



# **Non-Linear Dynamics - Phenomenology**

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**CERN Accelerator School**  
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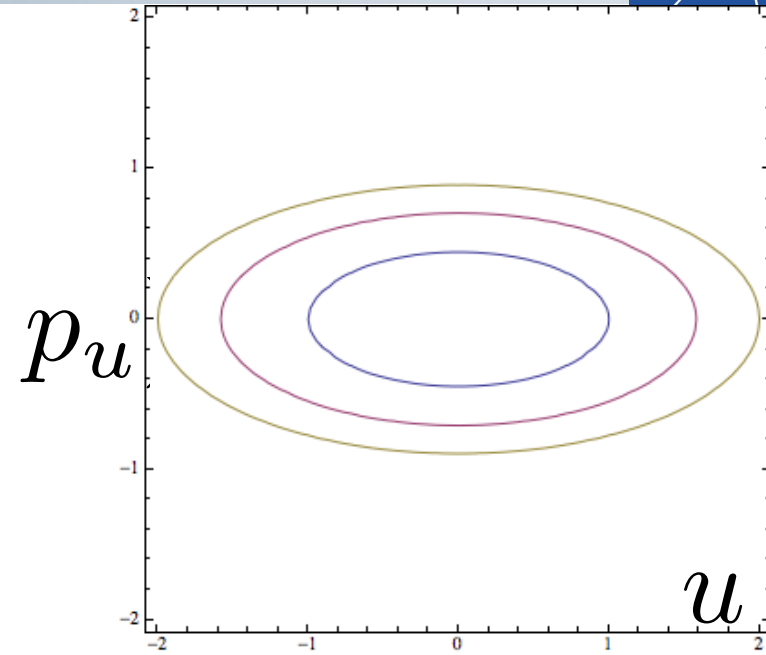
- Phase space dynamics – fixed point analysis
- Poincaré map
- Motion close to a resonance
- Onset of chaos
- Chaos detection methods
  - Dynamic Aperture
  - Lyapunov exponent
  - Frequency map analysis
  - Numerical applications

# Phase space dynamics - Fixed point analysis

- Valuable description when examining trajectories in **phase space** ( $u, p_u$ )
- Existence of integral of motion imposes geometrical constraints on phase flow
- For the simple **harmonic oscillator**

$$H = \frac{1}{2} (p_u^2 + \omega_0^2 u^2)$$

phase space curves are **ellipses** around the equilibrium point parameterized by the Hamiltonian (energy)

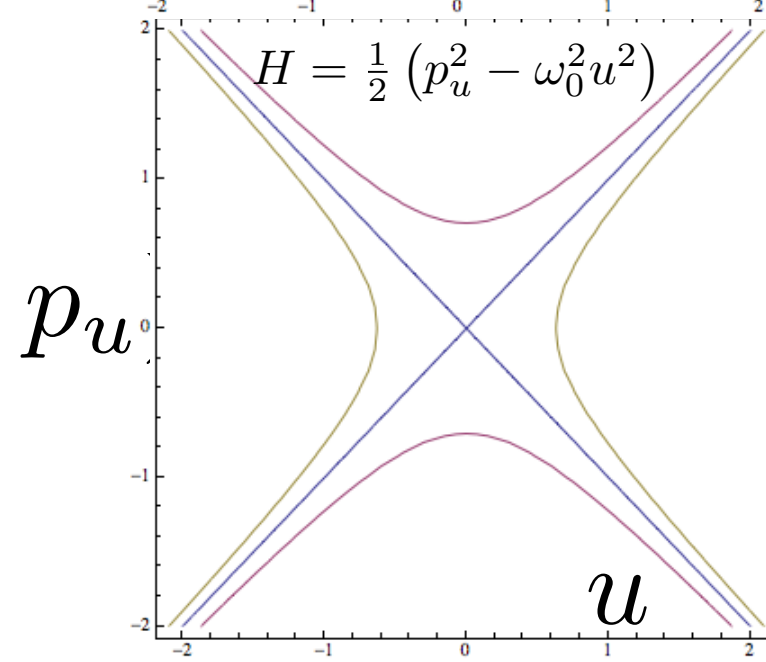
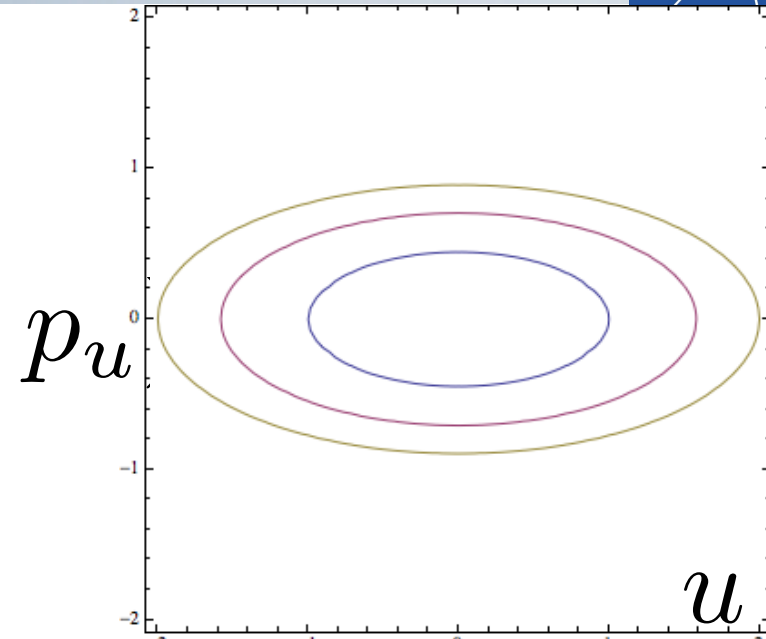


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- By simply **changing** the **sign** of the potential in the harmonic oscillator, the phase trajectories become **hyperbolas**, symmetric around the equilibrium point where two straight lines cross, moving towards and away from it

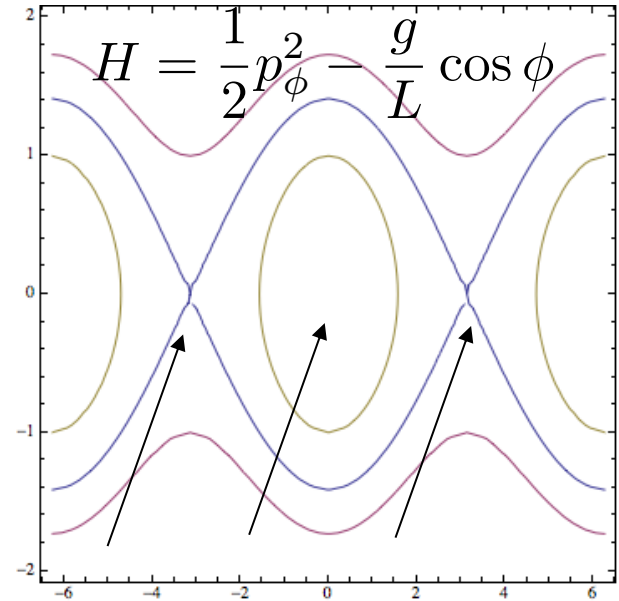
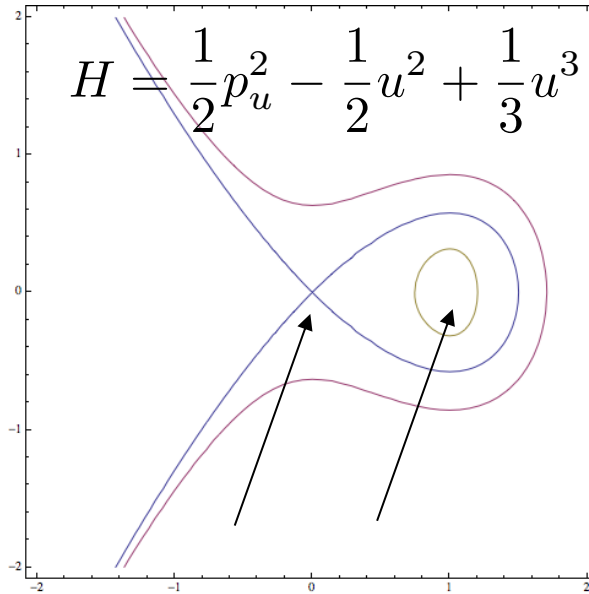
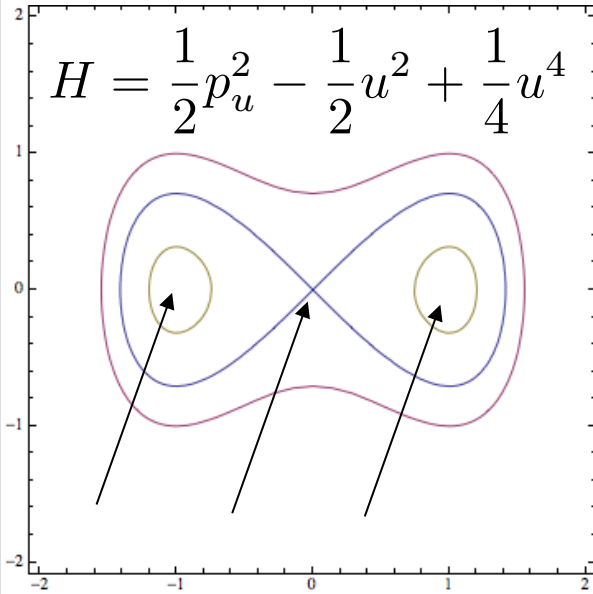


- Conservative non-linear oscillators have Hamiltonian

$$H = E = \frac{1}{2}p_u^2 + V(u)$$

with the potential being a general (polynomial) function of positions

- **Equilibrium points** are associated with extrema of the potential



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- Considering three non-linear oscillators
  - Quartic** potential (left): two minima and one maximum
  - Cubic** potential (center): one minimum and one maximum
  - Pendulum** (right): periodic minima and maxima



- Consider a general second order system
$$\frac{du}{dt} = f_1(u, p_u)$$
$$\frac{dp_u}{dt} = f_2(u, p_u)$$
- Equilibrium or “**fixed**” points  $f_1(u_0, p_{u0}) = f_2(u_0, p_{u0}) = 0$  are determinant for topology of trajectories at their vicinity



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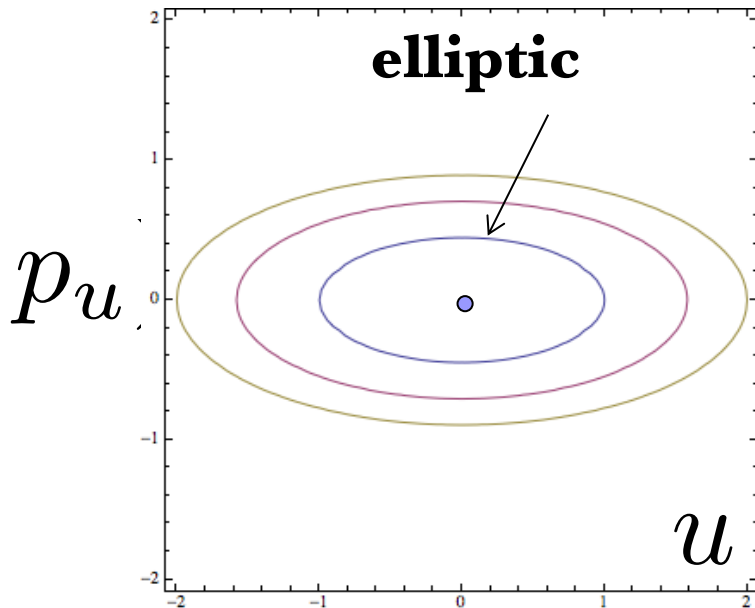
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- The **linearized equations** of motion at their vicinity are

$$\frac{d}{dt} \begin{bmatrix} \delta u \\ \delta p_u \end{bmatrix} = \mathcal{M}_J \begin{bmatrix} \delta u \\ \delta p_u \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1(u_0, p_{u0})}{\partial u} & \frac{\partial f_1(u_0, p_{u0})}{\partial p_u} \\ \frac{\partial f_2(u_0, p_{u0})}{\partial u} & \frac{\partial f_2(u_0, p_{u0})}{\partial p_u} \end{bmatrix}}_{\text{Jacobian matrix}} \begin{bmatrix} \delta u \\ \delta p_u \end{bmatrix}$$

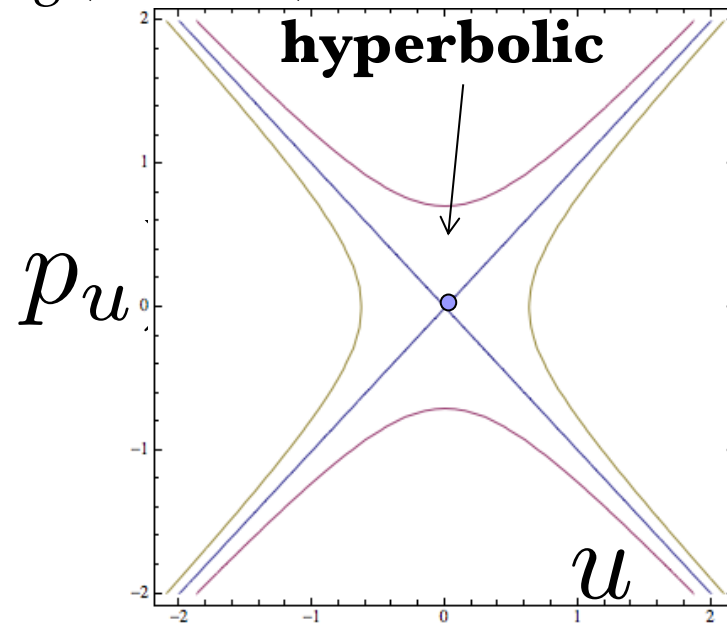
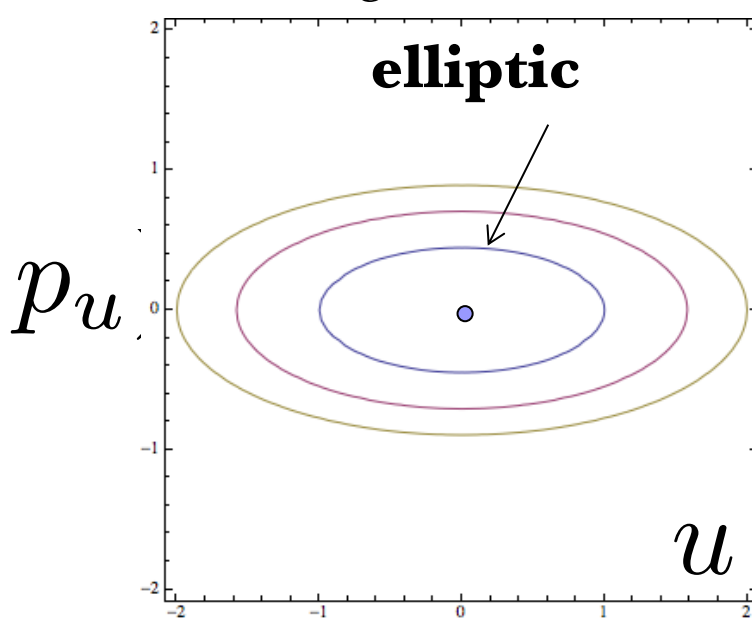
**Jacobian matrix**

