip\omega_0(z - z')}{c}\right]$$

$$\Delta E = -\frac{e^2\omega_0}{2\pi} \sum_{p=-\infty}^{\infty} Z_{\parallel}(p\omega_0) \underbrace{\int_{-\infty}^{\infty} \lambda(z) \exp\left(\frac{-ip\omega_0 z}{c}\right) dz}_{\hat{\lambda}(p\omega_0)} \underbrace{\int_{-\infty}^{\infty} \lambda(z') \exp\left(\frac{ip\omega_0 z'}{c}\right) dz'}_{\hat{\lambda}^*(p\omega_0)}$$

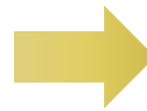
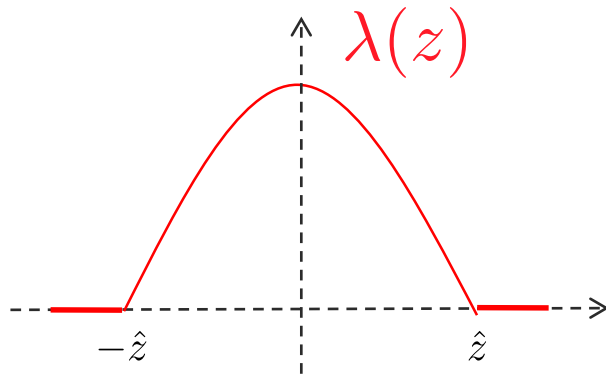
$$\Delta E = -\frac{e^2\omega_0}{2\pi} \sum_{p=-\infty}^{\infty} \left| \hat{\lambda}(p\omega_0) \right|^2 \operatorname{Re} [Z_{\parallel}(p\omega_0)]$$

Beam energy loss per turn

Replacing the **bunch spectrum with the beam spectrum**, we can calculate the energy loss from a beam

Bunch profile and spectrum

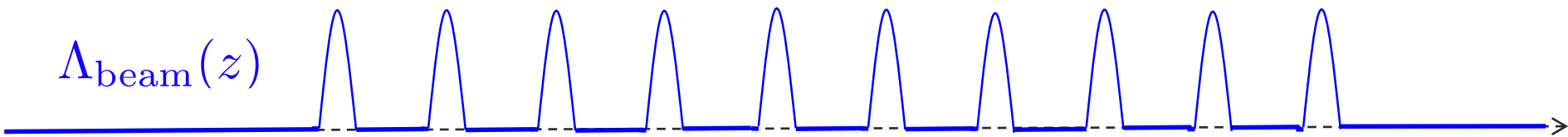
$$\lambda(z)$$



Beam profile and spectrum

$$\Lambda_{\text{beam}}(z)$$

$$\Lambda_{\text{beam}}(z)$$

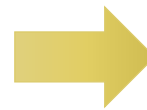


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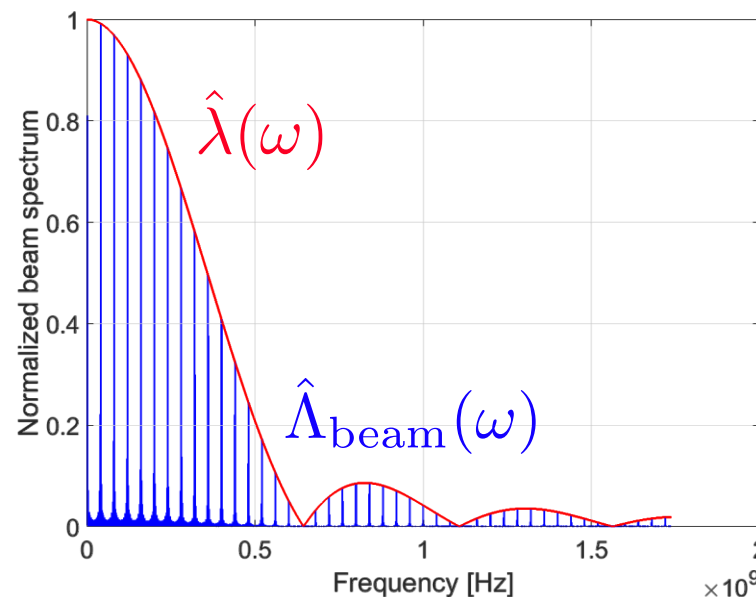
$$\lambda(z) \leftrightarrow \hat{\lambda}(\omega)$$



Beam profile and spectrum

$$\Lambda_{\text{beam}}(z)$$

Ex. parabolic, as shown in the previous slide



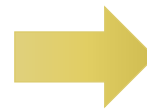
$$\Delta E_{\text{beam}} = -\frac{e^2 \omega_0}{2\pi} \sum_{p=-\infty}^{+\infty} |\Lambda_{\text{beam}}(p\omega_0)|^2 \text{Re} [Z_{||}(p\omega_0)]$$

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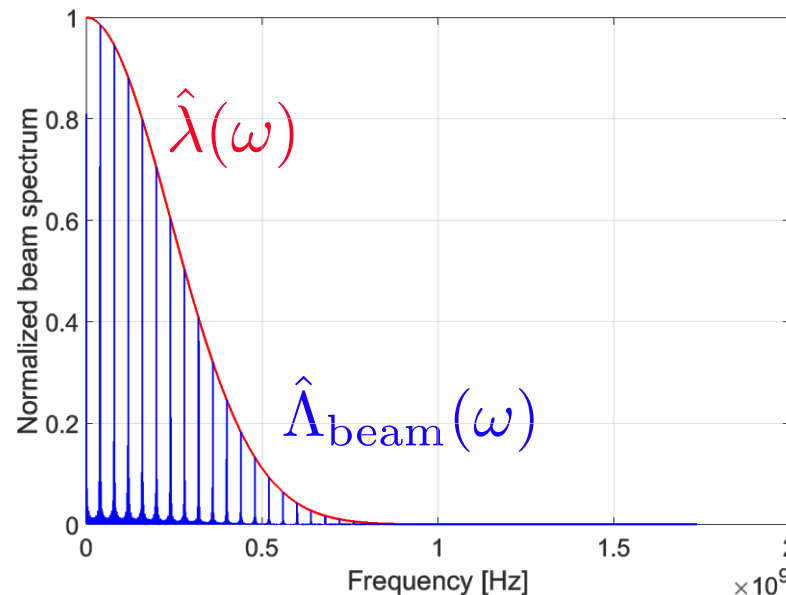
$$\lambda(z) \leftrightarrow \hat{\lambda}(\omega)$$



Beam profile and spectrum

$$\Lambda_{\text{beam}}(z) \leftrightarrow \hat{\Lambda}_{\text{beam}}(\omega)$$

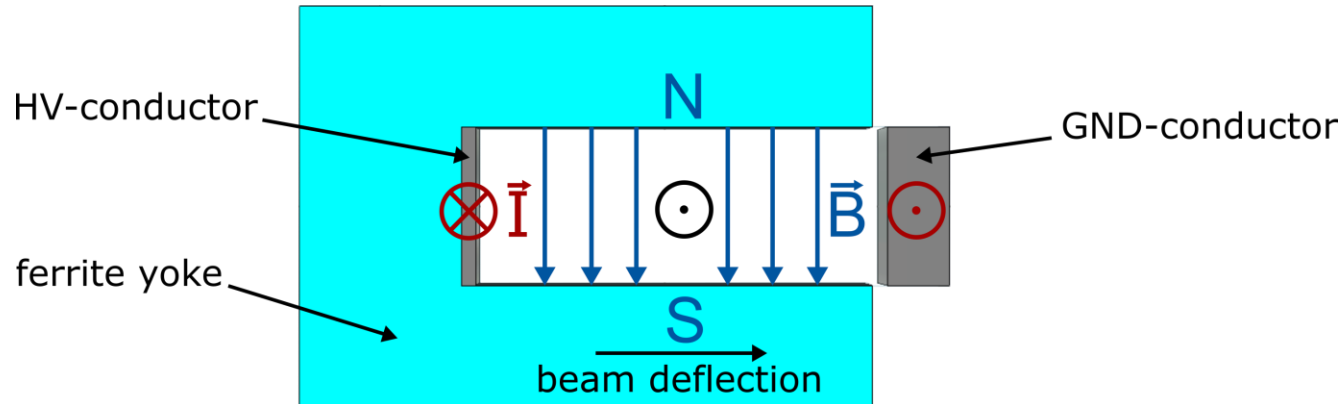
Or for a train of
Gaussian bunches



$$\Delta E_{\text{beam}} = -\frac{e^2 \omega_0}{2\pi} \sum_{p=-\infty}^{+\infty} |\Lambda_{\text{beam}}(p\omega_0)|^2 \text{Re} [Z_{||}(p\omega_0)]$$

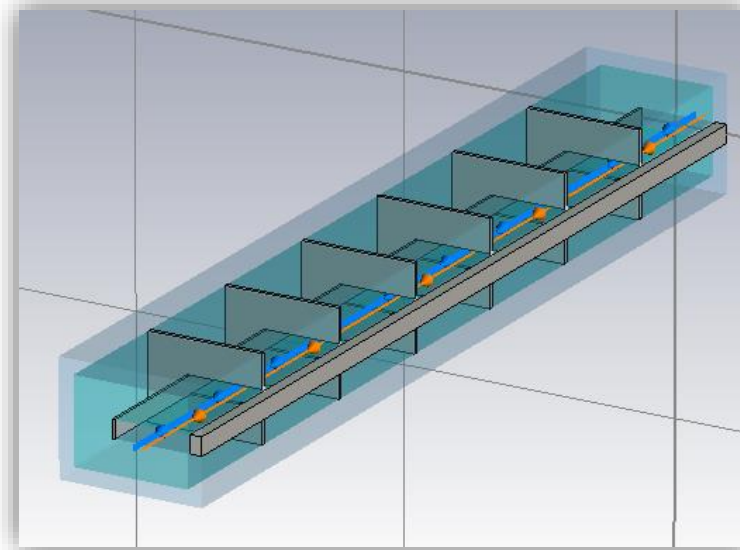
Application to the SPS extraction kickers

- Problem with SPS extraction kickers (MKE)
 - Extraction elements through which the beam passes every turn
 - Based on a fast pulsed magnet capable of deflecting the whole beam over one turn
 - Active only on turn in which beam has to be extracted, otherwise passive but with all its elements (ferrite, conductors) exposed to the beam



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 - Active only on turn in which beam has to be extracted, otherwise passive but with all its elements (ferrite, conductors) exposed to the beam
 - Use of beam for LHC filling (4x 200-ns spaced trains of 72x 25-ns spaced bunches) led to unacceptable heating of these elements)
 - Heating above Curie temperature leads to ferrite degradation → Beam cannot be extracted anymore from the SPS
 - Heating causes outgassing and strong pressure rise in the kicker sector, with consequent beam interlocking due to poor vacuum



Application to the SPS extraction kickers

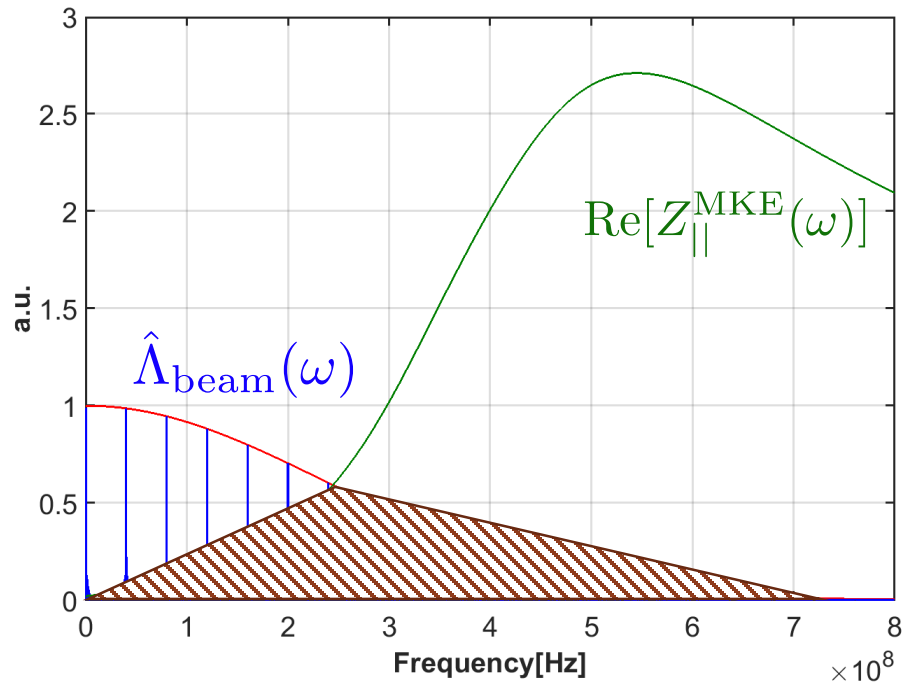
- We need to calculate the power loss in the kicker
 - Kicker impedance can be evaluated semi-analytically or via simulations
 - Then we apply the energy loss formula

$$\Delta E_{\text{beam}} = -\frac{e^2 \omega_0}{2\pi} \sum_{p=-\infty}^{+\infty} |\Lambda_{\text{beam}}(p\omega_0)|^2 \text{Re} [Z_{||}(p\omega_0)]$$

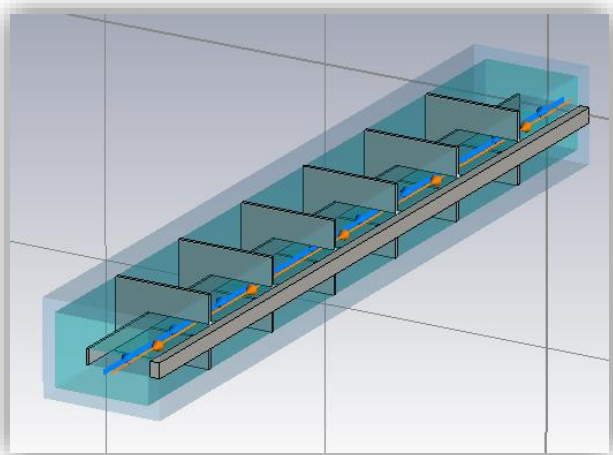
$$\Delta W = \frac{\Delta E_{\text{beam}}}{T_0}$$

Application to the SPS extraction kickers

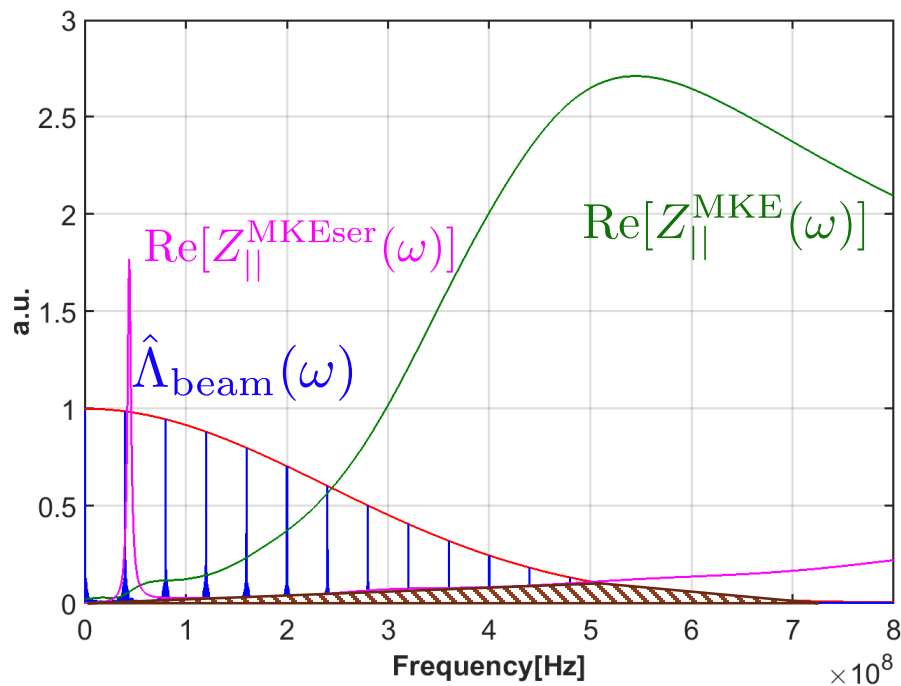
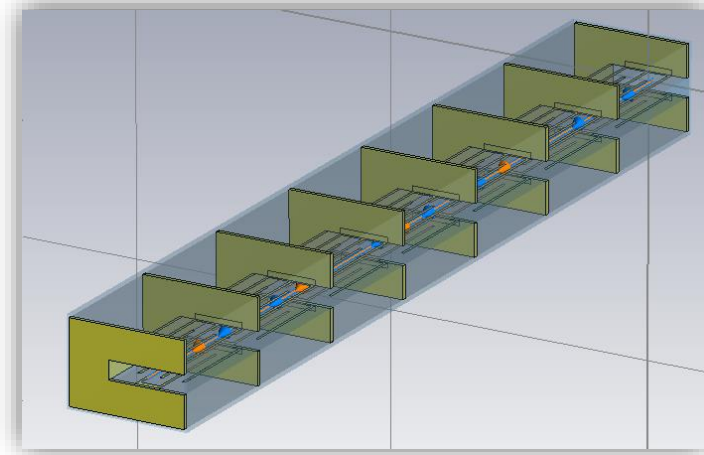
- We need to calculate the power loss in the kicker
 - Kicker impedance can be evaluated semi-analytically or via simulations
 - Then we apply the energy loss formula
- Kicker impedance already becomes significant at frequencies for which the beam spectrum has not fully decayed, causing the undesired heating
- We need to lower the kicker impedance \rightarrow Impedance dominated by losses in ferrite \rightarrow Ferrite shielding



Application to the SPS extraction kickers

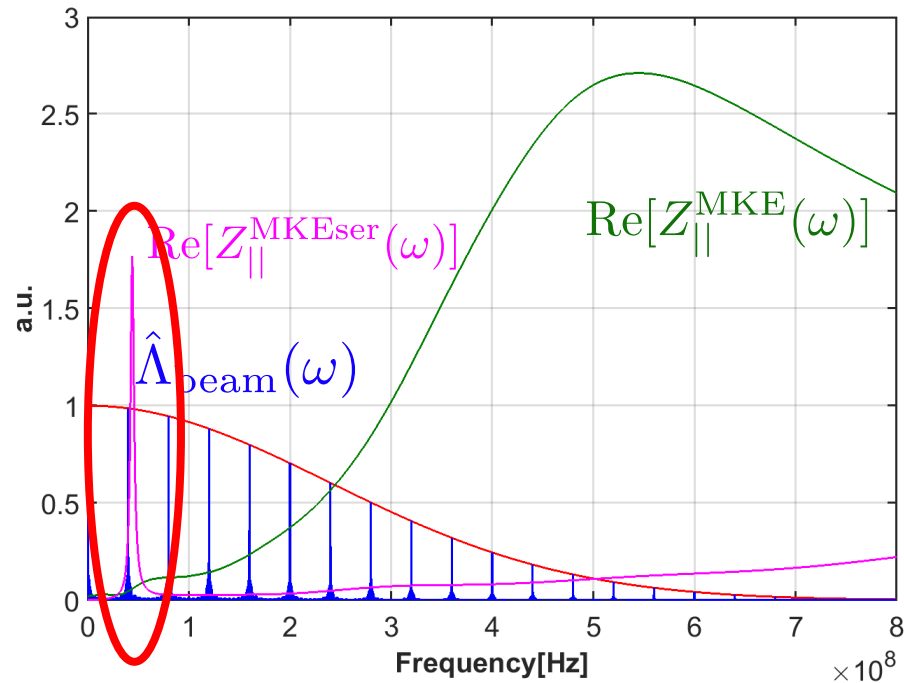


Print striped pattern of good conductor on ferrite (serigraphy)



