# Free Electron Lasers (FEL)

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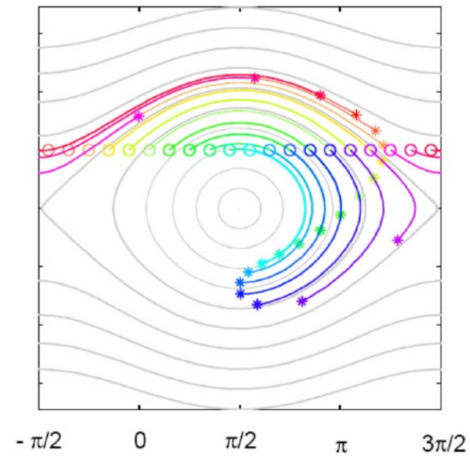
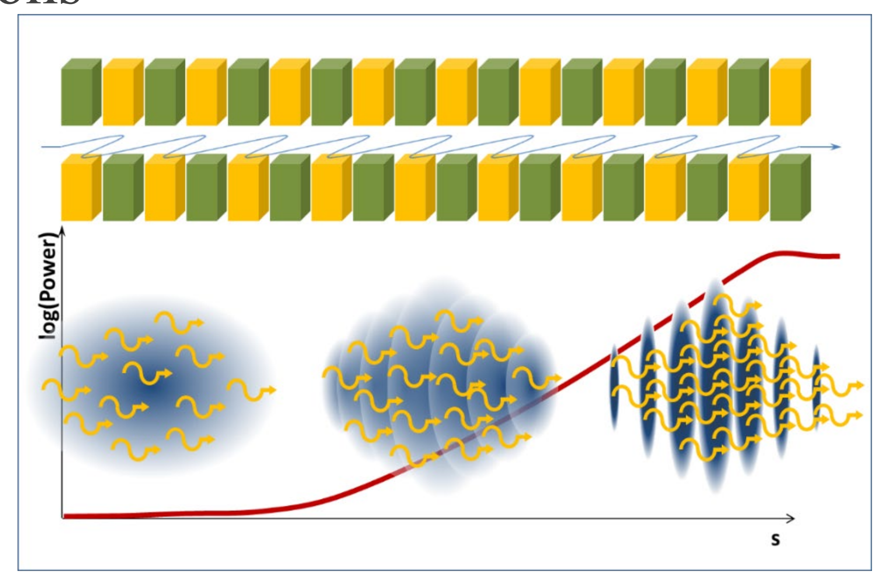


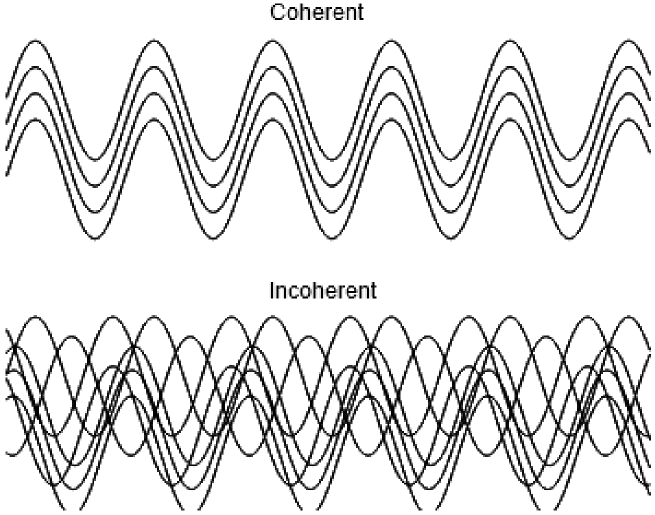

https://www.helmholtz-berlin.de/projects/berlinpro/erl-intro/linac-fel_en.html

# Coherence

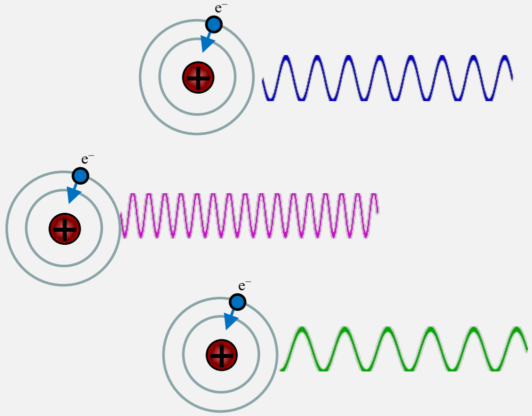
*2 waves are said to be coherent if they have a* ***constant relative phase!***

**Coherent light can interfere!**

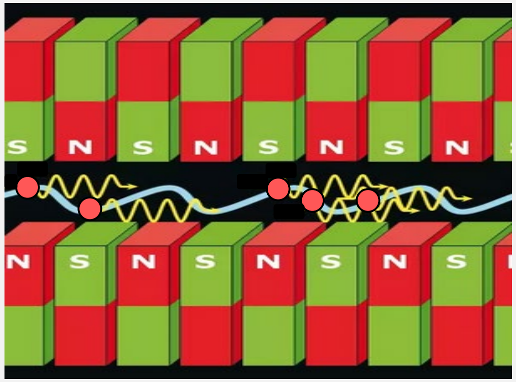
**Spontaneous emission** typically generates incoherent light:




**atoms / molecules**

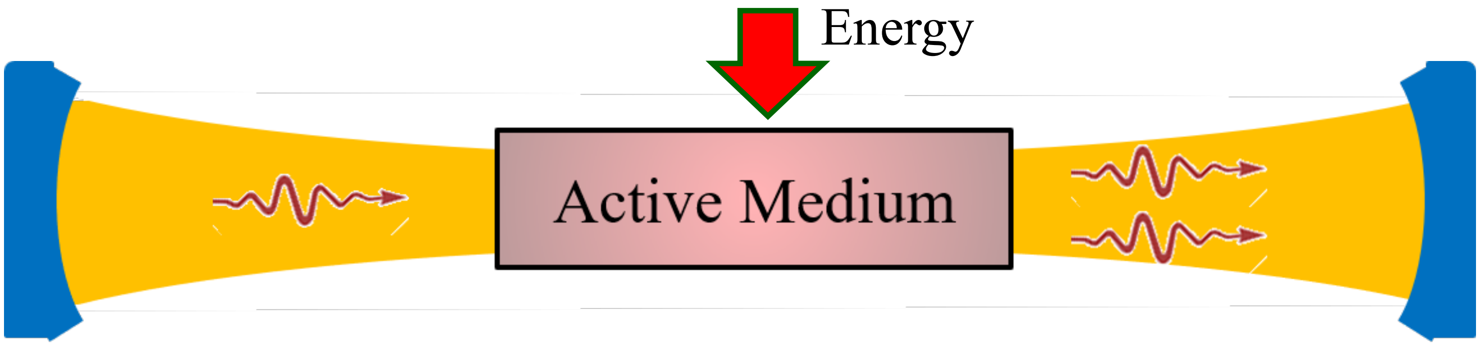


**many e⁻ in an undulator**



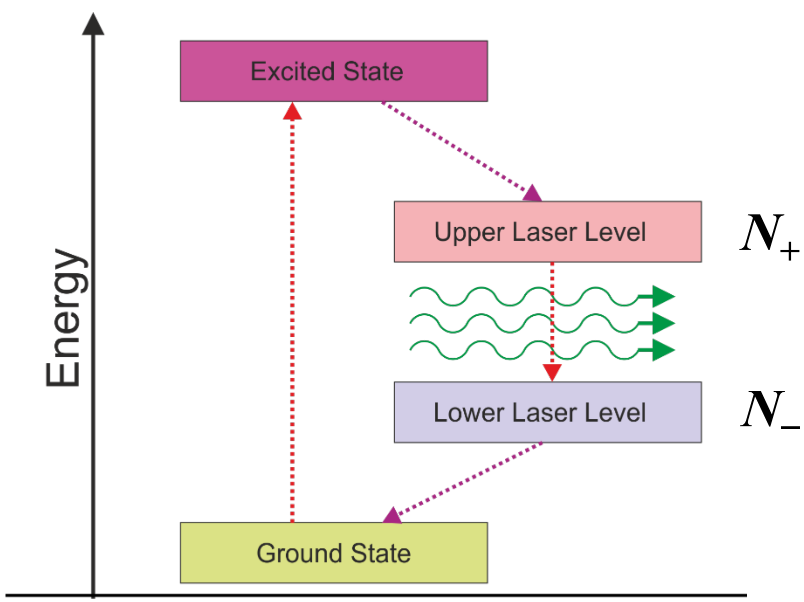
# Optical Laser

**Laser: L**ight **A**mplification by **S**timulated **E**mission of **R**adiation



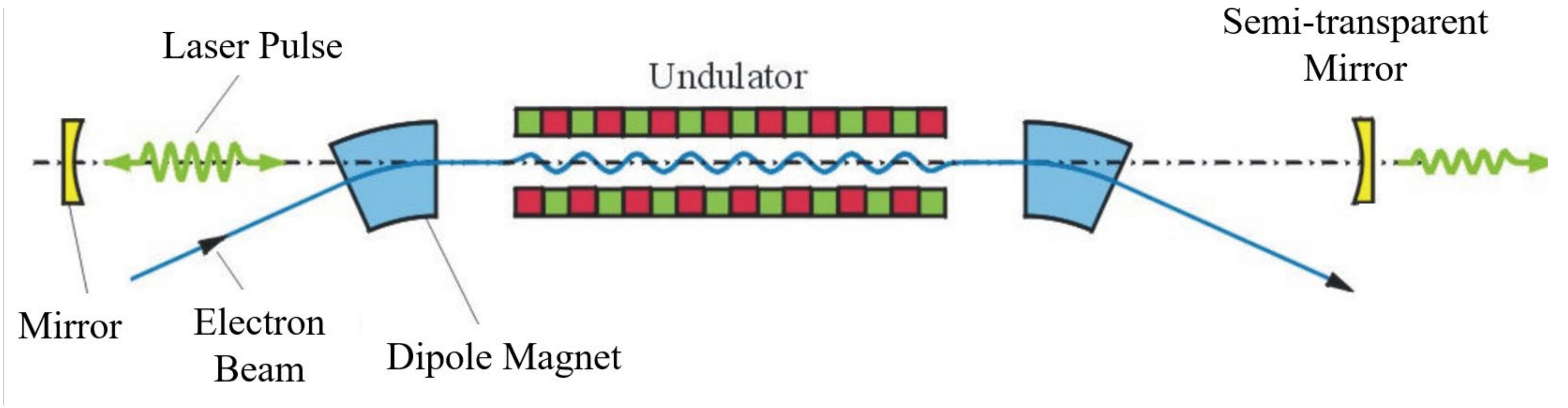

Amplification requires a **population inversion**:

→ 4 Level System



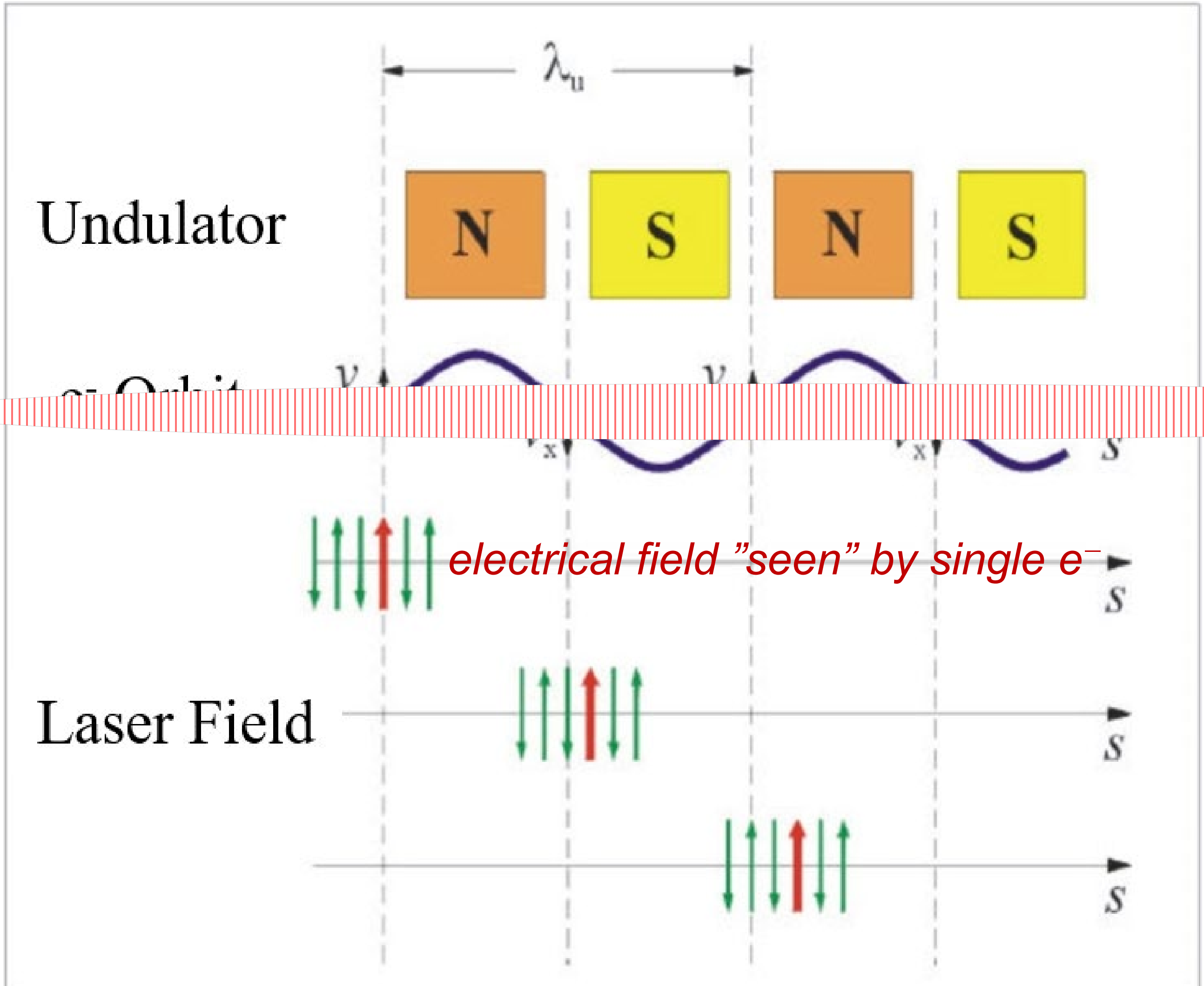
# Free Electron Laser

**Electron Beam in Undulator serves as Active Medium!**




# Population Inversion??

# FEL Amplification




**Typical Dimensions**

$\lambda_U \approx \text{cm}$

$\sigma_e > 0.1\ \text{mm}$

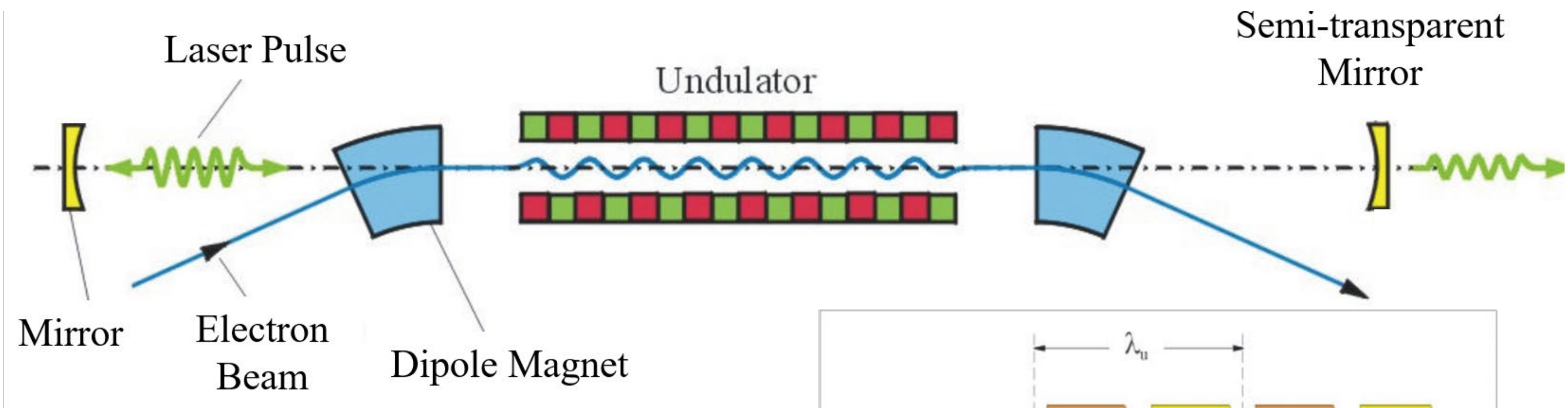
$\lambda_L \approx \text{nm}$

required bunching:

$$\sigma_e < \frac{1}{2}\lambda_L$$

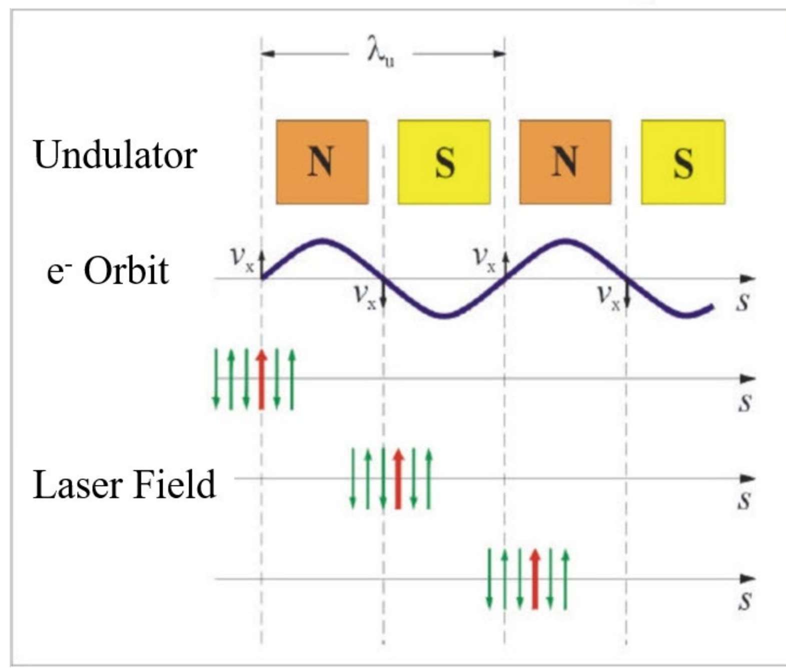
# Free Electron Laser

## Electron Beam in Undulator serves as Active Medium!




Amplification requires
**Microbunching!**

<u>and in addition:</u>
→ correct phase slippage
**→ correct relative phasing**



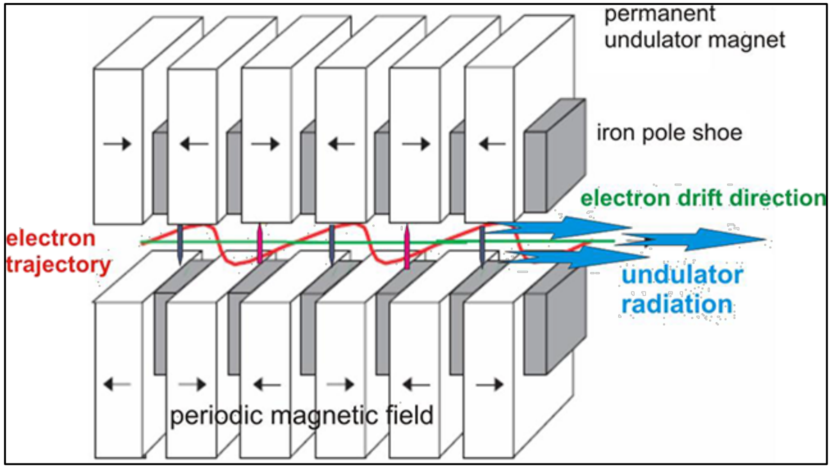
# Recap: Undulator Radiation

## Particle orbit in the undulator:

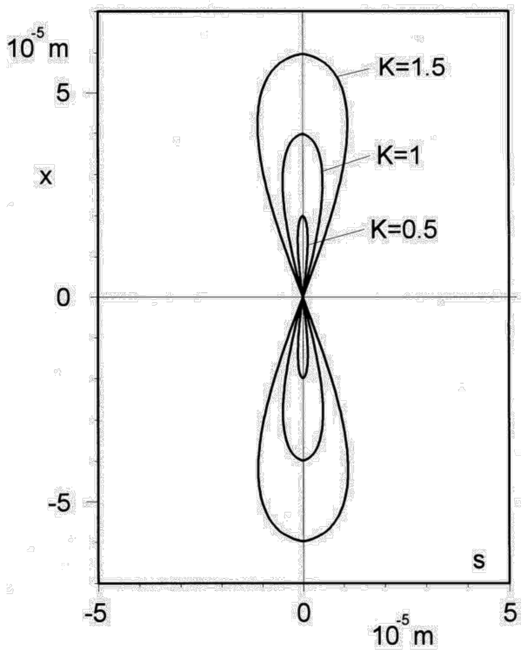
$$x(t) = \frac{K}{\gamma k_u} \cdot \sin(\omega_u t)$$

$$s(t) = \bar{\beta} ct - \frac{K^2}{8\gamma^2 k_u} \cdot \sin(2\omega_u t)$$

$$\bar{\beta} = 1 - \frac{1}{2\gamma^2}\left\{1 + \frac{K^2}{2}\right\}$$




Taken from Schmüser/Dohlus/Rossbach/Behrens




## Coherence condition in forward direction:

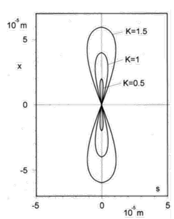
$$\lambda_L = \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) \cdot \lambda_u = \left(1 - \bar{\beta}\right) \cdot \lambda_u$$