- Fixed point nature is revealed by **eigenvalues** of  $\mathcal{M}_J$ , i.e. solutions of the characteristic polynomial  $\det |\mathcal{M}_J - \lambda \mathbf{I}| = 0$

- For **conservative systems** of 1 degree of freedom, the second order characteristic polynomial for any fixed point has two possible solutions:
  - Two **complex eigenvalues** with opposite sign, corresponding to **elliptic** fixed points. Phase space flow is described by **ellipses**, with particles evolving clockwise or anti-clockwise



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  - Two **real eigenvalues** with opposite sign, corresponding to **hyperbolic** (or saddle) fixed points. Flow described by two lines (or manifolds), incoming (stable) and outgoing (unstable)



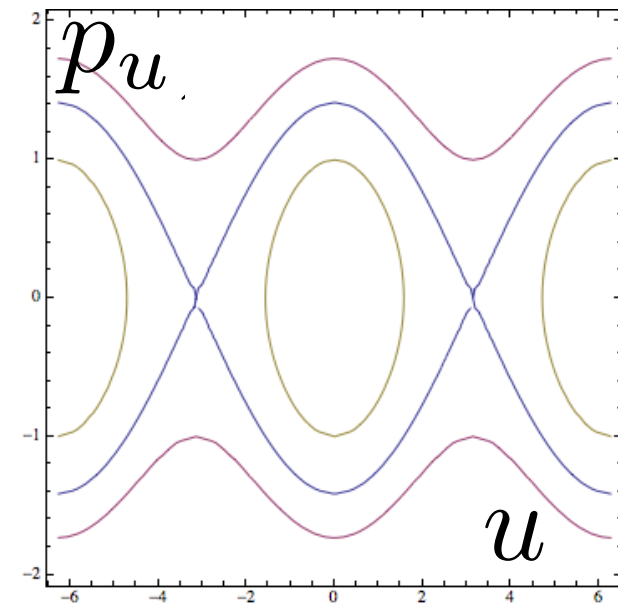


- The “fixed” points for a pendulum can be found at

$$(\phi_n, p_\phi) = (\pm n\pi, 0), \quad n = 0, 1, 2 \dots$$

- The Jacobian matrix is 
$$\begin{bmatrix} 0 & 1 \\ -\frac{g}{L} \cos \phi_n & 0 \end{bmatrix}$$

- The eigenvalues are 
$$\lambda_{1,2} = \pm i \sqrt{\frac{g}{L} \cos \phi_n}$$





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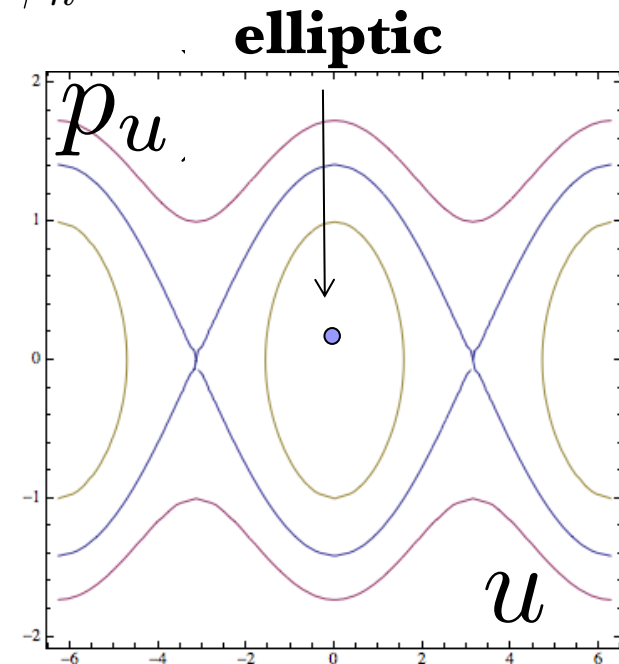
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- $\phi_n = 2n\pi$ , for which  $\lambda_{1,2} = \pm i \sqrt{\frac{g}{L}}$   
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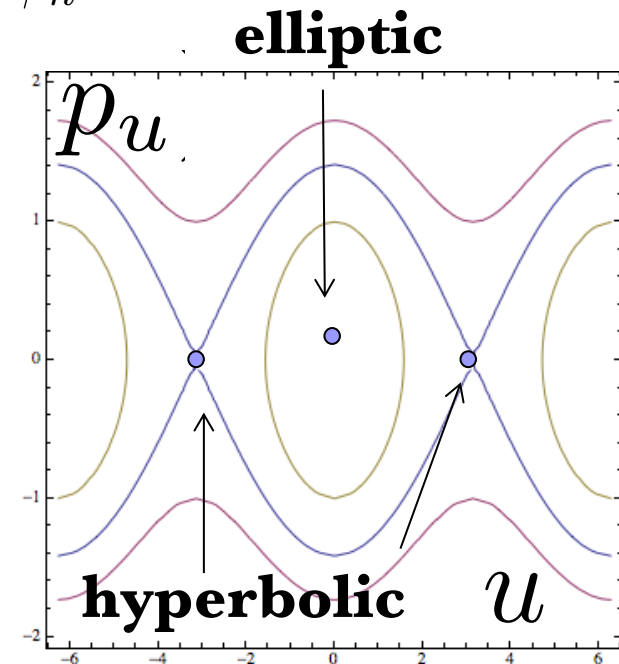
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- $\phi_n = (2n + 1)\pi$ , for which  $\lambda_{1,2} = \pm \sqrt{\frac{g}{L}}$   
corresponding to **hyperbolic** fixed points

- The **separatrix** are the stable and unstable manifolds through the hyperbolic points, separating bounded **librations** and unbounded **rotations**

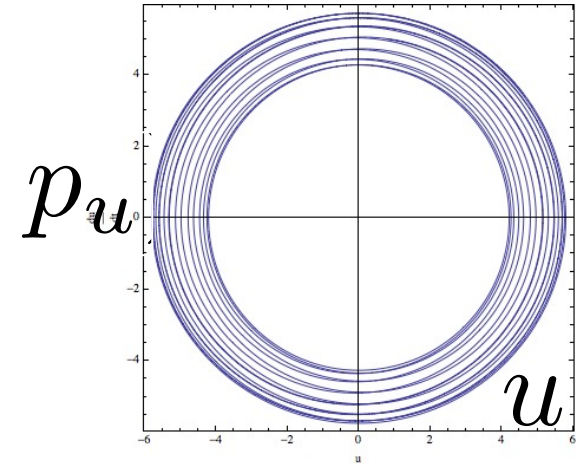




- Consider now a simple harmonic oscillator where the **frequency is time-dependent**

$$H = \frac{1}{2} (p_u^2 + \omega_0^2(t)u^2)$$

- Plotting the evolution in phase space, provides trajectories that **intersect** each other
- The phase space has **time** as **extra dimension**

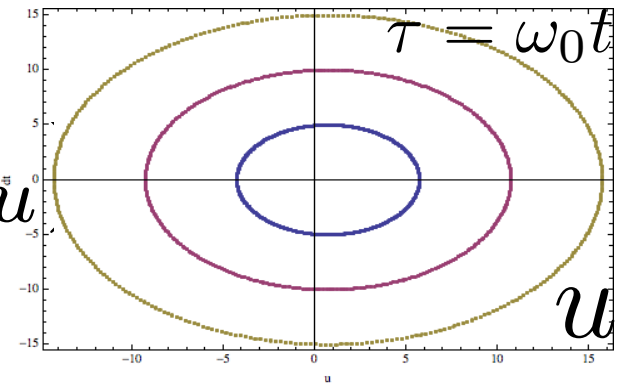
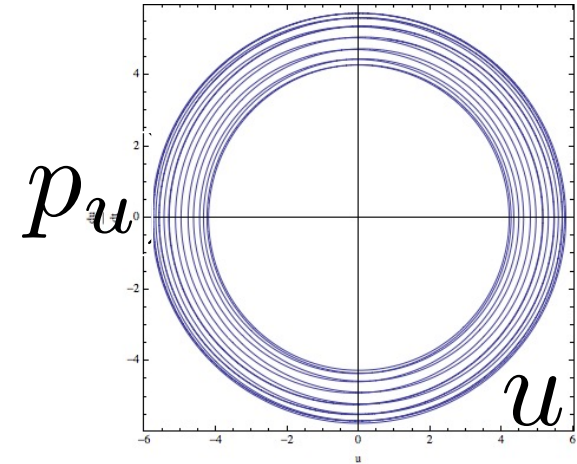




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- By **rescaling** the **time** to become  $\tau = \omega_0 t$  and considering every integer interval of the **new**  $p_u$  “**time**” variable, the **phase space** looks like the one of the **harmonic oscillator**
- This is the simplest version of a **Poincaré surface of section**, which is useful for studying geometrically phase space of multi-dimensional systems



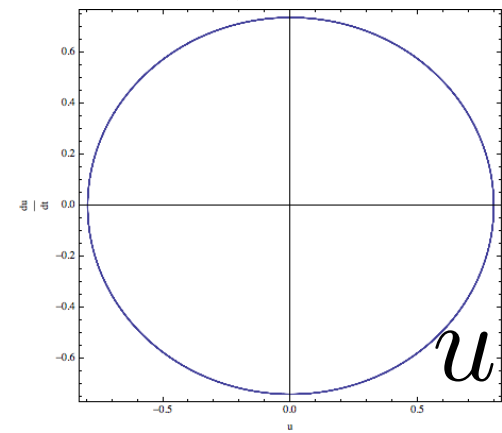
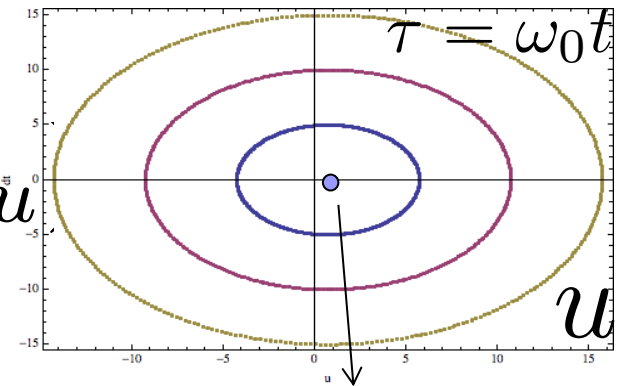
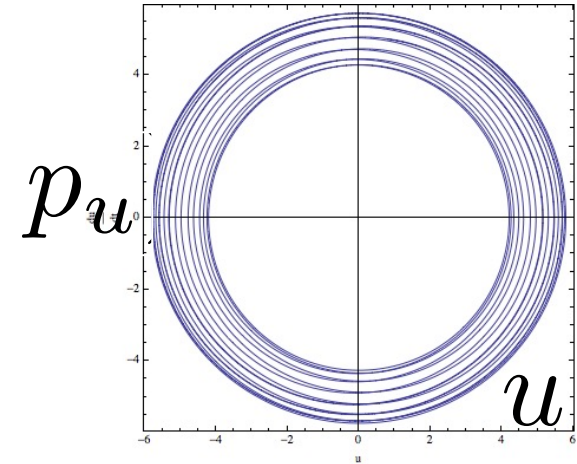




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- The **fixed point** in the surface of section is now a periodic orbit



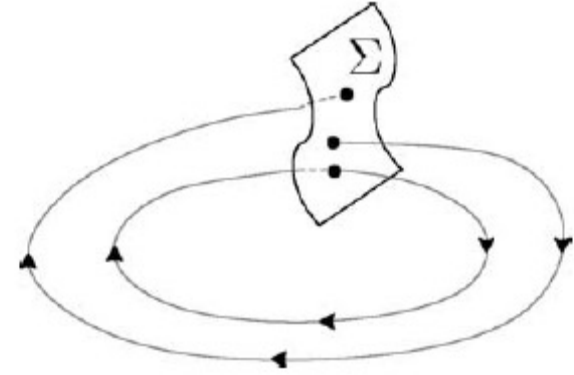
# Poincaré map



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■ **First recurrence or Poincaré map** (or surface of section) is defined by the intersection of trajectories of a dynamical system, with a fixed surface in phase space

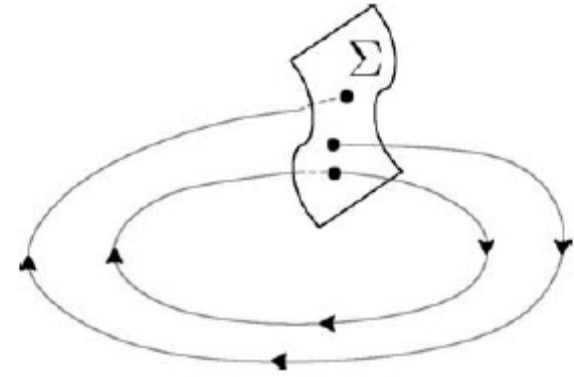




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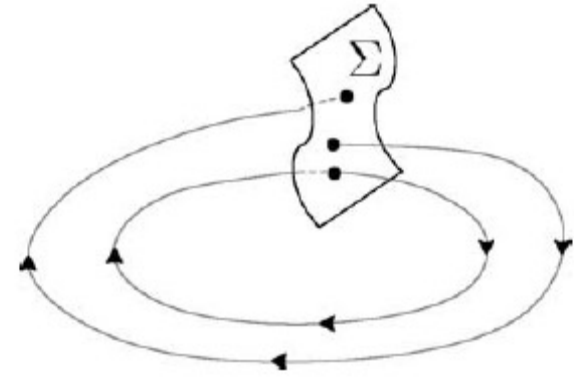
■ For an **autonomous** Hamiltonian system  $H(\mathbf{q}, \mathbf{p})$  (no **explicit** time dependence), it can be chosen to be any fixed surface in phase space, e.g.  $q_i = 0$

■ For a **non-autonomous** Hamiltonian system  $H(\mathbf{q}, \mathbf{p}, t)$  (explicit time dependence), which is **periodic**, it can be chosen as the period  $t = T_0$