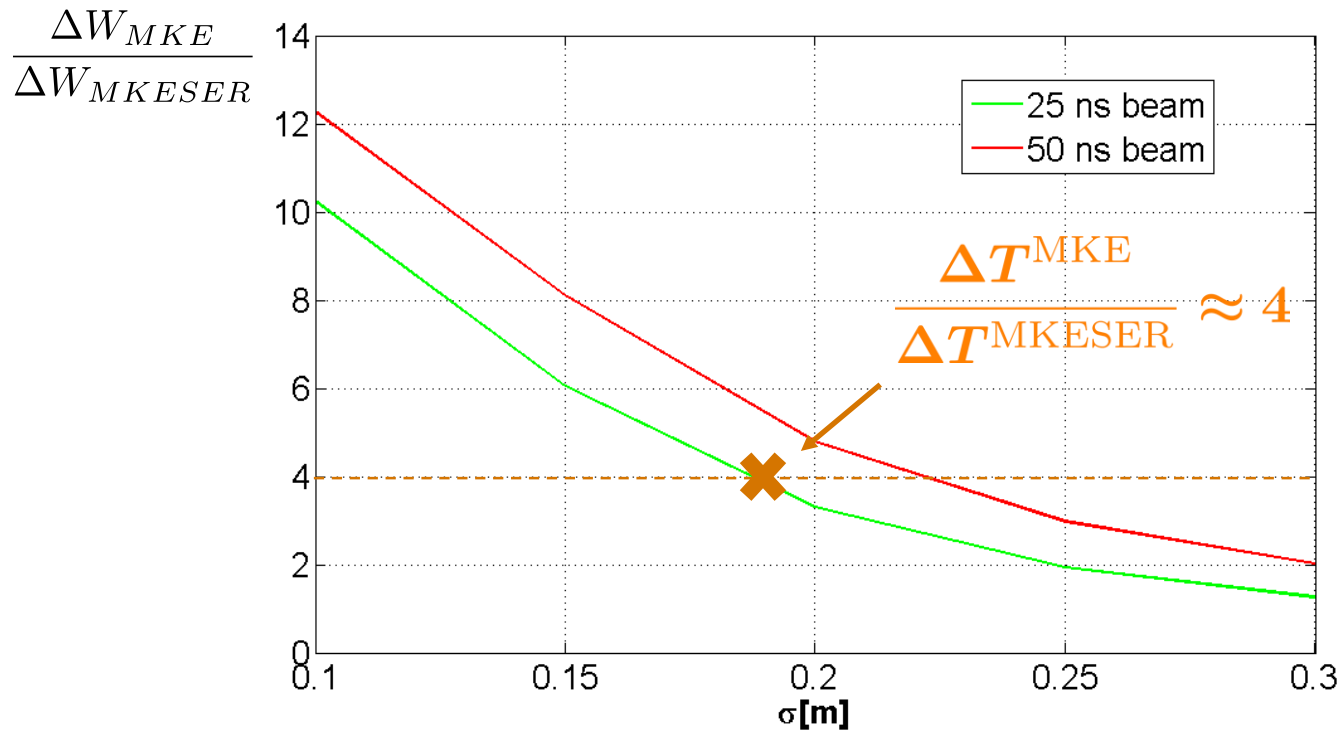
Application to the SPS extraction kickers

- This almost suppresses the impedance over the bunch spectrum
- It however introduces a low frequency peak, which needs to be kept far from beam spectral lines
 - Pay attention to do that for all needed bunch spacings



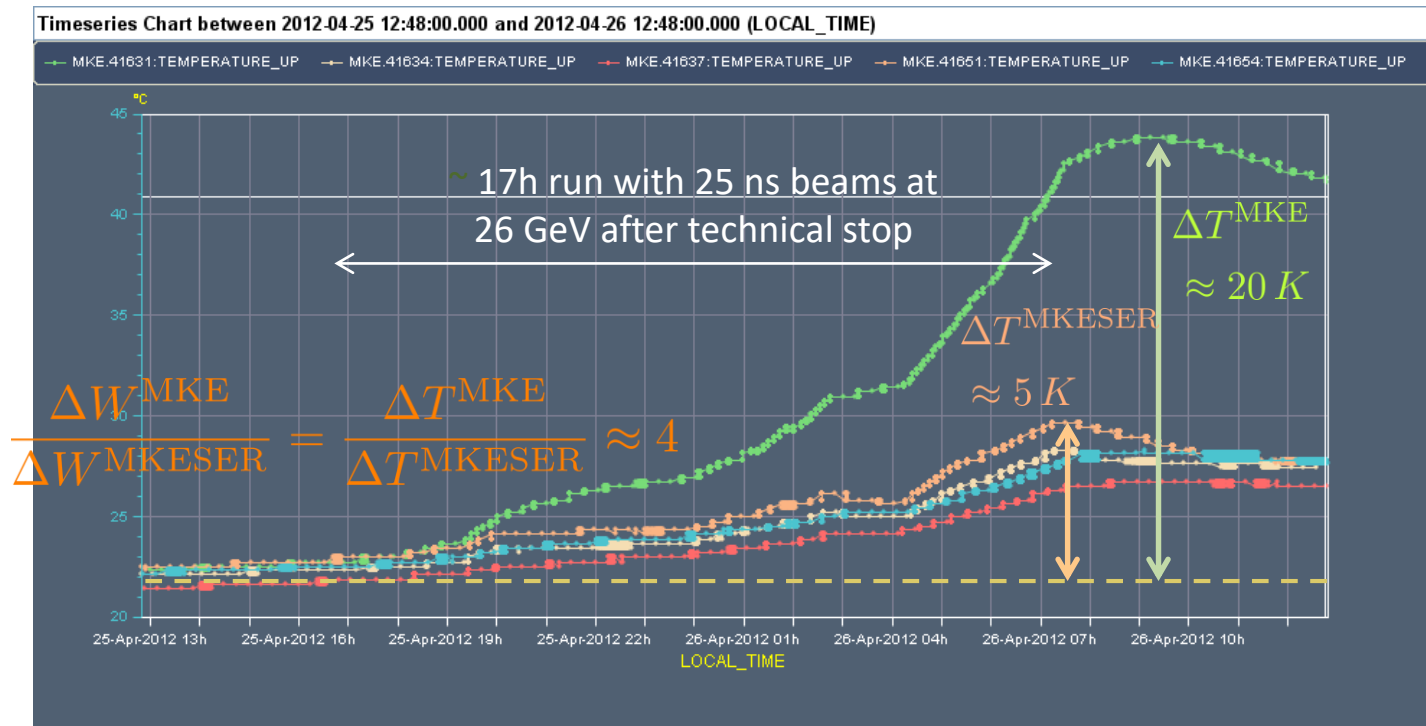
Application to the SPS extraction kickers

- This almost suppresses the impedance over the bunch spectrum
- It however introduces a low frequency peak, which needs to be kept far from beam spectral lines
- Expected up to 12-fold reduction of heat load, depending on bunch length and bunch spacing
 - Factor 4 for 25-ns LHC-type beam at 26 GeV



Application to the SPS extraction kickers

- This almost suppresses the impedance over the bunch spectrum
- It however introduces a low frequency peak, which needs to be kept far from beam spectral lines
- Expected up to 12-fold reduction of heat load, depending on bunch length and bunch spacing
 - Factor 4 for 25-ns LHC-type beam at 26 GeV → Experimentally measured!



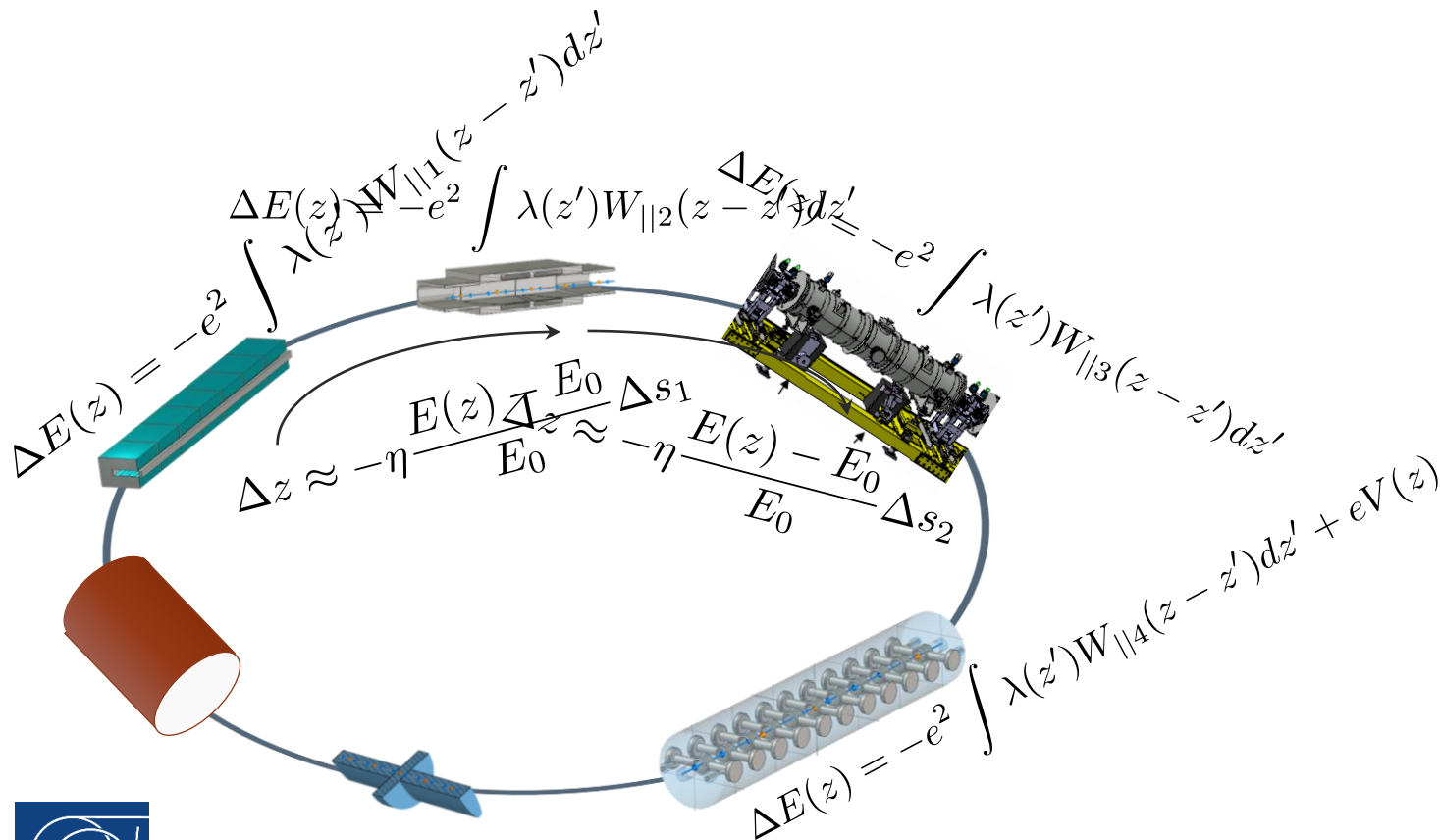
- We have further looked into the mechanism of energy loss and have seen the **impact of longitudinal impedances on machine elements** as these lead to **beam induced heating**.
- We have found that beam induced heating depends on the overlap of the **beam power spectrum** and the **impedance** of a given object.
- We have seen a **real world example** of the impact of an objects impedance on the beam induced heating.

Part 2: Multiparticle dynamics with wake fields – impact on machine elements and longitudinal beam dynamics

- General introduction to wake fields
- Longitudinal wake fields and the longitudinal wake function
- Energy loss – beam induced heating and stable phase shift
- Potential well distortion, bunch lengthening and microwave instability

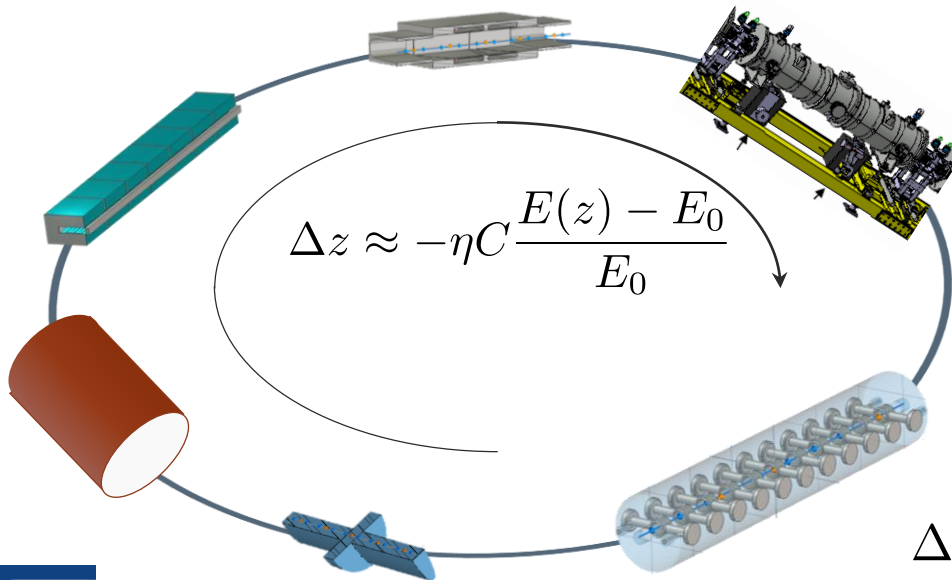
Longitudinal wakes in beam dynamics

- The effect of each localised wake/impedance on each particle in a beam can be described as an energy kick
- The accelerator is made of many components, each giving a small kick to the beam particles, which drift freely in between



Longitudinal wakes in beam dynamics

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- The accelerator is made of many components, each giving a small kick to the beam particles, which drift freely in between
- For simulations, the impedance is lumped in one place and kicks to beam particles are applied once per turn, with free drift over one turn



$$W_{\parallel}^{\text{Ring}}(z) = \sum W_{\parallel i}(z)$$

$$Z_{\parallel}^{\text{Ring}}(\omega) = \sum Z_{\parallel i}(\omega)$$

$$\Delta E(z) = -e^2 \int \lambda(z') W_{\parallel}^{\text{Ring}}(z - z') dz'$$

- The effect of each localised wake/impedance on each particle in a beam can be described as an energy kick
- The accelerator is made of many components, each giving a small kick to the beam particles, which drift freely in between
- For simulations, the impedance is lumped in one place and kicks to beam particles are applied once per turn, with free drift over one turn
- For analytical calculations, both global impedance and RF are smeared over the ring

$$\begin{cases} \frac{dz}{ds} = -\eta\delta \\ \frac{d\delta}{ds} = \frac{e}{m_0\gamma cC} \left[V_{\text{rf}}(z) - e \sum_k \int \lambda(z' + kC) W_{\parallel}^{\text{Ring}}(z - z' - kC) dz' \right] \end{cases}$$

$$H = -\frac{1}{2}\eta\delta^2 + \frac{e}{\beta^2 EC} U_{\text{rf}}(z) + \frac{e^2}{\beta^2 EC} \int_{-\infty}^z dz'' \sum_k \int \lambda(z' + kC) W_{\parallel}^{\text{Ring}}(z'' - z' - kC) dz'$$

- For a bunch under the effect of longitudinal wake fields, two different regimes can be found:
 - Regime of **potential well distortion**, i.e. due to the impedance a new equilibrium distribution can be found for the bunch
 - Stable phase shift
 - Synchrotron frequency shift
 - Different matching (→ bunch lengthening for lepton machines)
 - Regime of **longitudinal instability**, i.e. no equilibrium distribution can be found under the effect of the impedance, a perturbation grows exponentially
 - Dipole mode instabilities
 - Coupled bunch instabilities
 - Microwave instability (longitudinal mode coupling)

- The **equilibrium distribution** in the presence of a longitudinal wake field can be found analytically. The (linearized) **longitudinal Hamiltonian** with longitudinal wake fields is given as:

$$H = -\frac{1}{2}\eta\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \frac{e^2}{\beta^2 EC} \sum_k \int_0^z dz'' \int_{z''}^{\infty} dz' \lambda(z'+kC)W_{\parallel}(z''-z'-kC)$$

A simple Taylor expansion in z already qualitatively reveals some of the effects of the longitudinal wake fields onto the beam:

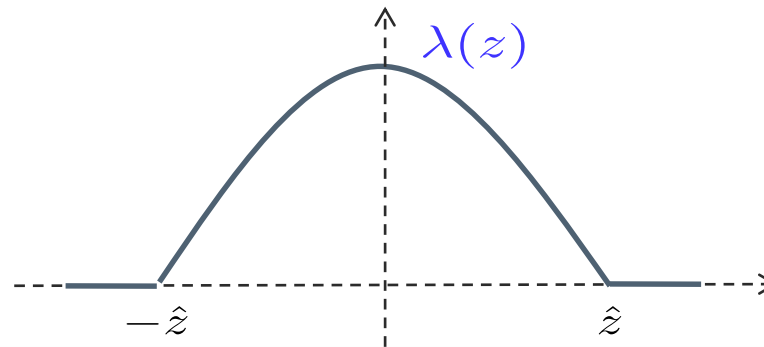
1. First order:
shift in the mean position (**stable phase shift**)
2. Second order:
change in bunch length accompanied by an (incoherent) **synchrotron tune shift**

- The equilibrium (matched) line charge density is then given by the self-consistency equation (**Haissinki equation**):

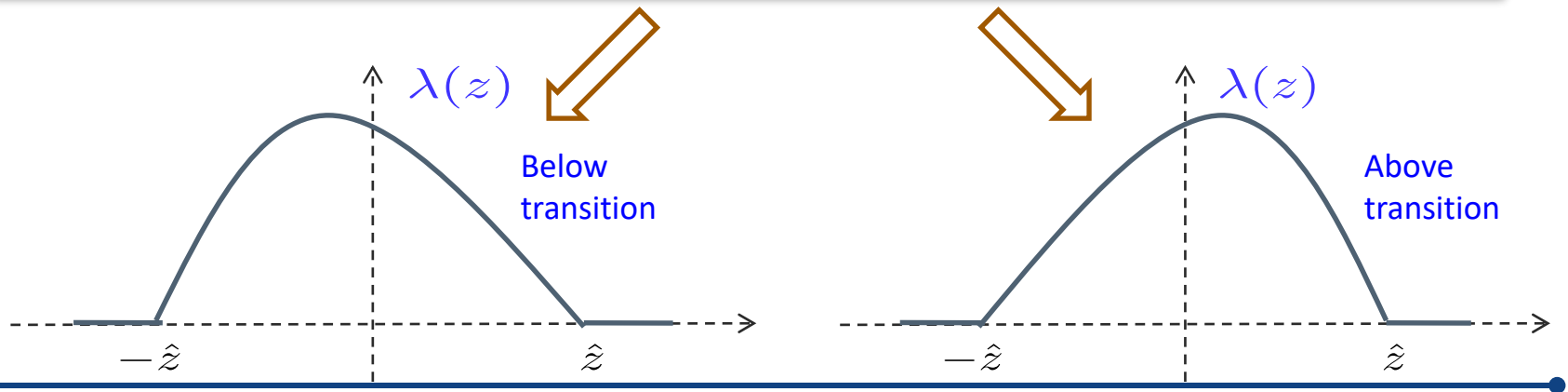
$$\lambda(z) = A \exp\left(-\frac{1}{2}\left(\frac{\omega_s z}{\eta\sigma_\delta\beta c}\right)^2 + \frac{e^2}{\eta\sigma_\delta^2\beta^2 EC} \sum_k \int_0^z dz'' \int_{z''}^{\infty} dz' \lambda(z'+kC)W_{\parallel}(z''-z'-kC)\right)$$

Bunch energy loss per turn and stable phase

- The **RF system compensates** for the energy loss by imparting a net acceleration to the bunch
- Therefore, the bunch readjusts to a **new equilibrium distribution** in the bucket and moves to an average synchronous angle $\Delta\Phi_s$

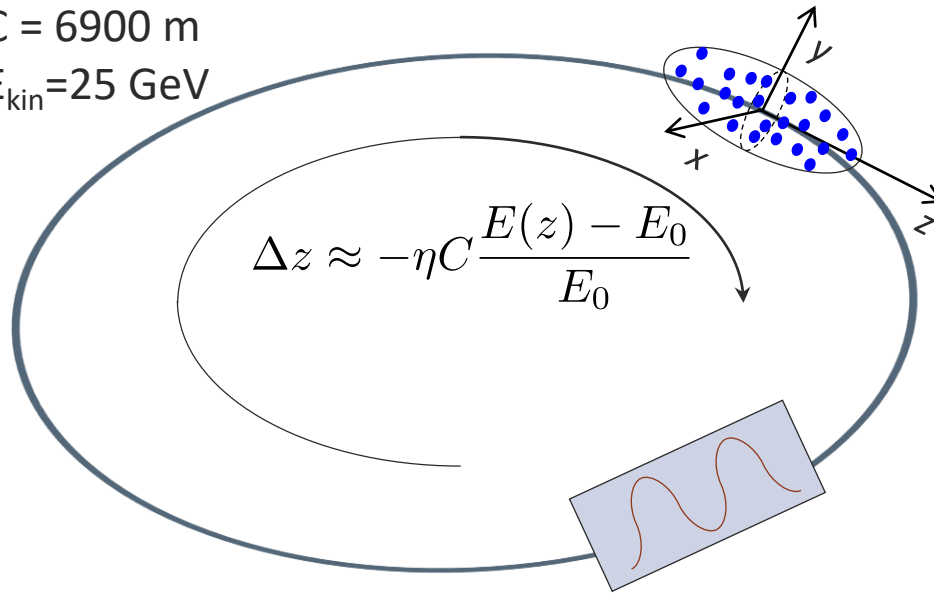


$$\sin \Delta\Phi_s = \frac{\Delta E_{\text{turn}}}{NeV_m} = -\frac{e\omega_0}{2\pi N V_m} \sum_{p=-\infty}^{\infty} \left| \hat{\lambda}(p\omega_0) \right|^2 \text{Re} (Z_{\parallel}(p\omega_0))$$