## Radiation power per e⁻ (1st harmonic):

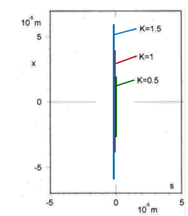
$$P = \frac{e^2 c \gamma^2 K^2 k_u^2}{12\pi\varepsilon_0 \left(1 + K^2/2\right)^2}$$

# Single Electron Energy Change with the Laser Field

Remark:
In the following, we want to neglect the longitudinal oscillation completely in order to achieve the aim (understanding!) preferably simply and fast. For a correct treatment, we then would have to modify the *K* parameter accordingly to (without proof):

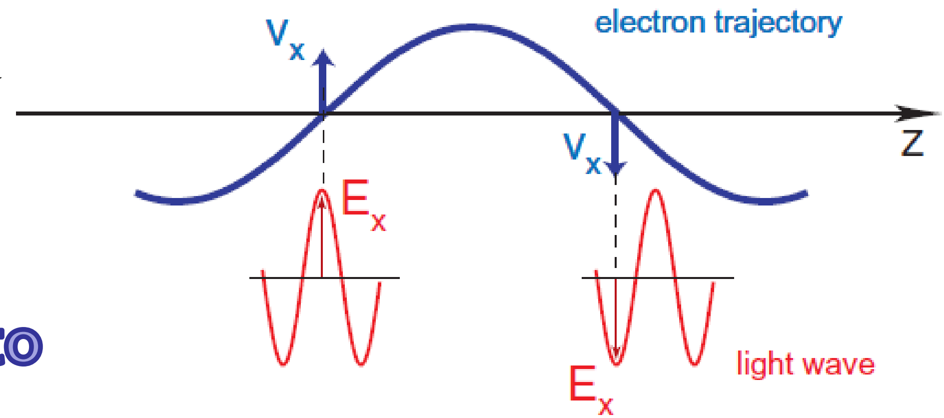



$$K \rightarrow K_{JJ} = K\left\{ J_0\left(\frac{K^2}{4+2K}\right) - J_1\left(\frac{K^2}{4+2K}\right)\right\}$$




Energy change of a single electron in the an externally generated laser field

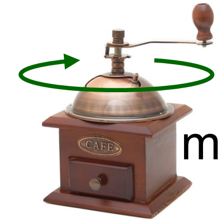
$$\frac{\mathrm{d}W}{\mathrm{d}t} = \vec{F}\cdot\vec{v} = -e\,E_x(t)\,v_x(t)$$




**add. energy gain/loss due to interaction with EM field**

Taken from Schmüser/Dohlus/Rossbach/Behrens


# Energy Exchange




We derived for the transverse electron orbit

$$v_x = \dot{x} = c \cdot \frac{K}{\gamma} \cos(k_u s), \qquad k_u = \frac{2\pi}{\lambda_u}$$

and the radiation field

relative phase to electron oscillation

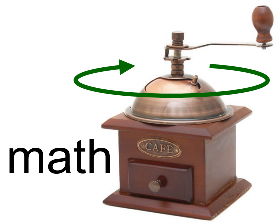
$$E_x(t) = E_0 \cos(\omega_L t - k_L s + \phi_L), \qquad k_L c = \omega_L$$

and with $\cos\alpha \cdot \cos\beta = \frac{1}{2}\{\cos(\alpha-\beta) + \cos(\alpha+\beta)\}$

$$\frac{\mathrm{dW}}{\mathrm{d}t} = -e E_x(t)\,\dot{x} = -e \frac{K_{JJ} c}{\gamma} \cos(k_u s)\, E_0 \cos(k_L s - \omega t + \phi_L)$$

$$= -e \frac{K_{JJ} c}{2\gamma} E_0 \left\{ \cos\Big(\underbrace{(k_L + k_u) s - \omega t + \phi_L}_{=\psi}\Big) + \cos\Big(\underbrace{(k_L - k_u) s - \omega t + \phi_L}_{=\chi}\Big) \right\}$$

→ **Definition of the two phases $\psi$ and $\chi$!**




# Energy Exchange


Energy variation is depending on 2 phases $\psi$ and $\chi$ :

$$\boxed{\frac{\mathrm{dW}}{\mathrm{d}t} = -e\frac{K_{JJ}\,c}{2\gamma}E_0\left(\cos\psi + \cos\chi\right)}$$

The phase $\psi$ is slowly varying and $\dot{\psi} = 0$ on resonance:

$$\boxed{\omega = k_L c}$$

$$\dot{\psi} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(k_L + k_u\right)s - \omega t + \phi_L\right\} = \left(k_L + k_u\right)\bar{\beta}c - \omega = \left(k_L + k_u\right)\bar{\beta}c - k_L c$$

$$= \left[k_u\bar{\beta} - \left(1 - \bar{\beta}\right)k_L\right]c$$

since for the resonant $k_L$ of the light wave (coherence condition!) we have

$$k_u = \left(1 - \bar{\beta}\right)\cdot k_L \qquad \rightarrow \qquad \dot{\psi} = k_u c\underbrace{\left(\bar{\beta} - 1\right)}_{\approx 0} \approx 0$$

The other phase $\chi$ is rapidly changing (by $4\pi$ over one undulator period!):

$$\dot{\chi} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(k_L - k_u\right)s - \omega t + \phi_L\right\} = \left[-k_u\bar{\beta} - \left(1 - \bar{\beta}\right)k_L\right]c = -2k_u c$$

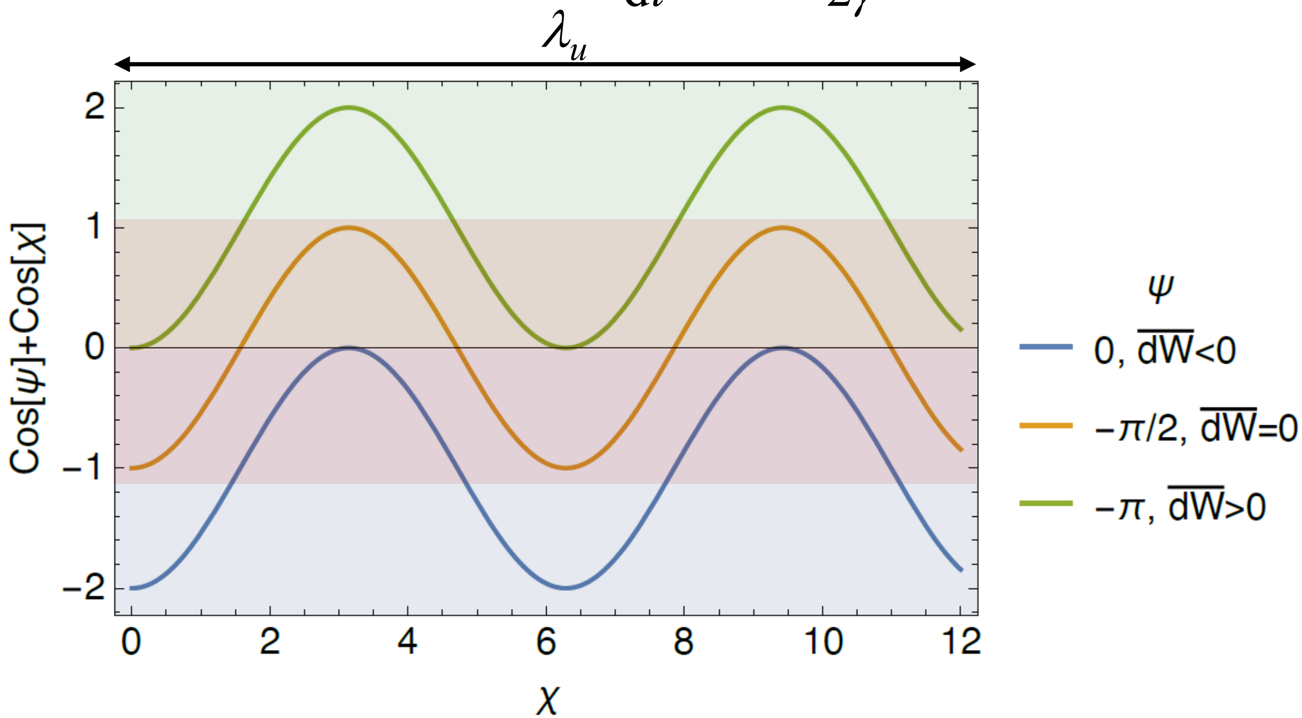
# Ponderomotive Phase θ

Phase dependency of the energy exchange: $\frac{\mathrm{dW}}{\mathrm{d}t} = -e\frac{K_{JJ}c}{2\gamma}E_0\left(\cos\psi + \cos\chi\right)$

***Electron:***

Energy Gain →

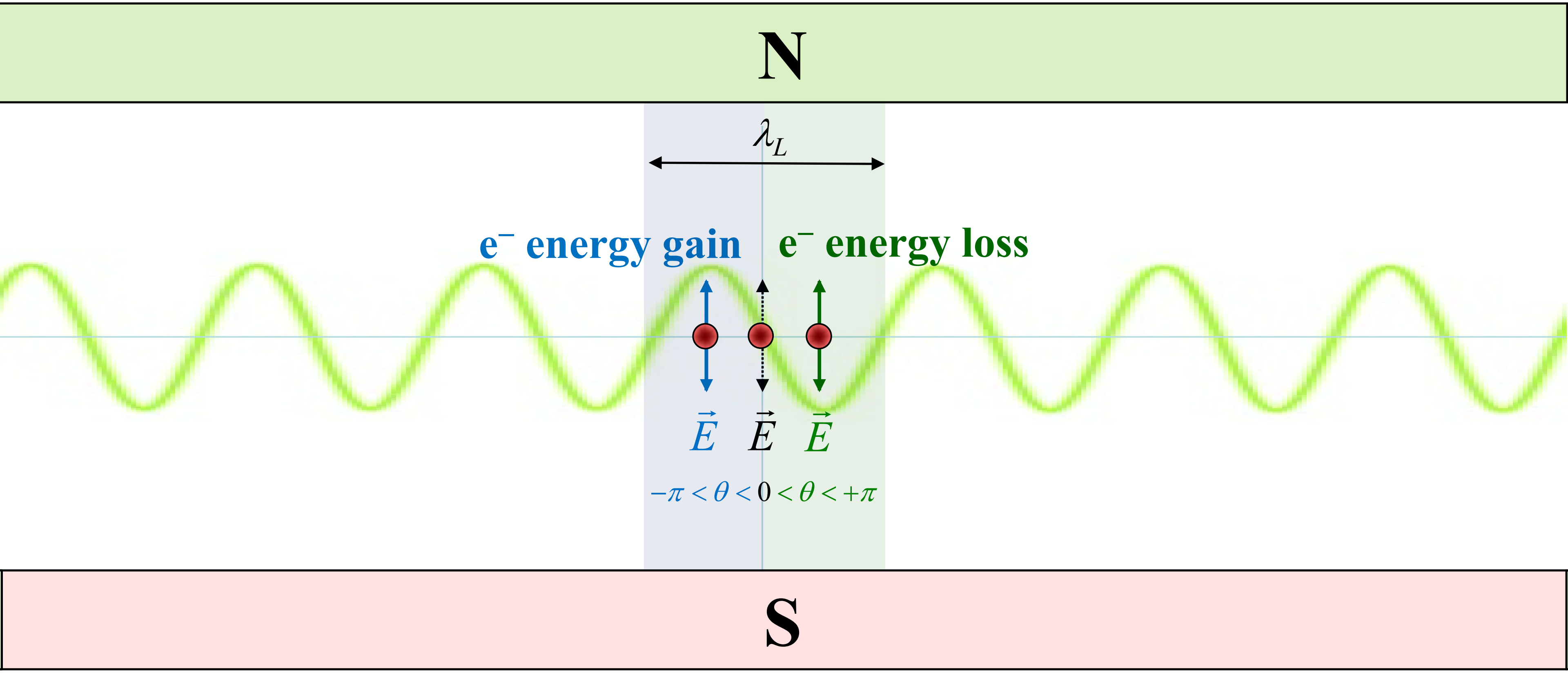
Energy Loss →




**Ponderomotive Phase:** $\theta = \psi + \pi/2$

- $-\pi < \theta < 0$: average energy transfer from EM field to electron
- $\theta = 0$: no average energy exchange
- $0 < \theta < +\pi$: average energy transfer from electron to EM field

# Electron Dynamics




**energy exchange depends on the ponderomotive phase θ**

# Key Parameters

<u>Findings so far:</u>

- average electron energy loss/gain: $\left\langle \frac{\mathrm{dW}}{\mathrm{d}t} \right\rangle = -e \frac{K_{JJ}\, c}{2\gamma} E_0 \sin\theta$
- on resonance ($\gamma = \gamma_{res}$), the ponderomotive phase is constant, $\dot{\theta} = 0$!

<u>But:</u>

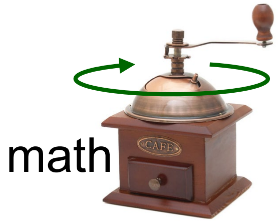
Electron energy loss or gain will cause

- change of electron's kinetic energy and Lorentz $\gamma$,
- change of the ponderomotive phase $\theta$.

interrelated!

<u>**Key parameters are therefore:**</u>

- **ponderomotive phase $\boldsymbol{\theta}$** with: $\theta = (k_L + k_u)s - \omega t + \phi_L + \pi/2$
- **relative energy deviation $\boldsymbol{\eta}$** with: $\eta = \frac{\gamma - \gamma_{res}}{\gamma_{res}}$
- **normalized field amplitude $\boldsymbol{\varepsilon}$** with: $\varepsilon = \frac{eE_0 K_{JJ}}{2m_0 c^2 \gamma_{res}^2}$




# Phase Equation


Change of the ponderomotive phase (cf. page 10):

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(k_L + k_u\right)s - \omega t + \phi_L + \pi/2\right\} = \;\ldots\; = c\left[k_u\bar{\beta} - \left(1-\bar{\beta}\right)k_L\right]$$

Now:

$$\bar{\beta} = 1 - \frac{1}{2\gamma^2}\left(1+\frac{K^2}{2}\right) = 1 - \frac{X}{\gamma^2} \qquad \text{with} \quad X = \frac{1}{2}\left(1+\frac{K^2}{2}\right) \qquad \approx 1$$

$$k_u = k_L \cdot \frac{1}{2\gamma_{res}^2}\left(1+\frac{K^2}{2}\right) = k_L\frac{X}{\gamma_{res}^2}$$

gives:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = c\Bigg[k_u\underbrace{\left(1-\frac{X}{\gamma^2}\right)}_{\bar{\beta}} - \underbrace{\frac{X}{\gamma^2}}_{1-\bar{\beta}}\underbrace{\frac{\gamma_{res}^2}{X}k_u}_{k_L}\Bigg] = ck_u\left[\left(1-\underbrace{\frac{X}{\gamma^2}}_{\approx 0}\right) - \frac{\gamma_{res}^2}{\gamma^2}\right] \approx ck_u\left(1-\frac{\gamma_{res}^2}{\gamma^2}\right)$$

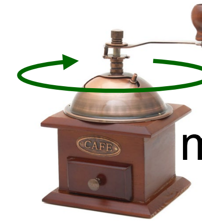
and with:

$$\frac{\gamma_{res}^2}{\gamma^2} = \frac{1}{\left(\eta+1\right)^2} \approx 1-2\eta \qquad \text{for} \quad \eta \ll 1$$

Finally:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 2ck_u\eta \qquad \rightarrow \qquad \frac{\mathrm{d}\theta}{\mathrm{d}s} = 2k_u\eta$$


# Energy Equation




We rewrite:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{1}{\gamma_{res}}\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{1}{\gamma_{res}}\frac{1}{m_0 c^2}\left\langle \frac{\mathrm{dW}}{\mathrm{d}t} \right\rangle$$

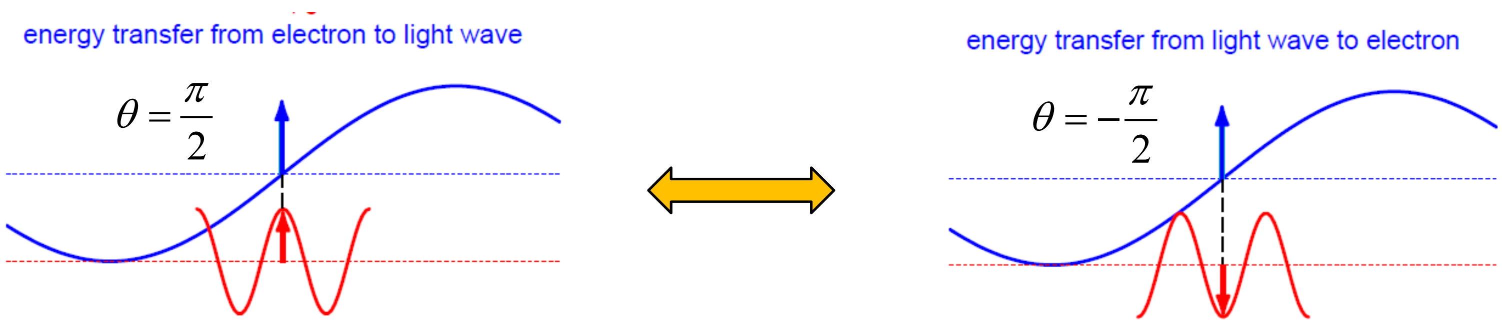
and with

$$\left\langle \frac{\mathrm{dW}}{\mathrm{d}t} \right\rangle = -e\frac{K_{JJ}\, c}{2\gamma} E_0 \sin\theta$$

one obtains:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -e\frac{K_{JJ}\, c}{2 m_0 c^2 \gamma_{res}^2} E_0 \sin\theta = -\varepsilon \cdot c \cdot \sin\theta$$