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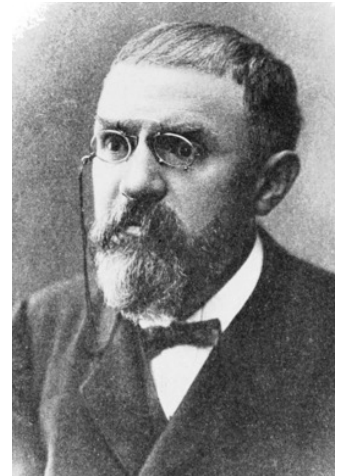


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■ In a system with  $n$  degrees of freedom (or  $n + 1$  including time), the phase space has  $2n$  (or  $2n + 2$ ) dimensions

■ By fixing the value of the Hamiltonian to  $H_0$ , the motion on a Poincaré map is reduced to  $2n - 2$  (or  $2n$ )





- Particularly useful for a system with **2 degrees of freedom**, or **1 degree of freedom + time**, as the motion on Poincaré map is described by 2-dimensional curves
- For continuous system, numerical techniques exist to compute the surface exactly (e.g. M.Henon Physica D 5, 1982)

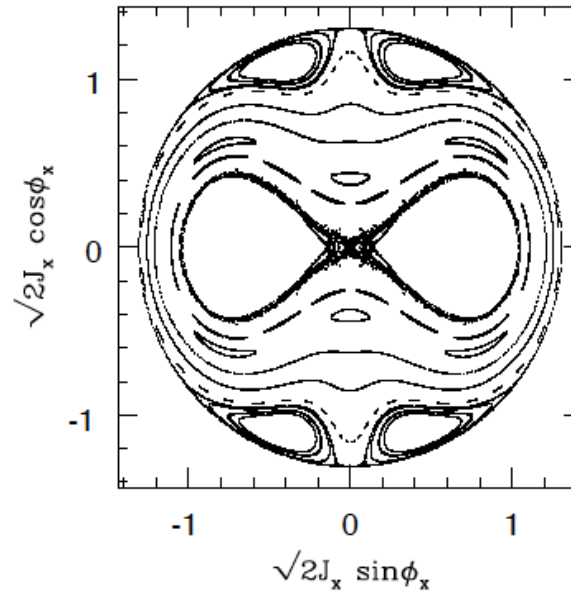
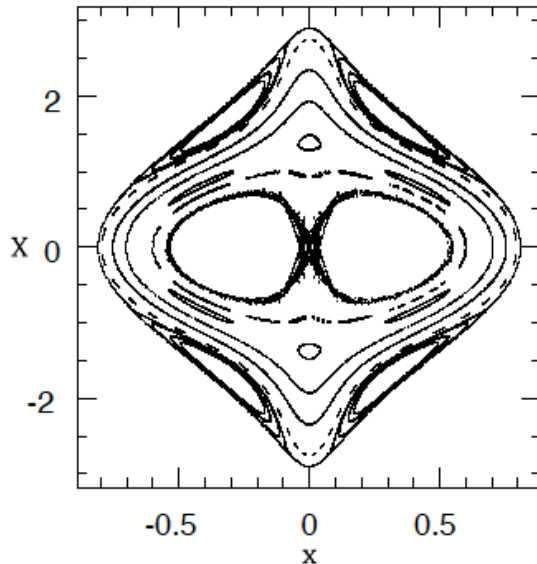


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- Example from Astronomy: the logarithmic galactic potential

$$H_q(x, y, X, Y) = \frac{1}{2}(X^2 + Y^2) + \ln\left(x^2 + \frac{y^2}{q^2} + R_c^2\right)$$

$$(x, y, X, Y) \mapsto (\phi_x, \phi_y, J_x, J_y)$$

$y = 0$



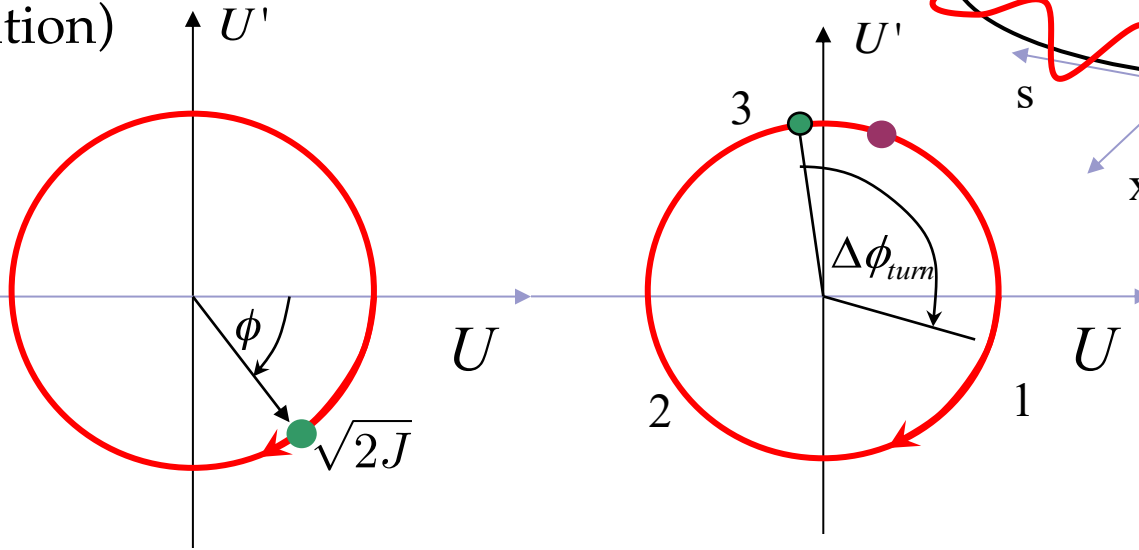
$\phi_y = 0$



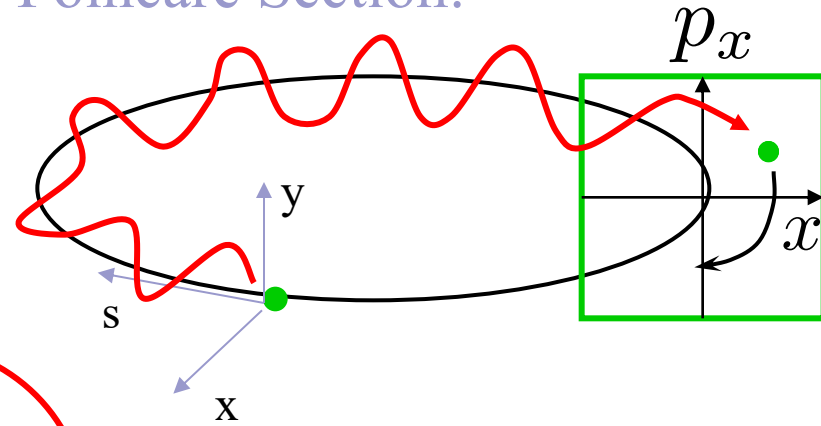
# Poincaré Section for a ring



- Record the particle coordinates at one location in a ring
- Unperturbed motion lies on a circle in normalized coordinates (simple rotation)



## Poincaré Section:



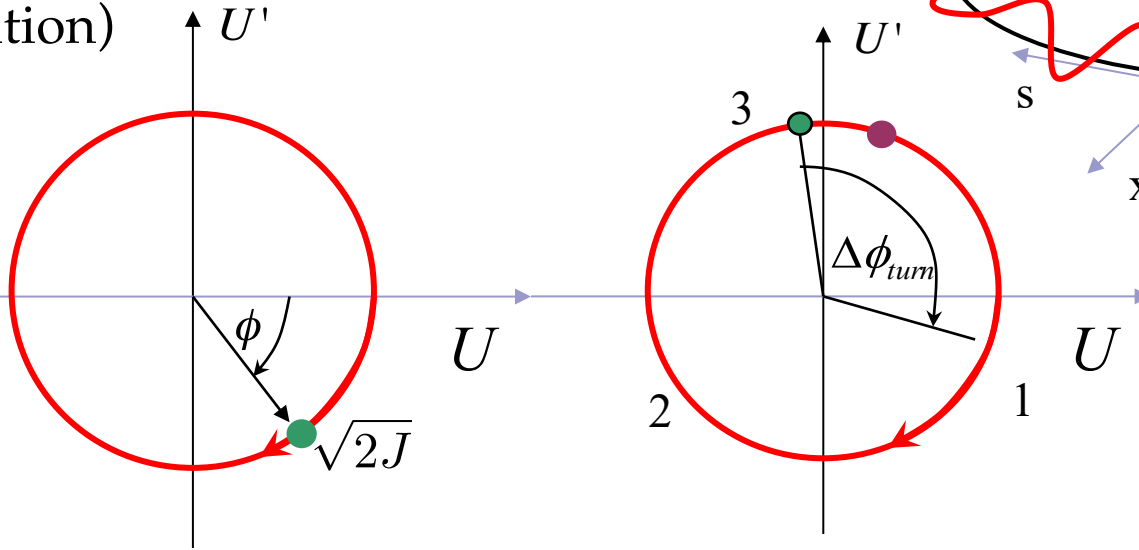




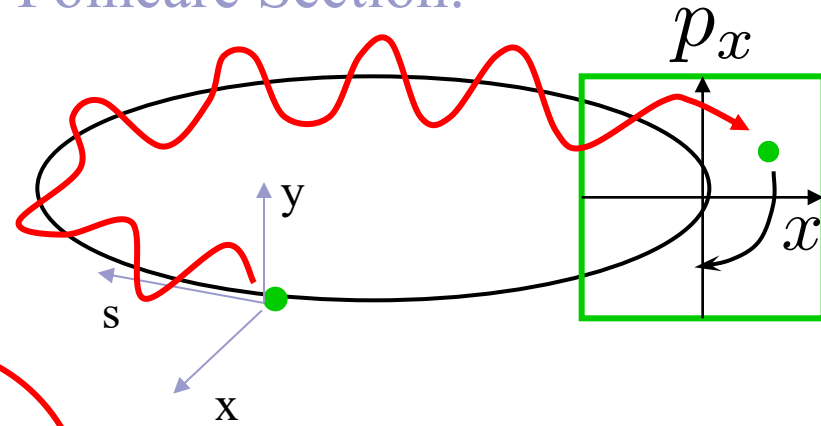
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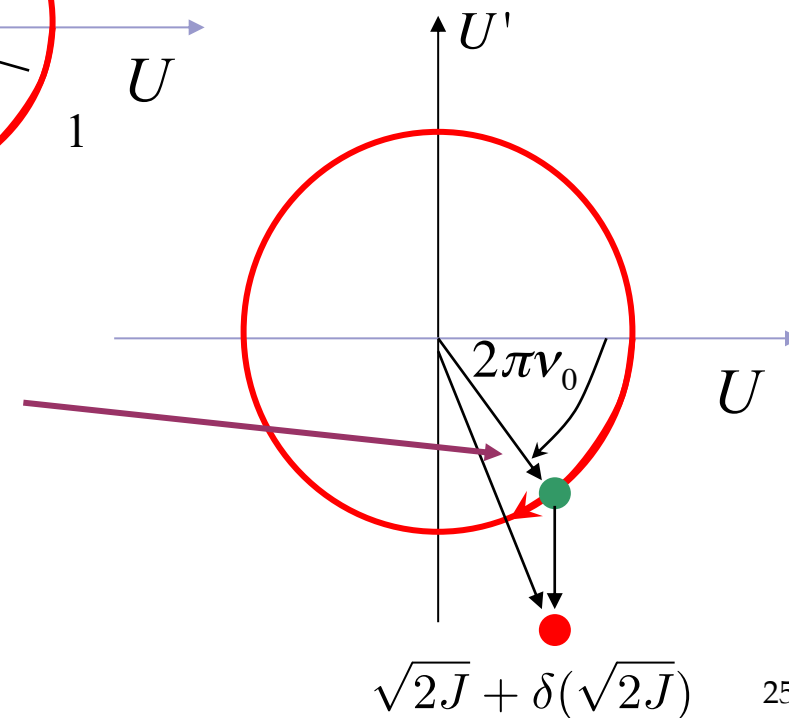


## Poincaré Section:



- Resonance condition corresponds to a periodic orbit or fixed points in phase space

- For a non-linear kick, the radius will change by  $\delta(\sqrt{2J})$  and the particles stop lying on circles

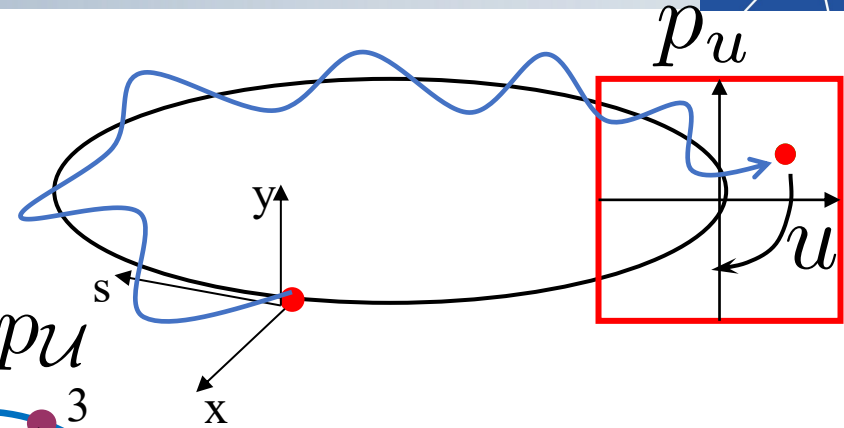
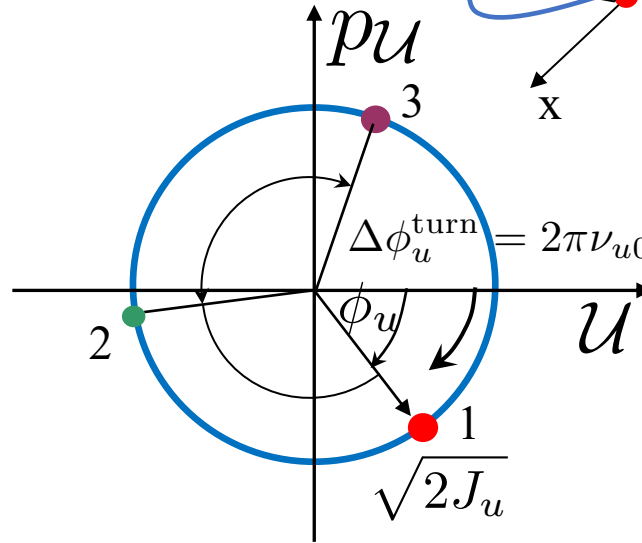
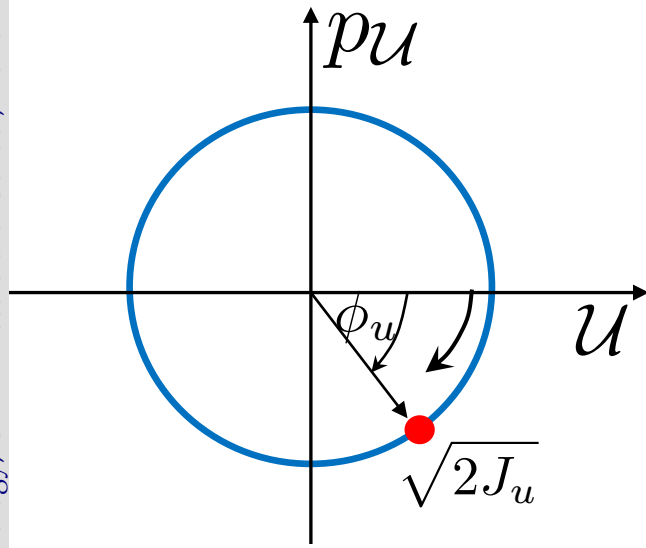