Bunch lengthening and μW instability

SPS ring
 $C = 6900 \text{ m}$
 $E_{\text{kin}} = 25 \text{ GeV}$



Single Gaussian bunch
 $\sigma_z = 0.2 \text{ m (0.67 ns)}$

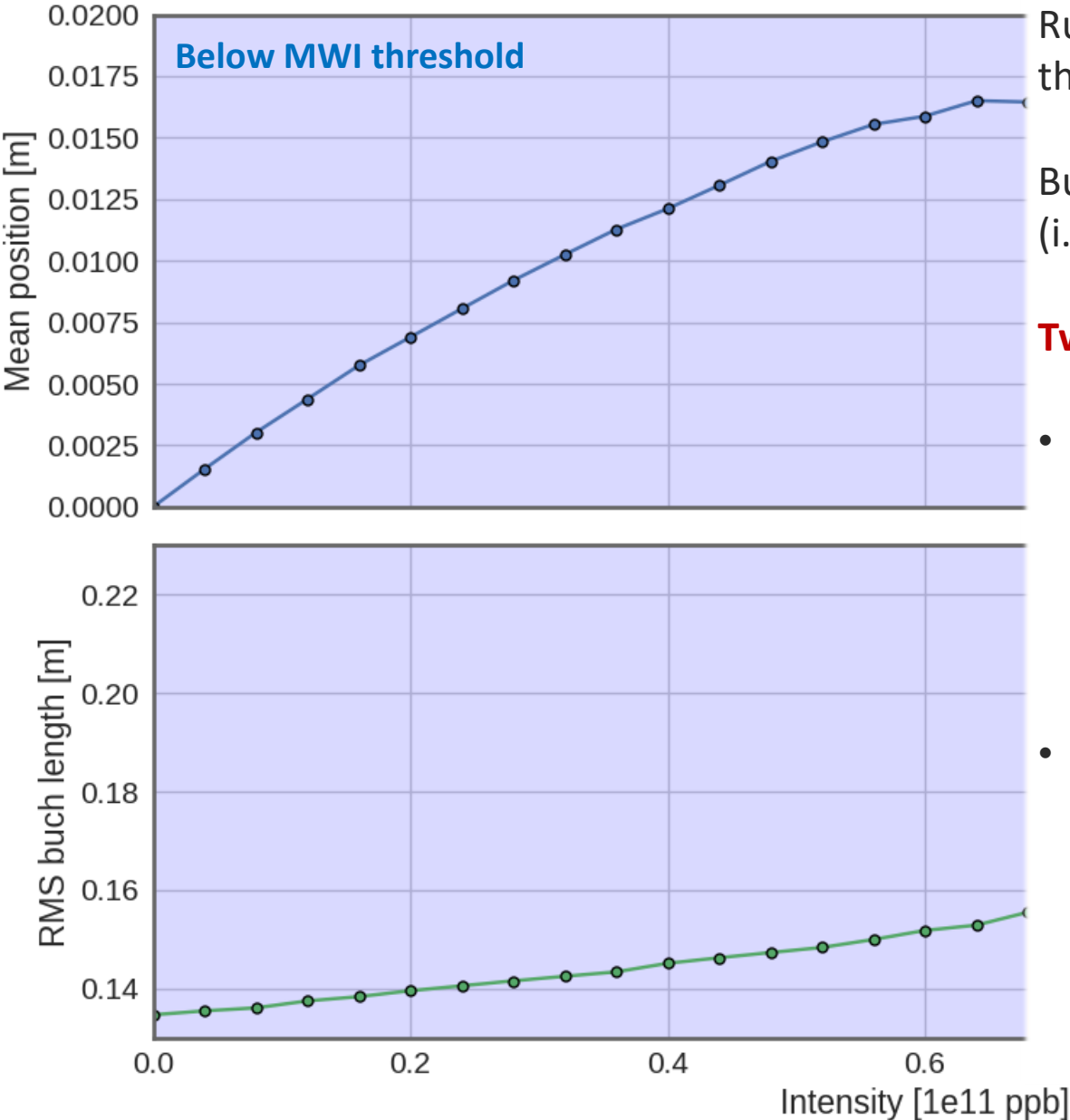
Ring impedance modeled as
 broad band resonator with
 $\omega_r = 700 \text{ MHz}$
 $Q=1$
 $R_s =$

Single RF system
 $\omega_{\text{rf}} = 200 \text{ MHz}$
 $V_{\text{rf}}^{\text{max}} = 3 \text{ MV}$

$$\Delta E(z) = -e^2 \int \lambda(z') W_{||}^{\text{Res}}(z - z') dz' + eV_{\text{rf}}(z)$$

$$Z_{||}^{\text{Res}}(\omega) = \frac{R_{s||}}{1 + iQ \left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega} \right)}$$

Bunch lengthening and μW instability



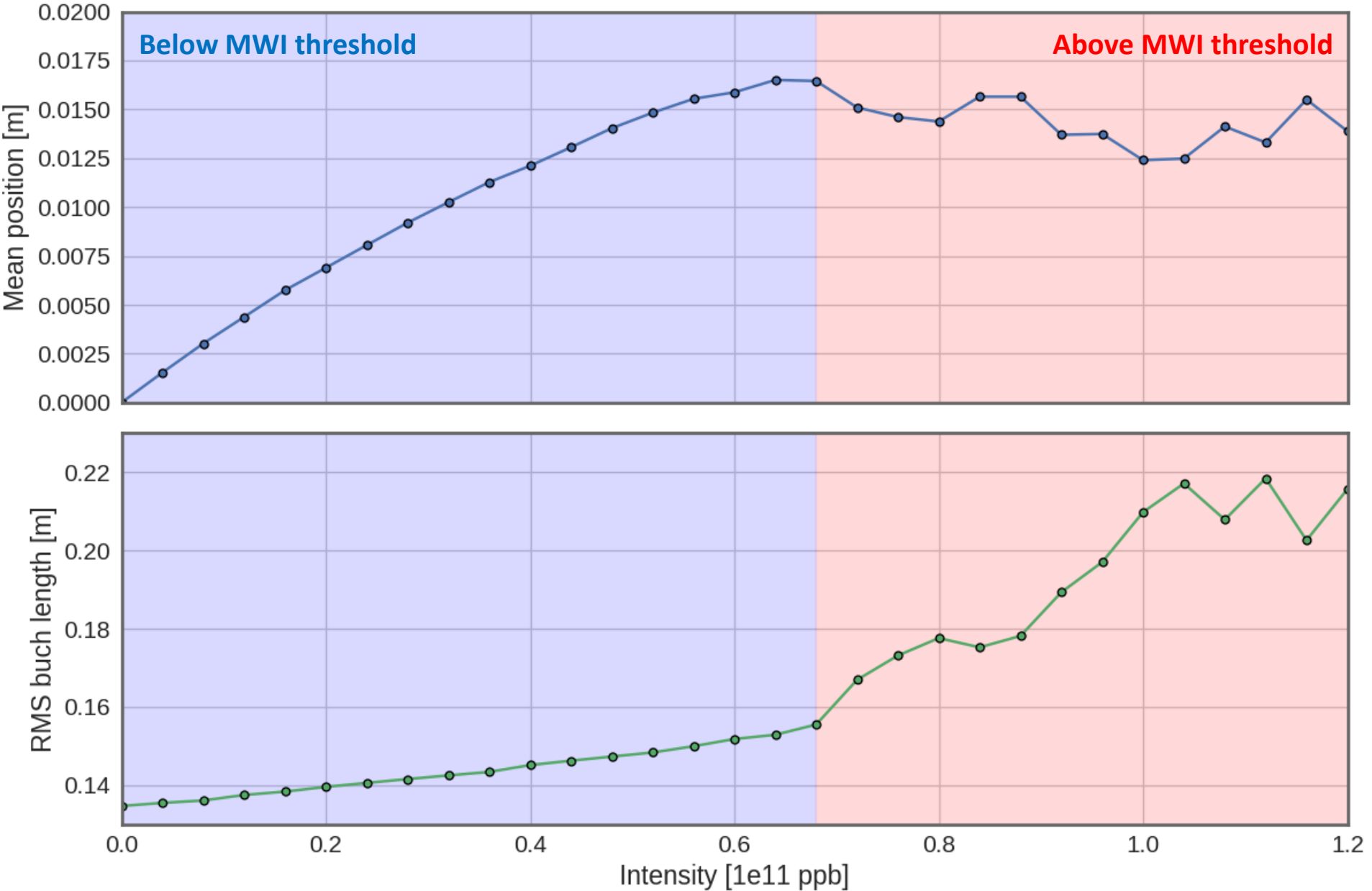
Running the numerical simulation for this case:

Bunch is matched at low intensity (i.e. without impedance)

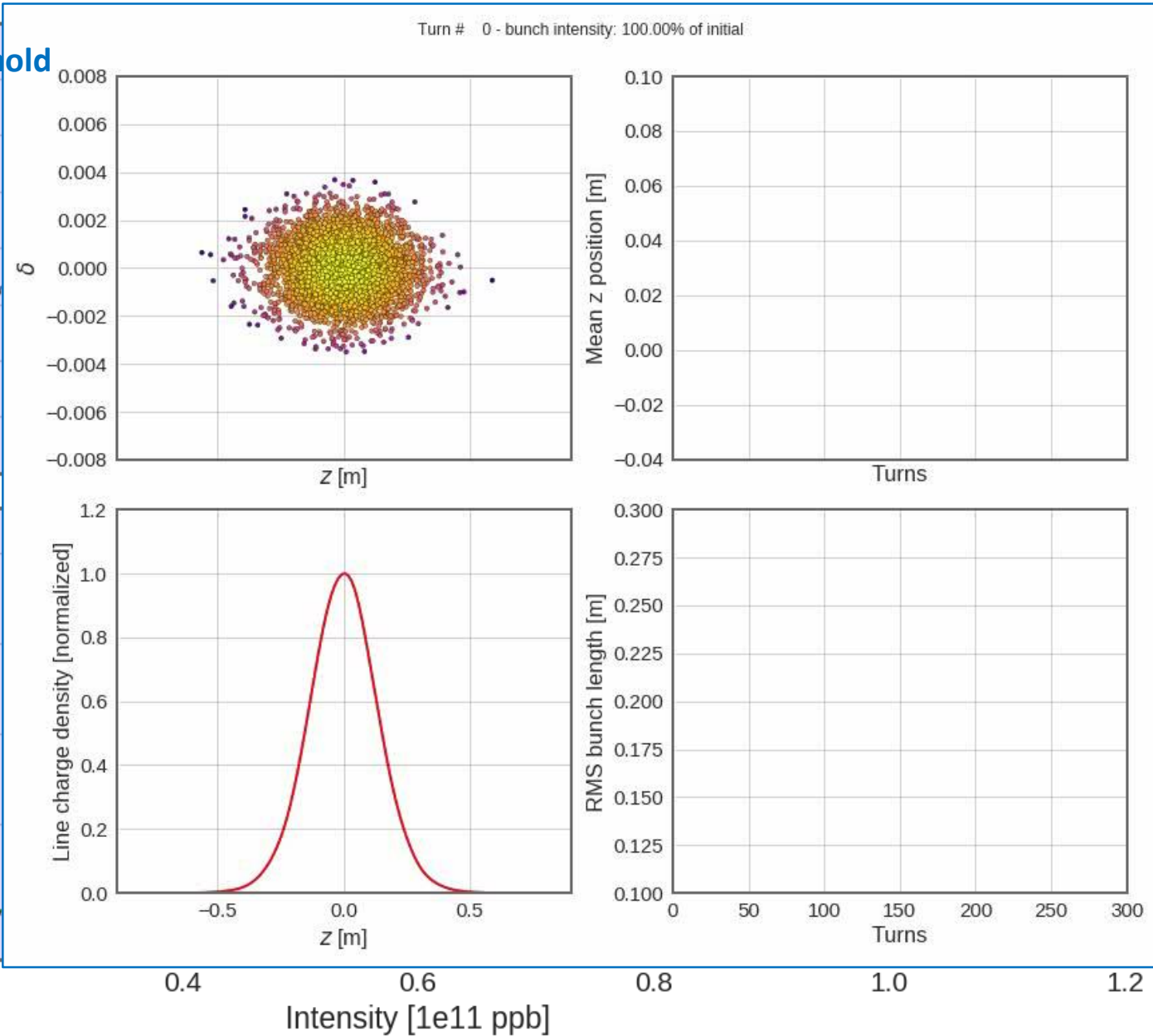
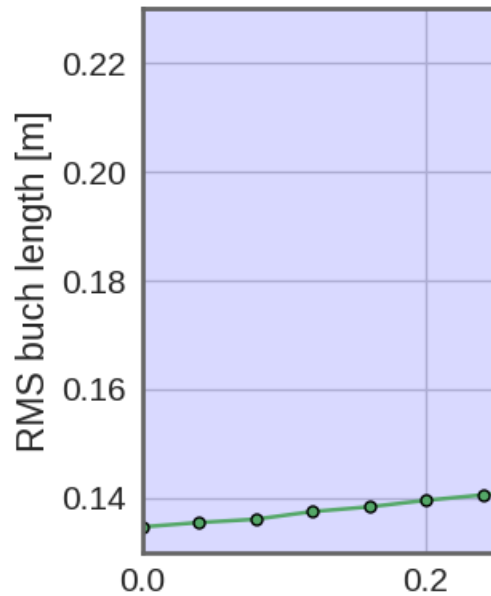
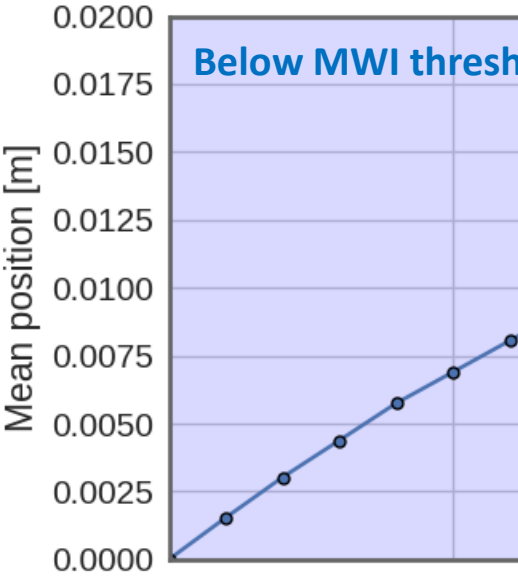
Two regimes are found:

- Bunch lengthening/emittance blow up regime with roughly linear increase of the **synchronous phase** and **bunch length** with intensity
- Unstable regime (**turbulent bunch lengthening**)

Bunch lengthening and μW instability

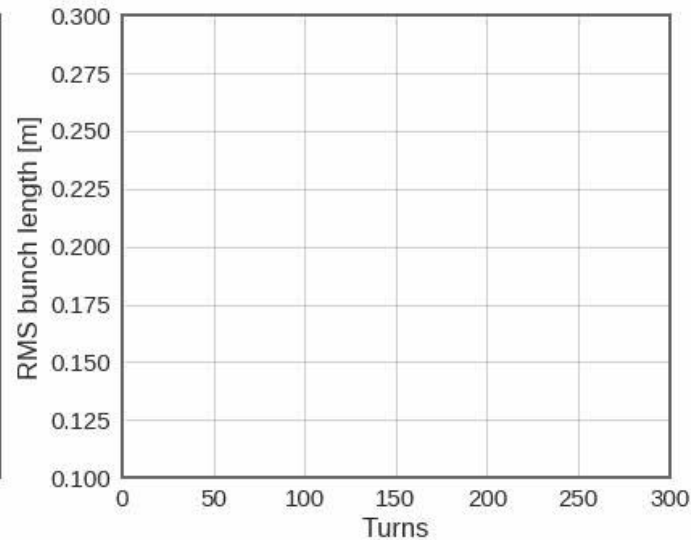
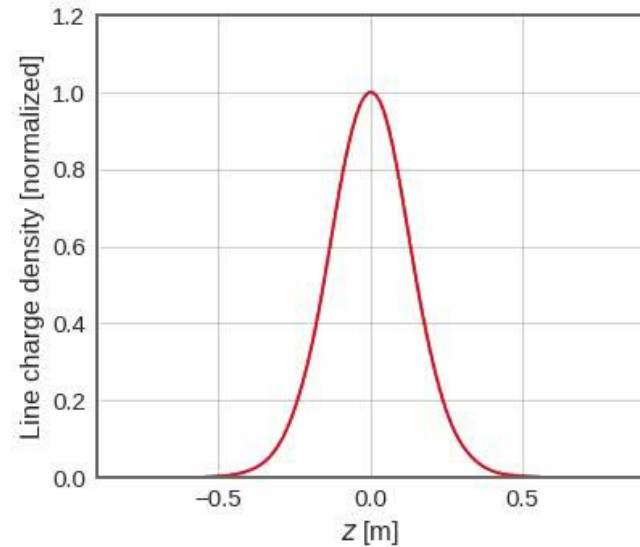
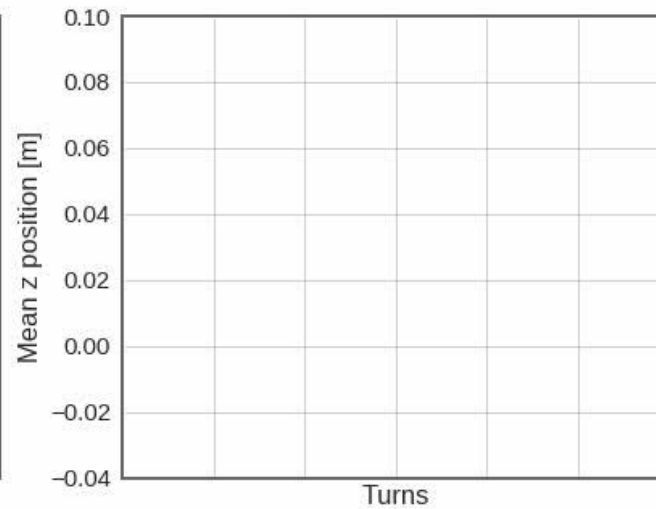
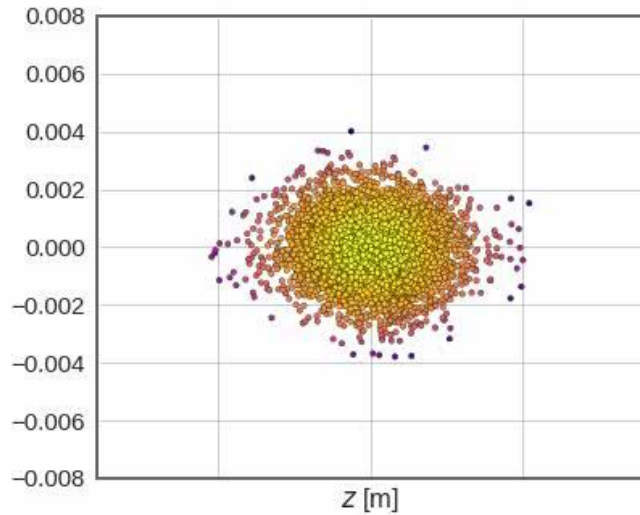


Bunch lengthening and μW instability

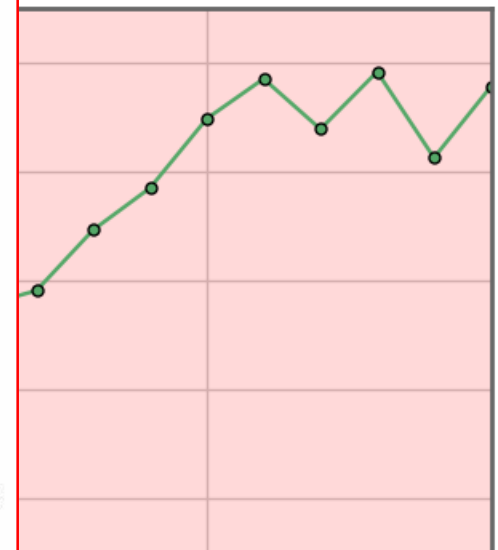
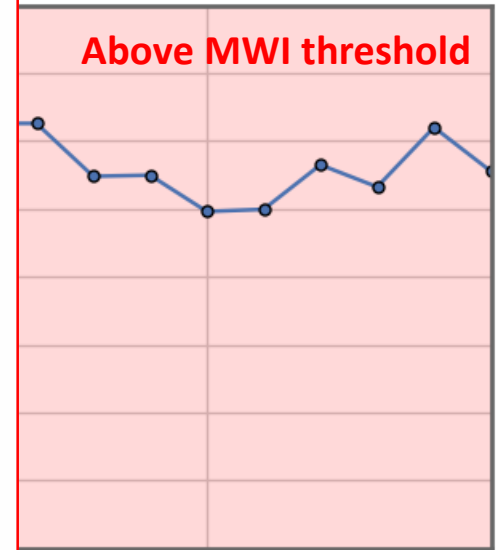


Bunch lengthening and μW instability

Turn # 0 - bunch intensity: 100.00% of initial



Above MWI threshold



0.0

0.2

0.4

0.6

0.8

1.0

1.2

Intensity [1e11 ppb]

- We have **discussed longitudinal wake fields** and impedances and examples of their impact on both the machine as well as the beam.
- We have learned about **beam induced heating** and how it is related to the beam power spectrum and the machine impedance.
- We have discussed the effects of **potential well distortion** (stable phase and synchrotron tune shifts, bunch lengthening and shortening).
- We have seen one example of **longitudinal instability** (microwave).