**Finally:** $\dfrac{\mathrm{d}\eta}{\mathrm{d}t} = -\varepsilon c \sin\theta \quad \rightarrow \quad \dfrac{\mathrm{d}\eta}{\mathrm{d}s} = -\varepsilon \sin\theta$



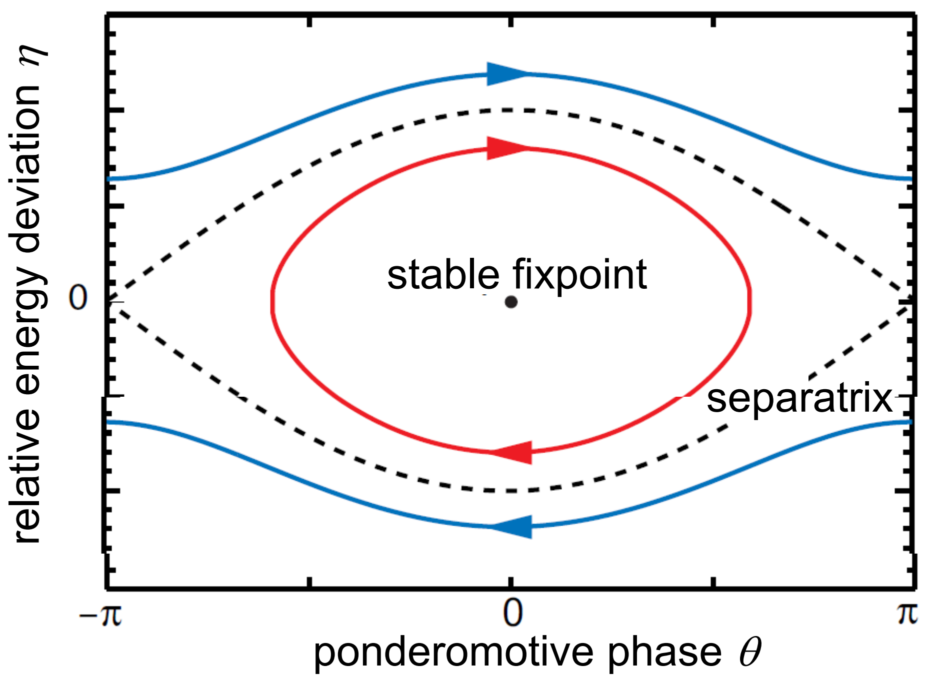

Taken from Schmüser/Dohlus/Rossbach/Behrens

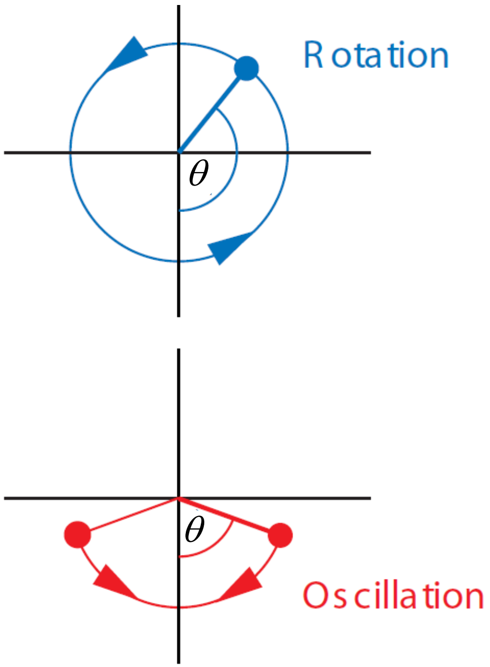
# Pendulum Equations

Phase equation: $\frac{\mathrm{d}\theta}{\mathrm{d}s} = 2k_u\eta$

Energy equation: $\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\varepsilon \sin\theta$

combined: $\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + 2k_u\varepsilon\sin\theta = 0$



math

# Stable Area ↔ Separatrix

Integrating the pendulum DGL

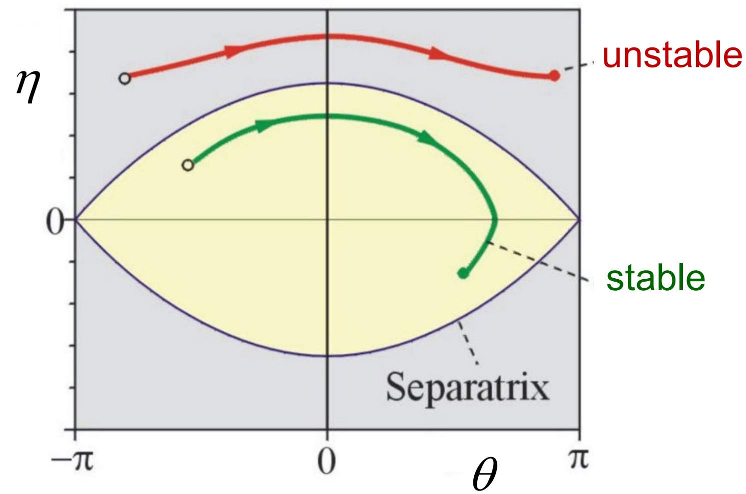
$$\theta''(s) + 2k_u\varepsilon \sin\theta(s) = 0 \quad \Big|\cdot\theta'$$

reveals

$$\frac{1}{2}\theta'^2 - 2k_u\varepsilon\cos\theta = \text{const.}$$

or with $\theta' \leftarrow \eta$:

$$k_u\eta^2 - \varepsilon\cos\theta = H$$




## Separatrix:

Trajectory $\eta_s(\theta)$ limiting the stable area of bound oscillations going through $\theta = \pm\pi$ where $\eta_s = 0$, thus $H = \varepsilon$ $\quad\rightarrow\quad k_u\eta_s^2 - \varepsilon\cos\theta = \varepsilon$

and therewith:

- maximum $\eta$ allowed for trapped motion $\boxed{\eta_{s,\max} = \sqrt{\frac{2\varepsilon}{k_u}}}$ ← *depends on intensity of laser field!*
- curve of separatrix $\eta_s(\theta) = \eta_{s,\max}\left\{\pm\sqrt{\frac{1}{2}(1+\cos\theta)}\right\}$

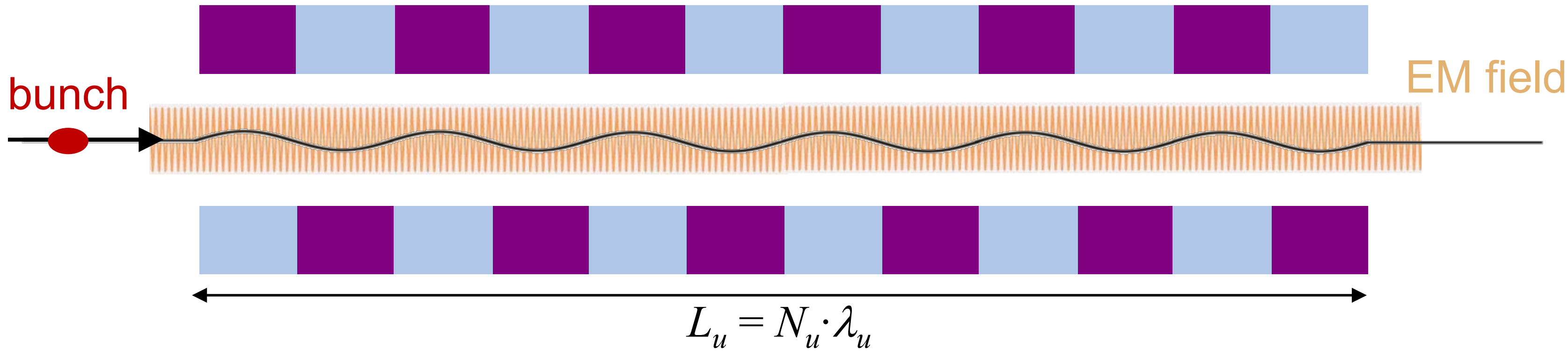
# Electron Bunch ↔ Laser Field

So far:
Interaction of a single electron with an externally generated laser field when co-propagating through an undulator

Now:
Consider an electron bunch of length $\sigma_b >> \lambda_L$

## Simplifying assumptions:

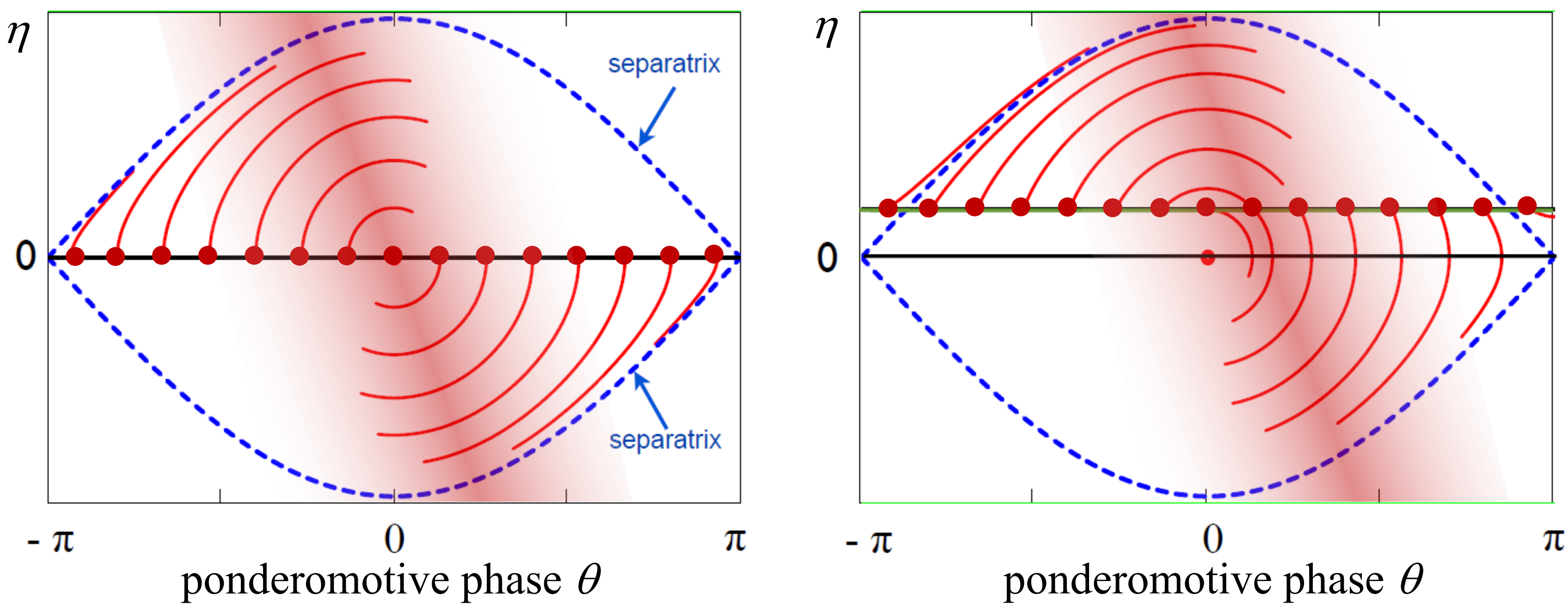
- laser field does not change significantly during bunch passage ($E$ = const.)
- "ideal" electron bunch with vanishing energy spread ($\sigma_\gamma = 0$)
- simple quasi 1D treatment of the problem ($\sigma_x$, $\sigma_y \rightarrow 0$)
- neglect spontaneous emission of undulator radiation



# Electron Bunch ↔ Laser Field

## Electron Injection

**on resonance: $\gamma = \gamma_{res}$** **above resonance: $\gamma > \gamma_{res}$**




→ energy modulation
→ density modulation
**no net energy transfer!**

→ energy modulation
→ density modulation
**net energy transfer!**

# Gain Function

**FEL gain function *G*** defined as relative growth of laser light intensity:

$$G = \frac{\Delta I_L}{I_L} \qquad \text{with} \qquad I_L = \varepsilon_0 E^2 \cdot V$$

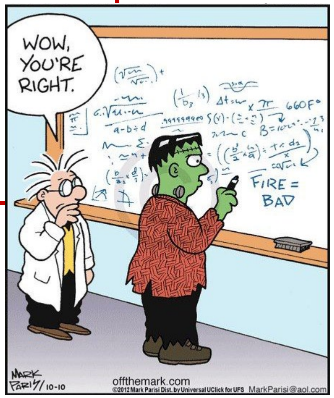
Since amplification = growth of laser light intensity is caused by energy transfer from $N_e$ electrons to the laser field, we have (with $n_e = N_e / V$)

$$G = -\frac{m_0 c^2 N_e \langle \Delta\gamma \rangle}{I_L} = -\frac{\gamma_{res} m_0 c^2 n_e \langle \eta(s = L_u) \rangle}{\varepsilon_0 E_0^2}$$

A somehow lengthy Taylor expansion up to second order for small $\varepsilon$ ($\varepsilon << k_u^{-1} L_u^{-2}$) gives

$$G = \frac{\pi e^2 L_u^3 E_0^2 K_{JJ}^2 n_e}{4\gamma_{res}^3 m_0 c^2 \varepsilon_0 \lambda_u} \frac{\mathrm{d}}{\mathrm{d}\xi_0} \left( \frac{\sin \xi_0}{\xi_0} \right)^2$$

where $\xi_0 = k_u L_u \eta_0$ at the undulator entrance ($s = 0$)



# Madey Theorem

$$\frac{\Delta\omega}{\omega_{res}} = \frac{\xi_0}{\pi N_u}$$

$$G = \frac{\pi}{2}\frac{K_{JJ}^2 N_u^3}{\gamma_r^3}\frac{I_{\text{beam}}}{I_{\text{Alfvén}}}\left(\frac{\lambda_u}{\sigma_r}\right)^2 \frac{\mathrm{d}}{\mathrm{d}\xi_0}\left(\frac{\sin\xi_0}{\xi_0}\right)^2$$

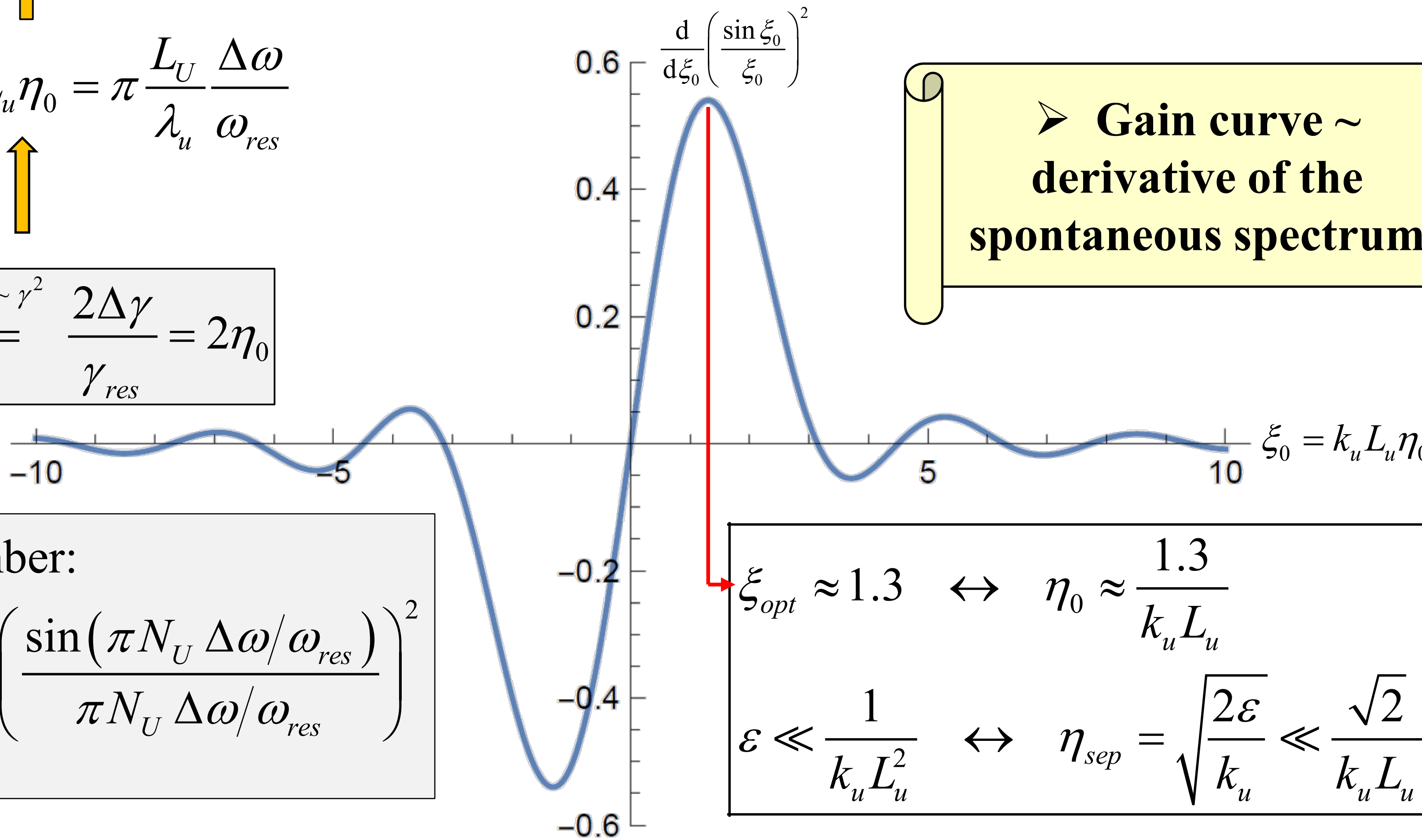
$$I_{\text{Alfvén}} = \frac{4\pi\varepsilon_0 c^3}{e} \approx 17\,\text{kA}$$

$$\xi_0 = k_u L_u \eta_0 = \pi \frac{L_U}{\lambda_u}\frac{\Delta\omega}{\omega_{res}}$$

$$\frac{\Delta\omega}{\omega_{res}} \overset{\omega\sim\gamma^2}{=} \frac{2\Delta\gamma}{\gamma_{res}} = 2\eta_0$$

remember:

$$I(\omega) \sim \left(\frac{\sin\left(\pi N_U \,\Delta\omega/\omega_{res}\right)}{\pi N_U\,\Delta\omega/\omega_{res}}\right)^2$$




- **Gain curve ~ derivative of the spontaneous spectrum**

$$\xi_{opt} \approx 1.3 \quad \leftrightarrow \quad \eta_0 \approx \frac{1.3}{k_u L_u}$$

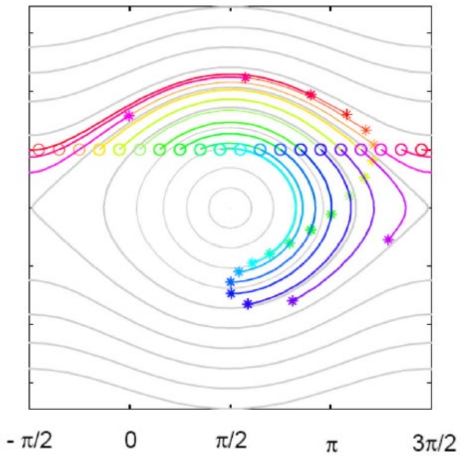
$$\varepsilon \ll \frac{1}{k_u L_u^2} \quad \leftrightarrow \quad \eta_{sep} = \sqrt{\frac{2\varepsilon}{k_u}} \ll \frac{\sqrt{2}}{k_u L_u} \approx \eta_0$$