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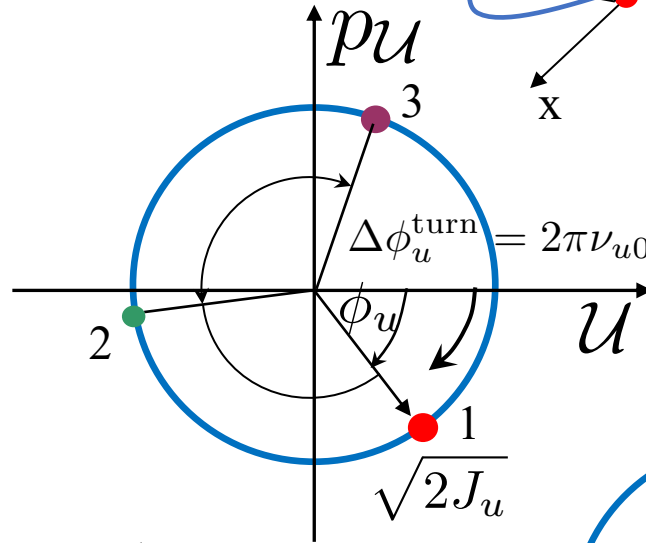
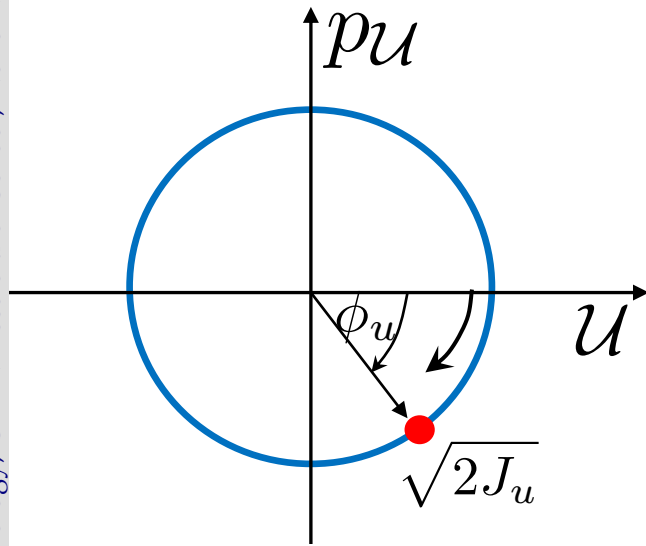
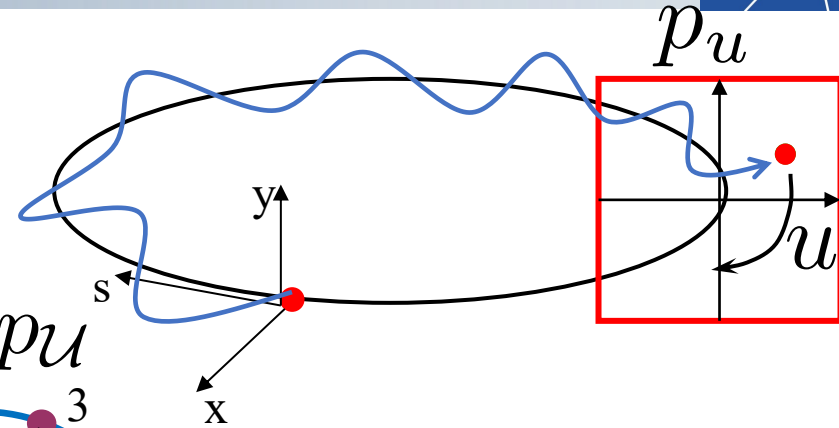




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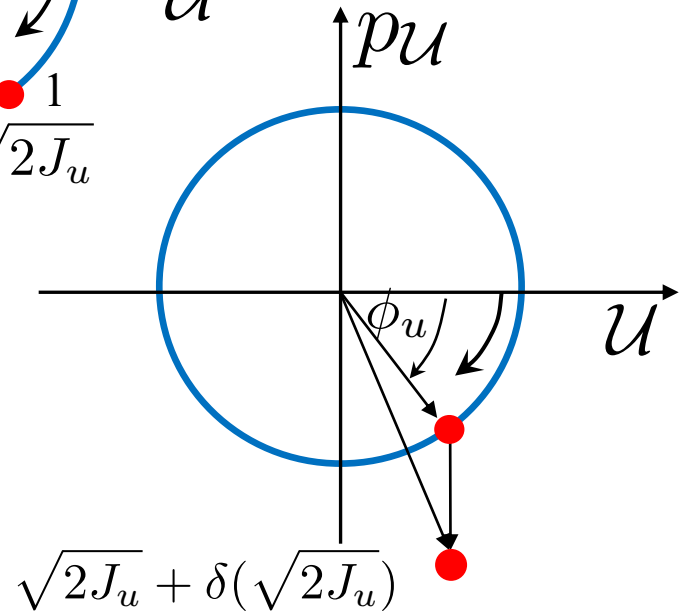


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- Simple **map** with single octupole kick with integrated strength  $k_3$  + rotation with phase advances  $(\mu_x, \mu_y)$

```
def OctupoleMap(k3,x,px,y,py):  
    x1 = x  
    px1 = px - k3*(x**3-3*x*y**2)  
    y1 = y  
    py1 = py - k3*(-3*x**2*y+y**3)  
    return x1,px1,y1,py1  
  
def Rotation(mux,muy,x,px,y,py):  
    x1 = cos(mux)*x+sin(mux)*px  
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    y1 = cos(muy)*y+sin(muy)*py  
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- Restrict motion in  $(x, p_x)$  plane i.e.  $y_0 = p_{y0} = 0$
- Iterate for a number of “turns” (here 1000)



# Example: Single Octupole

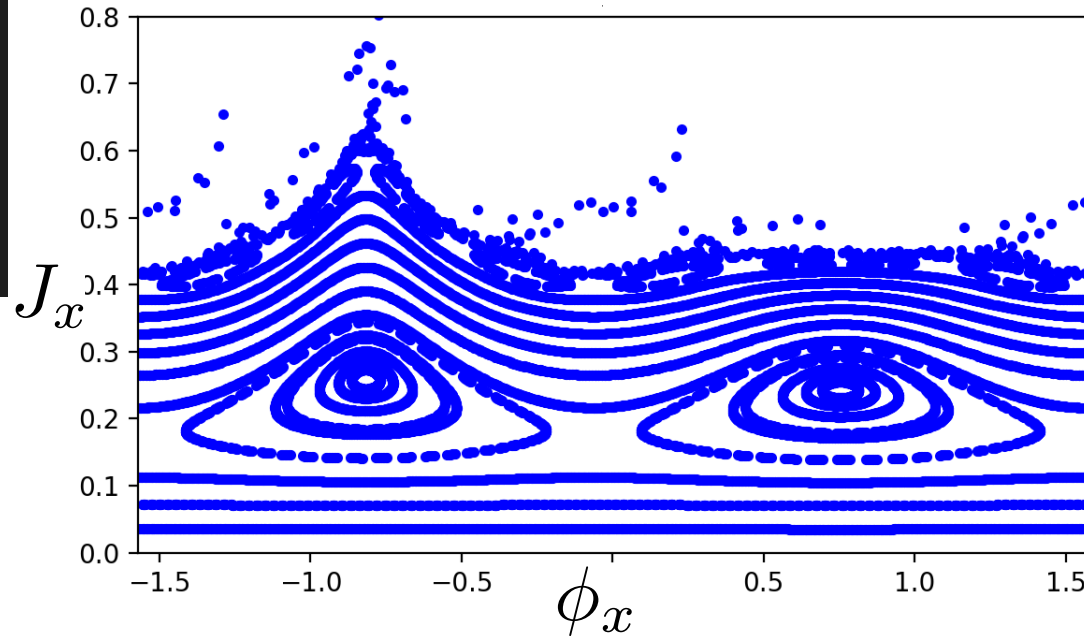
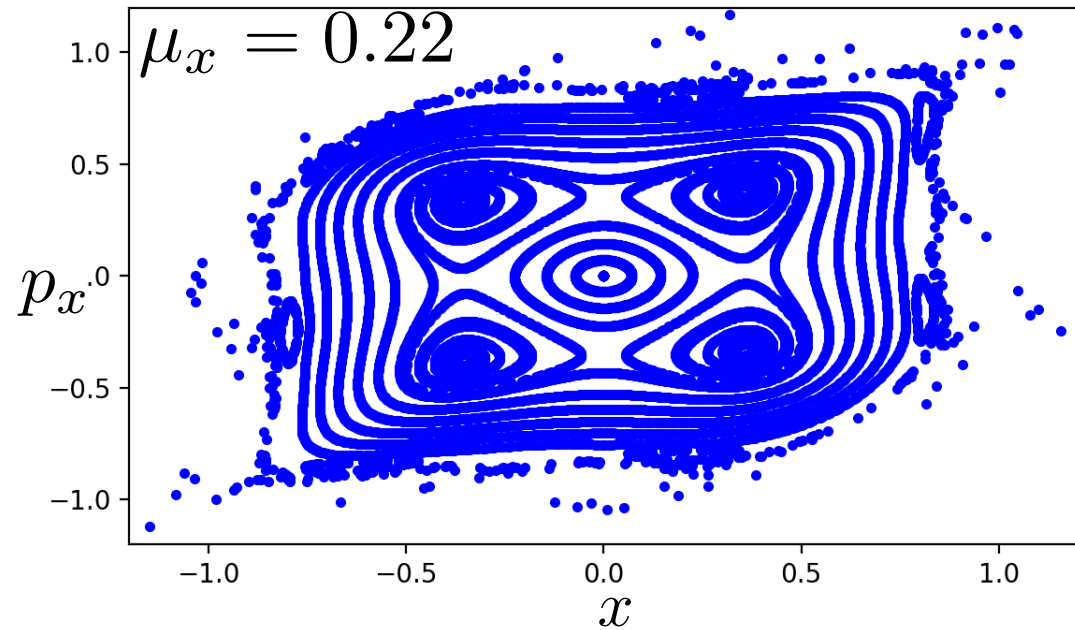


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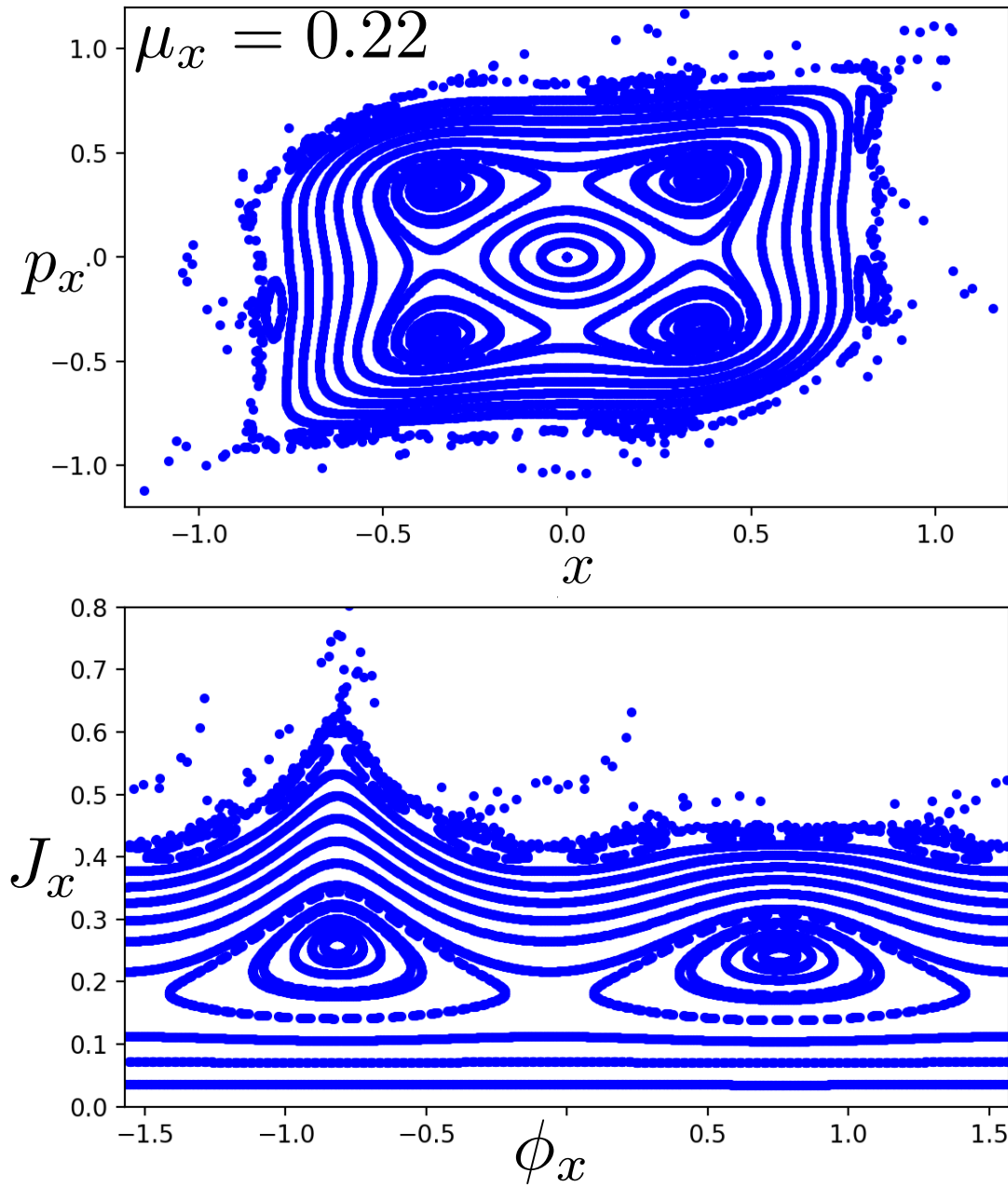




# Example: Single Octupole



- Appearance of **invariant curves** (“distorted” circles), where “action” is an integral of motion
- **Resonant islands** with stable and separatrices with unstable fixed points
- **Chaotic motion**
- Electromagnetic fields coming from multi-pole expansions (polynomials) do not bound phase space and chaotic trajectories may eventually escape to infinity (**Dynamic Aperture**)
- For some fields like beam-beam and space-charge this is not true, i.e. chaotic motion leads to halo formation



# Motion close to a resonance



- The vicinity of a resonance  $n_1\omega_1 + n_2\omega_2 = 0$  , can be studied through **secular perturbation theory** (see appendix) or transforming the 1-turn map (see Etienne's lectures)
- A canonical transformation is applied such that the new variables are in a frame remaining **on top** of the **resonance**
- If one frequency is slow, one can average the motion and remain only with a **1 degree of freedom Hamiltonian** which looks like the one of the **pendulum**
- Thereby, one can find the location and nature of the fixed points measure the width of the resonance





- For any **polynomial perturbation** of the form  $x^k$  the “resonant” Hamiltonian is written as

$$\hat{H}_2 = \delta J_2 + \alpha(J_2) + J_2^{k/2} A_{kp} \cos(k\psi_2)$$

- With the **distance** to the resonance defined as  $\nu = \frac{p}{3} + \delta$ ,  $\delta \ll 1$
- The non-linear shift of the tune is described by the term  $\alpha(J_2)$

- The conditions for the fixed points are

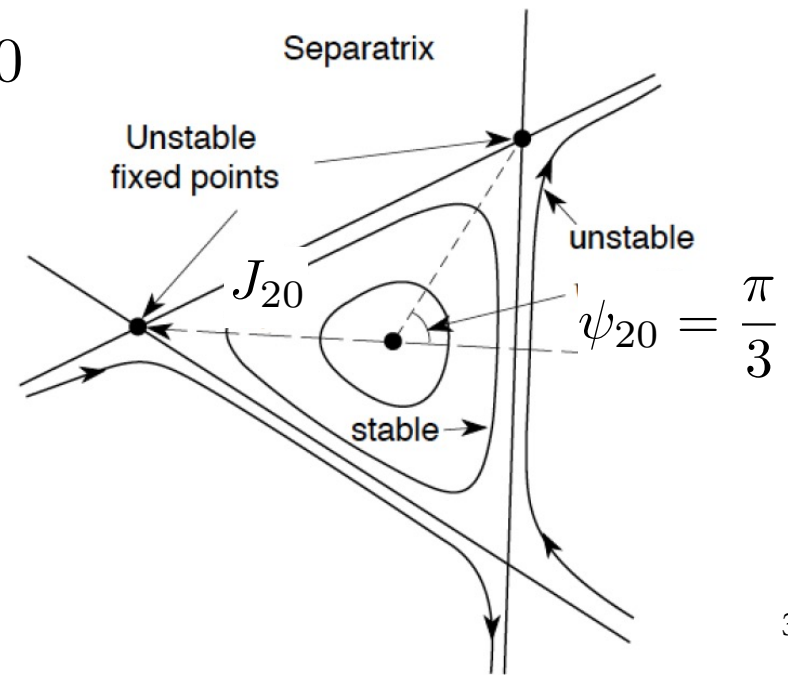
$$\sin(k\psi_2) = 0, \quad \delta + \frac{\partial\alpha(J_2)}{\partial J_2} + \frac{k}{2} J_2^{k/2-1} A_{kp} \cos(k\psi_2) = 0$$

- There are **fixed points** for which  $\cos(k\psi_{20}) = -1$  and the fixed points are **stable** (elliptic). They are surrounded by ellipses

- There are also **fixed points** for which  $\cos(k\psi_{20}) = 1$  and the fixed points are **unstable** (hyperbolic). The trajectories are hyperbolas



- The Hamiltonian for a sextupole close to a third order resonance is  $\hat{H}_2 = \delta J_2 + J_2^{3/2} A_{3p} \cos(3\psi_2)$
- Note the absence of the non-linear tune-shift term (in this 1<sup>st</sup> order approximation!)
- By setting the Hamilton's equations equal to zero, three fixed points can be found at  $\psi_{20} = \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, J_{20} = \left(\frac{2\delta}{3A_{3p}}\right)^2$
- For  $\frac{\delta}{A_{3p}} > 0$  all three points are unstable
- Close to the elliptic one at  $\psi_{20} = 0$  the motion in phase space is described by circles that they get more and more distorted to end up in the “triangular” **separatrix** uniting the unstable fixed points
- The tune separation from the resonance is  $\delta = \frac{3A_{3p}}{2} J_{20}^{1/2}$





- Simple **map** with single **sextupole kick** with integrated strength  $k_2$  + rotation with phase advances  $(\mu_x, \mu_y)$