Tomorrow Part 3

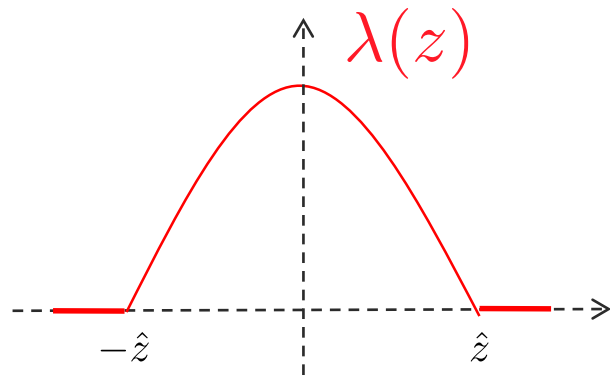
→ Transverse wake fields and impedances and their effects on the beam

End part 2

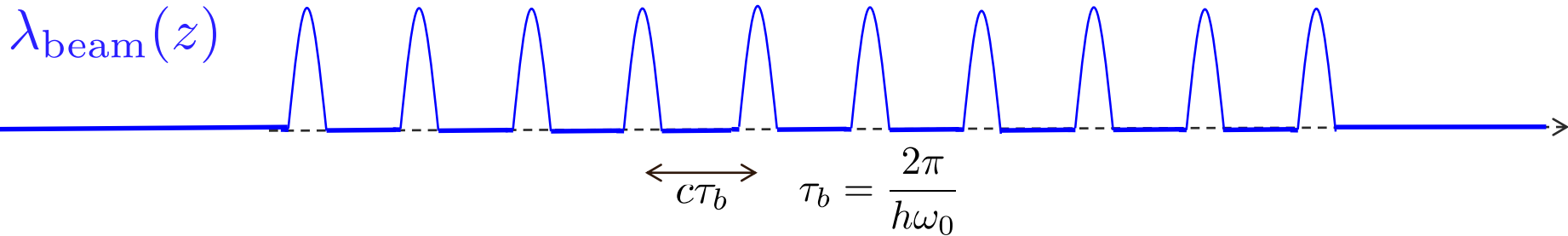


Backup - wakefields

Energy loss of a train of M identical bunches

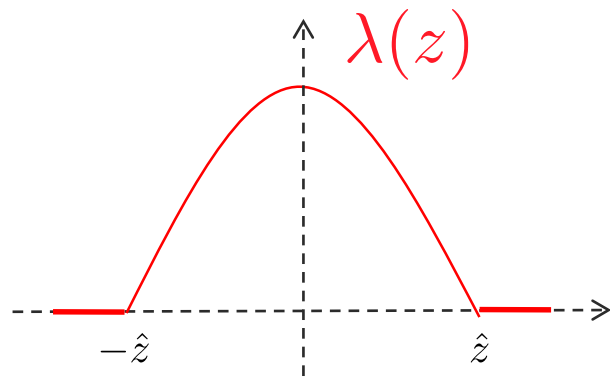


A train of M identical equally spaced bunches circulating in a ring

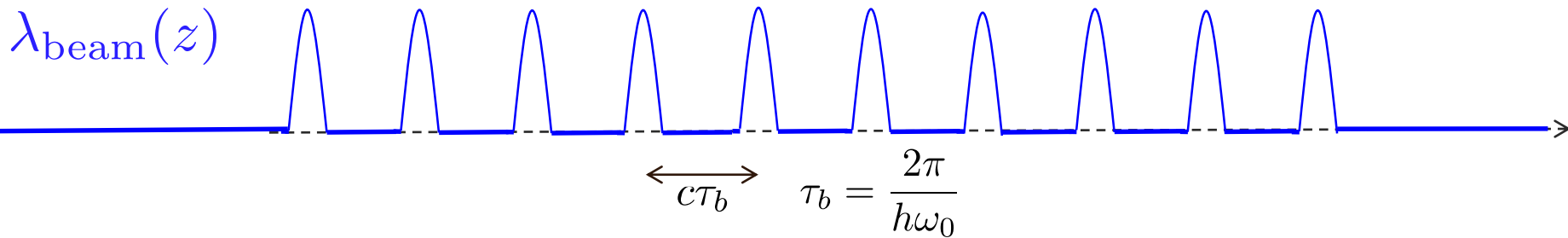


$$\lambda_{\text{beam}}(z) = \sum_{n=0}^{M-1} \lambda(z - n c \tau_b) \quad \stackrel{\mathcal{F}}{\iff} \quad \Lambda_{\text{beam}}(\omega) = \hat{\lambda}(\omega) \sum_{n=0}^{M-1} \exp(-i n \omega \tau_b)$$

Energy loss of a train of M identical bunches



A train of M identical equally spaced bunches circulating in a ring



$$\Delta E_{\text{beam}} = \frac{e^2 \omega_0}{2\pi} \sum_{p=-\infty}^{+\infty} \left| \hat{\lambda}(p\omega_0) \right|^2 \text{Re} [Z_{\parallel}(p\omega_0)] \cdot \left[\frac{1 - \cos\left(\frac{2\pi M p}{h}\right)}{1 - \cos\left(\frac{2\pi p}{h}\right)} \right]$$

Energy loss of a train of M identical bunches

$$\Delta E_{\text{beam}} = \frac{e^2 \omega_0}{2\pi} \sum_{p=-\infty}^{+\infty} \left| \hat{\lambda}(p\omega_0) \right|^2 \text{Re} [Z_{||}(p\omega_0)] \cdot \left[\frac{1 - \cos \left(\frac{2\pi M p}{h} \right)}{1 - \cos \left(\frac{2\pi p}{h} \right)} \right]$$

- The potential leading terms in the summation are those with $p = k \cdot h$, as the ratio in brackets tends to M^2 .
- Narrow-band impedances peaked around multiples of the harmonic number of the accelerator are **the most efficient to drain energy** from the beam → beam induced heating, instabilities.
- This type of impedances, usually **associated to the RF systems** and their higher order modes (HOMs), **need mitigation** in the accelerator design (e.g. detuners, HOM absorbers).

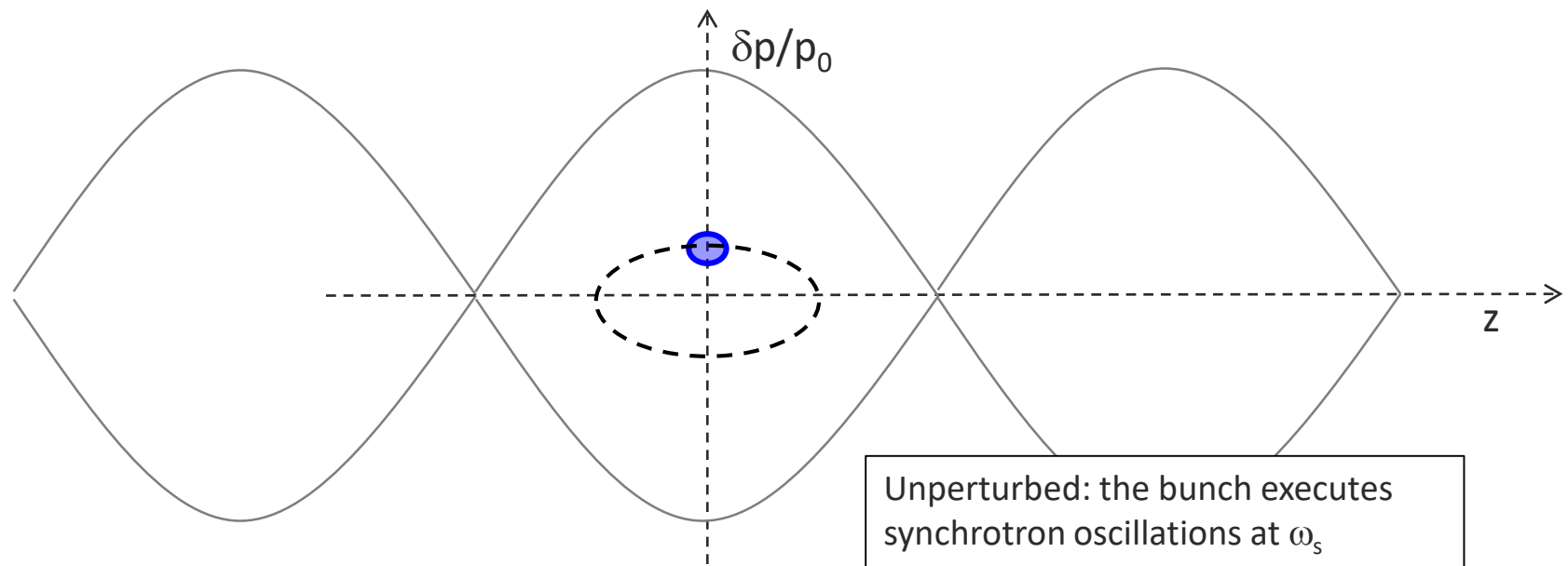
- We have learned about the **impact of the longitudinal impedance on the beam**.
- We found the **Haissinki equation** and discussed the **potential well distortion** along with the **stable phase shift** and **synchrotron tune shift**.
- We looked at some generic wake fields and the **two regimes** of potential well distortion with **bunch lengthening** and its transition to the **microwave instability**.
- We will now look specifically at multi-turn wake fields and the phenomenon of **Robinson instability** and damping.

Part 2: Longitudinal wakefields – impact on machine elements and beam dynamics

- Longitudinal wake fields and the longitudinal wake function
- Energy loss – beam induced heating and stable phase shift
- Potential well distortion, bunch lengthening and microwave instability
- Robinson instability

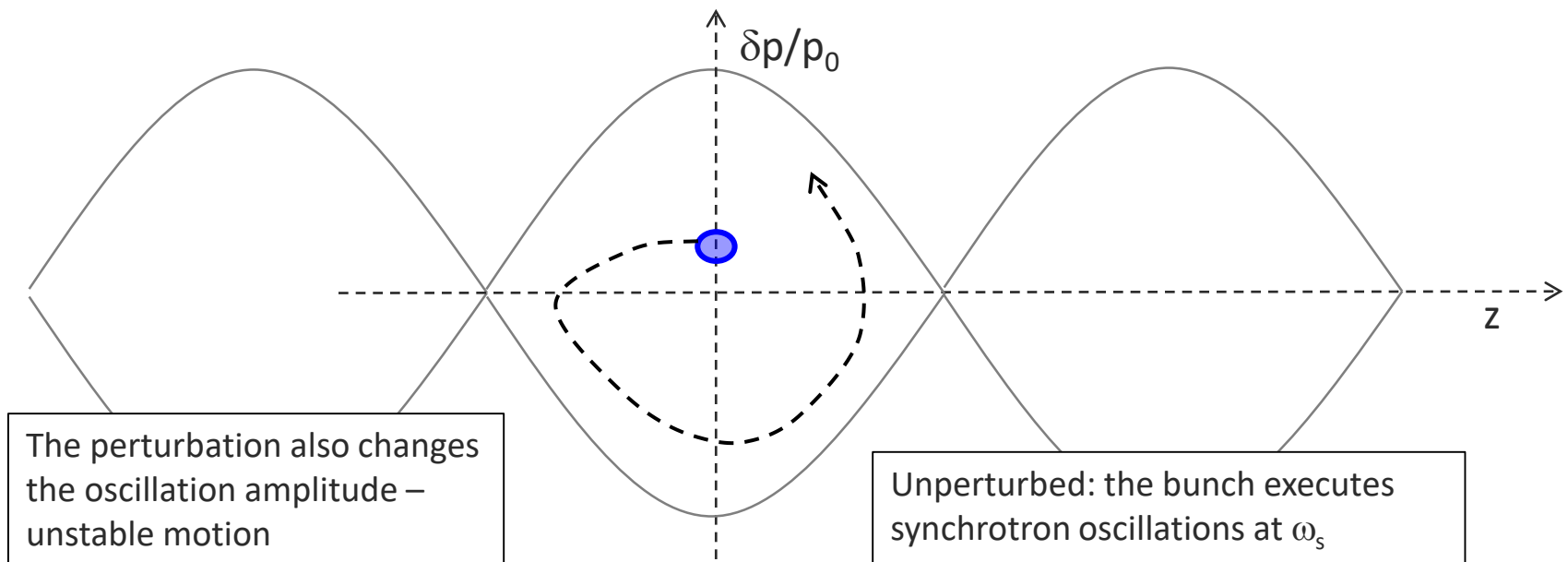
The Robinson instability

- To illustrate the Robinson instability we will use some simplifications:
 - The bunch is **point-like** and feels an external linear force (i.e. it would execute linear synchrotron oscillations in absence of the wake forces)
 - The bunch additionally feels the effect of a **multi-turn wake**



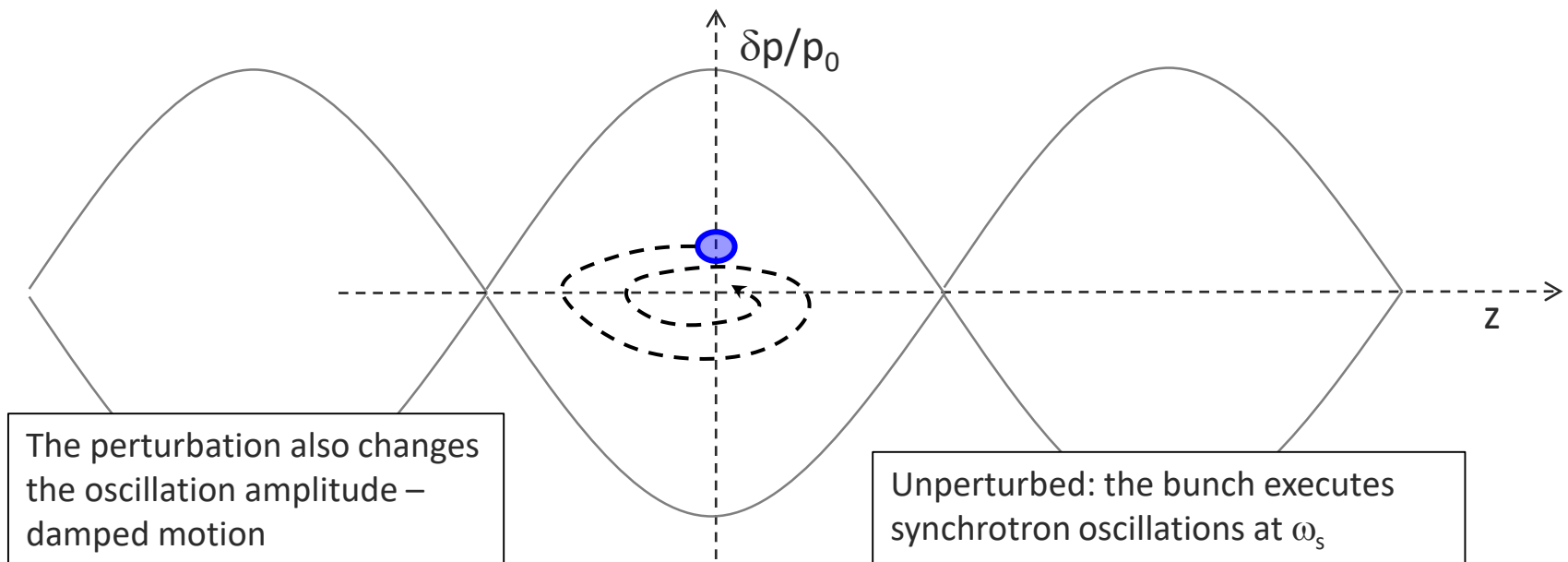
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 - The bunch is **point-like** and feels an external linear force (i.e. it would execute linear synchrotron oscillations in absence of the wake forces)
 - The bunch additionally feels the effect of a **multi-turn wake**
- Longitudinal Hamiltonian

$$\begin{aligned} H &= -\frac{1}{2}\eta\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \frac{e^2}{\beta^2 EC} \sum_k \int_0^z dz'' \int_{z''}^{\infty} dz' \lambda(z' + kC) W_{\parallel}(z'' - z' - kC) \\ &= -\frac{1}{2}\eta\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \frac{Ne^2}{\beta^2 EC} \sum_k \int_0^z dz'' W_{\parallel}(z(t) - z(t - kT_0) - kC) \end{aligned}$$

- Expansion of wake field (we assume that the wake can be linearized on the scale of a synchrotron oscillation)

$$\begin{aligned} W_{\parallel}(z(t) - z(t - kT_0) - kC) &\approx W_{\parallel}(kC) + W'_{\parallel}(kC) \left(z(t) - z(t - kT_0) \right) \\ &\approx W_{\parallel}(kC) + W'_{\parallel}(kC) kT_0 \frac{dz(t)}{dt} \end{aligned}$$

The Robinson instability

- The **first term** only contributes as a constant term in the solution of the equation of motion, i.e. the synchrotron oscillation will be executed around a certain z_0 and not around 0. This term represents the **stable phase shift** that compensates for the energy loss
 - The **second term** is a dynamic term introduced as a **“friction” term** in the equation of the oscillator, which can **lead to instability!**
-
- Equations of motion

$$\frac{d^2 z}{dt^2} + \omega_s^2 z^2 = \frac{Ne^2 \eta}{Cm_0 \gamma} \sum_{k=-\infty}^{\infty} \cancel{W_{\parallel}(kC)} + W'_{\parallel}(kC) kT_0 \frac{dz}{dt}$$

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- Ansatz

$$z(t) \propto \exp(-i\Omega t)$$

$$\frac{i}{C} \sum_{p=-\infty}^{\infty} \left(p\omega_0 Z_{\parallel}(p\omega_0) - (p\omega_0 + \Omega) Z_{\parallel}(p\omega_0 + \Omega) \right)$$

Expressed in terms of impedance

- Solution

$$(\Omega^2 - \omega_s^2) = -\frac{Ne^2 \eta}{Cm_0 \gamma} \sum_{k=-\infty}^{\infty} \left(1 - \exp(-ik\Omega T_0) \right) W'_{\parallel}(kC)$$

The Robinson instability

- We assume a small deviation from the synchrotron tune:
 - $\text{Re}(\Omega - \omega_s) \rightarrow$ **Synchrotron tune shift**
 - $\text{Im}(\Omega - \omega_s) \rightarrow$ **Growth/damping rate**, only depends on the dynamic term, if it is positive there is an instability!