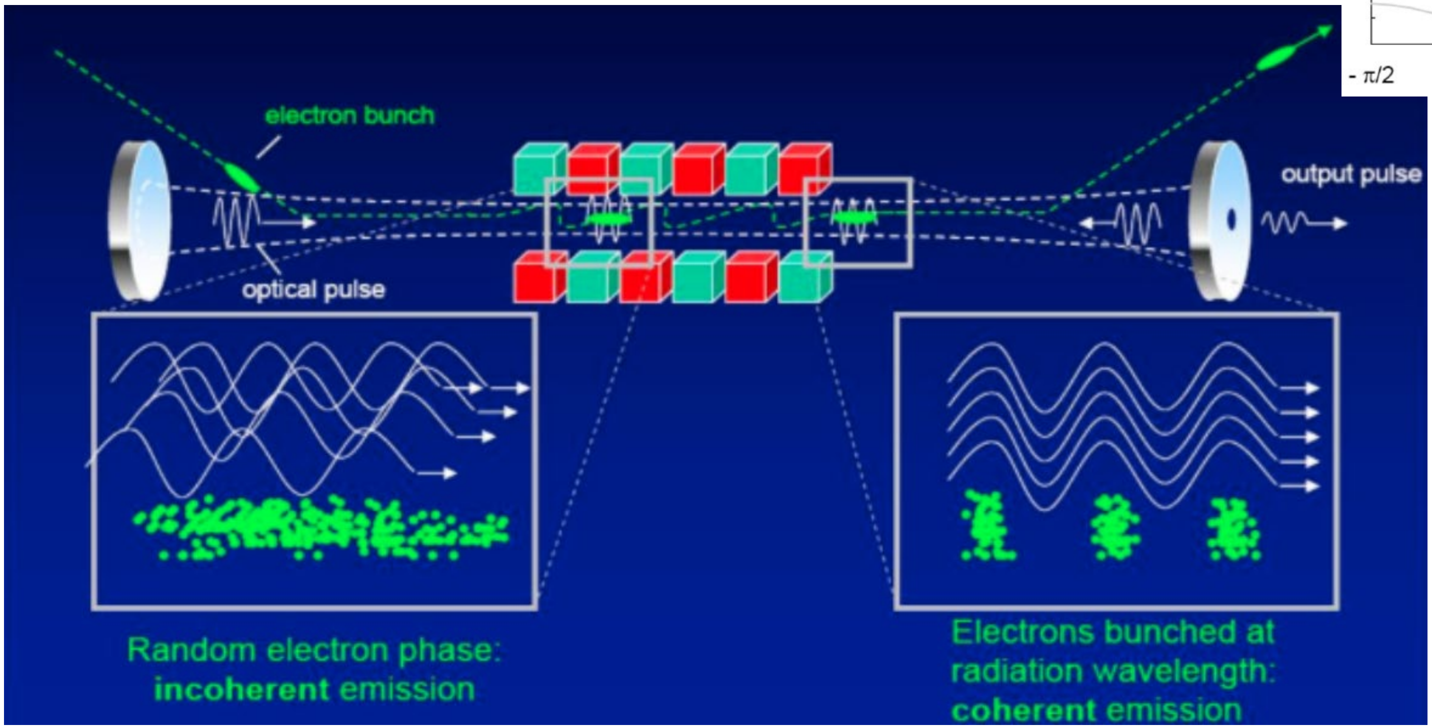
# Low Gain FEL

## Injection with energy above resonance energy:

- Energy modulation
  - Density modulation
    - Energy transfer

Intensity build-up over many passes!




Taken from http://www.stfc.ac.uk/astec/17452.aspx

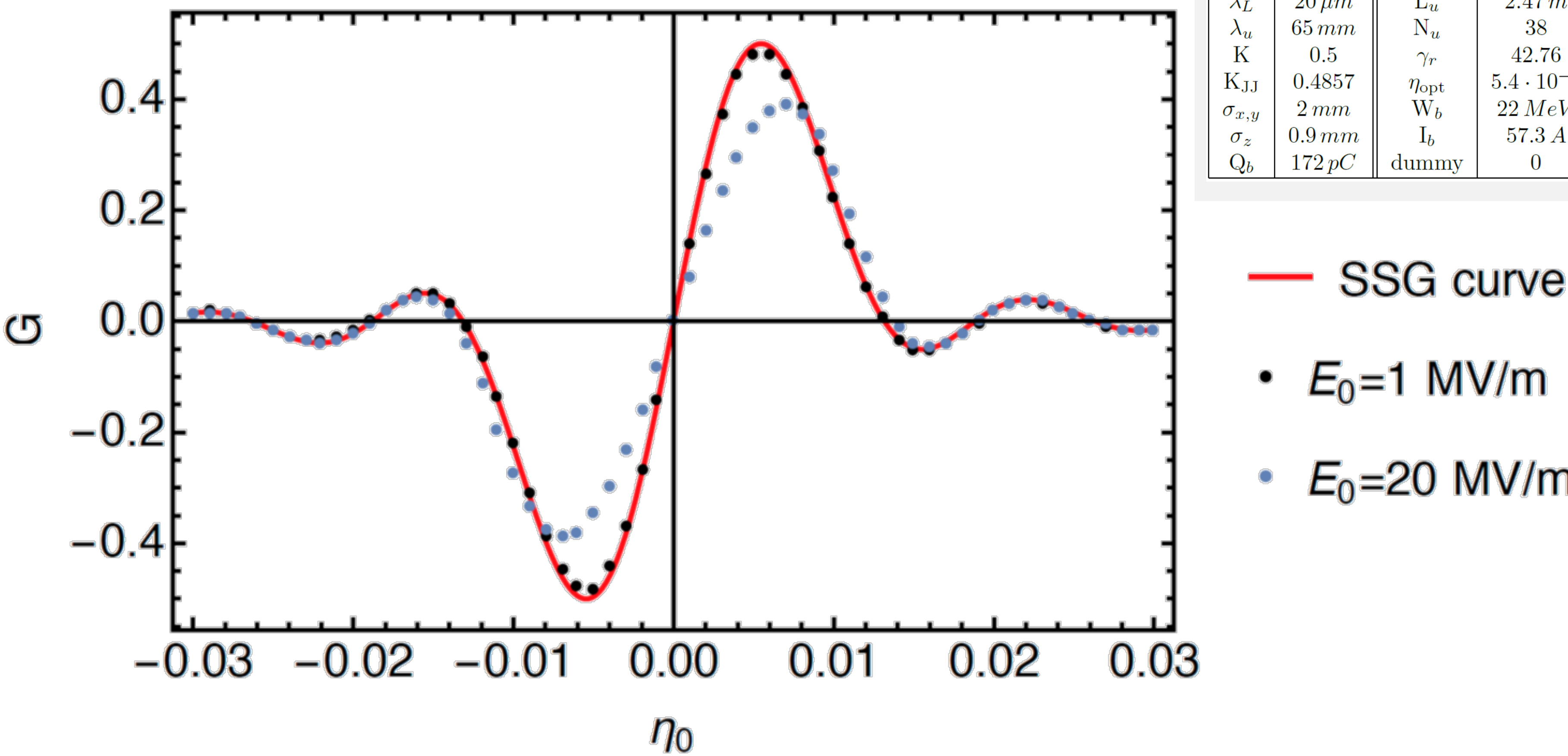
# Gain Curve

## Gain curve ↔ Madey Theorem:

Deviation for strong radiation fields

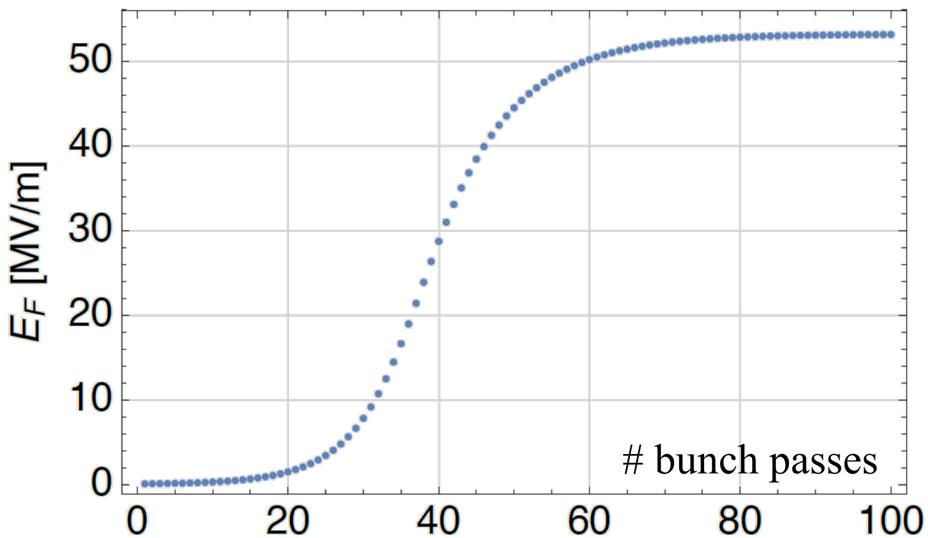
Example:
IR-FEL FELIX
( U Nijmegen)

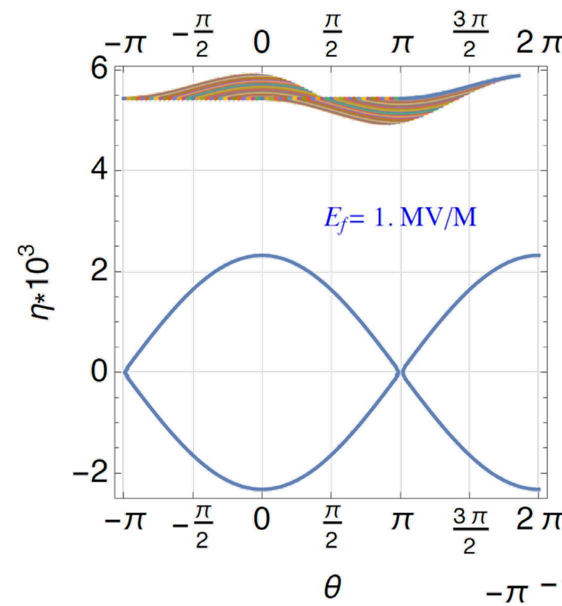
| | | | |
|---|---|---|---|
| $\lambda_L$ | $20\,\mu m$ | $L_u$ | $2.47\,m$ |
| $\lambda_u$ | $65\,mm$ | $N_u$ | 38 |
| K | 0.5 | $\gamma_r$ | 42.76 |
| $K_{JJ}$ | 0.4857 | $\eta_{opt}$ | $5.4 \cdot 10^{-3}$ |
| $\sigma_{x,y}$ | $2\,mm$ | $W_b$ | $22\,MeV$ |
| $\sigma_z$ | $0.9\,mm$ | $I_b$ | $57.3\,A$ |
| $Q_b$ | $172\,pC$ | dummy | 0 |

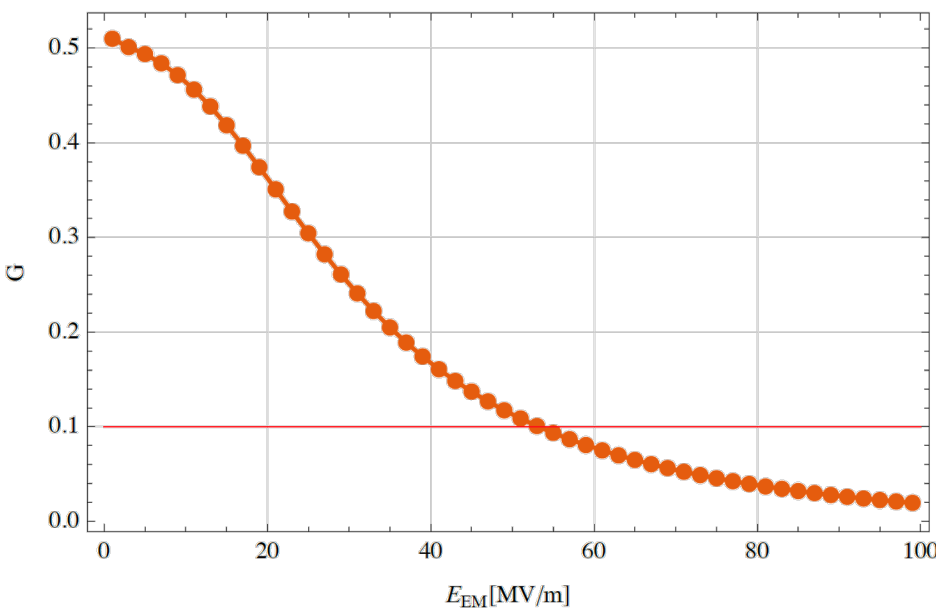


# Saturation

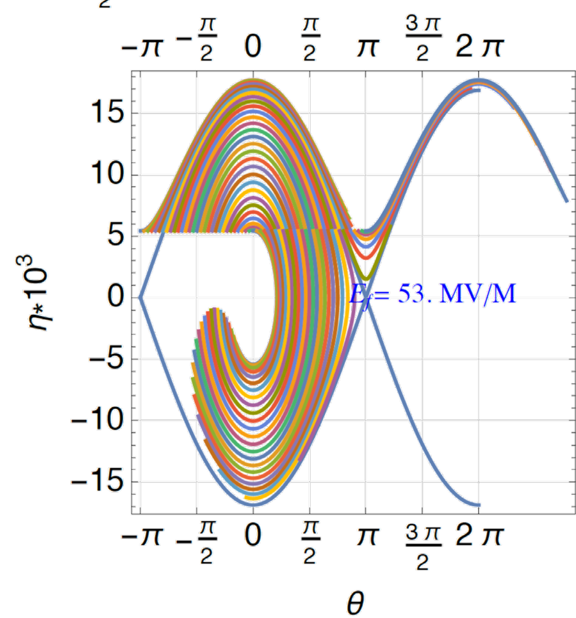
## Resonator losses are compensated by gain



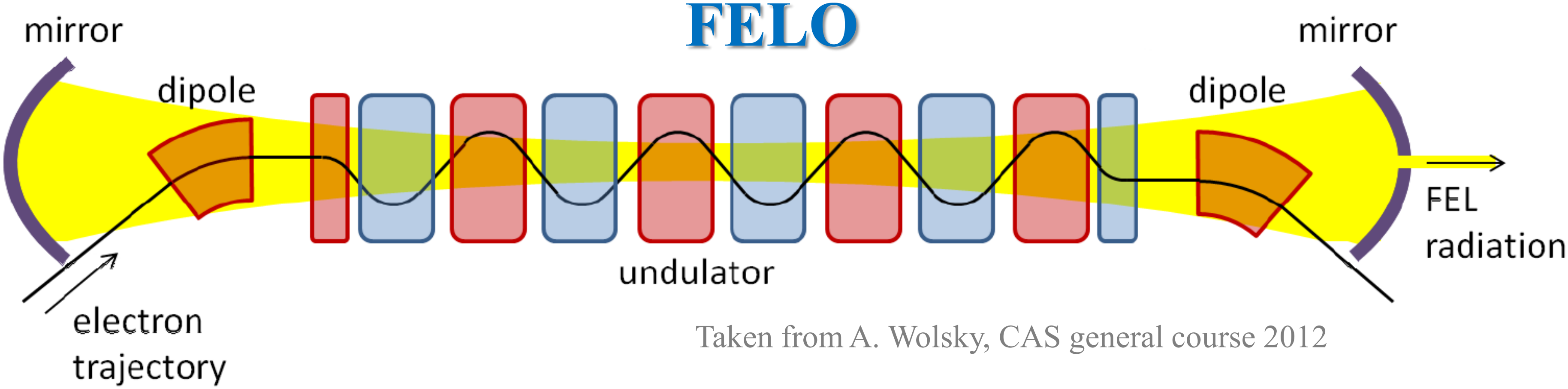




**Saturation!**



# Efficiency

## FELO




Taken from A. Wolsky, CAS general course 2012

**Optimum undulator length for FELO:**

- sufficient gain to compensate resonator losses: $G \sim N_u{}^3$
- high efficiency of energy transfer: $\Delta\eta_{sat} \approx 3/N_u$

*→ some ‰ of the beam energy is transferred to the radiation*

**How can we produce XUV and hard X-rays where no suitable mirrors are available?**

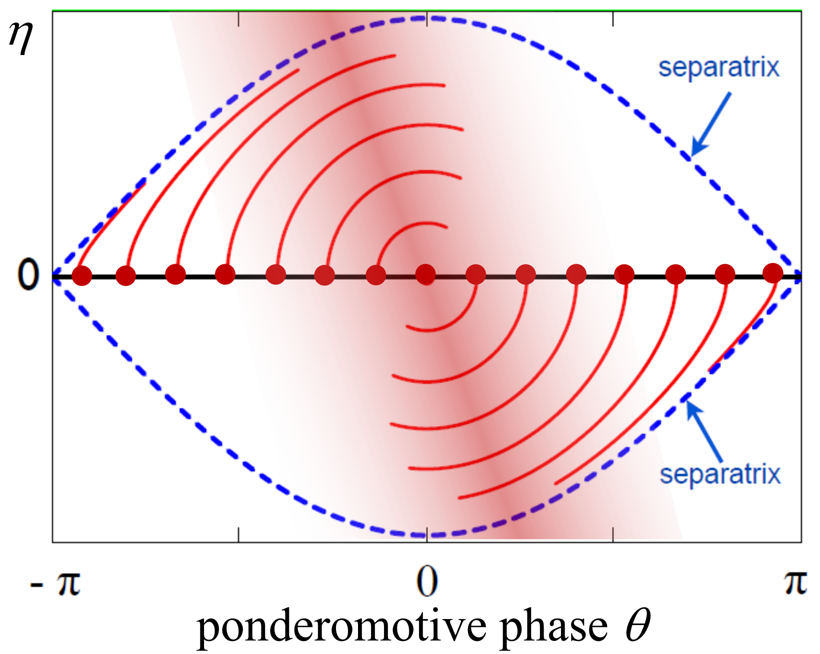
# So far…

… we have neglected that the energy exchange between electrons and the laser field will cause a change of the EM field intensity and set $E = E_0 = \text{const.}$ for a single passage of the undulator!

**This might be wrong for a "long" undulator!**

What happens if we make the undulator "longer" and consider a slowly varying field intensity?