```
def SextupoleMap(k2,x,px,y,py):  
    x1 = x  
    px1 = px - k2*(x**2-y**2)  
    y1 = y  
    py1 = py - k2*(-2*x*y)  
    return x1,px1,y1,py1  
  
def Rotation(mux,muy,x,px,y,py):  
    x1 = cos(mux)*x+sin(mux)*px  
    px1 =-sin(mux)*x+cos(mux)*px  
    y1 = cos(muy)*y+sin(muy)*py  
    py1 =-sin(muy)*y+cos(muy)*py  
    return x1,px1,y1,py1
```

- Restrict motion in  $(x, p_x)$  plane i.e.  $y_0 = p_{y0} = 0$
- Iterate for a number of “turns” (here 1000)

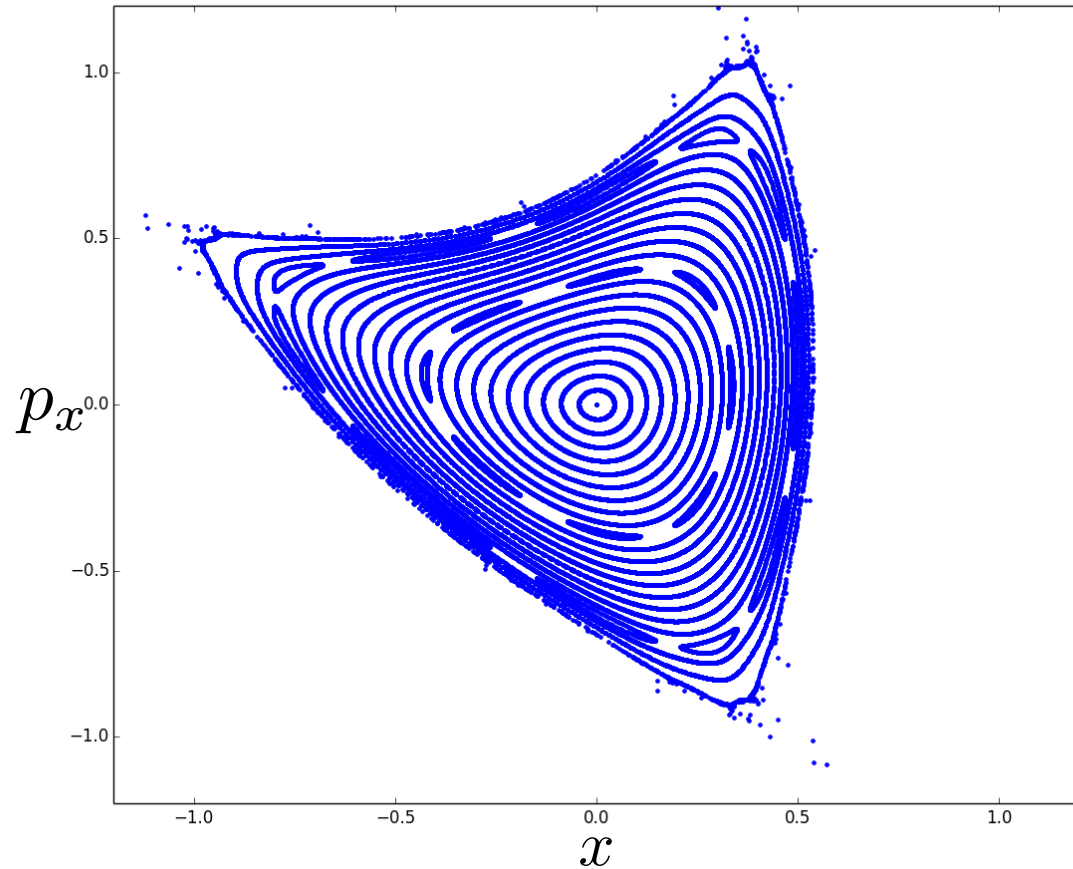


- Simple **map** with single **sextupole kick** with integrated strength  $k_2$  + rotation with phase advances  $(\mu_x, \mu_y)$

$$\mu_x = 0.38$$

```
def SextupoleMap(k2,x,px,y,py):  
    x1 = x  
    px1 = px - k2*(x**2-y**2)  
    y1 = y  
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    return x1,px1,y1,py1  
  