- **Solution:**

$$\begin{aligned}(\Omega^2 - \omega_s^2) &= -\frac{iNe^2\eta}{C^2m_0\gamma} \sum_{p=-\infty}^{\infty} \left(p\omega_0 Z_{\parallel}(p\omega_0) - (p\omega_0 + \Omega) Z_{\parallel}(p\omega_0 + \Omega) \right) \\ &\approx 2\omega_s (\Omega - \omega_s)\end{aligned}$$

- **Tune shift:**

$$\begin{aligned}\Delta\omega_s = \text{Re}(\Omega - \omega_s) &= \frac{e^2}{m_0c^2} \frac{N\eta}{2\omega_s\gamma T_0^2} \\ &\quad \sum_{p=-\infty}^{\infty} \left(p\omega_0 \text{Im}[Z_{\parallel}](p\omega_0) - (p\omega_0 + \omega_s) \text{Im}[Z_{\parallel}](p\omega_0 + \omega_s) \right)\end{aligned}$$

- **Growth rate:**

$$\tau^{-1} = \text{Im}[\Omega - \omega_s] = \frac{e^2}{m_0c^2} \frac{N\eta}{2\omega_s\gamma T_0^2} \sum_{p=-\infty}^{\infty} \left((p\omega_0 + \omega_s) \text{Re}[Z_{\parallel}](p\omega_0 + \omega_s) \right)$$

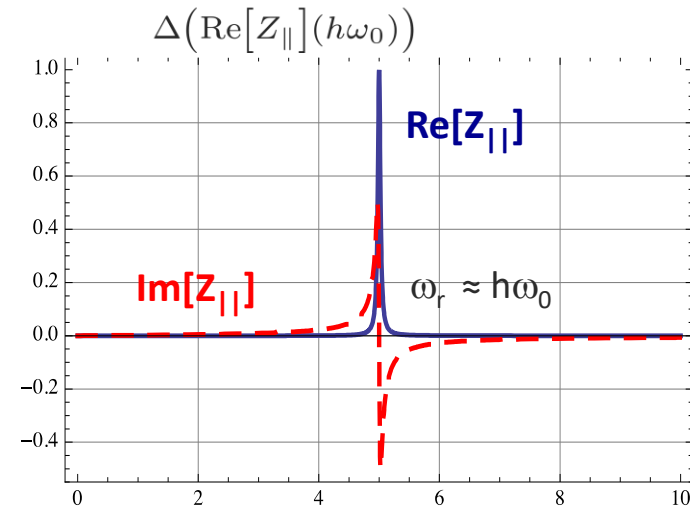
The Robinson instability

- We assume the impedance to be peaked at a frequency ω_r close to $h\omega_0 \gg \omega_s$ (e.g. RF cavity fundamental mode or HOM)
- Only two dominant terms are left in the summation at the RHS of the equation for the growth rate
- Stability requires that η and $\Delta \text{Re} [Z_{\parallel}] (p\omega_0)$ have different signs
- **Solution:**

$$\begin{aligned} \tau^{-1} = \text{Im} (\Omega - \omega_s) &= \frac{e^2}{m_0 c^2} \frac{N\eta}{2\omega_s \gamma T_0^2} \sum_{p=-\infty}^{\infty} \left((p\omega_0 + \omega_s) \text{Re}(Z)_{\parallel} (p\omega_0 + \omega_s) \right) \\ &= \frac{e^2}{m_0 c^2} \frac{N\eta h\omega_0}{2\omega_s \gamma T_0^2} \underbrace{\left(\text{Re} [Z_{\parallel}] (h\omega_0 + \omega_s) - \text{Re} [Z_{\parallel}] (h\omega_0 - \omega_s) \right)}_{\Delta(\text{Re}[Z_{\parallel}](h\omega_0))} \end{aligned}$$

- **Stability criterion:**

$$\eta \cdot \Delta(\text{Re} [Z_{\parallel}] (h\omega_0)) < 0$$



The Robinson instability

- Stability criterion:** $\eta \cdot \Delta(\text{Re}[Z_{\parallel}](h\omega_0)) < 0$

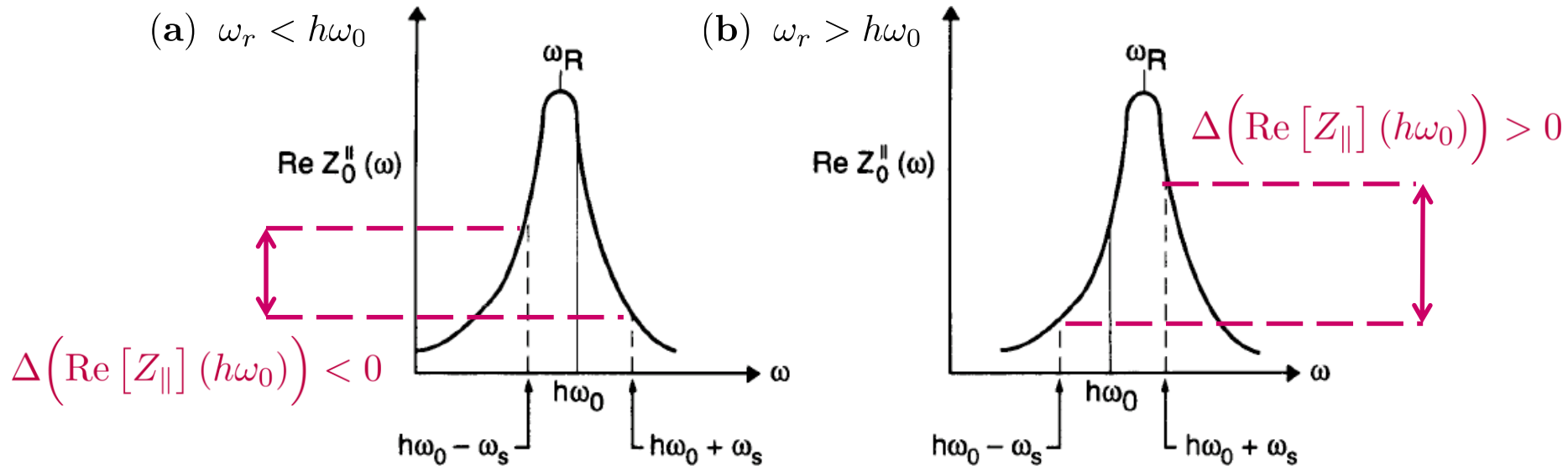
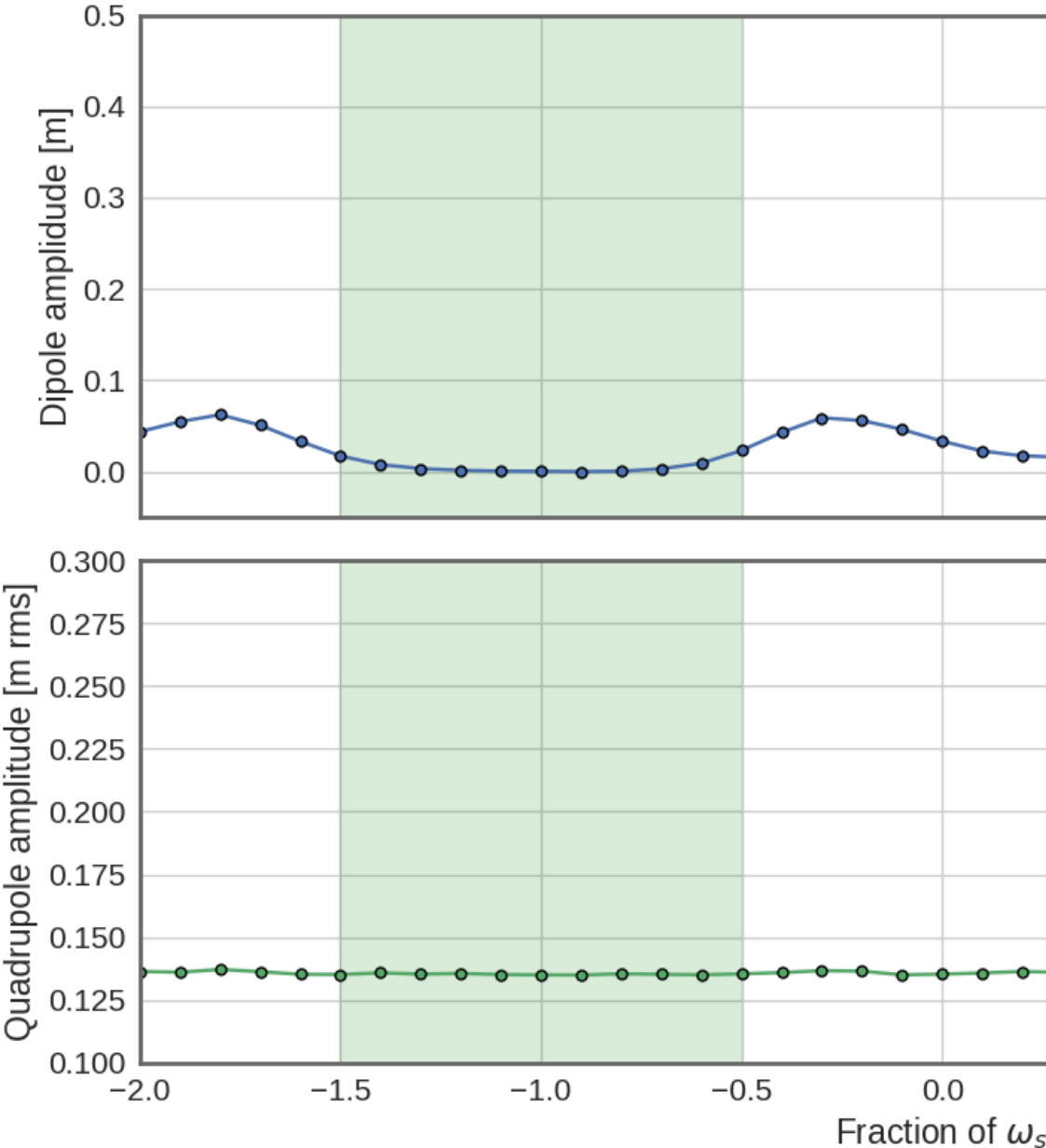


Figure 4.4. Illustration of the Robinson stability criterion. The rf fundamental mode is detuned so that ω_r is (a) slightly below $h\omega_0$ and (b) slightly above $h\omega_0$. (a) is Robinson damped above transition and antidamped below transition. (b) is antidamped above transition and damped below transition.

	$\omega_r < h\omega_0$	$\omega_r > h\omega_0$
Above transition ($\eta > 0$)	stable	unstable
Below transition ($\eta < 0$)	unstable	stable

Robinson damping and instability

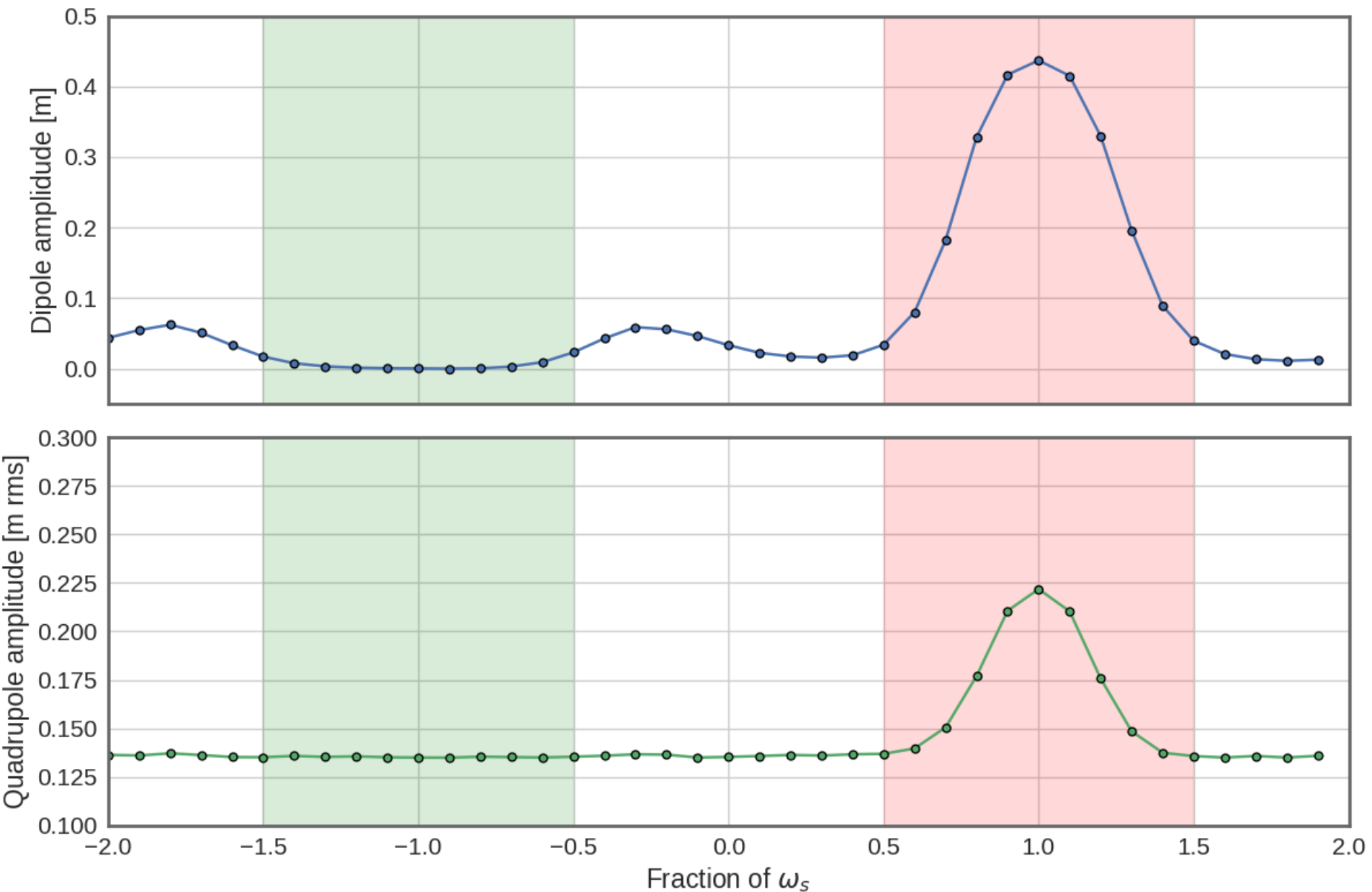


Examples of numerical simulations – SPS bunch with **single narrow-band resonator** wake:

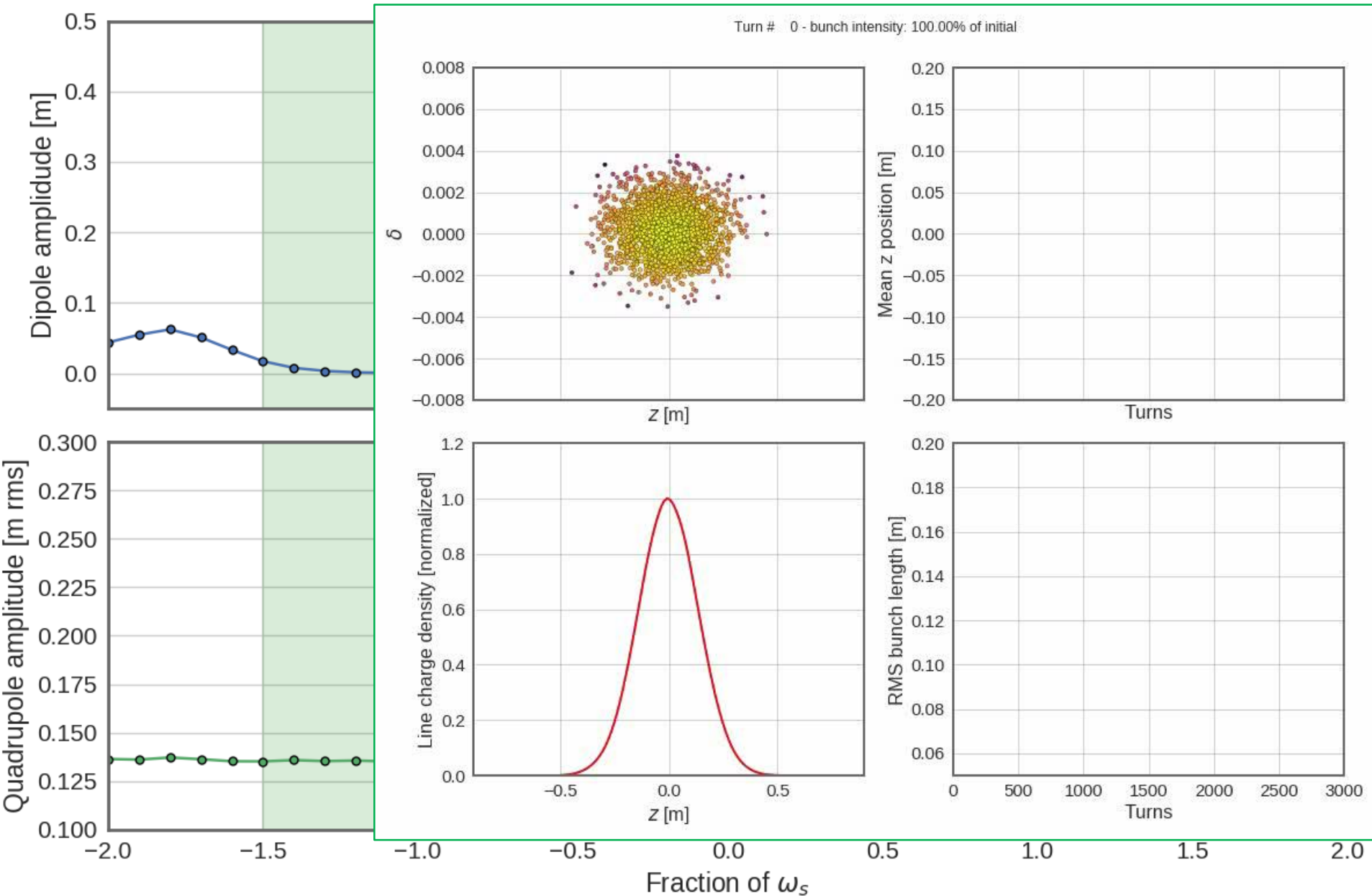
Initializing an otherwise matched bunch with a slight momentum error, **two regimes are found**:

- Regime of **Robinson damping** when the resonator is **detuned to $h\omega_0 - \omega_s$** . Initial dipole oscillations are damped.
- Regime of **Robinson instability** when the resonator is **detuned to $h\omega_0 + \omega_s$** . Initial dipole oscillations start to grow exponentially.

Robinson damping and instability

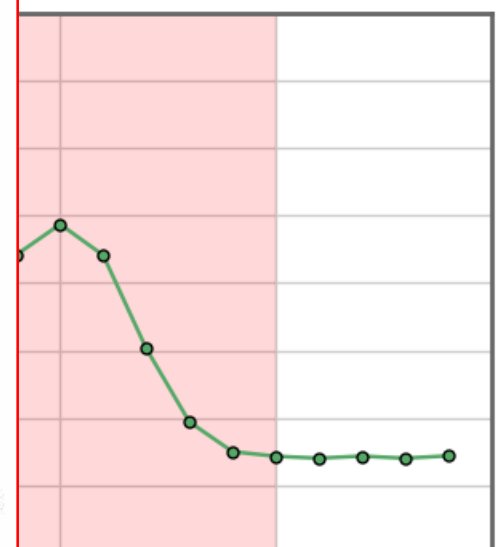
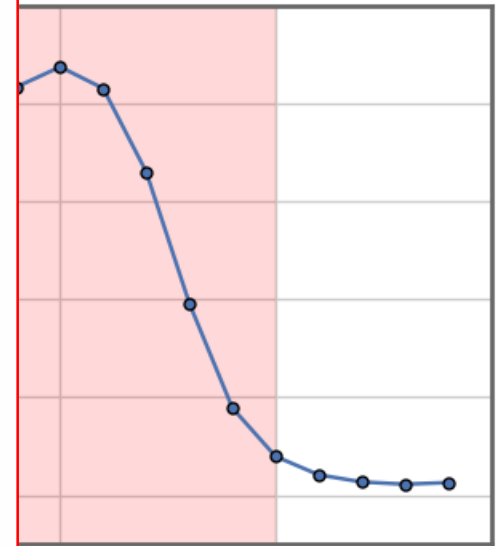
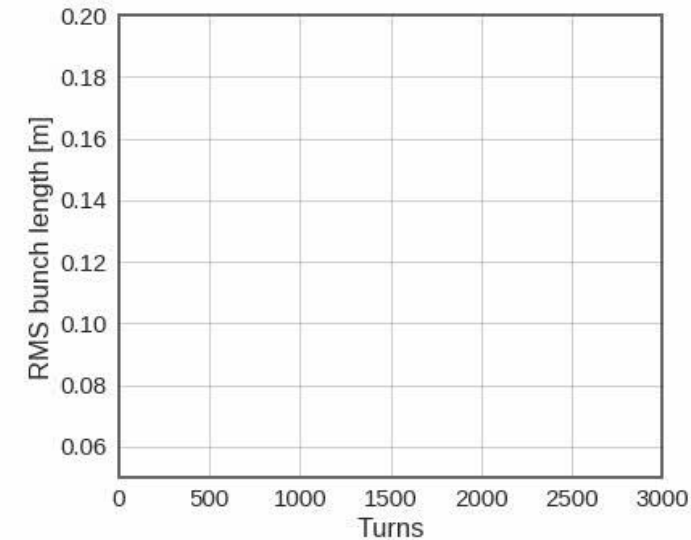
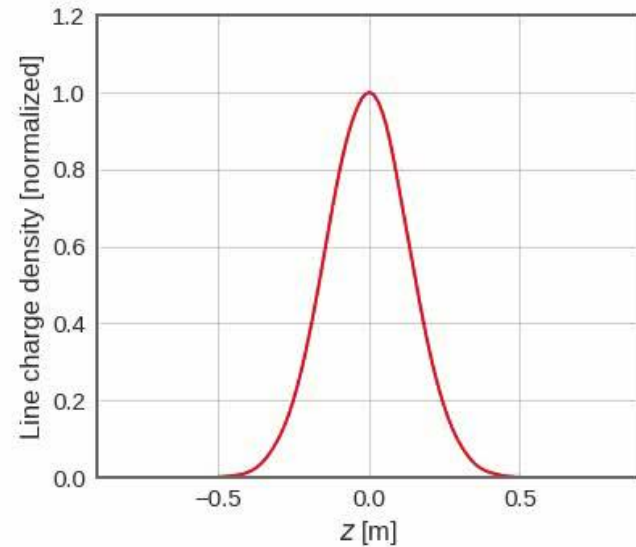
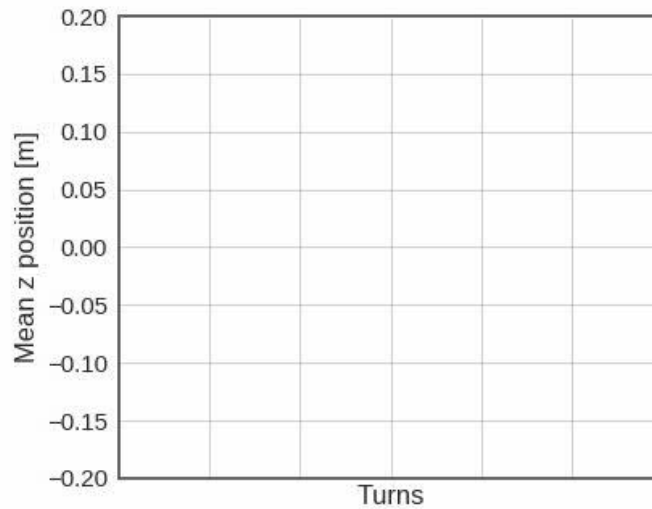
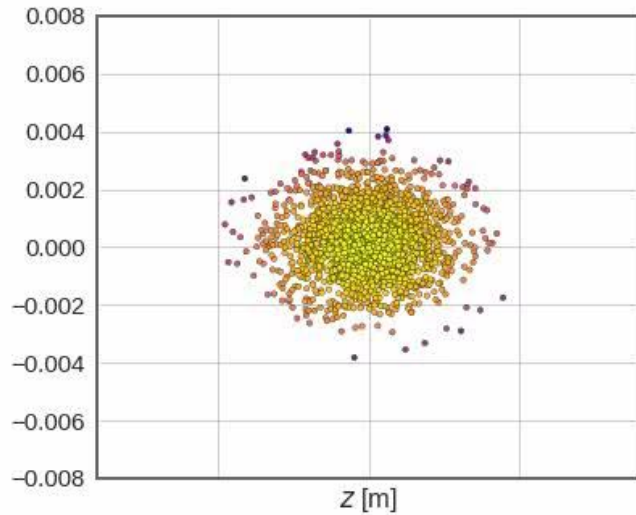