**Remember – injection on resonance:**




→ energy modulation
→ density modulation
**no net energy transfer!**

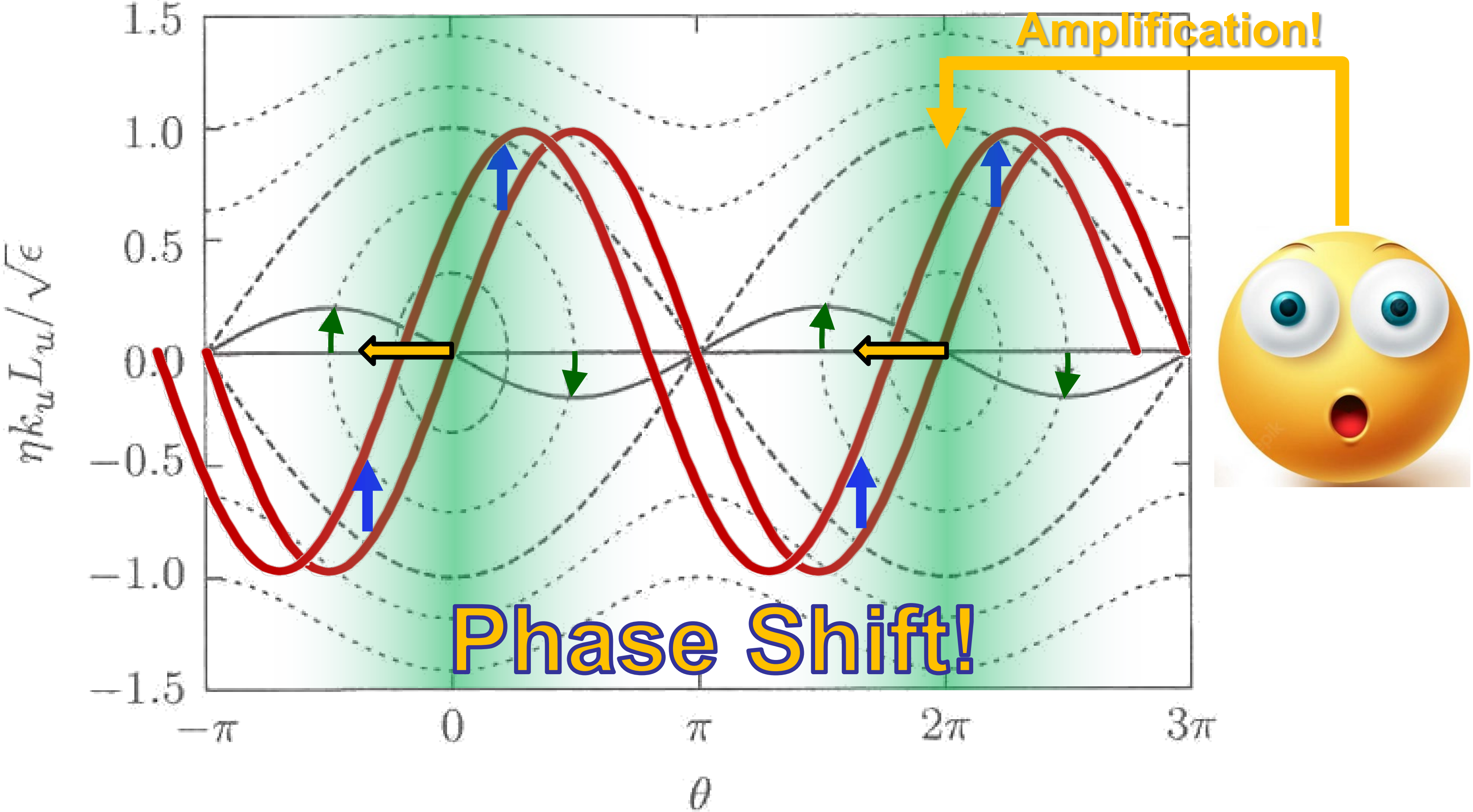
**really??**

→ ***be carefull…***

# Slow Variation of Laser Field

**Injection on resonance!**

**Interaction with external generated laser field**



# Extended Pendulum Equations

We have to extend the existing pendulum equations

- phase equations $\frac{\mathrm{d}\theta_j(s)}{\mathrm{d}s} = 2k_u\eta_j(s)$
- energy equations $\frac{\mathrm{d}\eta_j(s)}{\mathrm{d}s} = -\varepsilon\sin\theta_j(s)$

} $2\,N_e$ equations!

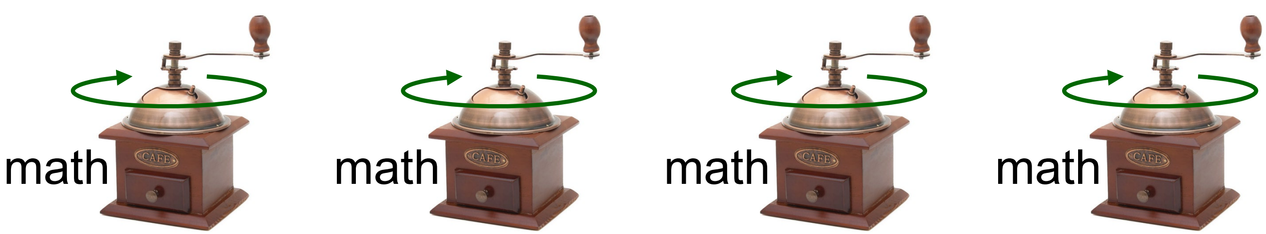
by an additional equation describing the slowly varying EM field

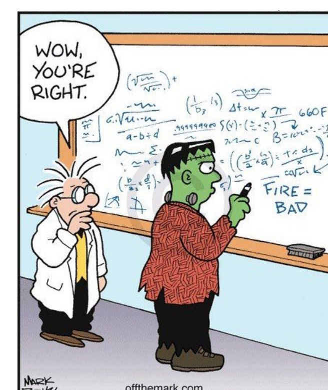
- field equation $\frac{\mathrm{d}E}{\mathrm{d}s} = ???$ (remember: $\varepsilon = \frac{eE_0K_{JJ}}{2m_0c^2\gamma_{res}^2}$)

**and to consider a slowly varying amplitude and phase (→ complex $E$ )!**

---

**Warning: What follows is a condensed version of the somehow lengthy math (show how to get there)!**



# Field Change in 1D Approx.

Slowly varying **amplitude** and **phase** ("*S*" means *slowly varying*):

$$E_x(s,t) = \hat{E}_S(s,t) \cdot \cos\left(k_L s - \omega t + \phi_S(s,t)\right)$$

Change to **complex field amplitude** defined by:

$$\tilde{E}(s,t) = \frac{1}{2}\hat{E}_S(s,t) \cdot e^{i\phi_S(s,t)}$$

$$\rightarrow \quad E_x(s,t) = \tilde{E}(s,t) \cdot e^{i(k_L s - \omega t)} + \tilde{E}^*(s,t) \cdot e^{-i(k_L s - \omega t)} = 2\,\mathrm{Re}\left\{\tilde{E}(s,t) \cdot e^{i(k_L s - \omega t)}\right\}$$

Wave equation links laser field and electron current in the undulator:

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right\}\vec{E} = -\mu_0 \frac{\partial \vec{j}}{\partial t}$$

1-dim

$$\frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} - \frac{\partial^2 E_x}{\partial s^2} = -\frac{1}{\varepsilon_0 c^2}\frac{\partial j_x}{\partial t}$$

# Field Change in 1D Approx.

*First Trick*: decompose the wave operator using

$$\partial_\pm = \frac{1}{c}\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x} \qquad \rightarrow \qquad \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} = \partial_+ \cdot \partial_-$$

Slowly varying complex field amplitude then means:

$$\left|\partial_\pm \tilde{E}\right| \cdot \lambda_L \ll \left|\tilde{E}\right| \qquad \text{or} \qquad \left|\partial_\pm \tilde{E}\right| \ll \left|\tilde{E}\right| \cdot k_L$$

We now have to compute

$$\partial_+ \cdot \partial_- E_x(s,t) = \partial_+ \cdot \partial_- \left( \tilde{E} \cdot e^{i(k_L s - \omega t)} + \tilde{E}^* \cdot e^{-i(k_L s - \omega t)} \right)$$

$= 0 \quad \leftrightarrow \quad \omega = k_L c$

$\approx 0 \quad \leftrightarrow \quad \left|\partial_- \tilde{E}\right| \ll k_L \left|\tilde{E}\right|$

Using

$$\partial_+ e^{i(k_L s - \omega t)} = -i\left(\frac{\omega}{c} - k_L\right) e^{i(k_L s - \omega t)} = 0 \qquad \partial_- e^{i(k_L s - \omega t)} = -i\left(\frac{\omega}{c} + k_L\right) e^{i(k_L s - \omega t)} = -2i\, k_L e^{i(k_L s - \omega t)}$$

we first get, since $\tilde{E}$ is slowly varying

$$\partial_+ \cdot \partial_- E_x(s,t) = -\partial_+ \left[ \left(2ik_L \tilde{E}\right) \cdot e^{i(k_L s - \omega t)} - \left(2ik_L \tilde{E}^*\right) \cdot e^{-i(k_L s - \omega t)} \right]$$

and finally

$$\partial_+ \cdot \partial_- E_x(s,t) = -2ik_L \left[ \left(\partial_+ \tilde{E}\right) \cdot e^{i(k_L s - \omega t)} - \left(\partial_+ \tilde{E}^*\right) \cdot e^{-i(k_L s - \omega t)} \right]$$

# Field Change in 1D Approx.

We insert the result in the wave equation

$$2ik_L\left[\left(\partial_+\tilde{E}\right)\cdot e^{i\left(k_L s-\omega t\right)}-\left(\partial_+\tilde{E}^*\right)\cdot e^{-i\left(k_L s-\omega t\right)}\right]=\frac{1}{\varepsilon_0 c^2}\frac{\partial j_x}{\partial t}\qquad \Big|\cdot e^{-i\left(k_L s-\omega t\right)}$$

multiply with the phase factor and obtain

$$2ik\cdot\left(\partial_+\tilde{E}\right)-2ik\cdot\left(\partial_+\tilde{E}^*\right)\cdot e^{-2i\left(k_L s-\omega t\right)}=\frac{1}{\varepsilon_0 c^2}\frac{\partial j_x}{\partial t}\cdot e^{-i\left(k_L s-\omega t\right)}$$

<u>*Second Trick*</u>: Since the field amplitude is slowly varying, we average over a small number $n$ of the rapidly oscillating periods $T$, thus $\Delta t = 2n\pi/\omega$ and use

$$\rightarrow\quad \frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2} 2ik\left(\partial_+\tilde{E}\right)\mathrm{d}t\approx 2ik\left(\partial_+\tilde{E}\right),\qquad \frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2} 2ik\left(\partial_+\tilde{E}^*\right)e^{-2i\left(k_L s-\omega t\right)}\mathrm{d}t\approx 0$$

yielding

$$\boxed{2ik\cdot\left(\partial_+\tilde{E}\right)=\frac{1}{\varepsilon_0 c^2}\frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2}\frac{\partial j_x}{\partial t}\cdot e^{-i\left(k_L s-\omega t\right)}\mathrm{d}t}$$

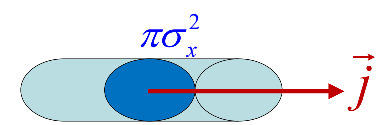
# Field Change in 1D Approx.

*Third Trick*: We integrate by parts and assume that $j_x$ is periodic in $\lambda_L$

$$2ik\cdot\left(\partial_+\tilde{E}\right)=\frac{1}{\varepsilon_0 c^2}\frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2}\underbrace{\frac{\partial j_x}{\partial t}}_{u'}\cdot\underbrace{e^{-i\left(k_L s-\omega t\right)}}_{v}\mathrm{d}t=\frac{i\omega}{\varepsilon_0 c^2}\frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2}\underbrace{j_x}_{u}\cdot\underbrace{e^{-i\left(k_L s-\omega t\right)}}_{v'/i\omega}\mathrm{d}t$$

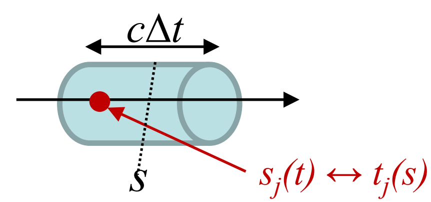
The current density is generated by single electrons (at positions $s_j$) having a transverse velocity from the undulator motion. Assuming that the bunch "fills" a transverse area $\pi\sigma_x^2$ and $\gamma_j\approx\gamma_{res}$ we obtain

$$j_x=\frac{-e}{\pi\sigma_x^2}\underbrace{\frac{cK}{\gamma_{res}}\cos\left(k_u s\right)}_{=\dot{x}}\sum_{j=1}^{N_e}\delta\left(s-s_j(t)\right)$$




and therewith

$$2ik\cdot\left(\partial_+\tilde{E}\right)=\frac{i\omega}{\varepsilon_0 c^2}\frac{-e}{\pi\sigma_x^2}\frac{cK}{\gamma_{res}}\frac{1}{c\Delta t}\sum_{j=1}^{N_\Delta}\cos\left(k_u s\right)e^{-i\left(k_L s-\omega t_j\right)}$$




which yields with replacing the sum by the average over all $N_\Delta=n_e\left(\pi\sigma_x^2\right)\left(c\Delta t\right)$ electrons in the slice $\Delta t$:

$$\partial_+\tilde{E}=-\frac{eKn_e}{2\varepsilon_0\gamma_{res}}\left\langle\cos\left(k_u s\right)e^{-i\left(k_L s-\omega t_j\right)}\right\rangle$$

# Field Change in 1D Approx.

We express $\cos(k_u s)$ by its complex representation

$$\partial_+ \tilde{E} = -\frac{eKn_e}{2\varepsilon_0 \gamma_{res}} \left\langle \frac{e^{ik_u s} + e^{-ik_u s}}{2} e^{-i\left(k_L s - \omega t_j\right)} \right\rangle = -\frac{eKn_e}{4\varepsilon_0 \gamma_{res}} \left\langle e^{i\left(k_u s - k_L s + \omega t_j\right)} + e^{-i\left(k_u s + k_L s - \omega t_j\right)} \right\rangle$$

*<u>Fourth Trick</u>*: We now use the definition of the phases $\psi$ and $\chi$ (cf. page 9) and neglect again the longitudinal oscillation by replacing $K \rightarrow K_{JJ}$:

$$\partial_+ \tilde{E} = -\frac{eK_{JJ}n_e}{4\varepsilon_0 \gamma_{res}} \left\langle e^{-i\chi_j} + e^{-i\psi_j} \right\rangle \approx -\frac{eK_{JJ}n_e}{4\varepsilon_0 \gamma_{res}} \left\langle e^{-i\psi_j} \right\rangle = -\frac{eK_{JJ}n_e}{4\varepsilon_0 \gamma_{res}} \left\langle e^{-i\theta_j} \right\rangle$$

Last step

$$\frac{1}{c}\frac{\partial \theta}{\partial t} = 2k_u$$

$$\partial_+ \tilde{E} = \frac{\partial \tilde{E}(s,t)}{\partial s} + \frac{1}{c}\frac{\partial \tilde{E}(s,t)}{\partial t} = \frac{\partial \tilde{E}(s,\theta)}{\partial s} + \underbrace{2k_u \frac{\partial \tilde{E}(s,\theta)}{\partial \theta}}_{\approx 0 \text{ (for fundamental)}} \approx \frac{\mathrm{d}\tilde{E}(s,\theta)}{\mathrm{d}s}$$

and finally:

$$\boxed{\frac{\mathrm{d}\tilde{E}}{\mathrm{d}s} = -\frac{eK_{JJ}n_e}{4\varepsilon_0 \gamma_{res}} \left\langle e^{-i\theta_j} \right\rangle}$$

# Coupled 1D Equations

**→ Extension of the pendulum equations to a system of coupled differential equations:**

| | | |
|---|---|---|
| $\frac{\mathrm{d}\tilde{E}}{\mathrm{d}s} = -\kappa_2 n_e \left\langle e^{-i\theta_j} \right\rangle$ | with bunching factor | $b = \left\langle e^{-i\theta_j} \right\rangle$ |
| $\frac{\mathrm{d}\theta_j}{\mathrm{d}s} = 2k_u \eta_j$ | with ponderomotive phases | $\theta_j$ |
| $\frac{\mathrm{d}\eta_j}{\mathrm{d}s} = \kappa_1 \left( \tilde{E} \cdot e^{i\theta_j} + \tilde{E}^* \cdot e^{-i\theta_j} \right)$ | with rel. energy deviations | $\eta_j$ |

## Assumptions:

- one-dimensional treatment
- slowly varying field amplitude and phase
- restriction to the fundamental harmonic
- no space charge effects considered (which are small)

Abbreviations:

$$\kappa_1 = \frac{eK_{JJ}}{2\gamma_{res}^2 m_0 c^2}$$

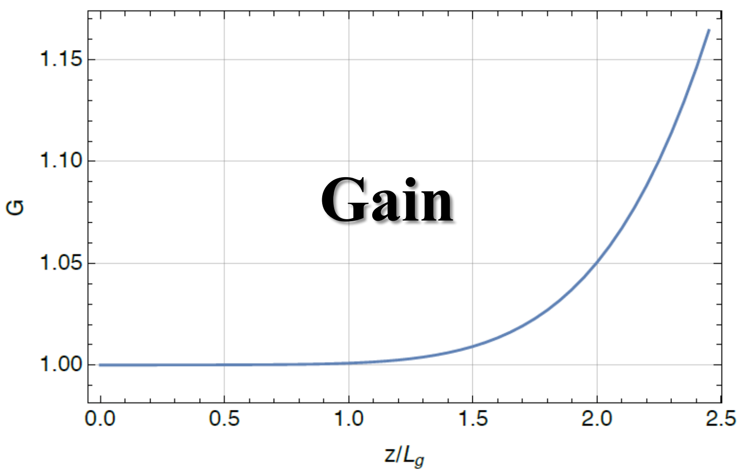
$$\kappa_2 = \frac{eK_{JJ}}{4\varepsilon_0 \gamma_{res}}$$

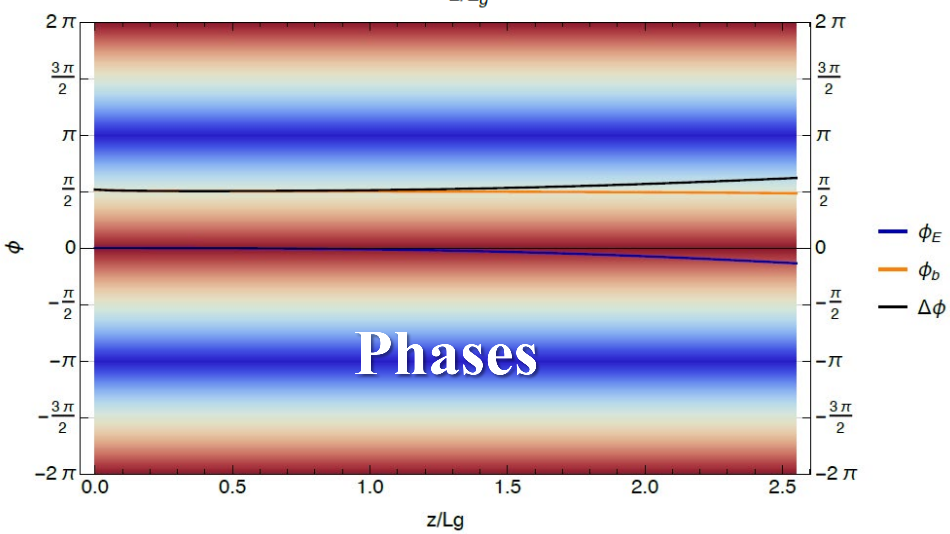
Set of $2N_e$+1 coupled differential equations!!