def Rotation(mux,muy,x,px,y,py):  
    x1 = cos(mux)*x+sin(mux)*px  
    px1 = -sin(mux)*x+cos(mux)*px  
    y1 = cos(muy)*y+sin(muy)*py  
    py1 = -sin(muy)*y+cos(muy)*py  
    return x1,px1,y1,py1
```

- Restrict motion in  $(x, p_x)$  plane i.e.  $y_0 = p_{y0} = 0$
- Iterate for a number of “turns” (here 1000)





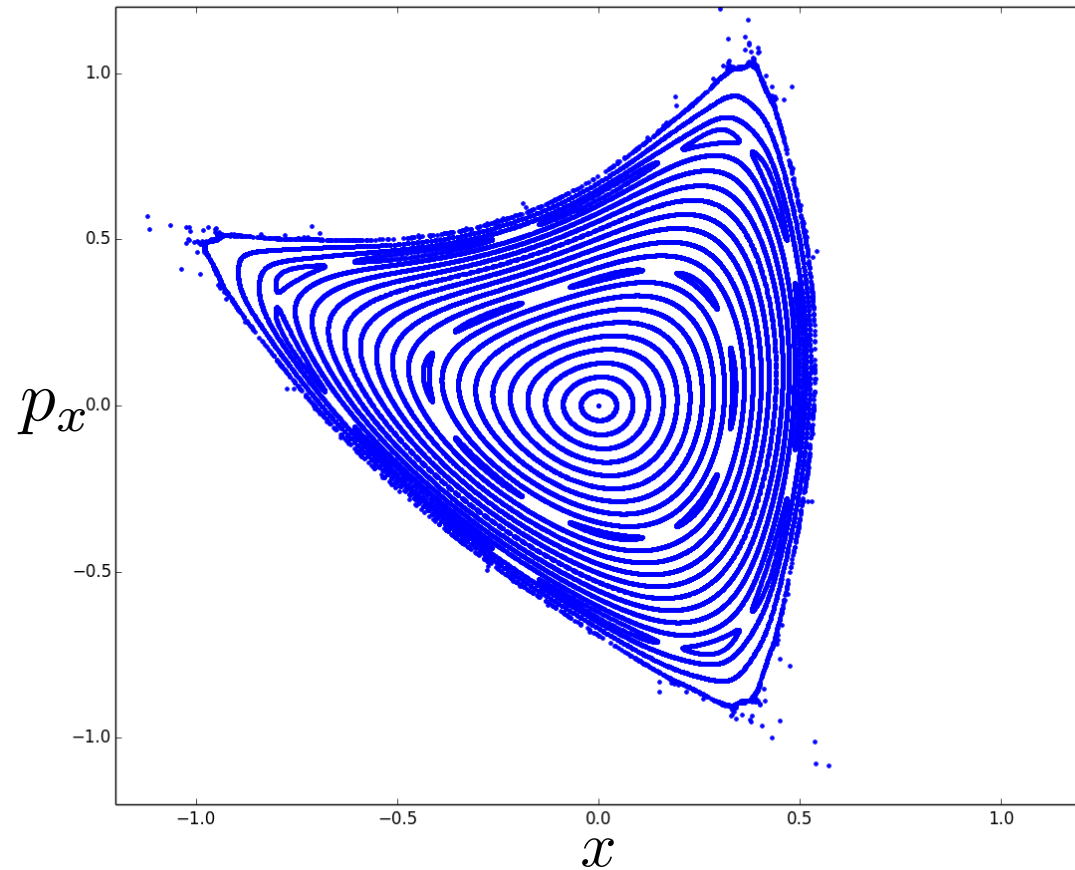
# Example: Single Sextupole



- Appearance of 3<sup>rd</sup> order resonance for certain phase advance

- ... but also 4<sup>th</sup> order resonance

$$\mu_x = 0.38$$





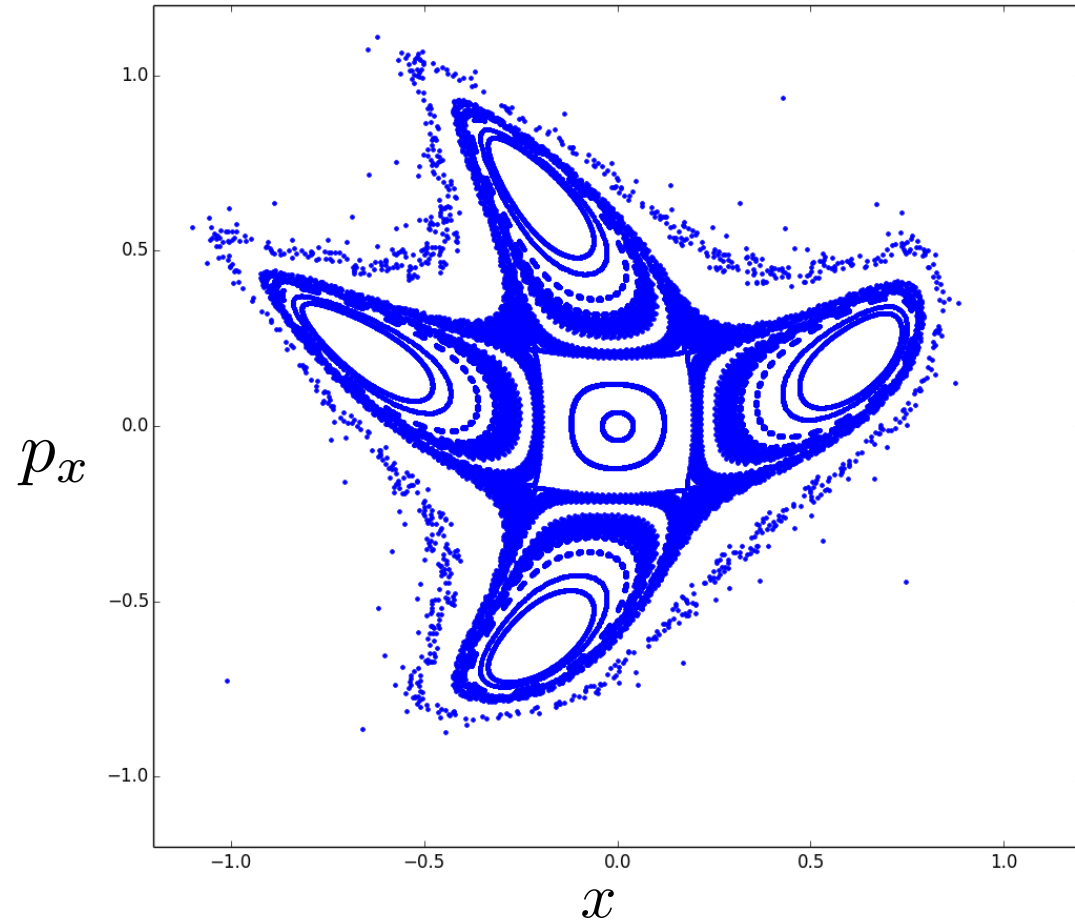
# Example: Single Sextupole



- Appearance of 3<sup>rd</sup> order resonance for certain phase advance

- ... but also 4<sup>th</sup> order resonance

$$\mu_x = 0.253$$





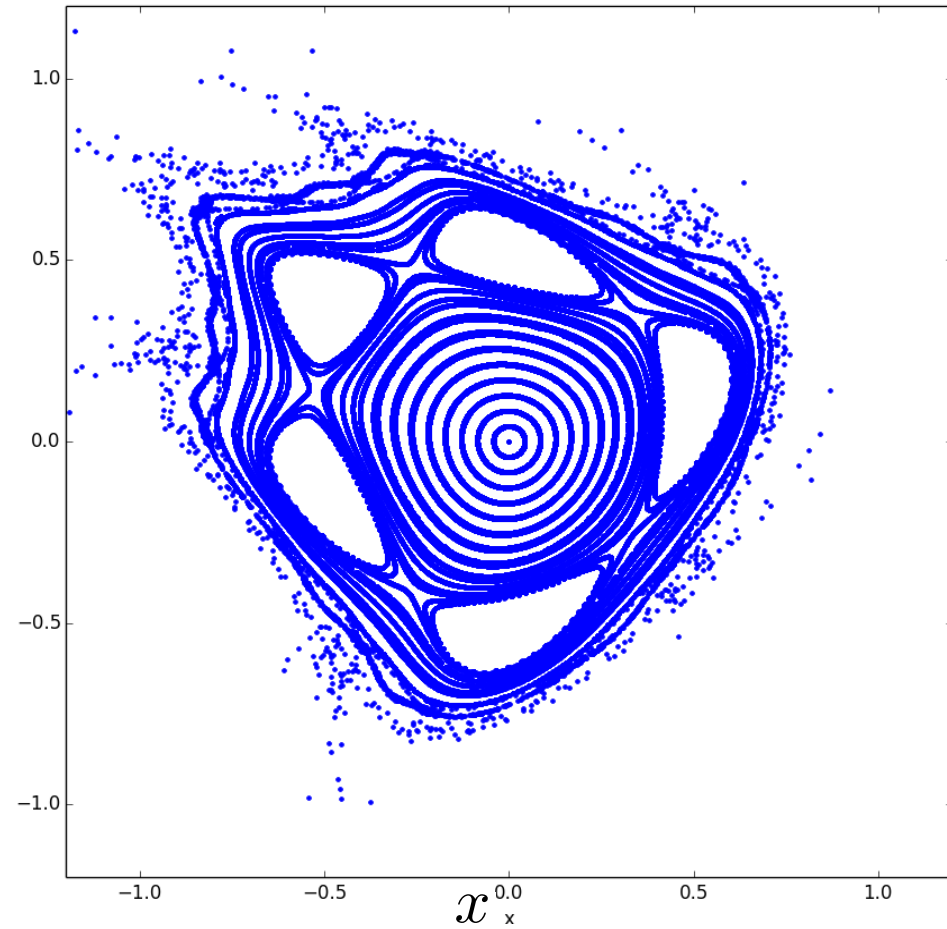
- Appearance of 3<sup>rd</sup> order resonance for certain phase advance

- ... but also 4<sup>th</sup> order resonance

- ... and 5<sup>th</sup> order resonance

$$\mu_x = 0.21$$

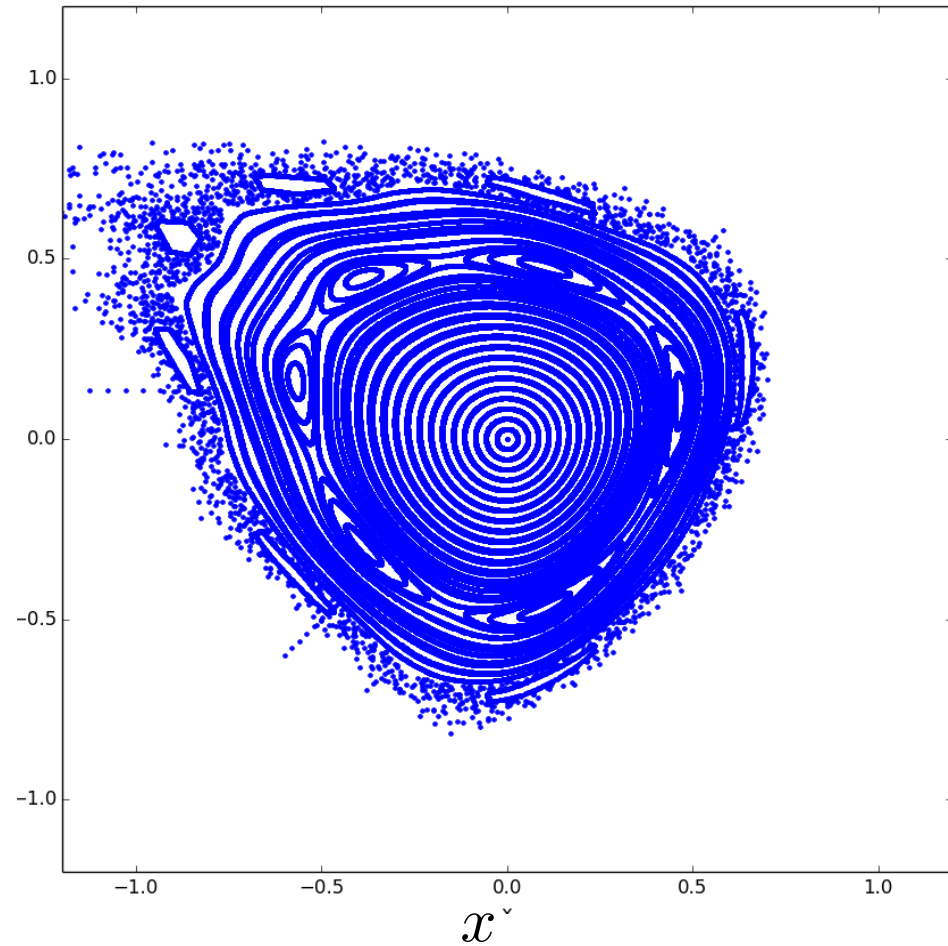
$p_x$





- Appearance of 3<sup>rd</sup> order resonance for certain phase advance
- ... but also 4<sup>th</sup> order resonance
- ... and 5<sup>th</sup> order resonance
- ... and 6<sup>th</sup> order and 7<sup>th</sup> order and several higher orders...

$$\mu_x = 0.18$$







- The resonant Hamiltonian close to the 4<sup>th</sup> order resonance is written as

$$\hat{H}_2 = \delta J_2 + cJ_2^2 + J_2^2 A_{4p} \cos(4\psi_2)$$

- The **fixed points** are found by taking the derivative over the two variables and setting them to zero, i.e.

$$\sin(4\psi_2) = 0, \quad \delta + 2cJ_2 + 2J_2 A_{kp} \cos(4\psi_2) = 0$$

- The fixed points are at

$$\psi_{20} = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$$

- For **half** of them, there is a minimum in the potential as  $\cos(4\psi_{20}) = -1$  and they are **elliptic** and **half** of them they are **hyperbolic** as  $\cos(4\psi_{20}) = 1$

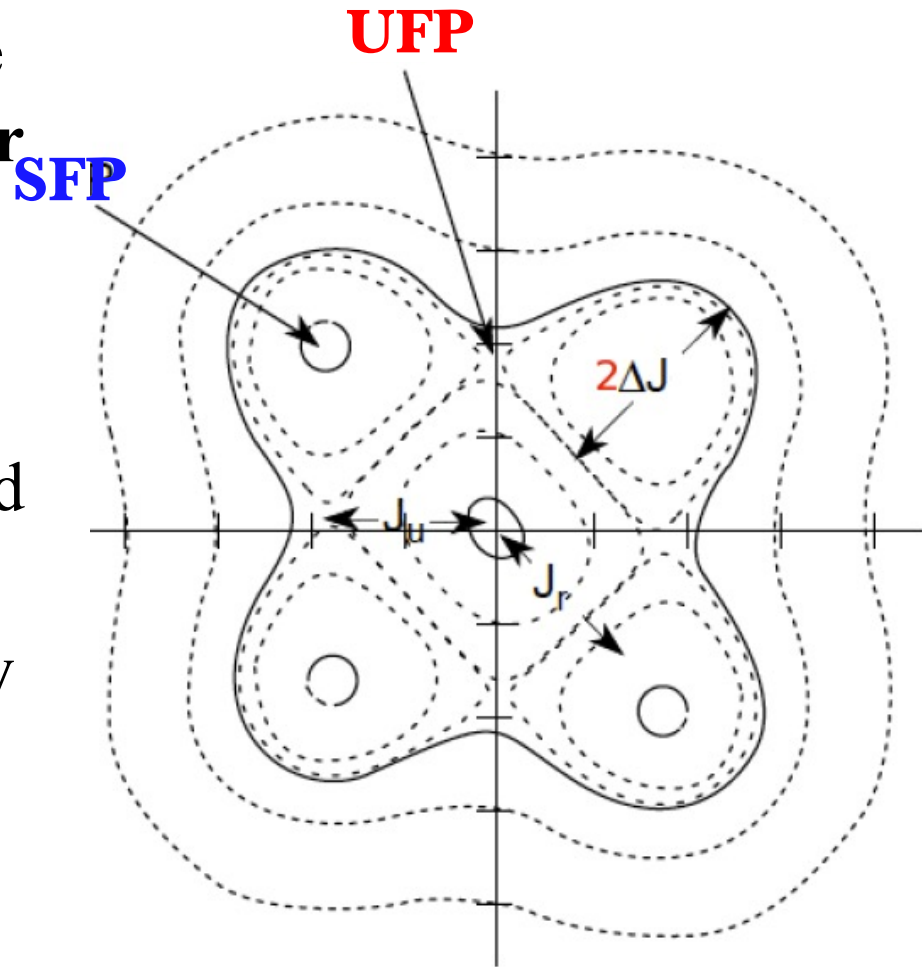


- **Regular motion** near the center, with curves getting more deformed towards a **rectangular shape**

- The **separatrix** passes through 4 unstable fixed points, but motion seems well contained

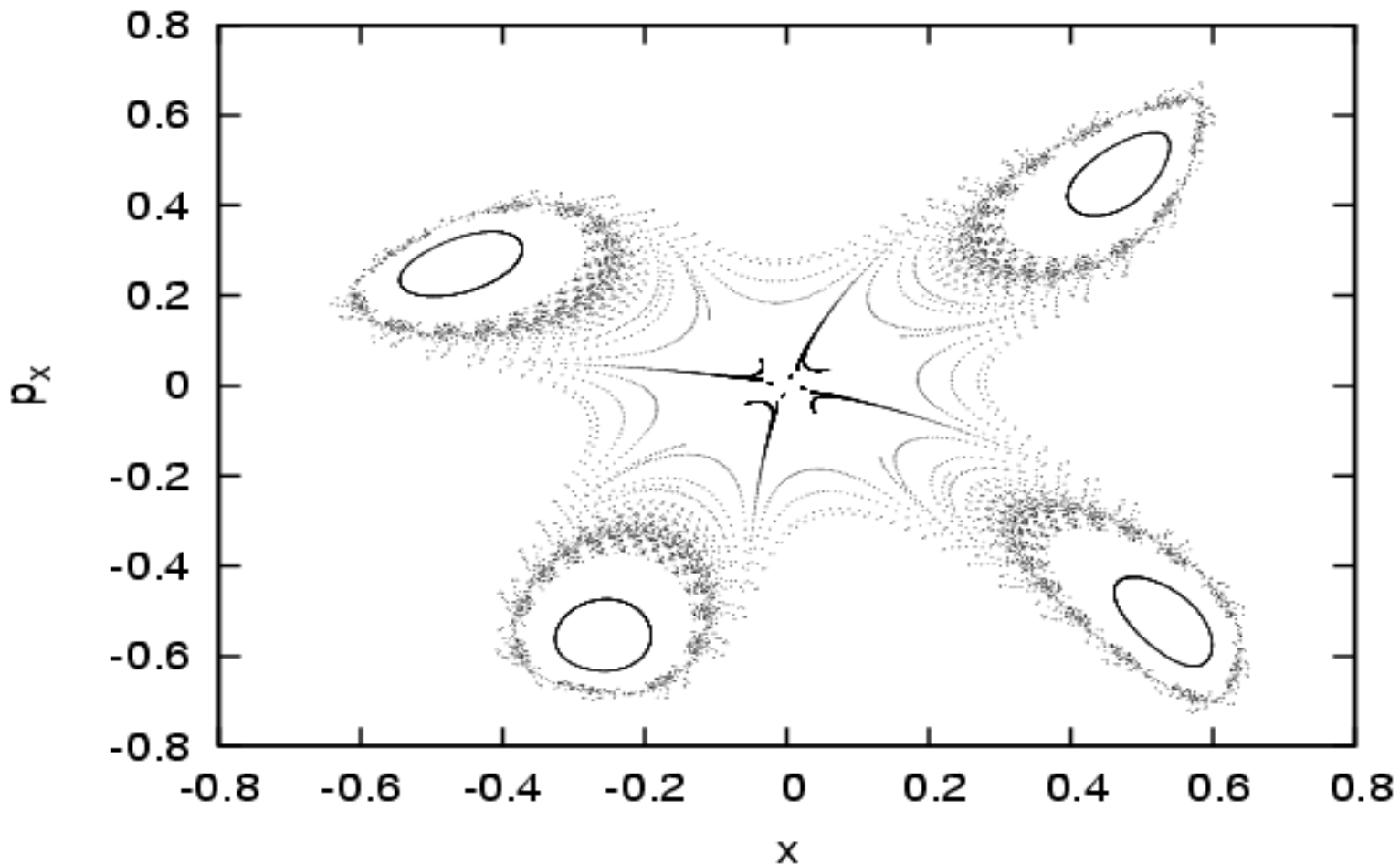