Robinson damping and instability



Robinson damping and instability

Turn # 0 - bunch intensity: 100.00% of initial



-2.0 -1.5 -1.0

-0.5 0.0 0.5

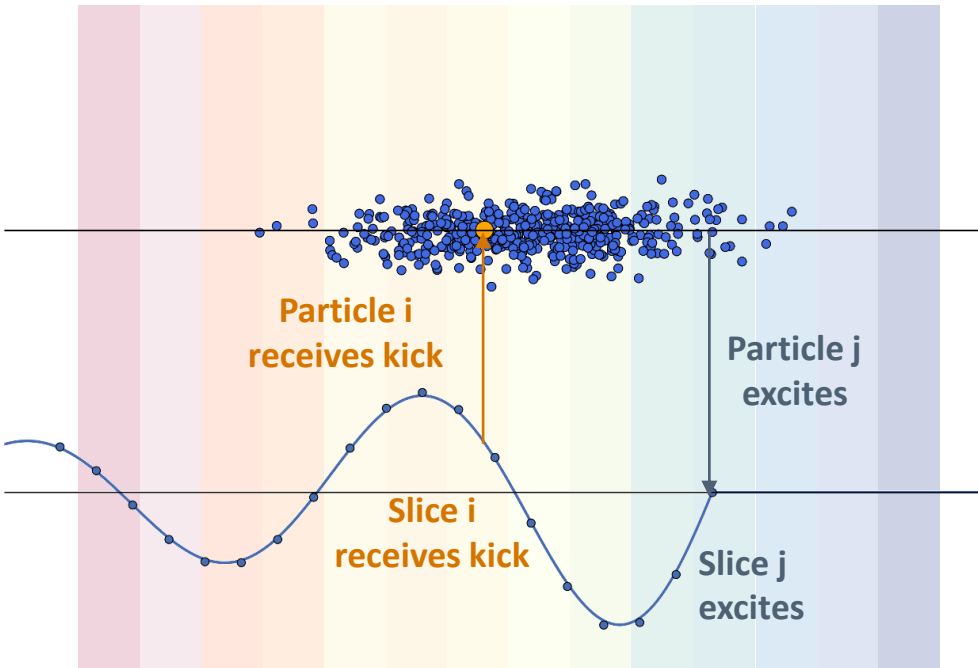
Fraction of ω_s

1.0 1.5 2.0

- The **Robinson instability** occurs for a single bunch under the action of a **multi-turn wake field**
 - It contains a term of coherent synchrotron tune shift which depends only on the imaginary part of the longitudinal impedance
 - It results into an unstable rigid bunch dipole oscillation where the growth rate depends on the real part of the longitudinal impedance
- Other **important collective effects** can affect a bunch in a beam – some of them of which we have also seen
 - Potential well distortion (resulting in synchronous phase shift, bunch lengthening or shortening, synchrotron tune shift/spread)
 - High intensity single bunch instabilities (e.g. microwave instability)
 - Coasting beam instabilities (e.g. negative mass instability)
 - Coupled bunch instabilities
- To be able to study these effects we would need to resort to a **more detailed description** of the bunch(es)
 - Vlasov equation (kinetic model)
 - Macroparticle simulations

Bunch energy loss per turn

- Single traversal of a bunch through an impedance source



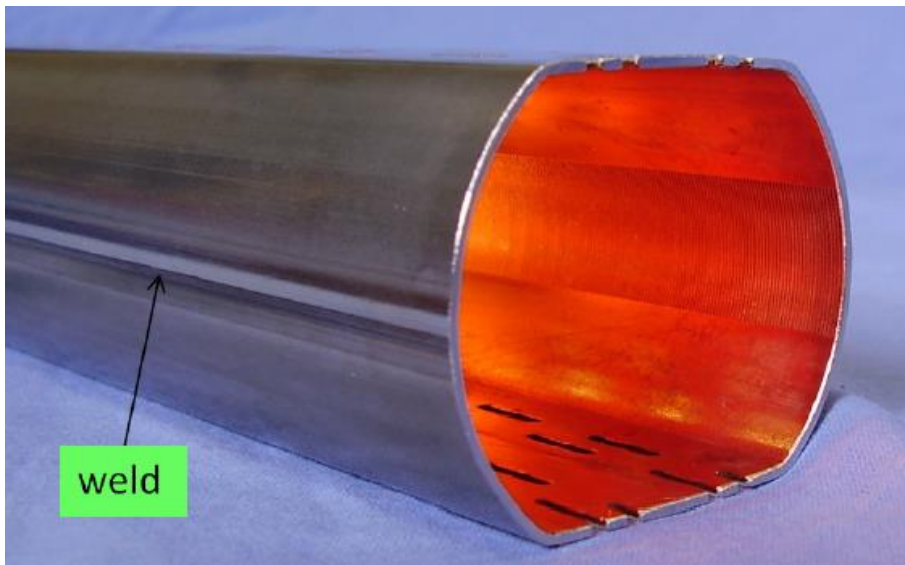
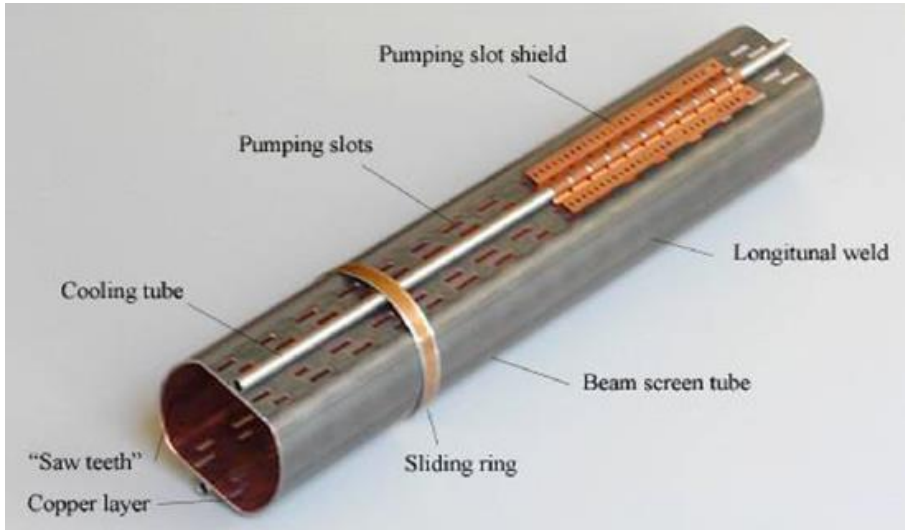
$$\Delta E_{ij} = -e^2 W_{||}(z_{ij})$$

$$\Delta E_{bunch} = -e^2 \sum_{j=1}^{N_b} \sum_{i=1}^{N_b} W_{||}(z_{ij})$$

$$\Delta E_{ij} = -e^2 N[j] N[i] W_{||}[(i - j) \Delta z]$$

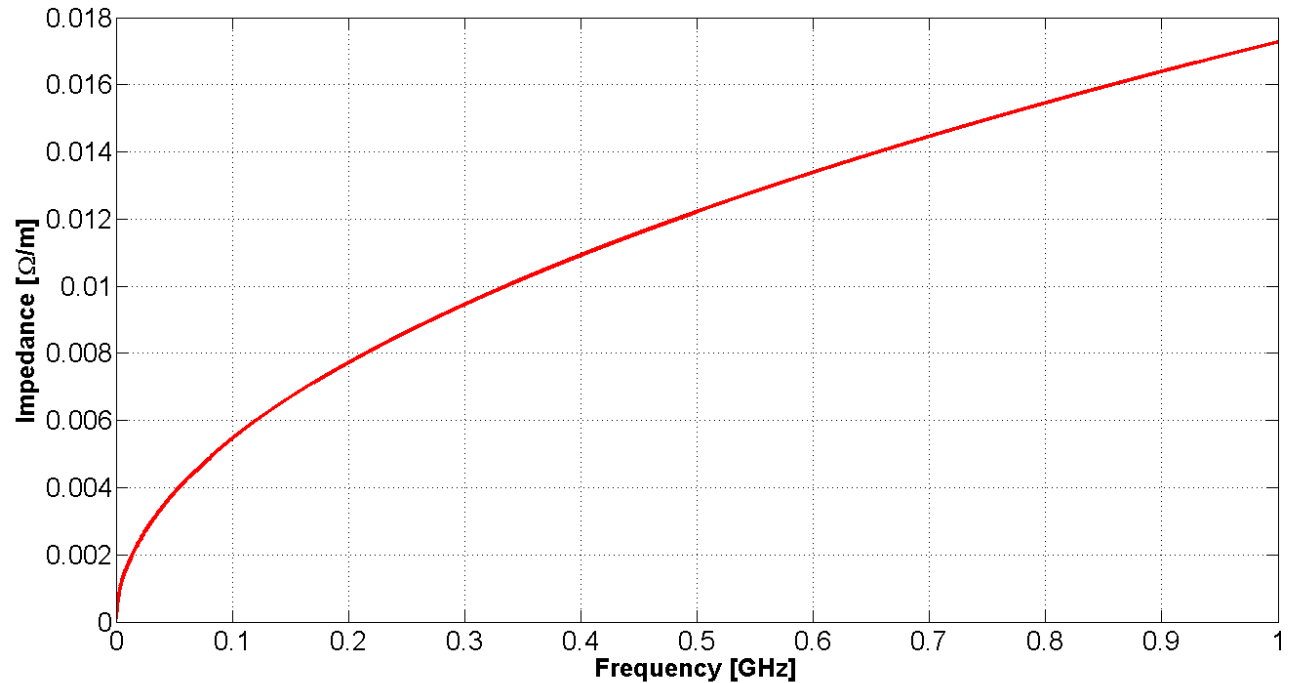
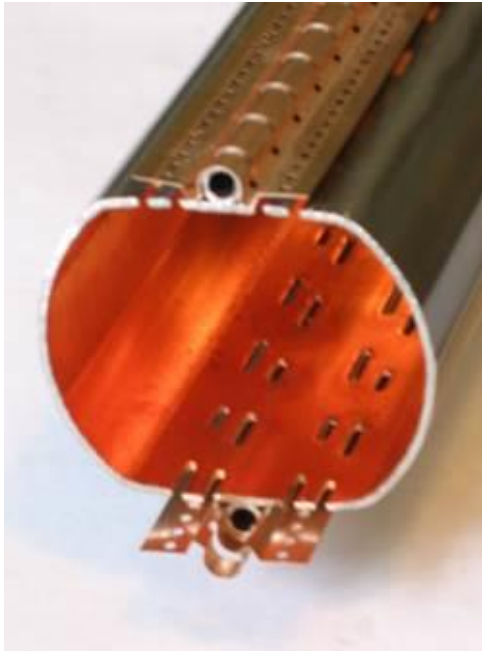
$$\Delta E_i = -e^2 N[i] \sum_{j=0}^i N[j] W_{||}[(i - j) \Delta z]$$

Application to the LHC beam screen



- All along the arcs and in other cold regions of the LHC, a beam screen is interposed between the beam and the cold bore
- The LHC beam screen is made of stainless steel with a layer of **few mm of co-laminated copper**
- Due to the production procedure, there is **a stainless steel weld** on one side of the beam screen that remains exposed to the beam.
- The screen has **holes for pumping** on top and bottom

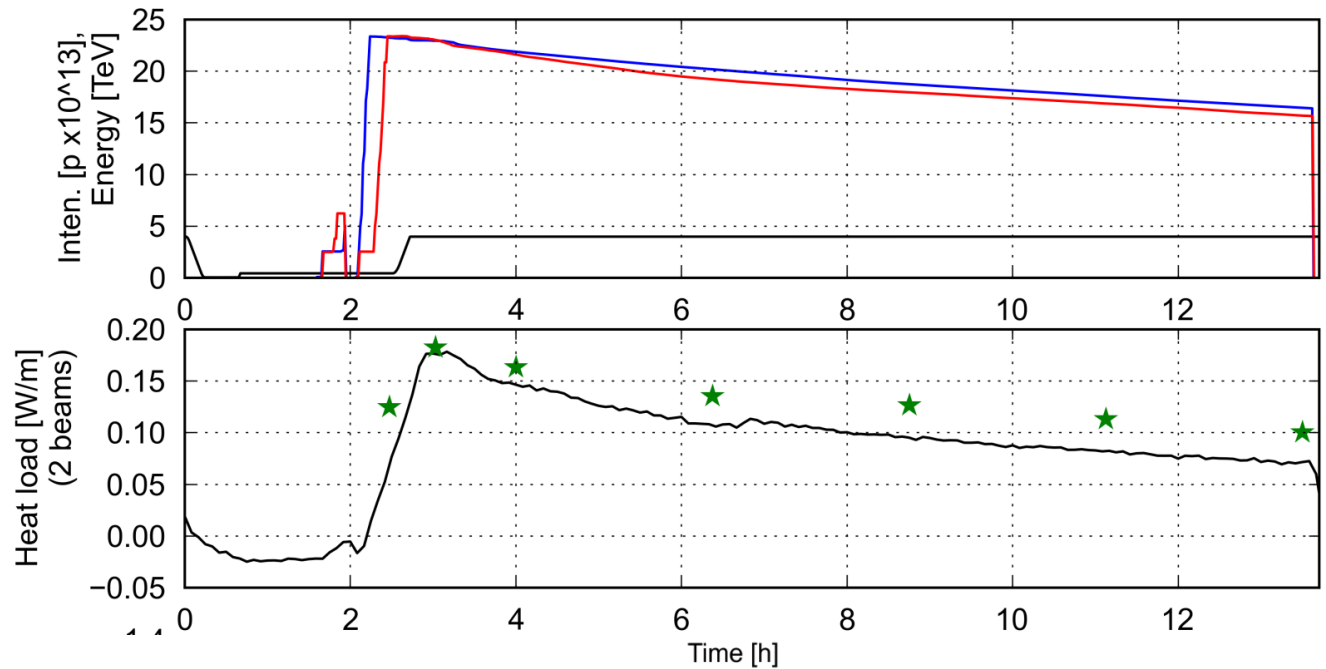
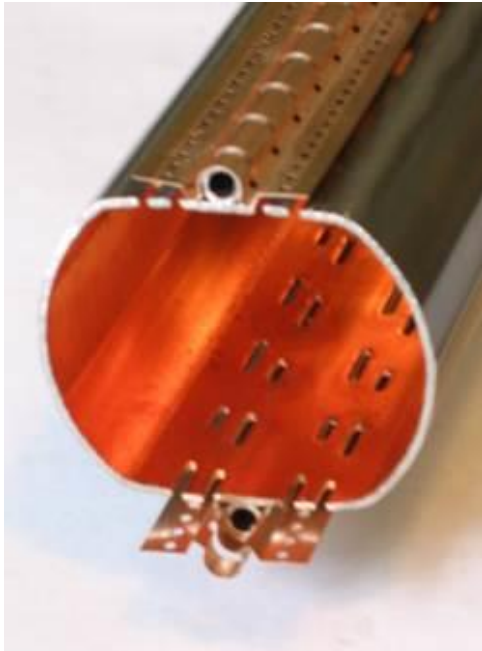
Application to the LHC beam screen



- The impedance model includes the **weld on one side of the beam screen**, which means a small longitudinal stripe of exposed StSt, as well as **the pumping holes**

Application to the LHC beam screen

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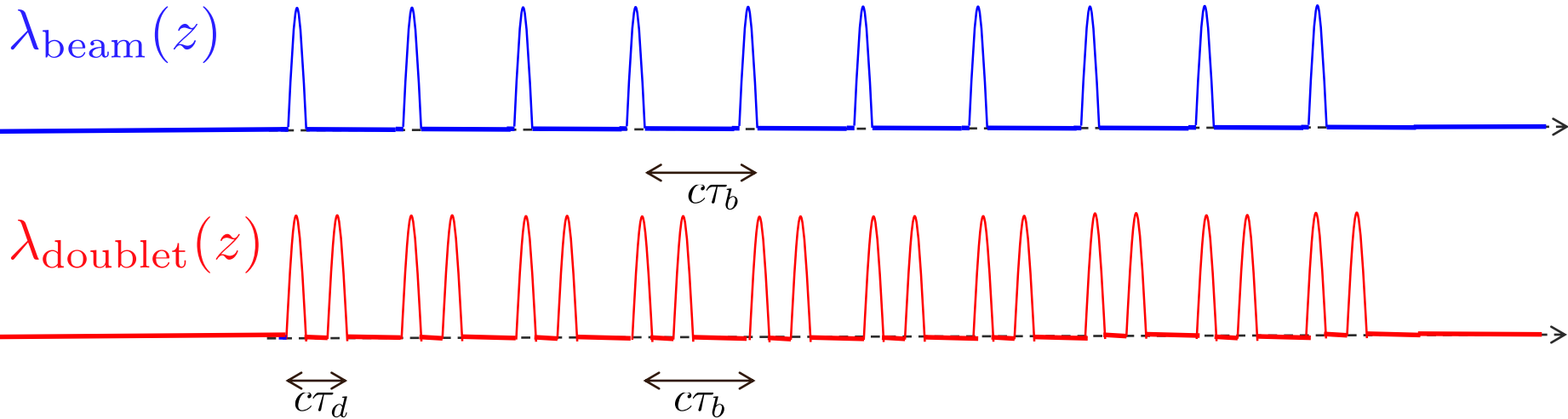
★ Estimation (impedance + synchrotron rad.)