# Numerical Solution

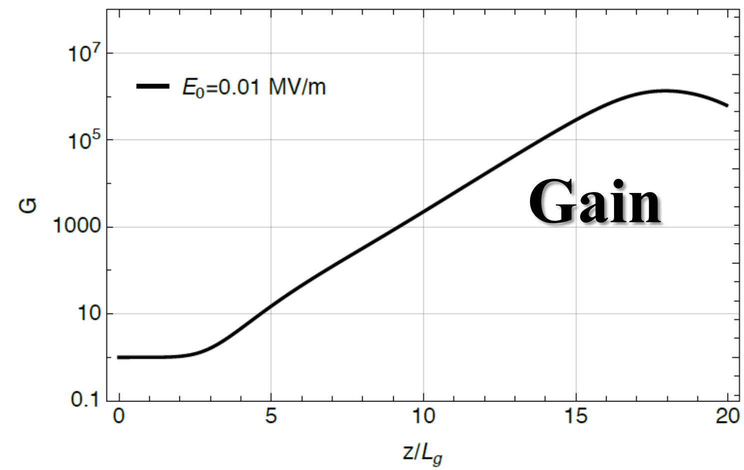
**Redefinition of the gain: $G = G_{old} + 1$**

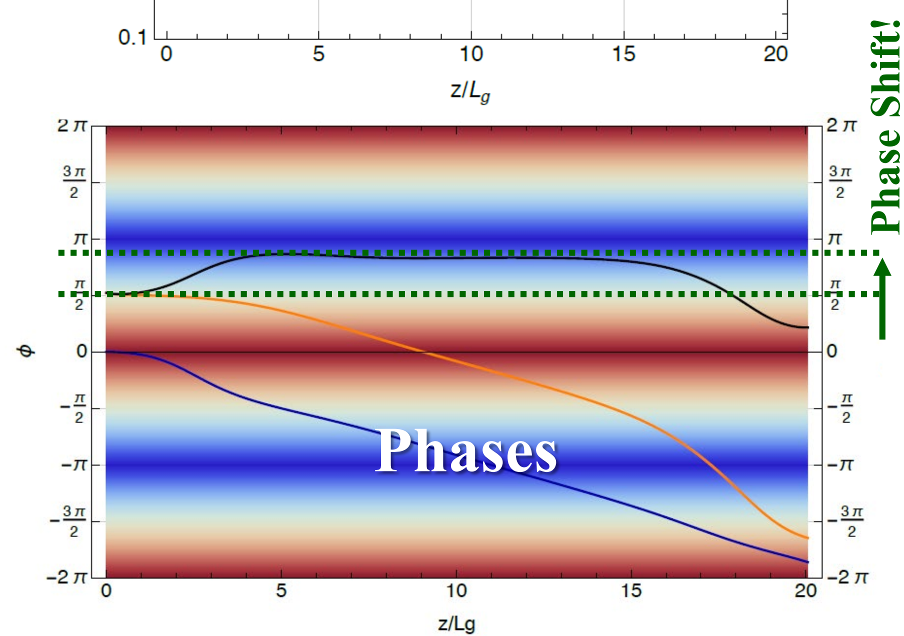
## "Short" Undulator




## "Long" Undulator



# Normalized Parameters

**Deeper understanding of the differential equations by defining normalized scale parameters:**

longitudinal coordinate: $\hat{s} = 2k_u \rho s$ $\qquad \dfrac{\mathrm{d}\theta_j}{\mathrm{d}\hat{s}} = \hat{\eta}_j$

rel. energy deviation: $\hat{\eta} = \dfrac{\eta}{\rho}$ $\qquad \dfrac{\mathrm{d}\hat{\eta}_j}{\mathrm{d}\hat{s}} = a e^{i\theta_j} + a^* e^{-i\theta_j}$

norm. field amplitude: $a = \dfrac{\kappa_1}{2k_u \rho^2}\tilde{E}$ $\qquad \dfrac{\mathrm{d}a}{\mathrm{d}\hat{s}} = -\dfrac{\kappa_1 \kappa_2 n_e}{4k_u^2 \rho^3}\left\langle e^{-i\theta_j}\right\rangle = -\left\langle e^{-i\theta_j}\right\rangle$

Pierce parameter:

$$\rho = \sqrt[3]{\frac{\kappa_1 \kappa_2 n_e}{4k_u^2}} = \sqrt[3]{\frac{1}{8\pi}\left(\frac{I_{\text{beam}}}{I_{\text{Alfvén}}}\right)\left(\frac{K_{JJ}}{1+K^2/2}\right)^2\left(\frac{\gamma\lambda^2}{2\pi\sigma_x^2}\right)}$$

Coupled equations simplify to

$$\frac{\mathrm{d}a}{\mathrm{d}\hat{s}} = -\left\langle e^{-i\theta_j}\right\rangle = -b, \qquad \frac{\mathrm{d}b}{\mathrm{d}\hat{s}} = -i\left\langle \hat{\eta}_j \cdot e^{-i\theta_j}\right\rangle = -iP, \qquad \frac{\mathrm{d}P}{\mathrm{d}\hat{s}} = a + a^*\left\langle e^{-2i\theta_j}\right\rangle - i\left\langle \hat{\eta}_j^2 e^{-i\theta_j}\right\rangle$$

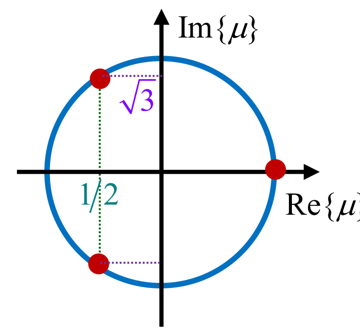
*norm. field amplitude* *bunching factor* *collective momentum*

# Cubic Differential Equation

Combination yields a differential equation of 3rd order:

$$\frac{\mathrm{d}^3 a}{\mathrm{d}\hat{s}^3} = ia \qquad \text{with Ansatz} \quad a = C \cdot e^{-i\mu\hat{s}} \qquad \rightarrow \qquad \mu^3 = 1$$

which has 3 solutions of the characteristic polynomial:




$$\mu_1 = 1, \qquad \mu_2 = -\frac{1}{2}\left(1 + i\sqrt{3}\right), \qquad \mu_3 = -\frac{1}{2}\left(1 - i\sqrt{3}\right)$$

yielding the **general solution:**

$$a(\hat{s}) = C_1 e^{-i\hat{s}} + C_2 e^{\frac{1}{2}\left(i-\sqrt{3}\right)\hat{s}} + C_3 e^{\frac{1}{2}\left(i+\sqrt{3}\right)\hat{s}}$$

**exp. increase**

with the initial values:

- normalized field amplitude $\quad a(0) = \sum C_i$
- bunching factor $\quad b(0) = -\left.\frac{\mathrm{d}a}{\mathrm{d}\hat{s}}\right|_0 = i\sum \mu_i C_i$
- collective momentum $\quad P(0) = i\left.\frac{\mathrm{d}b}{\mathrm{d}\hat{s}}\right|_0 = i\sum \mu_i^2 C_i$

# Cubic Differential Equation

Initial values are determined from following system of equations:

$$\begin{pmatrix} a_0 \\ b_0 \\ P_0 \end{pmatrix} = \mathbf{M}_\mu \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ i\mu_1 & i\mu_2 & i\mu_3 \\ i\mu_1^2 & i\mu_2^2 & i\mu_3^2 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

which yields after matrix inversion:

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \mathbf{M}_\mu^{-1} \cdot \begin{pmatrix} a_0 \\ b_0 \\ P_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{i}{3} & -\frac{i}{3} \\ \frac{1}{3} & \frac{1}{6}(i+\sqrt{3}) & \frac{1}{3}(-1)^{5/6} \\ \frac{1}{3} & \frac{1}{6}(i-\sqrt{3}) & \frac{1}{3}(-1)^{5/6} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ b_0 \\ P_0 \end{pmatrix}$$

Considering an initial energy shift $\hat{\eta}_0 = \eta_0 / \rho$ :

$$P(\hat{s}) \rightarrow \left\langle \hat{\eta}_j e^{-i\theta_j} \right\rangle + \hat{\eta}_0 \qquad \rightarrow \qquad \mu^3 - 2\hat{\eta}_0 \mu^2 + \hat{\eta}_0^2 \mu - 1 = 0$$

# Cubic Differential Equation

## Case 1: start from already existing radiation field

Starting conditions:

- no density modulation $\rightarrow b_0 = 0$
- no energy offset and modulation $\rightarrow \eta_0 = 0 \ \rightarrow P_0 = 0$
- Incoming radiation field $\rightarrow \boldsymbol{a_0 > 0}$

$$\Rightarrow \quad C_1 = C_2 = C_3 = \frac{1}{3} a_0$$

**Field amplitude:**

$$a(\hat{s}) = \frac{a_0}{3} \left\{ e^{-i\hat{s}} + e^{\frac{1}{2}\left(i-\sqrt{3}\right)\hat{s}} + e^{\frac{1}{2}\left(i+\sqrt{3}\right)\hat{s}} \right\}$$

**Gain:**

$$G(\hat{s}) = \frac{|a|^2}{a_0^2} = \frac{1}{9} \left\{ 3 + e^{-\sqrt{3}\hat{s}} + e^{\sqrt{3}\hat{s}} + 2\cos\left(\frac{3}{2}\hat{s}\right) \cdot \left[ e^{-\frac{\sqrt{3}}{2}\hat{s}} + e^{\frac{\sqrt{3}}{2}\hat{s}} \right] \right\}$$

# Cubic differential Equation

## Case 1: start from existing radiation field

### Universal gain curve:

$$G(\hat{s}) = \frac{|a|^2}{a_0^2} = \frac{1}{9}\left\{3 + e^{-\sqrt{3}\hat{s}} + e^{\sqrt{3}\hat{s}} + 4\cos\left(\frac{3}{2}\hat{s}\right)\cosh\left(\frac{\sqrt{3}}{2}\hat{s}\right)\right\}$$

Asymptotical behavior for large $\hat{s}$:

$$G \approx \frac{1}{9} e^{\sqrt{3}\hat{s}} = \frac{1}{9} e^{2\sqrt{3}k_u\rho \cdot s} = \frac{1}{9} e^{s/L_G}$$

Definition of the 1 dim gain length (**power gain length**):

$$\sqrt{3}\hat{s} = 1 \quad \rightarrow \quad L_G = \frac{1}{2\sqrt{3}k_u\rho} = \frac{\lambda_u}{4\pi\sqrt{3}\rho}$$

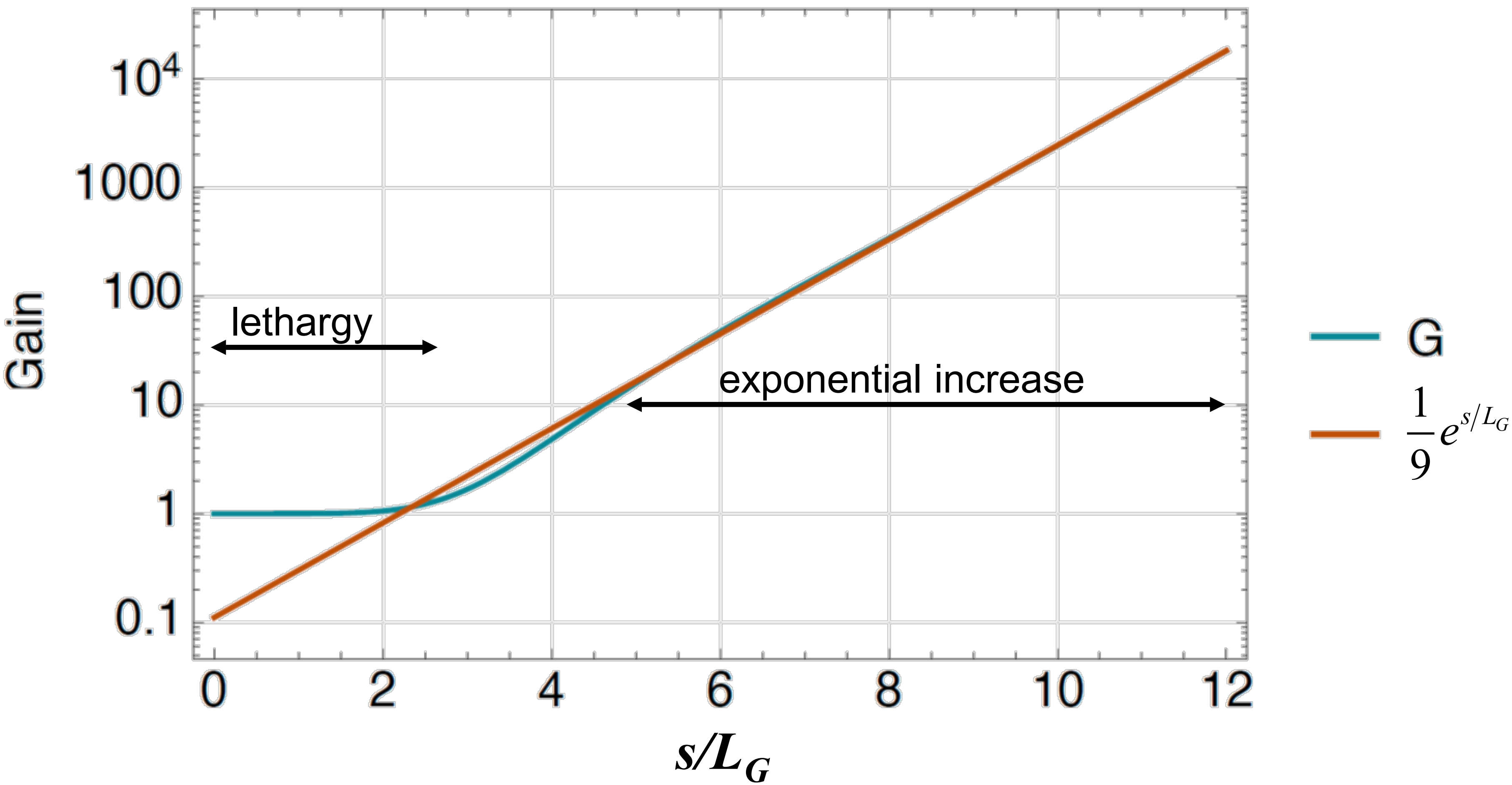
Behavior for small $s/L_G$ (Taylor expansion) ↔ "**Lethargy**"

$$G_{\text{leth}} = 1 + \frac{1}{1080}\left(\frac{s}{L_G}\right)^6 = 1 + \left(\frac{s}{3.2L_G}\right)^6$$

# Universal Gain Curve

$$G(\hat{s}) = \frac{|a|^2}{a_0^2} = \frac{1}{9}\left\{3 + 2\cosh\chi + 4\cos\left(\frac{\sqrt{3}}{2}\chi\right)\cosh\left(\frac{\chi}{2}\right)\right\}$$

$$\chi = \frac{s}{L_G}$$

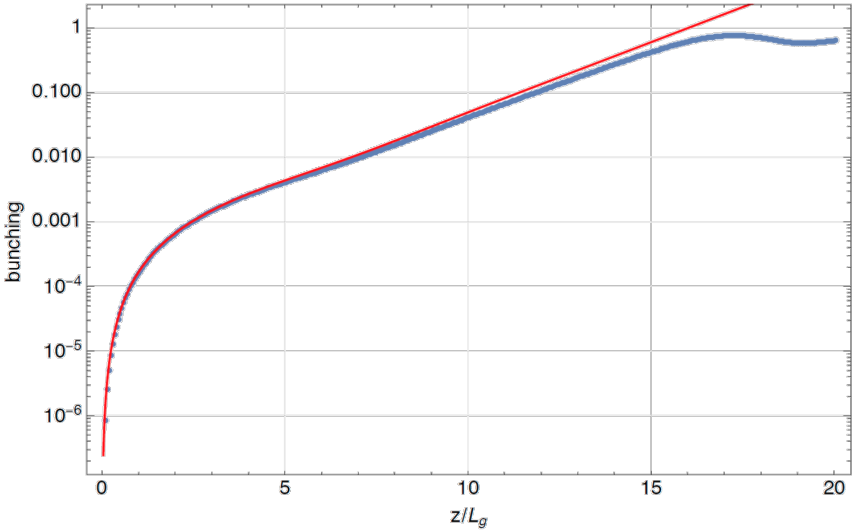


# Saturation

Region of exponential increase:

$$|a|^2 = \frac{a_0^2}{9} e^{-s/L_g} = \frac{4}{3}|b|^2 < \frac{4}{3} \approx 1$$

→ field amplitude cannot grow larger than $|a| \approx 1$




$$|a|^2 = \left| \frac{\kappa_1}{2k_u\rho^2} E \right|^2 \leq 1 \quad \rightarrow \quad \boxed{\tilde{E}_{\text{sat}} = \frac{2k_u\rho^2}{\kappa_1}}$$

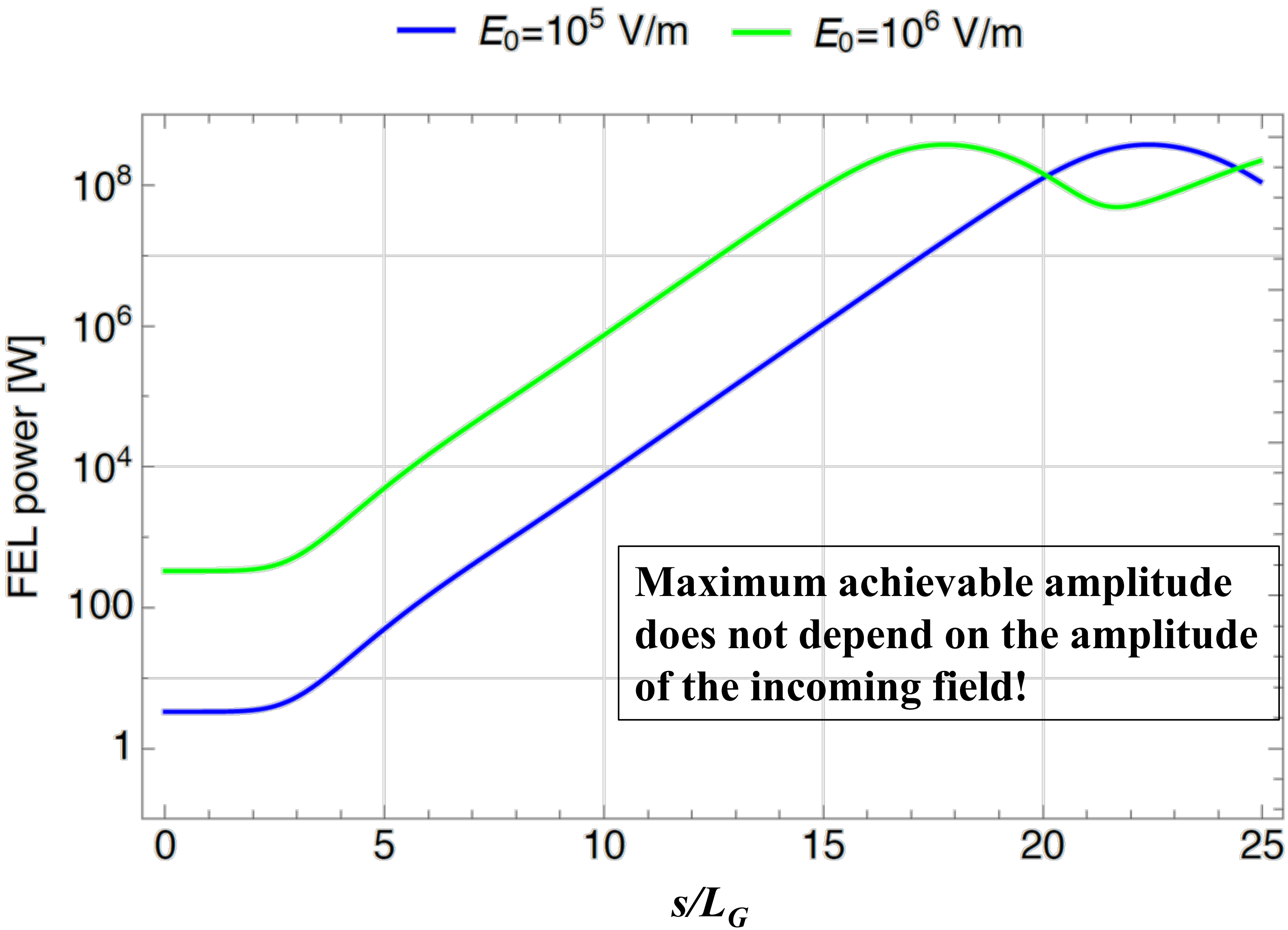
$$\rightarrow \quad \mathcal{W}_{\text{sat}} = \frac{1}{2}\varepsilon_0 E_{sat}^2 = 2\varepsilon_0 \left|\tilde{E}_{sat}\right|^2 = 2\varepsilon_0\rho\left(\frac{2k_u}{\kappa_1}\right)^2 \rho^3 = 2\varepsilon_0\rho n_e\left(\frac{\kappa_2}{\kappa_1}\right) = \rho\, n_e \gamma_{res} m_0 c^2 = \rho\, \mathcal{W}_{\text{beam}}$$