- **Four stable fixed points** exist and they are surrounded by stable motion (**islands of stability**)

- Question: Can the **central fixed point** become **hyperbolic** (answer in the appendix)





- Now, if  $c = 0$  the solution for the action is  $J_{20} = 0$
- So there is **no minima** in the potential, i.e. the central fixed point is **hyperbolic**

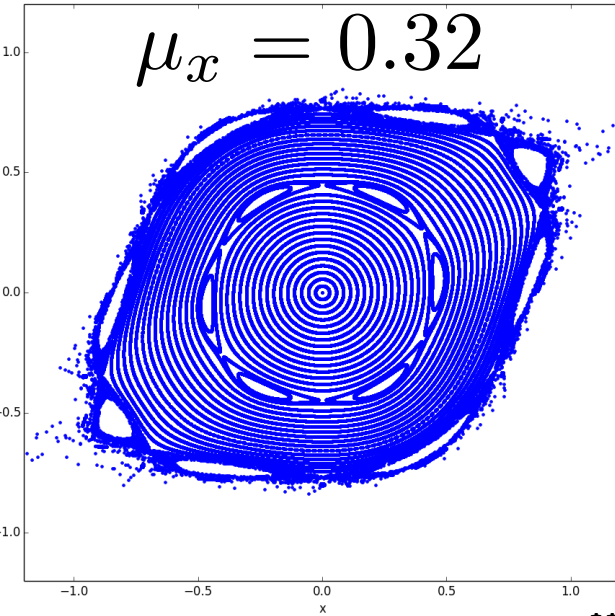
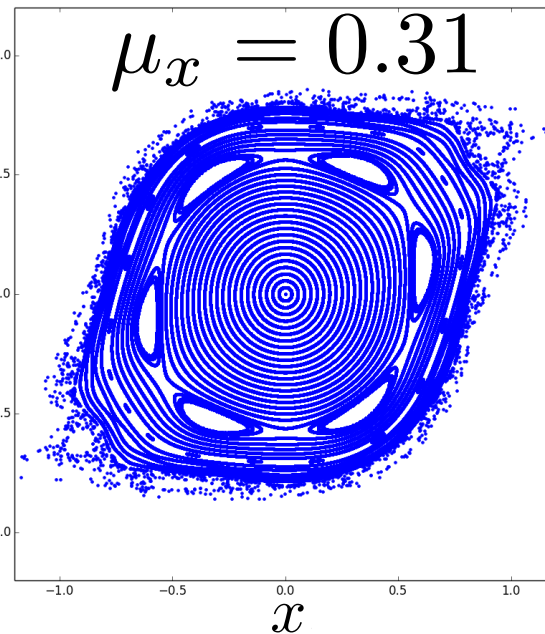
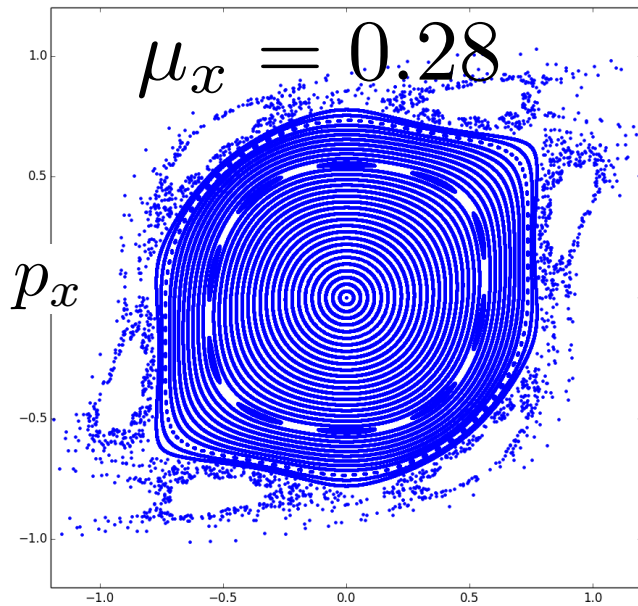
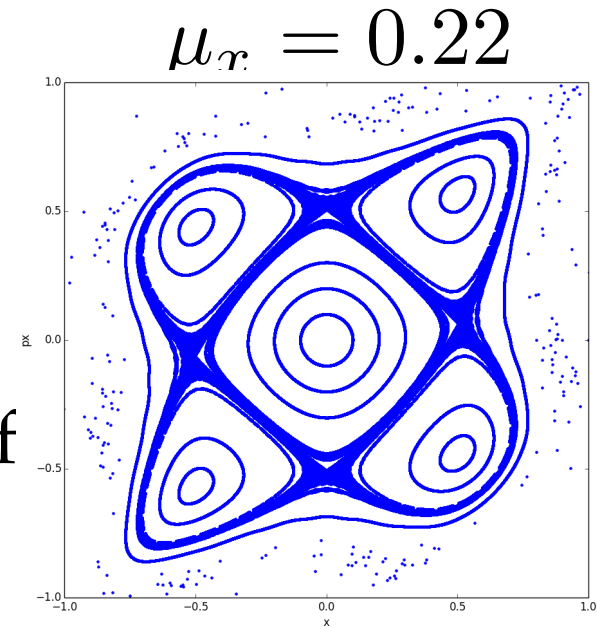




# Single Octupole



- As for the sextupole, the octupole can excite any resonance
- Multi-pole magnets can excite **any resonance order**
- It depends on the **tunes, strength of the magnet and particle amplitudes**

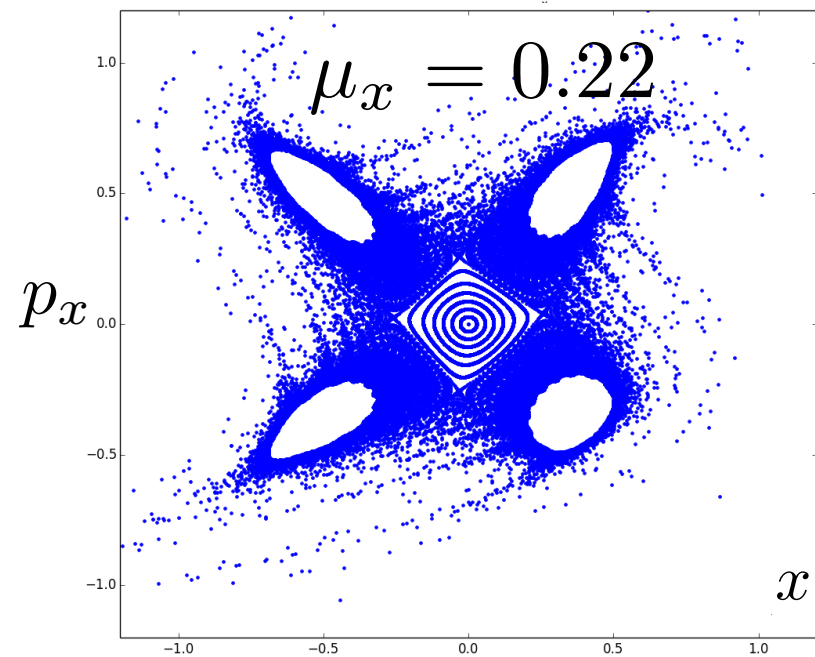
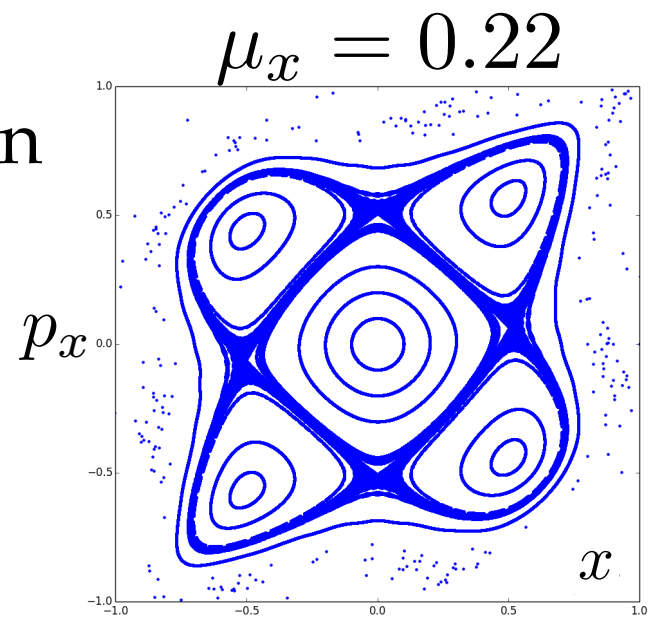




# Single Octupole + Sextupole



■ Adding a sextupole and an octupole increases the chaotic motion region, when close to the 4<sup>th</sup> order resonance



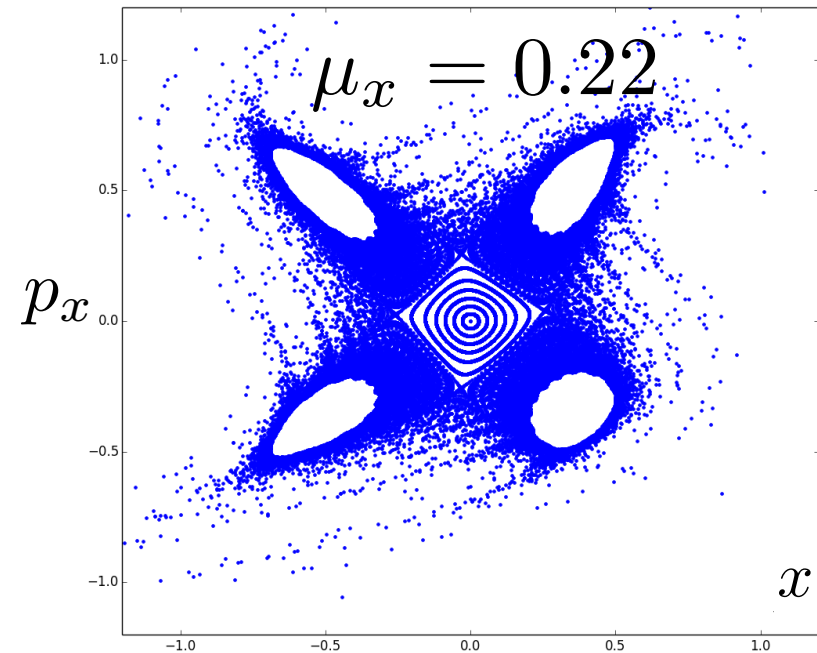
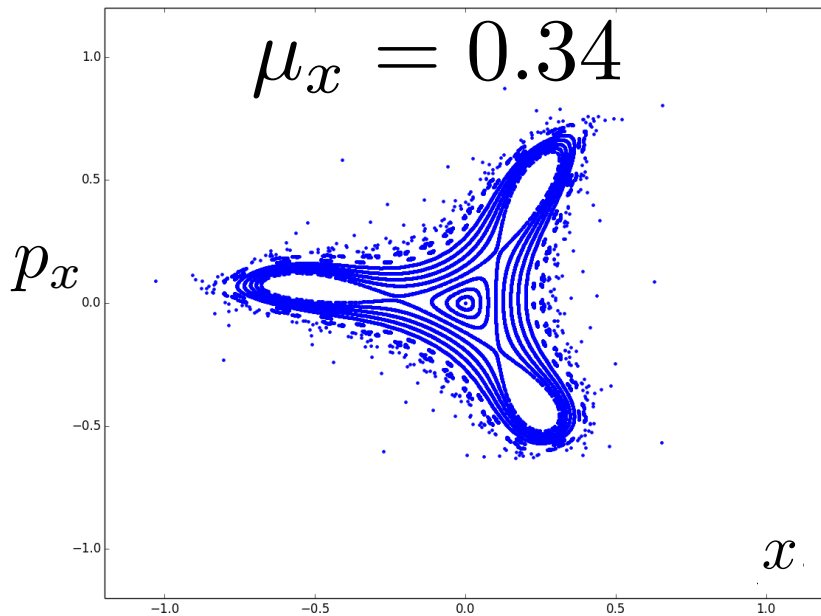
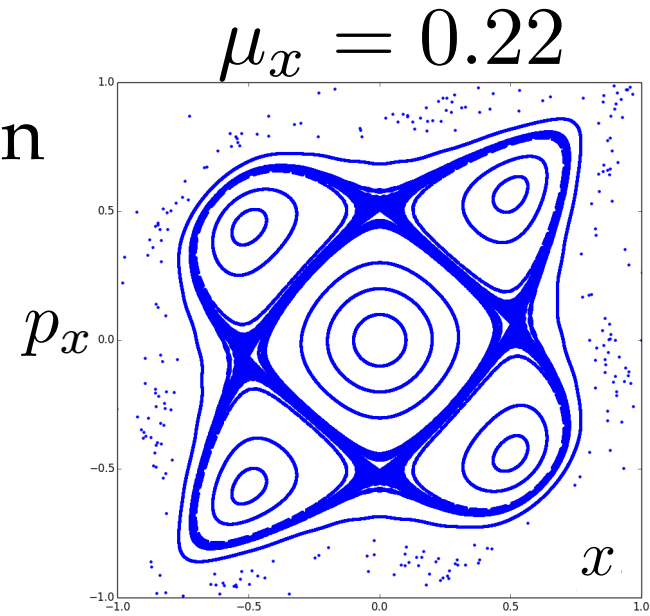


# Single Octupole + Sextupole



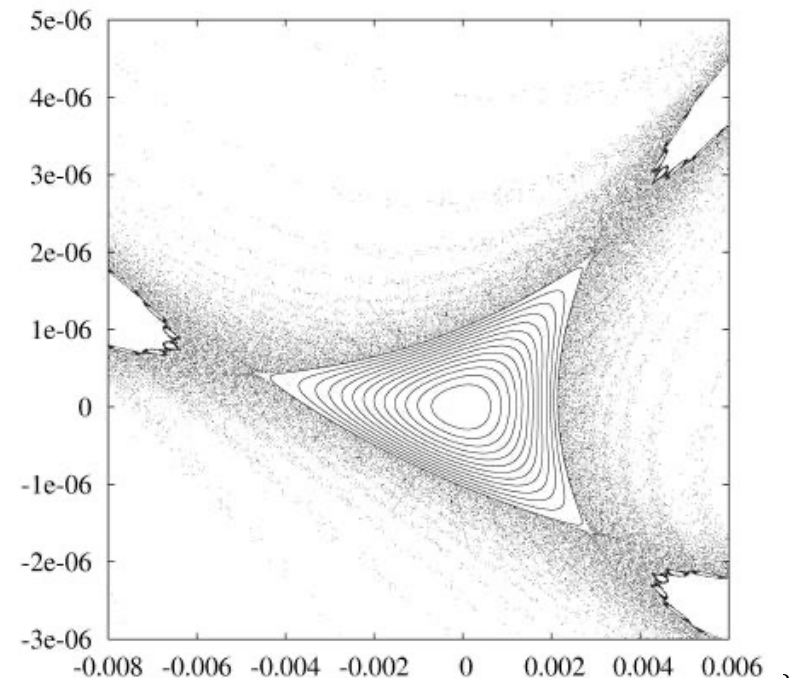
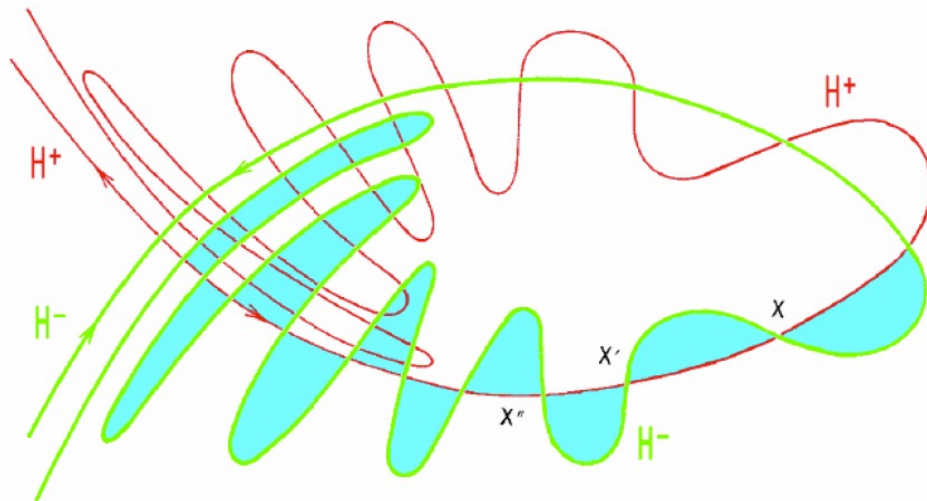
■ Adding a sextupole and an octupole increases the chaotic motion region, when close to the 4<sup>th</sup> order resonance

■ But also allows the appearance of 3<sup>rd</sup> order resonance stable fixed points



# Onset of chaos

- When **perturbation** becomes **higher**, motion around the **separatrix** becomes **chaotic** (producing **tongues** or **splitting** of the separatrix)
- **Unstable** fixed points are indeed the **source of chaos** when a perturbation is added

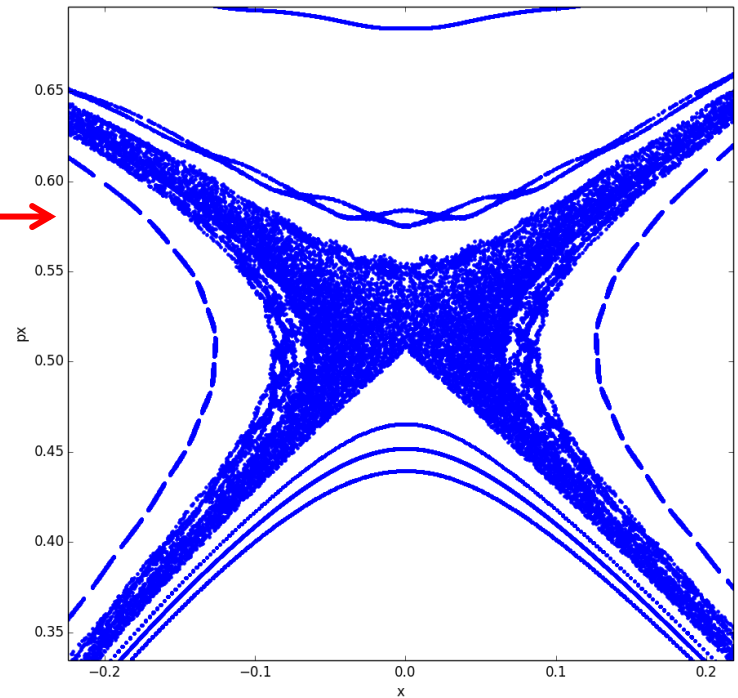
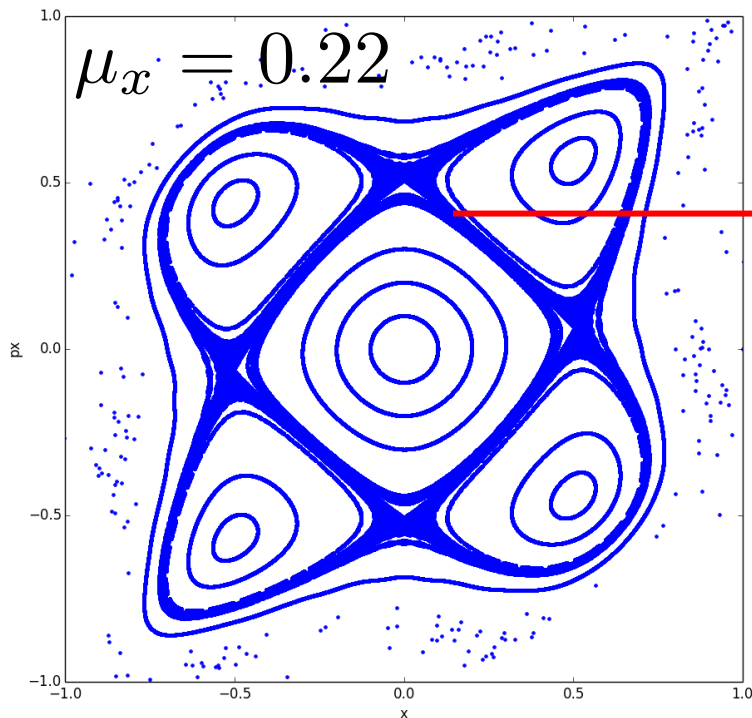
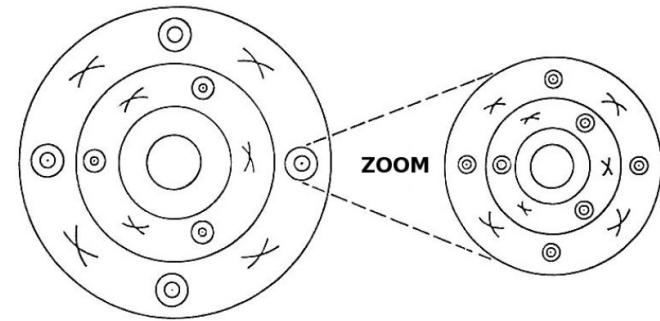




- **Poincare-Birkhoff theorem** states that under perturbation of a resonance only an **even number of fixed points** survives (**half stable** and the other **half unstable**)

- **Themselves get destroyed** when perturbation gets **higher**, etc. (**self-similar fixed points**)

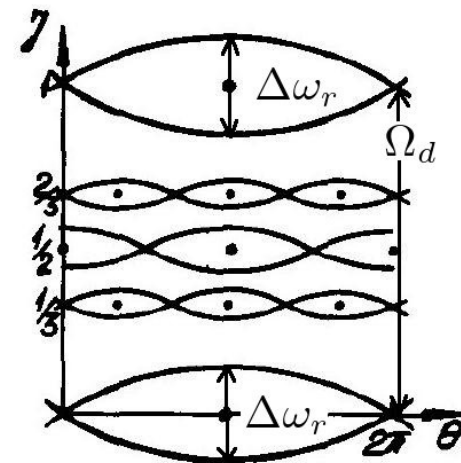
- Resonance **islands grow** and **resonances can overlap** allowing diffusion of particles