— Heat load measurement from cryogenics

$$\Delta E_{\text{beam}} = \frac{e^2 \omega_0}{2\pi} \sum_{p=-\infty}^{\infty} |\tilde{\lambda}(p\omega_0)|^2 \text{Re} [Z_{||}(p\omega_0)] \cdot \left[\frac{1 - \cos\left(\frac{2\pi M p}{h}\right)}{1 - \cos\left(\frac{2\pi p}{h}\right)} \right]$$

- The heat dissipated on the beam screen **can be calculated for a beam made of bunches spaced by 50 ns** and compared to the measurement from cryogenics

Beam energy loss: a doublet beam



$$\Lambda_{\text{beam}}(\omega) \rightarrow \Lambda_{\text{doublet}}(\omega) = \Lambda_{\text{beam}}(\omega) [1 + \exp(-i\omega\tau_d)]$$

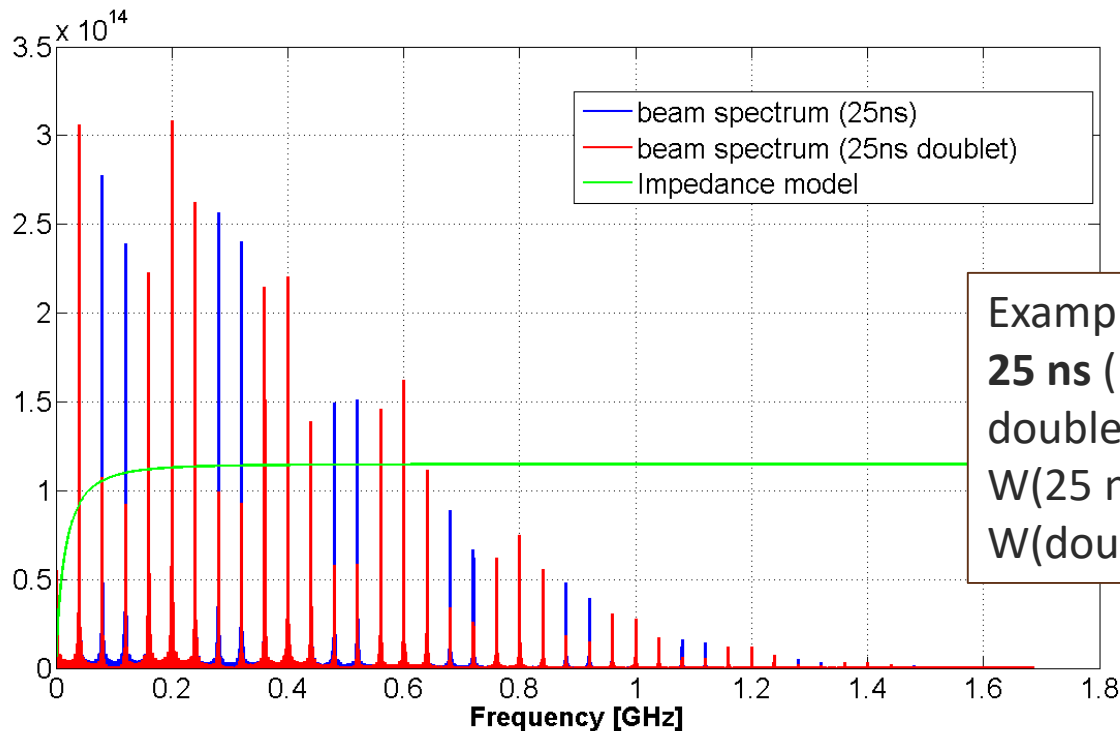


$$\Delta E_{\text{doublet}} = \frac{2e^2\omega_0}{\pi} \sum_{p=-\infty}^{\infty} |\Lambda_{\text{beam}}(p\omega_0)|^2 \cos^2\left(\frac{p\omega_0\tau_d}{2}\right) \text{Re} [Z_{\parallel}(p\omega_0)]$$

N.B. in this example the doublet has double total intensity than single beam

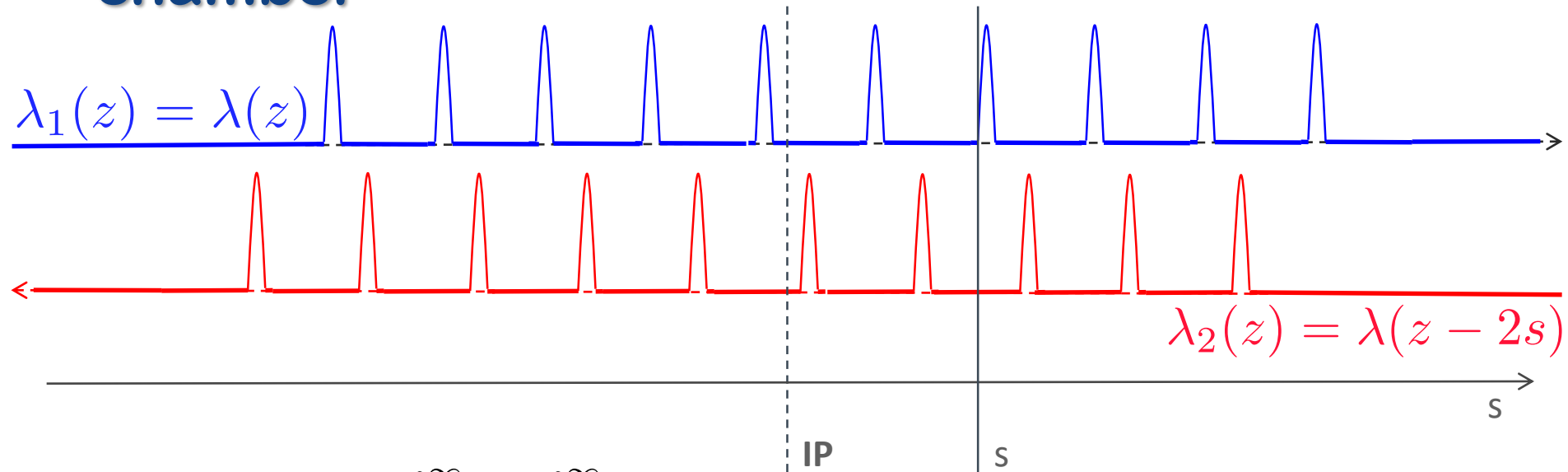
Beam energy loss: a doublet beam

- No additional impedance energy loss is expected with the doublet beam with respect to nominal beam for same total intensity
 - Beam power spectrum is modulated with \cos^2 function and lines are weakened by this modulation
 - For higher doublet intensity, global effect depends on the impedance spectrum
 - Example \rightarrow LHC injection beam stopper (TDI)

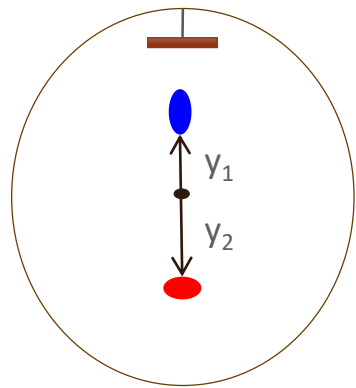


Example for LHC beam in TDI:
25 ns (1.2×10^{11} p/b) vs. **20+5 ns**
 doublet (1.5×10^{11} p/doublet)
 $W(25 \text{ ns}) = 456 \text{ W}$
 $W(\text{doublet}) = 338 \text{ W}$

Beam energy loss: collider's common chamber



$$\Delta E_{\text{beam1}}(s) = e^2 \int_{-\infty}^{\infty} \lambda(z) \int_{-\infty}^{\infty} [\lambda(z') W_{\parallel b1}(z - z') - \lambda(z' - 2s) W_{\parallel b2}(z - z')] dz' dz$$

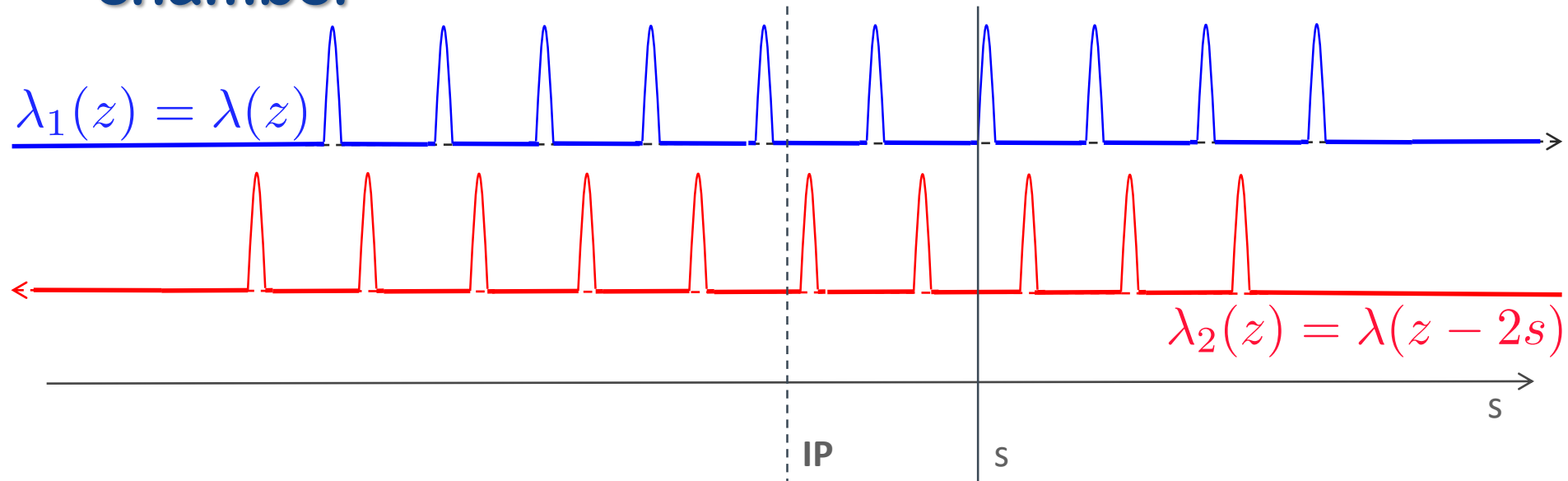


$$W_{\parallel b1}(z) = W_{\parallel}^{(0)}(z) + W_{\parallel}^{(1d)}(z)y_1 + W_{\parallel}^{(1q)}(z)y_1$$

$$W_{\parallel b2}(z) = W_{\parallel}^{(0)}(z) + W_{\parallel}^{(1d)}(z)y_2 + W_{\parallel}^{(1q)}(z)y_1$$

$$\text{with } W_{\parallel}^{1d}(z) = W_{\parallel}^{1q}(z)$$

Beam energy loss: collider's common chamber



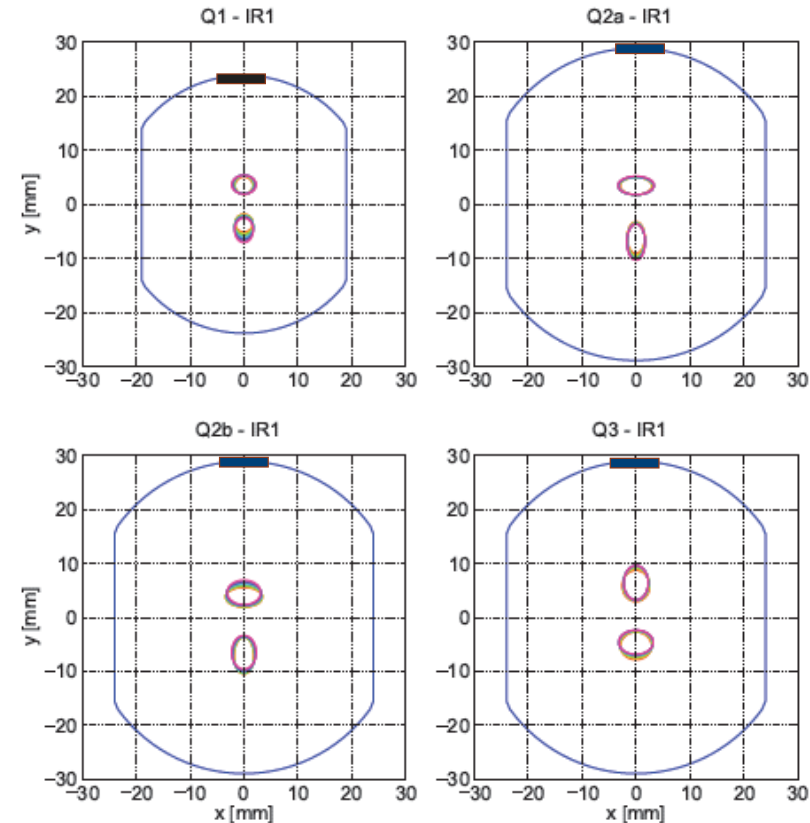
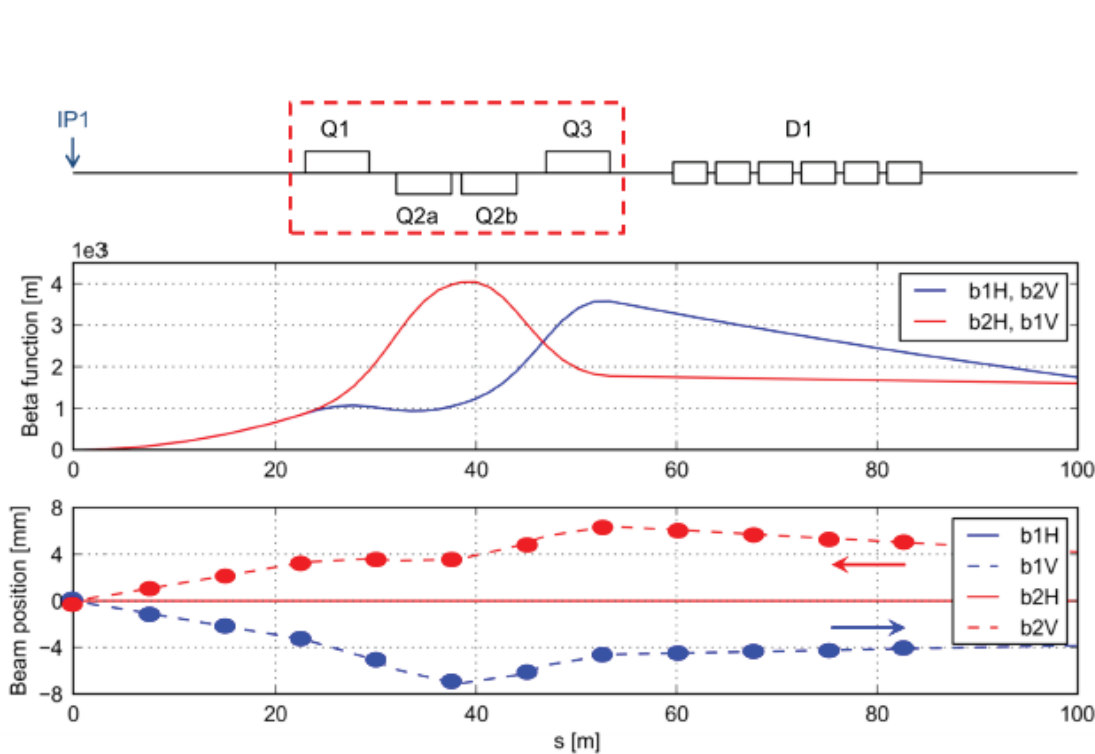
$$W_{||}^{1d}(z), W_{||}^{1q}(z) \xleftrightarrow{\mathcal{F}} Z_{||}^1(\omega) \quad W_{||}^0(z) \xleftrightarrow{\mathcal{F}} Z_{||}^0(\omega)$$

$$\Delta E_{\text{beam1}}(s) + \Delta E_{\text{beam2}}(s) =$$

$$\frac{4e^2\omega_0}{\pi} \sum_{p=0}^{\infty} |\Lambda(p\omega_0)|^2 \left\{ \text{Re} \left[Z_{||}^0(p\omega_0) \right] + [y_1(s) + y_2(s)] \text{Re} \left[Z_{||}^1(p\omega_0) \right] \right\} \cdot \sin^2 \left(\frac{p\omega_0 s}{c} \right)$$

$$\Delta W_{CC} = \frac{\omega_0}{2\pi} \int_{-s_0}^{s_0} [\Delta E_{\text{beam1}}(s) + \Delta E_{\text{beam2}}(s)] ds$$

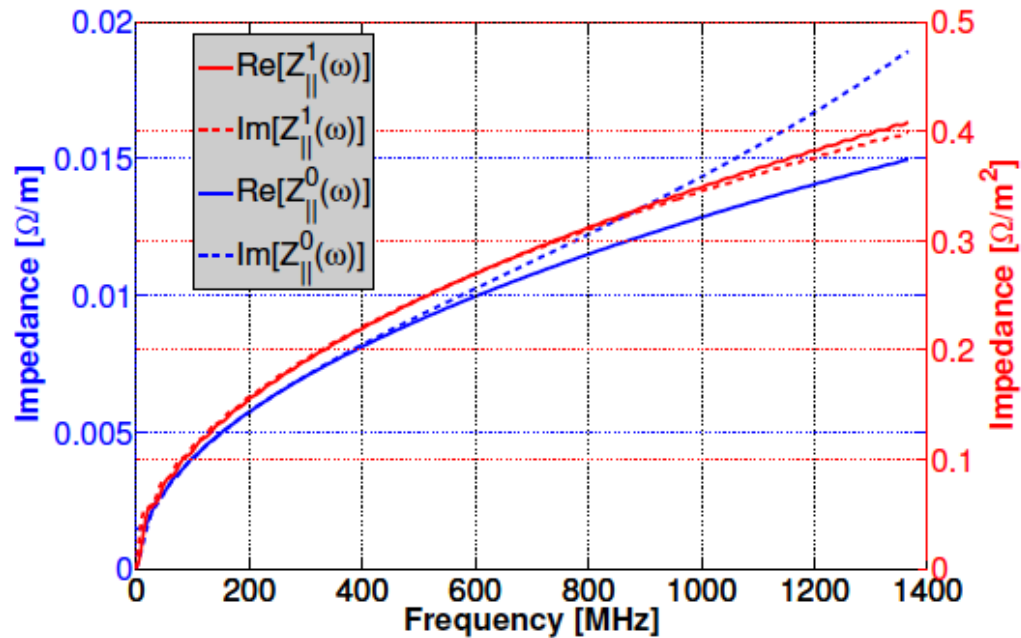
Beam energy loss in the LHC triplets



- Application to the LHC inner triplets

- Beams are separated vertically (IP1) or horizontally (IP5)
- Strongly off-axis for ~30m, all relative delays between beams swept
- Asymmetric chamber in the direction of separation because of the weld

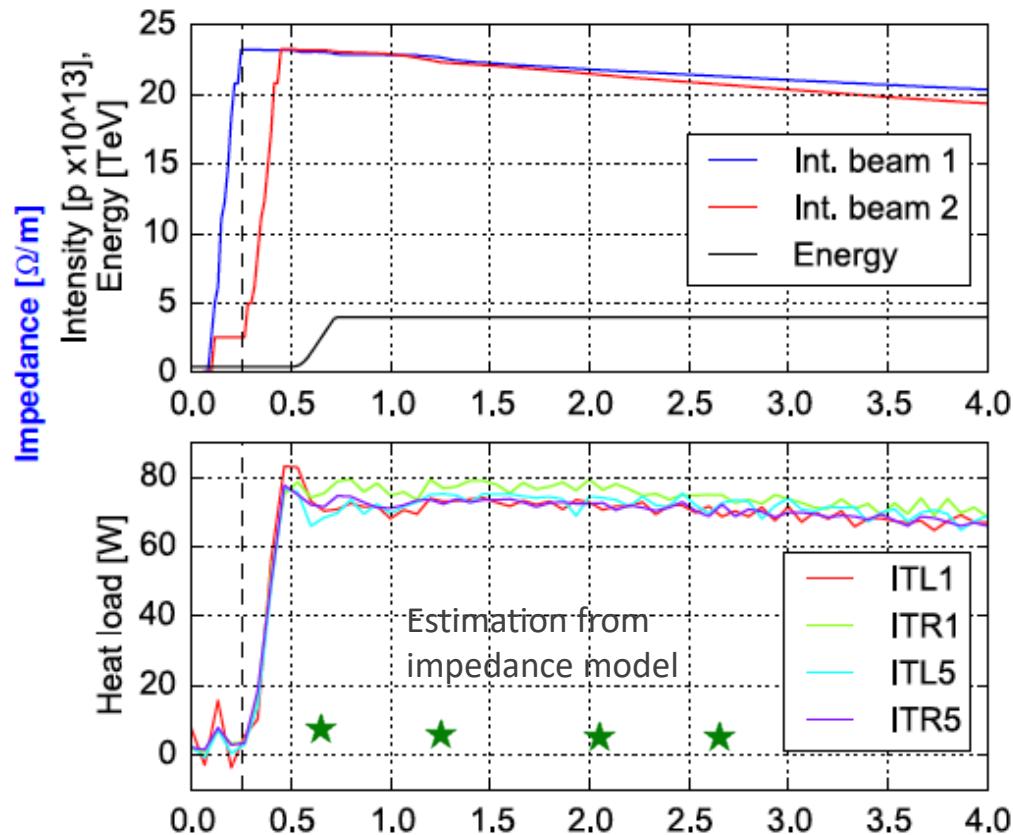
Beam energy loss in the LHC triplets



$$\Delta W_{IT} = 4 \text{ W}$$

for a typical 50 ns fill of the LHC

Beam energy loss in the LHC triplets



- Comparison with measured data (L. Taviani)

- Estimated heat load more than a factor 10 below measurement

- Indication of a dominant contribution from electron cloud, also enhanced by the two-beam effect

$$\Delta W_{IT} \approx 4 \text{ W}$$

for a typical 50 ns fill of the LHC

Panofsky-Wenzel Theorem

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Source terms (displaced point charge traveling along s with speed v) in Cartesian coordinates:

$$\rho(x, y, s, t) = q_1 \delta(x - x_1) \delta(y - y_1) \delta(s - vt)$$

$$\vec{j}(x, y, s, t) = \rho(x, y, s, t) \vec{v}$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Source terms (displaced point charge traveling along s with speed v) in cylindrical coordinates:

$$\begin{aligned} \rho(r, \theta, s, t) &= \frac{q_1}{r_1} \delta(r - r_1) \delta_P(\theta) \delta(s - vt) = \\ &= \frac{q_1}{r_1} \delta(r - r_1) \delta(s - vt) \sum_{m=0}^{\infty} \frac{\cos m\theta}{\pi(1 + \delta_{m0})} \end{aligned}$$

$$\vec{j}(r, \theta, s, t) = \rho(r, \theta, s, t) \vec{v}$$

$$v = \beta c \quad \text{with} \quad \beta \approx 1$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_s}{\partial s} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_s}{\partial s} = 0$$

$$\frac{\partial B_s}{\partial y} - \frac{\partial B_y}{\partial s} - \frac{1}{c^2} \frac{\partial E_x}{\partial t} = 0$$

$$\frac{\partial E_s}{\partial y} - \frac{\partial E_y}{\partial s} + \frac{\partial B_x}{\partial t} = 0$$

$$\frac{\partial B_x}{\partial s} - \frac{\partial B_s}{\partial x} - \frac{1}{c^2} \frac{\partial E_y}{\partial t} = 0$$

$$\frac{\partial E_x}{\partial s} - \frac{\partial E_s}{\partial x} + \frac{\partial B_y}{\partial t} = 0$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{1}{c^2} \frac{\partial E_s}{\partial t} = \mu_0 \rho c$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_s}{\partial t} = 0$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