and thus

$$P_{sat} = \rho \cdot I_{\text{beam}} \cdot U_{\text{beam}}$$

**‰ efficiency of energy transfer**

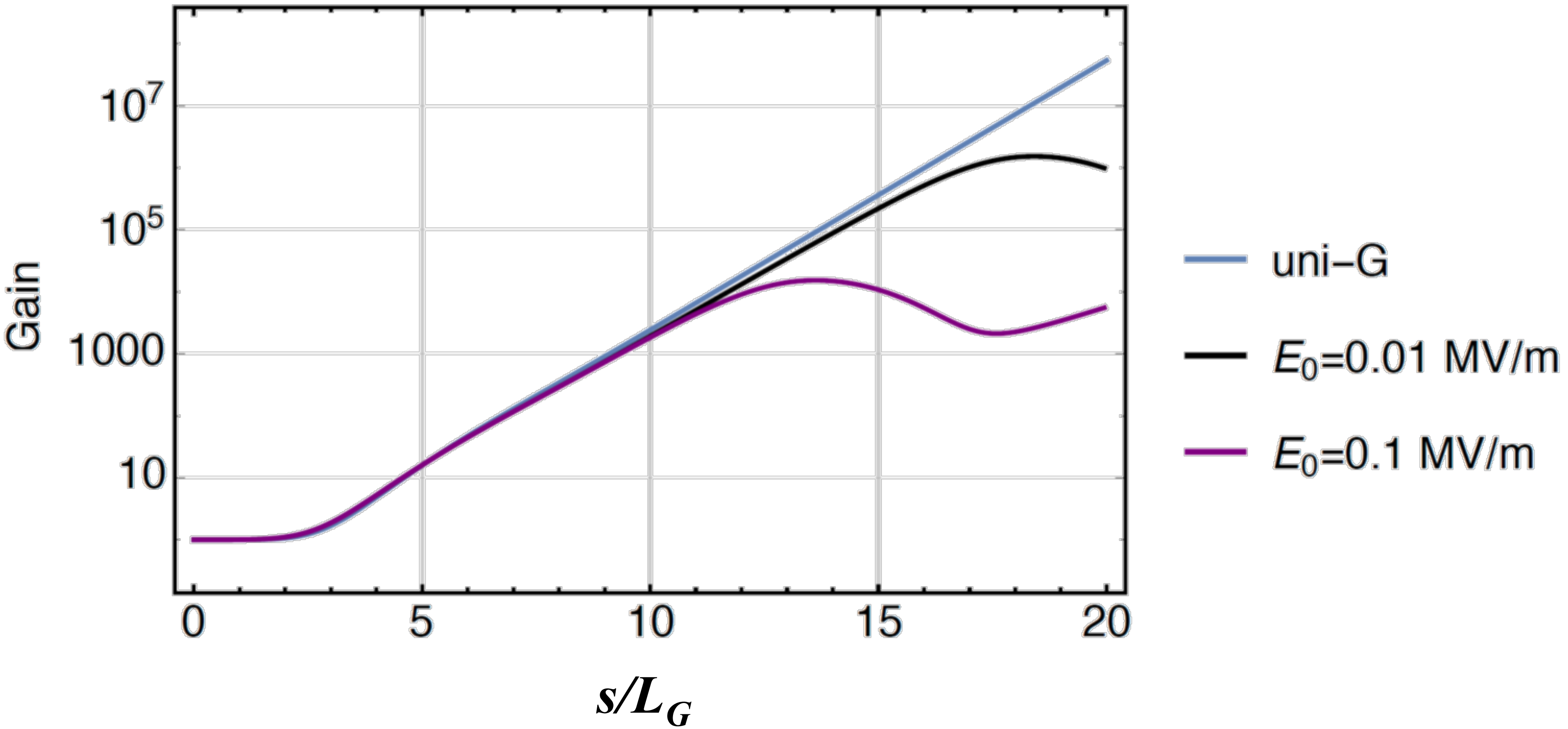
# Saturation



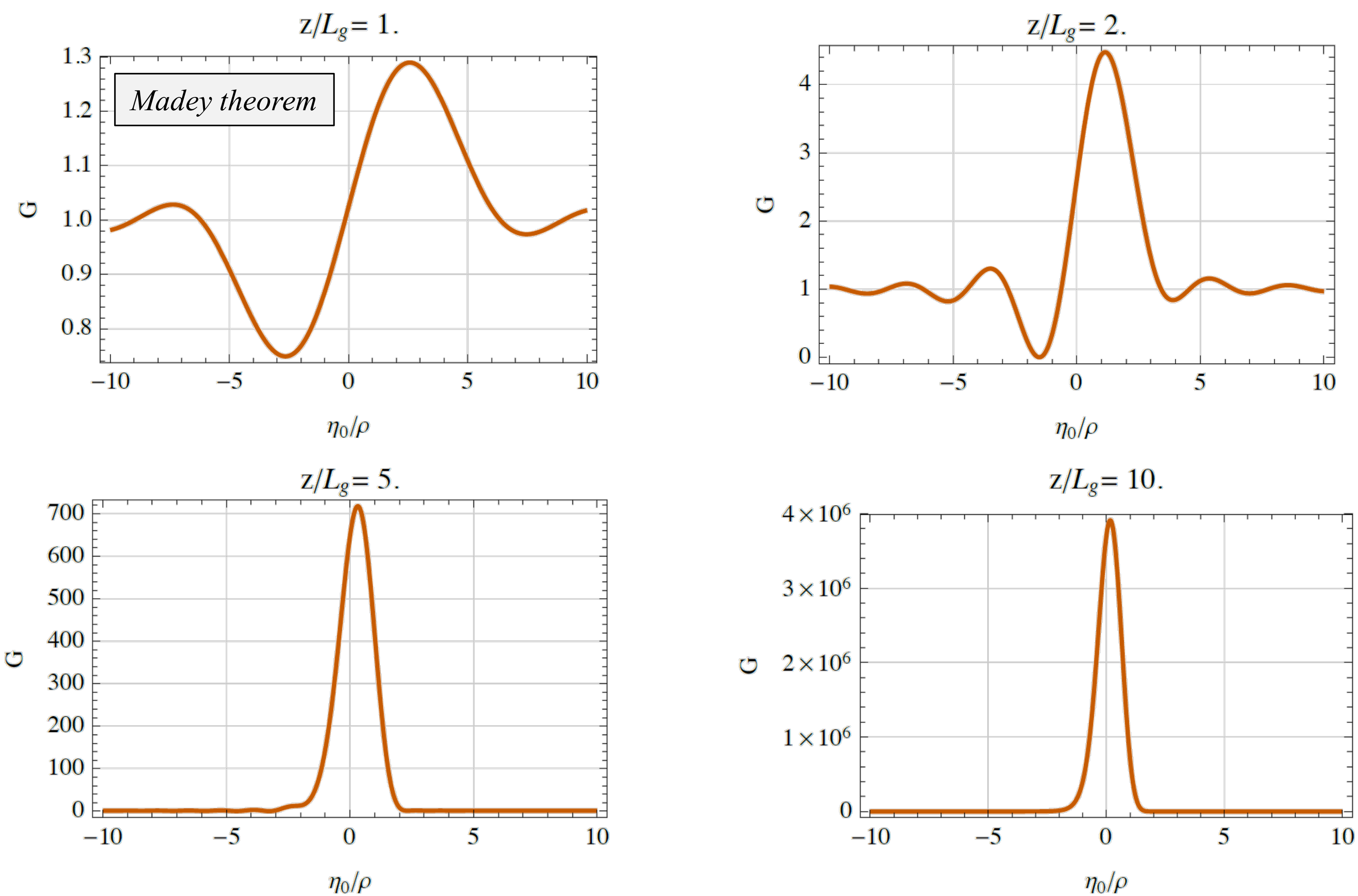

**Maximum achievable amplitude does not depend on the amplitude of the incoming field!**

# Saturation

**Maximum achievable gain factor depends on the amplitude of the incoming field**



# Gain and Bandwidth



# Cubic Differential Equation

## Case 2: Start from an existing density modulation

Starting conditions:

- Density modulation → $b_0 = \left\langle e^{-i\theta_j} \right\rangle\Big|_0$ at $\lambda_m \approx \lambda_r$
- Energy offset → coll. e. modulation! → $\eta_i = 0$, → $P_0 = ib'\big|_0 = \hat{\eta}_0 b_0$
- incoming radiation field → $a_0 = 0$

$$\Rightarrow \quad C_1 = -i\frac{b_0}{3}, \quad C_2 = (-1)^{5/6}\frac{b_0}{3}, \quad C_3 = (-1)^{1/6}\frac{b_0}{3}$$

for $\eta_0 = 0$

**Field energy:**

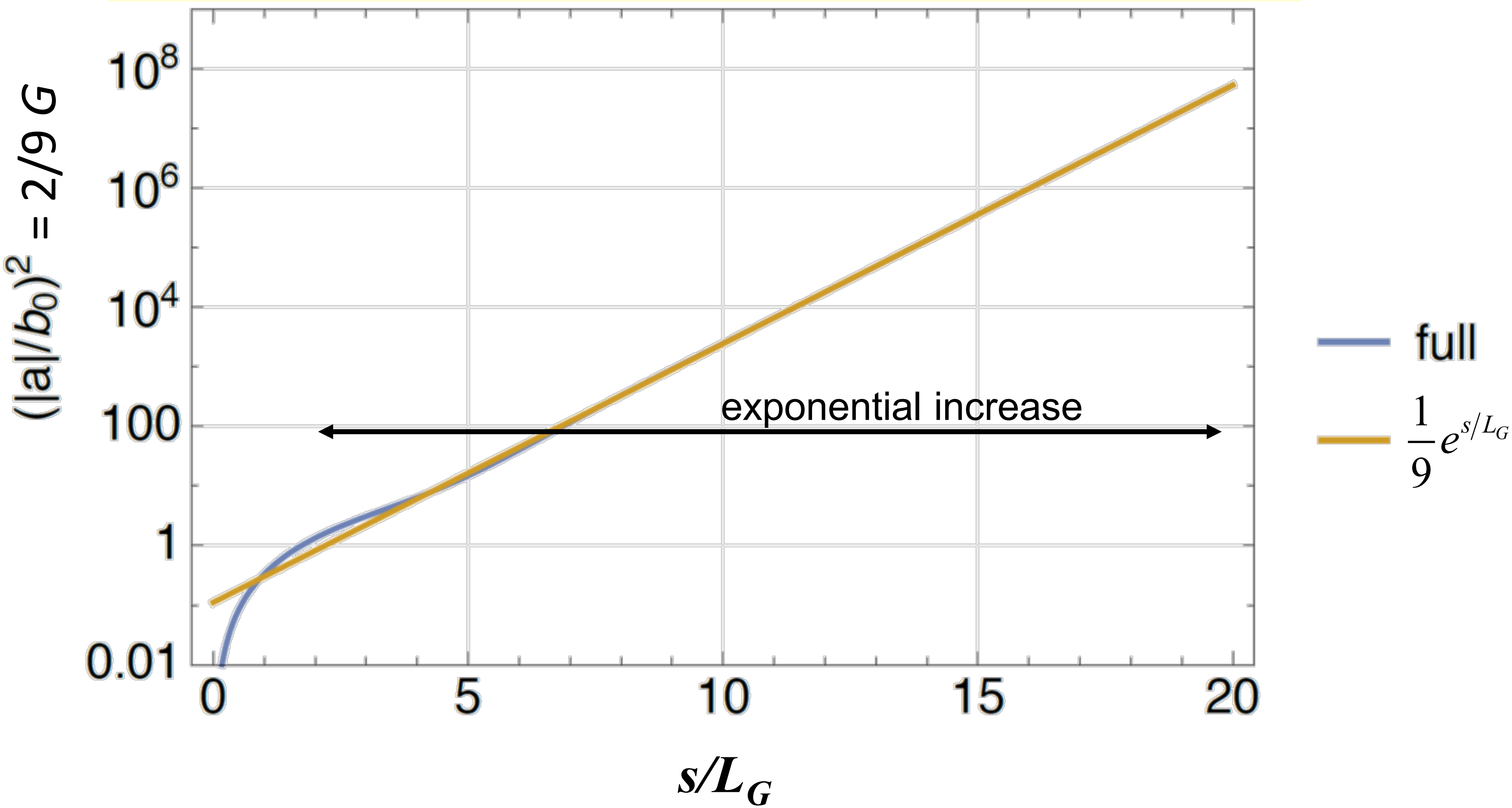
$$|a|^2 = \frac{2}{9} b_0^2 G(\chi), \qquad \chi = \frac{s}{L_g}$$

**Gain:**

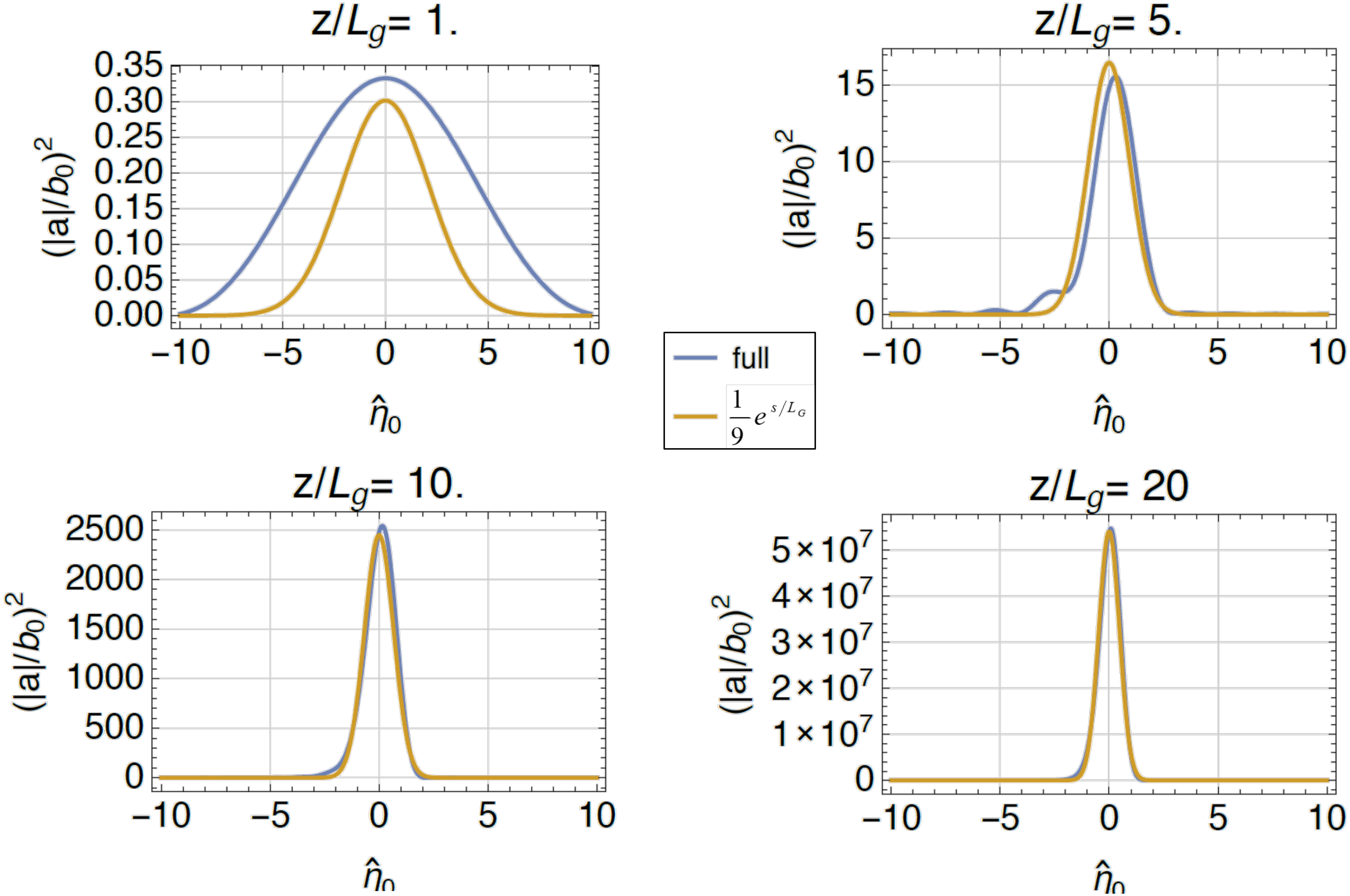
$$G(\hat{s}) = \boxed{\cosh\chi} + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\chi\right)\sinh\left(\frac{1}{2}\chi\right) - \cos\left(\frac{\sqrt{3}}{2}\chi\right)\cosh\left(\frac{1}{2}\chi\right)$$

# Universal Gain Curve

$$G(\hat{s}) = \cosh\chi + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\chi\right)\sinh\left(\frac{1}{2}\chi\right) - \cos\left(\frac{\sqrt{3}}{2}\chi\right)\cosh\left(\frac{1}{2}\chi\right) \qquad \chi = \frac{s}{L_G}$$

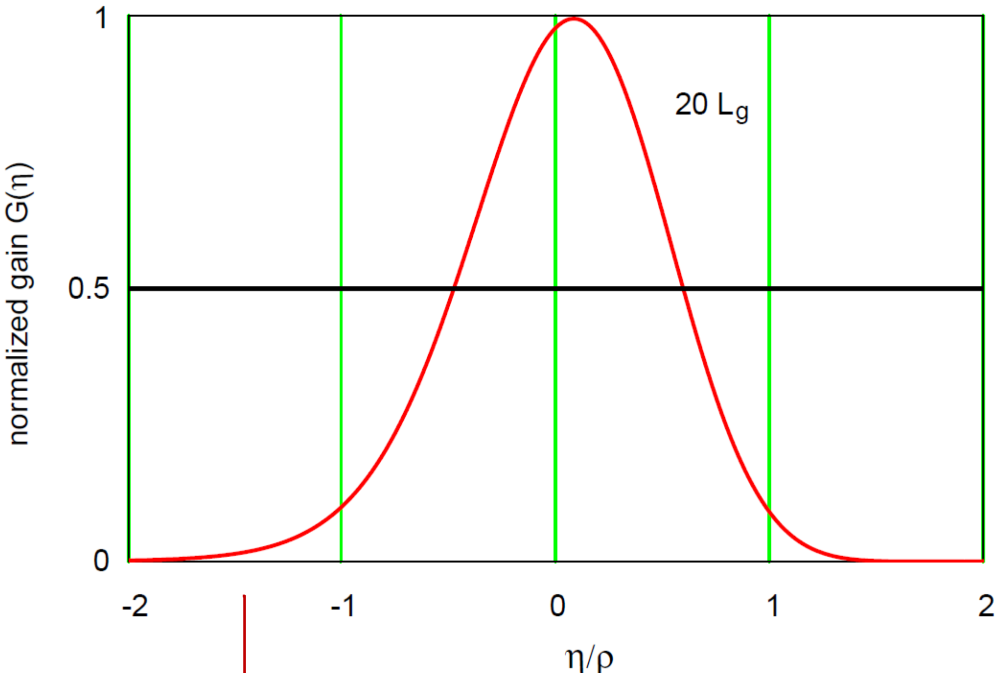


# Gain and Bandwidth



# Bandwidth

**normalized gain @ $s$ = 20 $L_g$:**




Finding:

FEL Gain drops significantly, when the relative energy variation $\eta$ exceeds the Pierce parameter $\rho$!

$s$-dependent energy bandwidth

$$\Delta\eta(s) = 3\sqrt{\pi}\rho\sqrt{\frac{L_g}{s}}$$

Gain curve has a FWHM ≈ $\rho$

**→ $\rho$ determines spectral width of the generated readiation!**

# SASE

## Self Amplified Spontaneous Emission (SASE)

Was proposed in the beginning of the 1980s to produce high power short wavelength FEL radiation. 2 ways of considering the start of the FEL process:

- **spontaneous emission** at the beginning of the undulator is amplified,
- **random longitudinal distribution** of electrons leads to bunching non-vanishing factor at resonant frequency starting the FEL process.

**Both pictures are fully equivalent!**

Time structure:

Not the full bunch is contributing to the SASE start-up! Number of contributing electrons are determined by the undulator amplification bandwidth $\sigma_\omega \approx \rho\omega$!