We want to find relations between the forces on the witness charge:

$$\vec{F}_{\perp} = q_2[(E_x - cB_y)\hat{x} + (E_y + cB_x)\hat{y}]$$

$$F_s = q_2 E_s$$

with

$$s - ct = z$$



$$\frac{\partial}{\partial s} = \frac{\partial}{\partial z} = -\frac{1}{c} \frac{\partial}{\partial t}$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_s}{\partial s} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_s}{\partial s} = 0$$

$$\frac{\partial B_s}{\partial y} - \frac{\partial B_y}{\partial s} - \frac{1}{c^2} \frac{\partial E_x}{\partial t} = 0$$

$$\frac{\partial B_x}{\partial s} - \frac{\partial B_s}{\partial x} - \frac{1}{c^2} \frac{\partial E_y}{\partial t} = 0$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{1}{c^2} \frac{\partial E_s}{\partial t} = \mu_0 \rho c$$

We first use this set of equations

$$\frac{\partial E_s}{\partial y} - \frac{\partial E_y}{\partial s} + \frac{\partial B_x}{\partial t} = 0$$

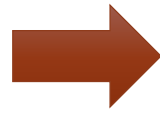
$$\frac{\partial E_x}{\partial s} - \frac{\partial E_s}{\partial x} + \frac{\partial B_y}{\partial t} = 0$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_s}{\partial t} = 0$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\frac{\partial F_x}{\partial z} - \frac{\partial F_s}{\partial x} = 0$$

$$\frac{\partial F_s}{\partial y} - \frac{\partial F_y}{\partial z} = 0$$



$$\frac{\partial \vec{F}_\perp}{\partial z} = \nabla_\perp F_s$$

$$\frac{\partial \int_0^L \vec{F}_\perp ds}{\partial z} = \nabla_\perp \int_0^L F_s ds$$

Result known as Panofsky-Wenzel theorem

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\frac{\partial \int_0^L \vec{F}_\perp ds}{\partial z} = \nabla_\perp \int_0^L F_s ds$$

$$\int_0^L F_x ds = W_x(z) \Delta x_1 + W_{Qx}(z) x$$



$$W'_x(z) = W_{||}^{(dq)}(z) \quad \overset{\mathcal{F}}{\iff} \quad \frac{\omega}{c} Z_x(\omega) = Z_{||}^{(dq)}(\omega)$$

$$W'_{Qx}(z) = 2W_{||}^{(2q)}(z) \quad \overset{\mathcal{F}}{\iff} \quad \frac{\omega}{c} Z_{Qx}(\omega) = 2Z_{||}^{(2q)}(\omega)$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\frac{\partial \int_0^L \vec{F}_\perp ds}{\partial z} = \nabla_\perp \int_0^L F_s ds$$

$$\int_0^L F_x ds = W_x(z) \Delta x_1 + W_{Qx}(z) x$$



$$W'_x(z) = V$$

The longitudinal and transverse wake functions are not independent, although in general no relation can be established between $W_{||}(z)$ and $W_{x,y}(z)$, which are the main wakes in the longitudinal and transverse planes, respectively.

$$\left. \begin{matrix} (dq) \\ || \end{matrix} \right) (\omega)$$

$$W'_{Qx}(z) = 2W_{||}^{(2q)}(z) \quad \xleftrightarrow{\mathcal{F}} \quad \frac{\omega}{c} Z_{Qx}(\omega) = 2Z_{||}^{(2q)}(\omega)$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_s}{\partial s} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_s}{\partial s} = 0$$

$$\frac{\partial B_s}{\partial y} - \frac{\partial B_y}{\partial s} - \frac{1}{c^2} \frac{\partial E_x}{\partial t} = 0$$

$$\frac{\partial E_s}{\partial s} - \frac{\partial E_y}{\partial t} + \frac{\partial B_x}{\partial t} = 0$$

$$\frac{\partial B_x}{\partial s} - \frac{\partial B_s}{\partial x} - \frac{1}{c^2} \frac{\partial E_y}{\partial t} = 0$$

We can now use also these two sets of equations to find additional properties of the wakes

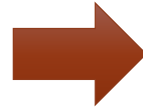
$$\frac{\partial E_s}{\partial s} - \frac{\partial E_x}{\partial t} + \frac{\partial B_y}{\partial t} = 0$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{1}{c^2} \frac{\partial E_s}{\partial t} = \mu_0 \rho c$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_s}{\partial t} = 0$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\frac{\partial F_x}{\partial x} = -\frac{\partial F_y}{\partial y}$$

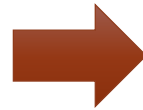


$$W_{Qx}(z) = -W_{Qy}(z)$$

This is an interesting result!
The quadrupolar wakes in x and y must be equal with opposite signs

$$\frac{\partial \int_0^L F_x ds}{\partial x} = -\frac{\partial \int_0^L F_y ds}{\partial y}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$



This relation means that the cross-wakes between x and y must be equal.
We have so far ignored these terms in our derivations.

$$\frac{\partial \int_0^L F_x ds}{\partial y} = \frac{\partial \int_0^L F_y ds}{\partial x}$$

Instabilities

- The **equilibrium distribution** in the presence of a longitudinal wake field can be found analytically. The (linearized) **longitudinal Hamiltonian** with longitudinal wake fields is given as:

$$H = -\frac{1}{2}\eta\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \frac{e^2}{\beta^2 EC} \sum_k \int dz'' \int dz' \lambda(z' + kC) W_0'(z'' - z' - kC)$$

- Remember the example of the harmonic oscillator:

$$H = \frac{1}{2} p^2 + \frac{1}{2} \boxed{W} q^2$$

↓

Coefficient determines frequency/tune

- The **equilibrium distribution** in the presence of a longitudinal wake field can be found analytically. The (linearized) **longitudinal Hamiltonian** with longitudinal wake fields is given as:

$$H = -\frac{1}{2}\eta\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \underbrace{\frac{e^2}{\beta^2 EC} \sum_k \int dz'' \int dz' \lambda(z' + kC) W_0'(z'' - z' - kC)}_{\text{we make an expansion in } z - \text{factor out } \frac{1}{2\eta\beta^2 c^2}}$$

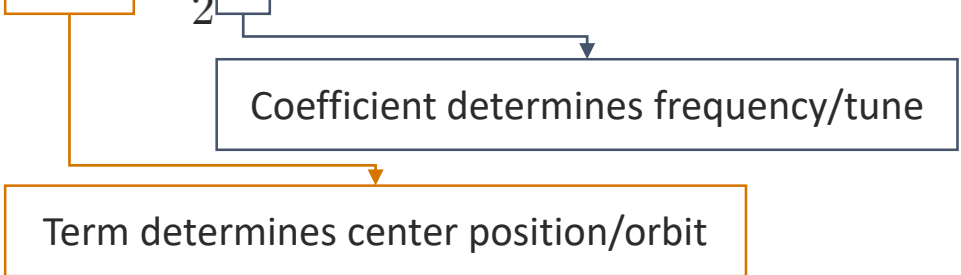
we make an expansion in z – factor out $\frac{1}{2\eta\beta^2 c^2}$

- Remember the example of the harmonic oscillator:

$$H = \frac{1}{2} p^2 + V q + \frac{1}{2} W q^2$$

Term determines center position/orbit

Coefficient determines frequency/tune



- The **equilibrium distribution** in the presence of a longitudinal wake field can be found analytically. The (linearized) **longitudinal Hamiltonian** with longitudinal wake fields is given as:

$$H = -\frac{1}{2}\eta\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \underbrace{\frac{e^2}{\beta^2 EC} \sum_k \int dz'' \int dz' \lambda(z'+kC)W'_0(z''-z'-kC)}_{\text{expansion in } z - \text{factor } \frac{1}{2\eta\beta^2 c^2}}$$

- It follows then quite easily that:

$$\begin{aligned}\Delta\omega_s &\approx -\frac{1}{2\omega_s} \frac{e^2 \eta c^2}{EC} \int dz' \lambda(z')W''_0(z-z') \\ &= -\frac{i}{4\pi} \frac{e^2 \eta c^2}{\omega_s EC} \int d\omega \hat{\lambda}(\omega)Z_0(\omega)\frac{\omega}{c}\end{aligned}$$

Remember, we make use of:
 $\Omega^2 - \omega_s^2 \approx 2\omega_s \Delta\omega_s$

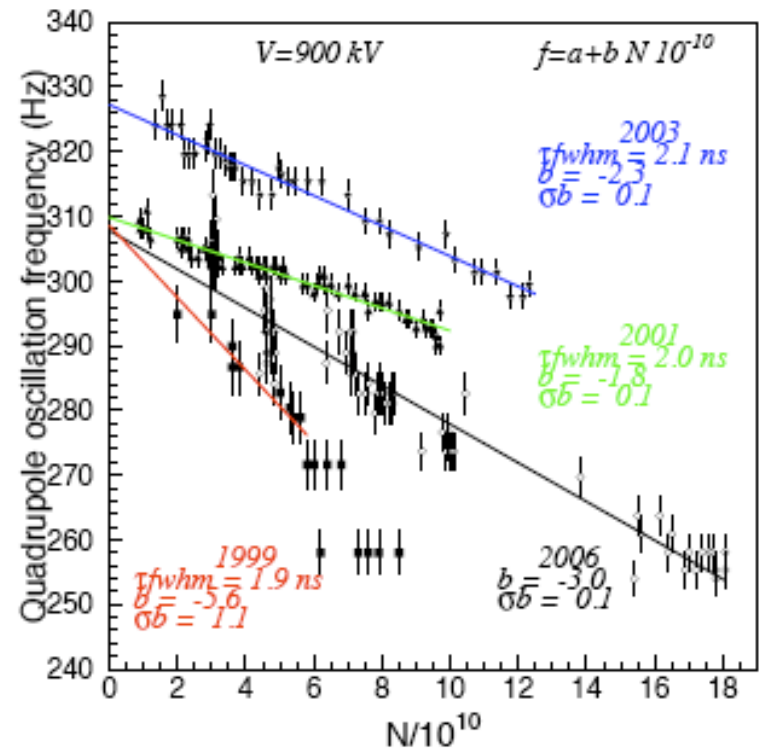
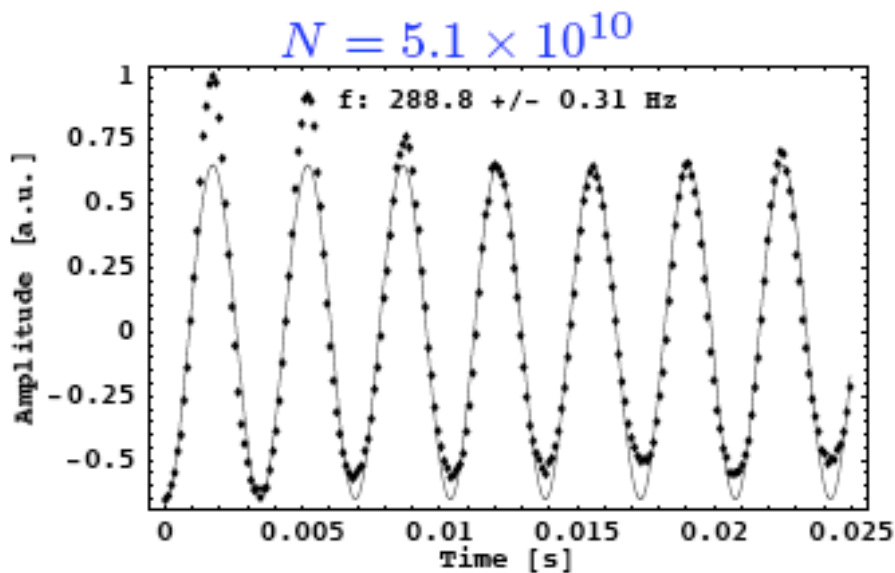
- The synchrotron tune shift from an impedance is, hence, given as:

$$\Delta Q_s = -\frac{1}{4\omega_s} \frac{e^2 \eta}{(2\pi^2)E} \int d\omega \omega \hat{\lambda}(\omega) \text{Im}[Z_0(\omega)]$$

Measurements of synchrotron tune shift at SPS

The CERN Accelerator School

- The slope of the **incoherent synchrotron tune shift with intensity**, measured in reproducible conditions over the years, shows the evolution of the **imaginary part of the machine impedance** (E. Shaposhnikova, T. Bohl, J. Tuckmantel)
 - The technique uses the quadrupole oscillations of a bunch injected with a mismatch
 - Qs can be extrapolated from bunch length or peak amplitude measurements



Measurements of potential well distortion

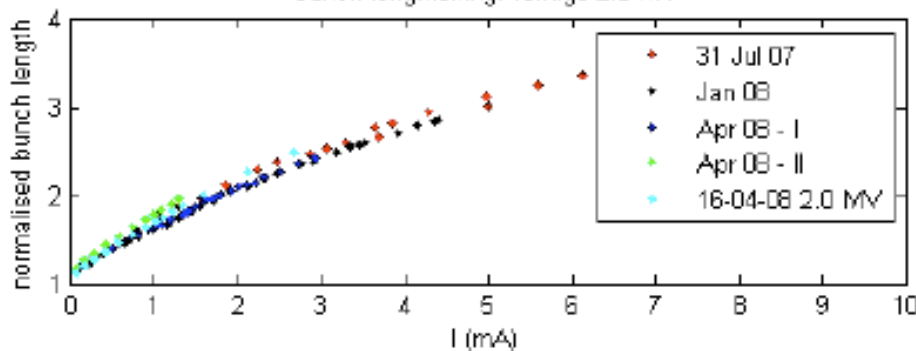
Stable phase and bunch lengthening

Measurements at light sources

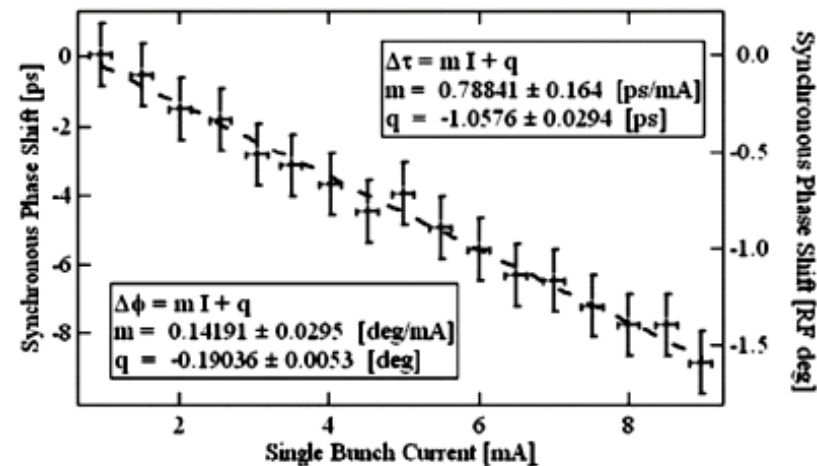
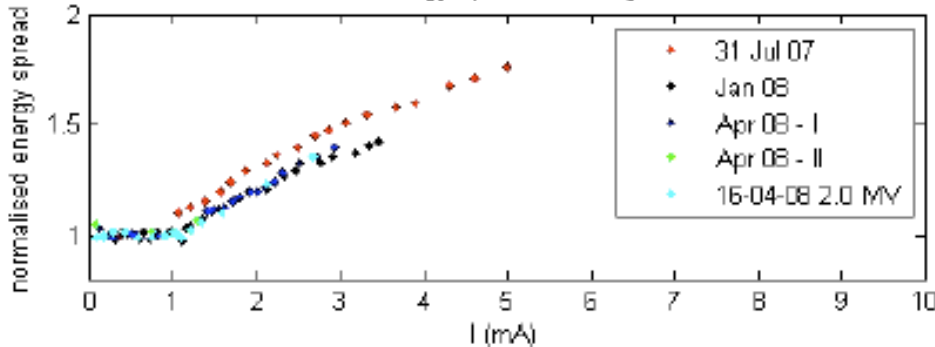
⇒ Bunch lengthening @DIAMOND (left, R. Bartolini)

⇒ Energy loss measured through the synchronous phase shift @Australian light source (right, R. Dowd, M. Boland, G. LeBlanc, M. Spencer, Y. Tan, PAC07)

bunch lengthening: voltage 2.0 MV



energy spread widening



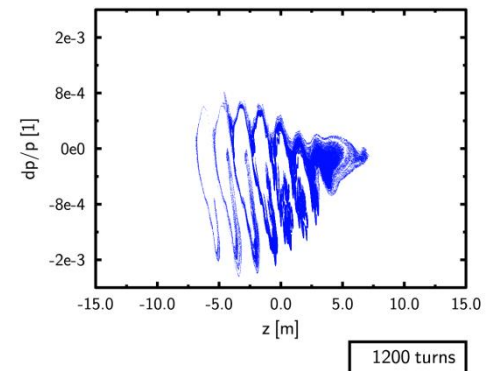
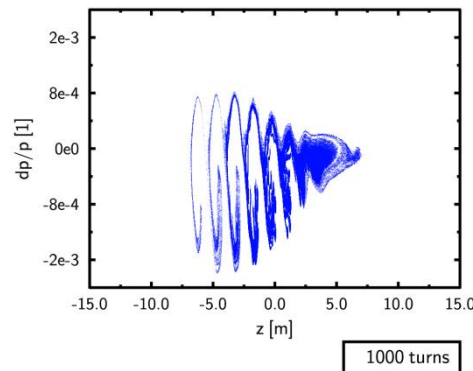
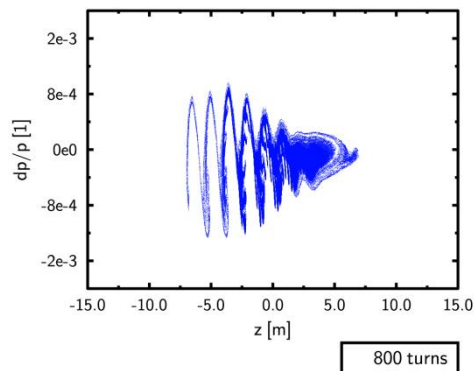
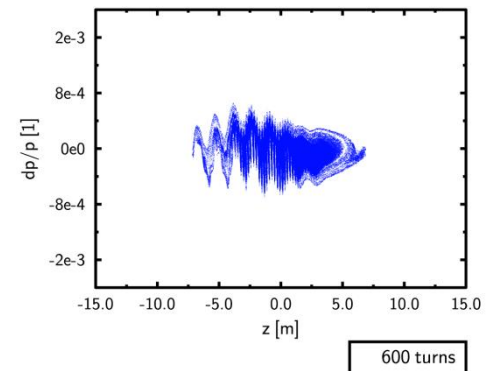
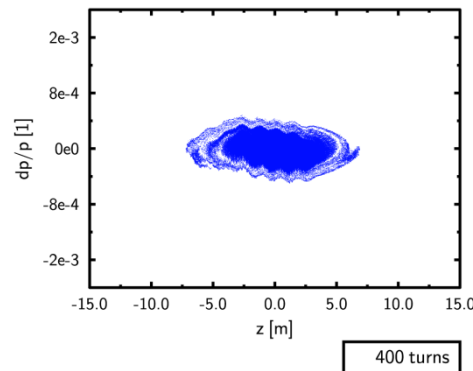
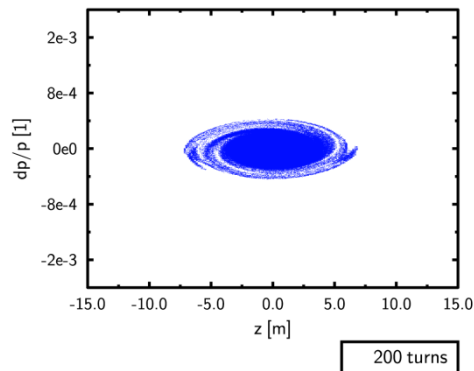
Synchronous phase shift measured with a streak camera in the Australian Synchrotron.

Examples of numerical simulations debunching bunch with SPS impedance model

Microwave instability on a debunching bunch is used at SPS for probing the machine impedance
(E. Shaposhnikova, T. Bohl, H. Timkó, et al.)

⇒ Long bunch injected into SPS with RF off slowly debunches due to low momentum spread

⇒ Spectrum of bunch profile reveals important components for the impedance



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- ⇒ Long bunch injected into SPS with RF off slowly debunches due to low momentum spread
- ⇒ Spectrum of bunch profile reveals important components for the impedance
- ⇒ Simulations with impedance model are used to match measured profile