## Coherence or cooperation length $L_C$

can be roughly determined from time-bandwidth product $\tau \cdot \sigma_\omega$:

$$\tau_c = \frac{\sqrt{\pi}}{\sigma_\omega} \approx \frac{\sqrt{\pi}}{\rho\omega} = \frac{\lambda_L}{2\sqrt{\pi}\rho c} \qquad \rightarrow \qquad \boxed{L_C = c\tau_c = \frac{\lambda_L}{2\sqrt{\pi}\rho} \approx 300\,\lambda_L}$$

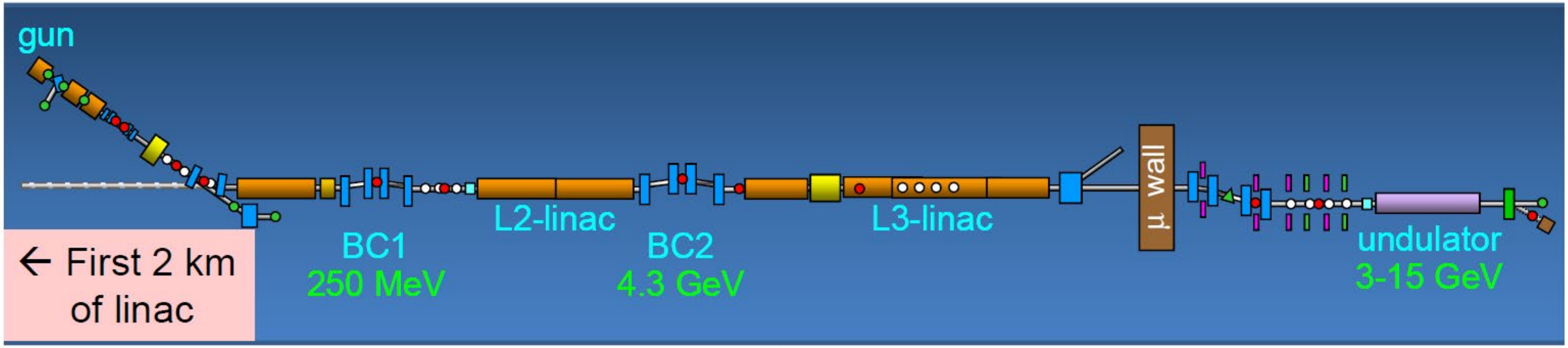
**Within the bunch, several areas can start a SASE process individually!**

# High Gain SASE FEL

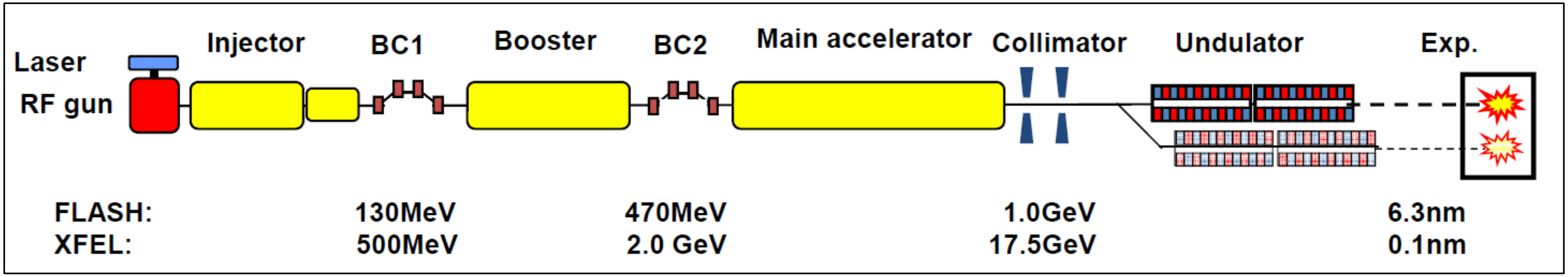
**e beam requirements:**

- transv. emittance $\varepsilon_{x,y} \le \lambda_L / 4\pi$
- energy spread $\sigma_\gamma / \gamma < \rho$
- energy, current $E_{beam} \approx \text{GeV}, \quad I_{peak} \approx \text{kA}$

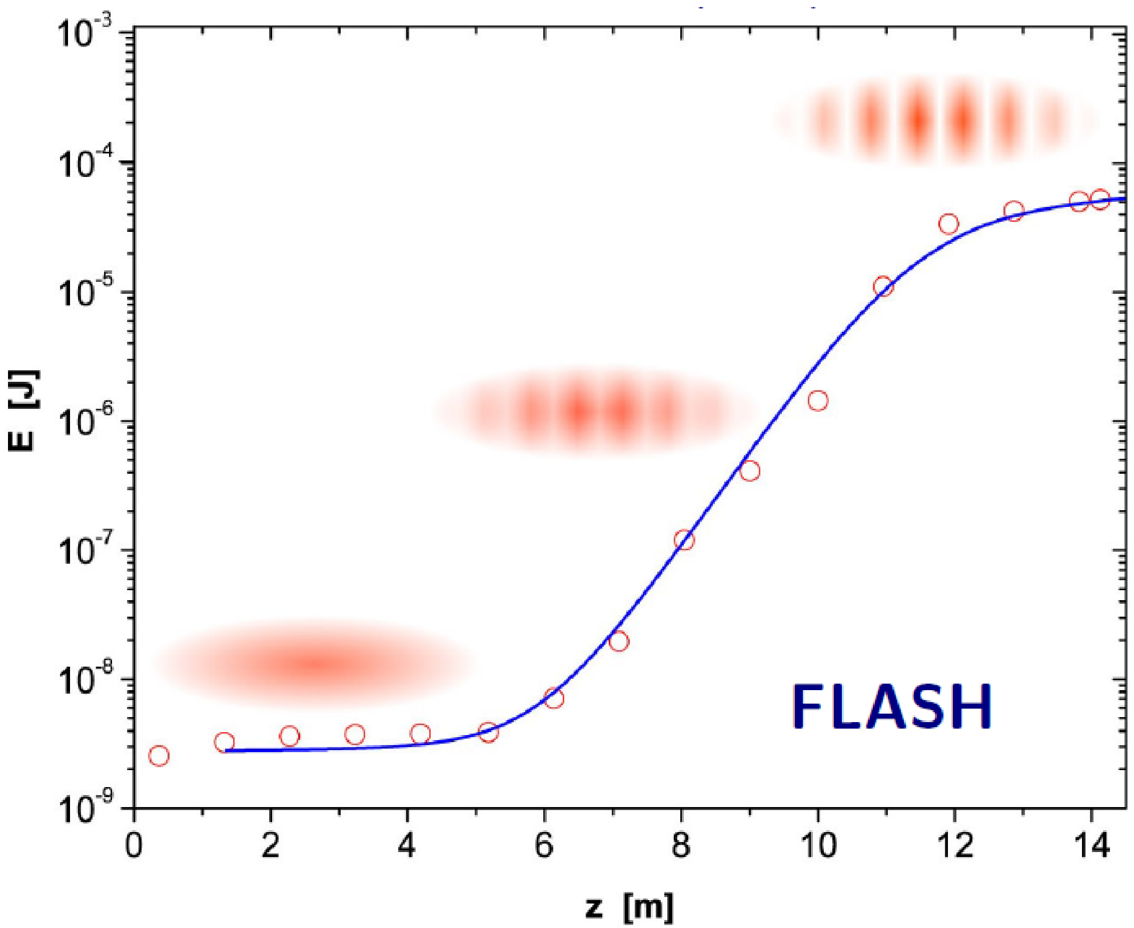
**Linac Coherent Light Source LCLS: the blue pint of all SASE FELs**

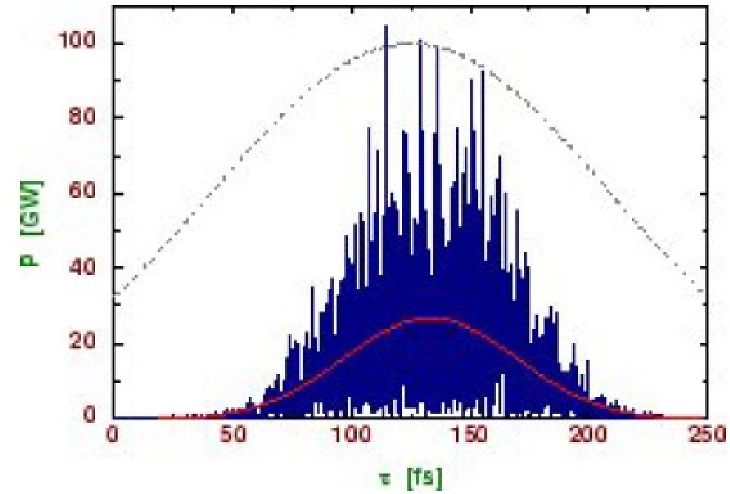


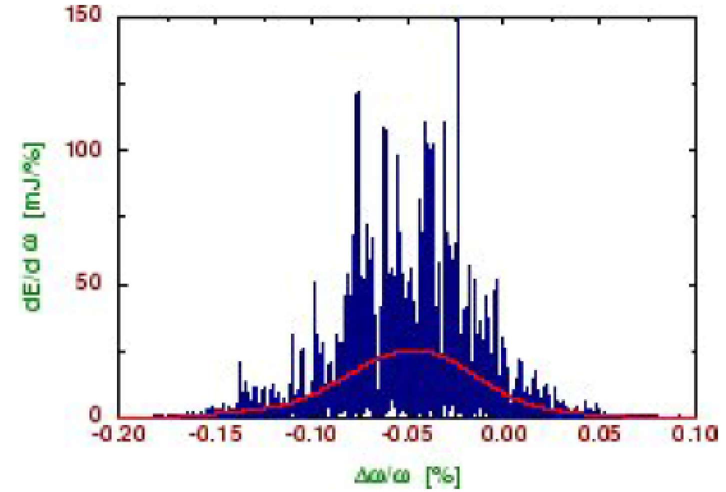

**FLASH and European XFEL: long pulse trains from s.c. Linacs**

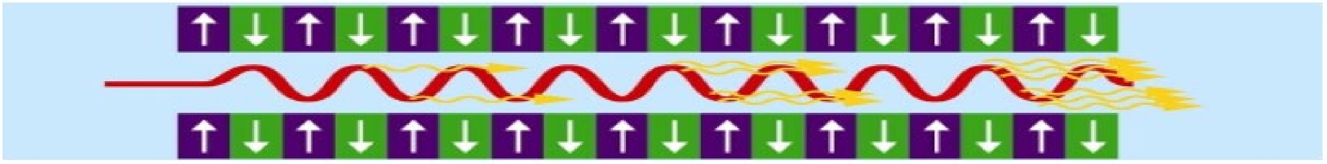


# High Gain SASE FEL



**Low temporal coherence!**

# Peak Brilliance

Figure of merit: peak brillance

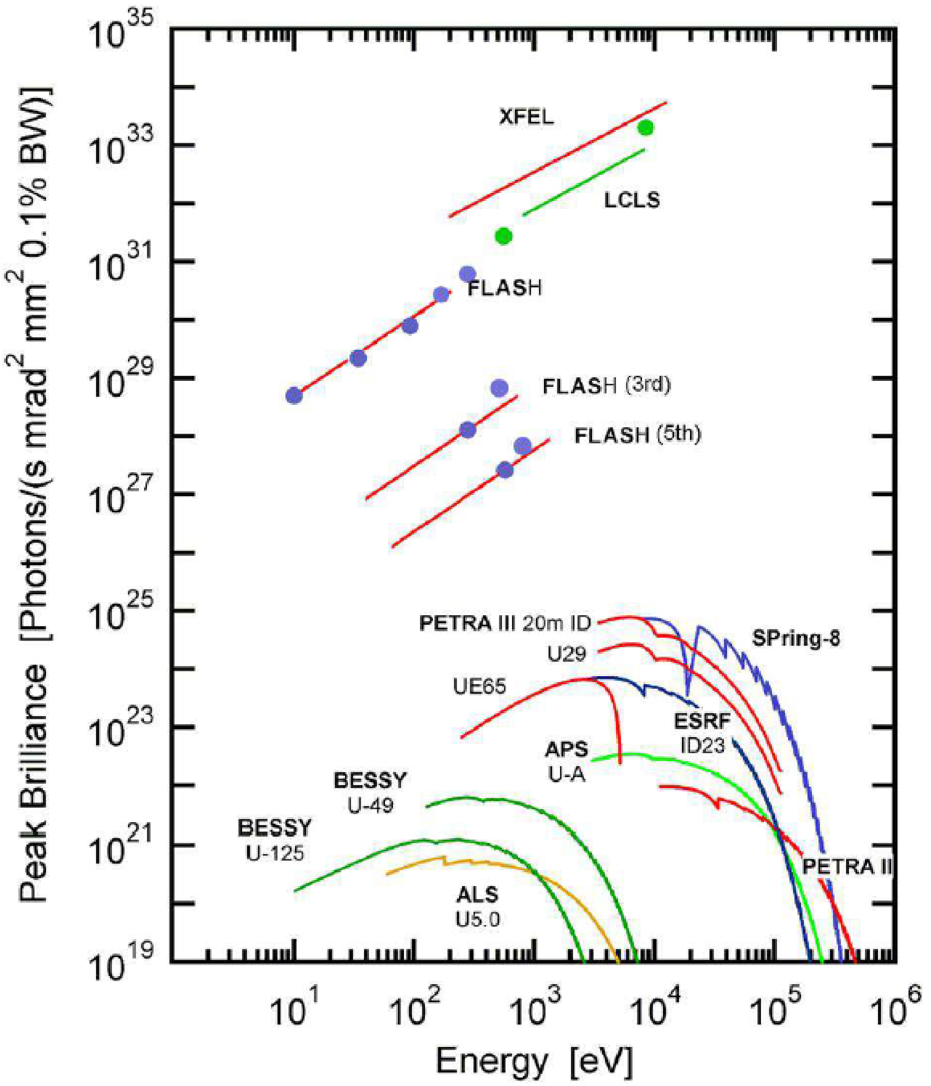
number of photons

$$B = \frac{\frac{d}{dt} n}{4\pi^2 \Sigma_x \Sigma_y \Sigma_{x'} \Sigma_{y'} \, d\omega / \omega}$$

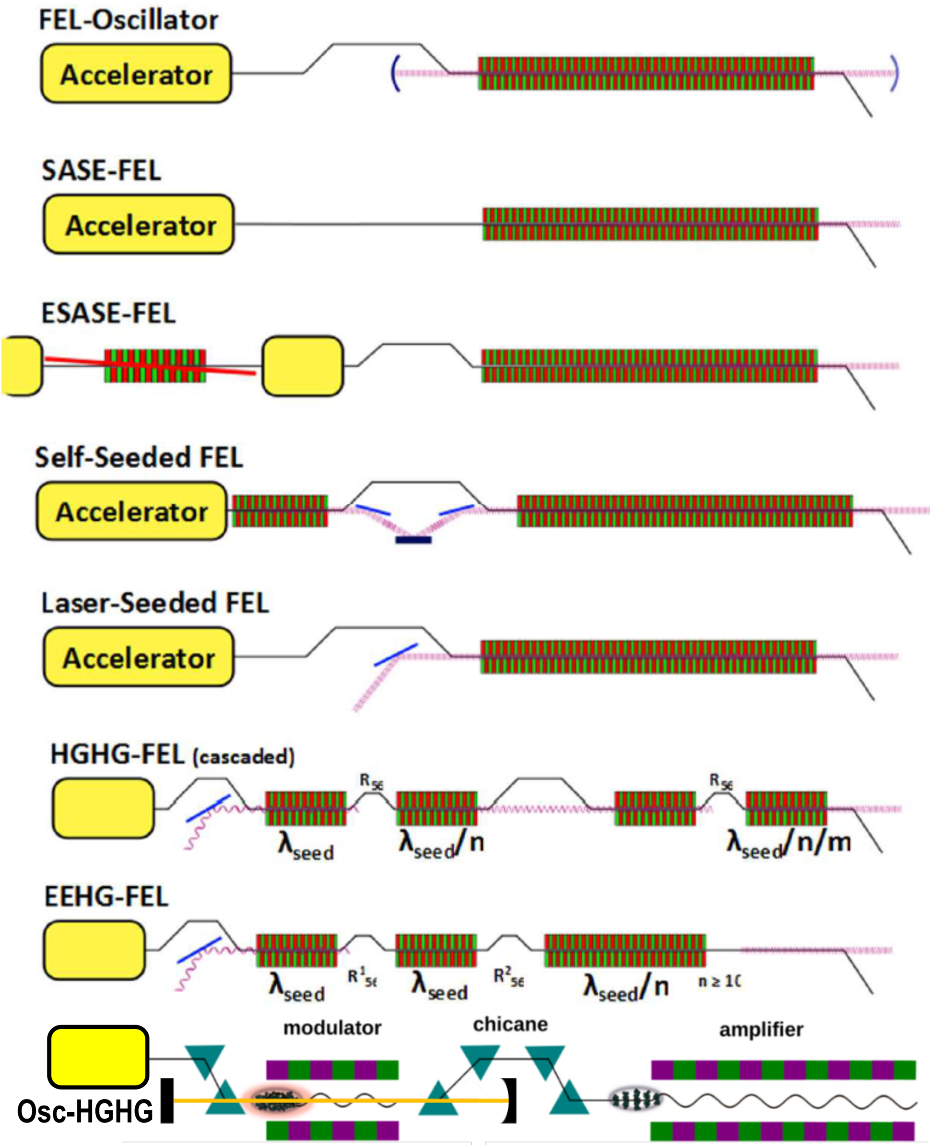
$$\Sigma^2 \approx \sigma_\gamma^2 + \sigma_e^2$$

Undulator beam lines:

$$B_{peak} \approx 10^{25} \mathrm{mm}^{-2} \mathrm{mrad}^{-2} \mathrm{s}^{-1} (0.1\%)^{-1}$$



# Outlook: Seeding




Requires MHz electron bunch repetition (storage ring or cw linac)
Bandwidth determined by mirror system

e.g. XFELO planned @ Eur. XFEL

„Seeding“ by spontaneous synchrotron radiation, i.e. by shot noise

e.g. Eur. XFEL $\lambda_L < 1$ Å

Increase peak current within mirco-bunches generated through laser modulation and subsequent compression

Cut out monochromatic portion from initial SASE FEL for seeding

e.g. LCLS Eur. XFEL

Generate coherent seeding pulse by external laser (synchronized to e-beam!)

FLASH: $\lambda_L = 38$ nm

Dto., but also produce higher FEL harmonics for further seeding stages.

Like HGHG, but generate very high harmonics by multiple compression and multiple seeding.

e.g. FERMI @ ELETTRA $\lambda_L \approx 4$ nm

**HGHG seeding with an oscillator starting from shot noise**

planned @ FLASH

# Literature

Recommended Textbooks:

- J.A. Clark, ***The Science and Technology of Undulators and Wigglers***, Oxford Science Publications, ISBN 019850855: *Synchrotron Radiation, Undulators and Wigglers, includes technical aspects and many details*
- P. Schmüser, M. Dohlus, J. Rossbach, C. Behrens, ***Free-Electron Lasers in the Ultraviolet and X-Ray Regime***, Second Edition (2014), Springer, ISBN 9783319040806: *The Hamburg Blue-Book on Free Electron Lasers*
- K.-J. Kim, Z. Huang, R. Lindberg, ***Synchrotron Radiation and Free-Electron Lasers***, Cambridge University Press (2017), ISB 9781107162617*: Excellent Book going deep into the theory of FEL way beyond the scope of this lecture*
- K. Wille, ***The Physics of Particle Accelerators. An Introduction***. Oxford University Press, Oxford (2001): *A compact book with some insights in LG FELs*
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