- When perturbation grows, the resonance island width grows
- **Chirikov** (1960, 1979) proposed a **criterion** for the **overlap** of two **neighboring resonances** and the onset of orbit diffusion

- The **distance** between two resonances is  $\delta \hat{J}_{1, n, n'} = \frac{2 \left( \frac{1}{n_1 + n_2} - \frac{1}{n'_1 + n'_2} \right)}{\left| \frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^2} \right|_{\hat{J}_1 = \hat{J}_{10}}}$
- The **simple overlap criterion** is  $\Delta \hat{J}_{n \max} + \Delta \hat{J}_{n' \max} \geq \delta \hat{J}_{n, n'}$



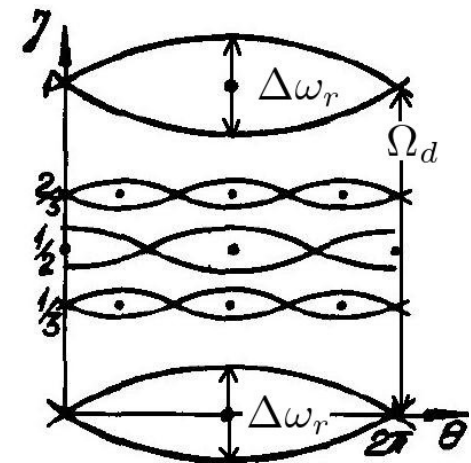
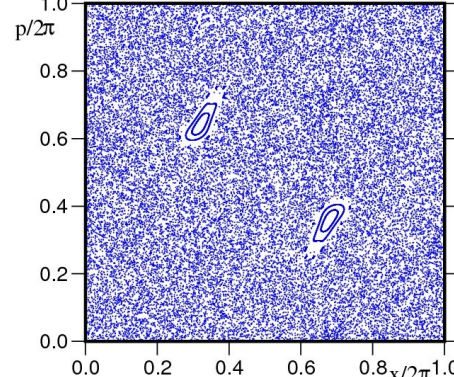
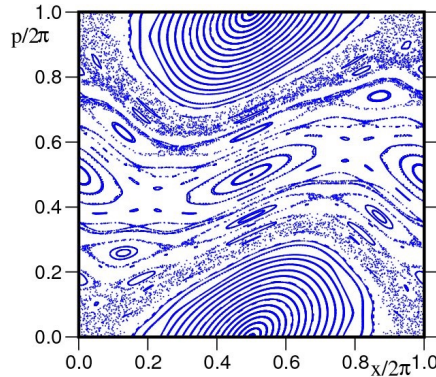
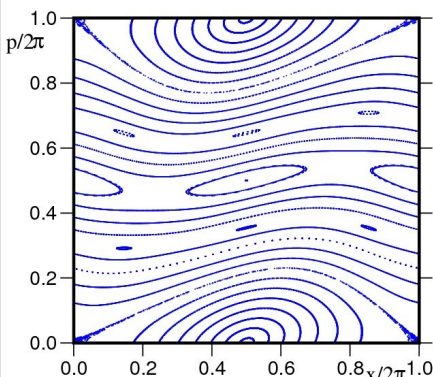


# Resonance overlap criterion



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- The **simple overlap criterion** is  $\Delta \hat{J}_{n, max} + \Delta \hat{J}_{n', max} \geq \delta \hat{J}_{n, n'}$
- Considering the **width of chaotic layer** and **secondary islands**, the “two thirds” rule apply  $\Delta \hat{J}_{n, max} + \Delta \hat{J}_{n', max} \geq \frac{2}{3} \delta \hat{J}_{n, n'}$
- Example: **Chirikov’s standard map**

$$p_{n+1} = p_n + K \sin(\theta_n) \quad \theta_{n+1} = \theta_n + p_{n+1}$$



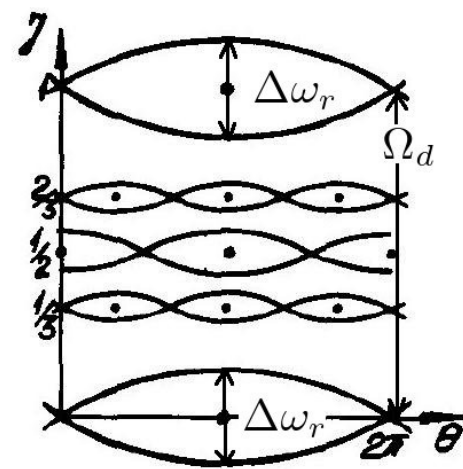
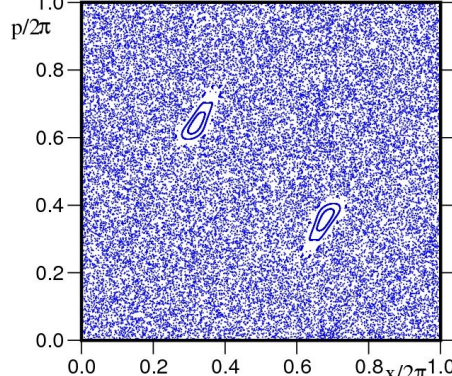
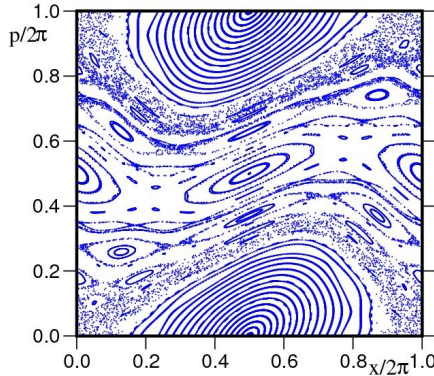
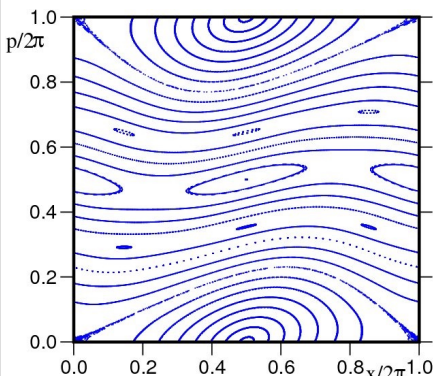


# Resonance overlap criterion



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- The **distance** between two resonances is  $\delta \hat{J}_{1, n, n'} = \frac{2 \left( \frac{1}{n_1 + n_2} - \frac{1}{n'_1 + n'_2} \right)}{\left| \frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^2} \right|_{\hat{J}_1 = \hat{J}_{10}}}$
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- Considering the **width of chaotic layer** and **secondary islands**, the “two thirds” rule apply  $\Delta \hat{J}_{n, max} + \Delta \hat{J}_{n', max} \geq \frac{2}{3} \delta \hat{J}_{n, n'}$
- The main limitation is the **geometrical nature** of the criterion (**difficulty to be extended for > 2 degrees of freedom**)

$$p_{n+1} = p_n + K \sin(\theta_n) \quad \theta_{n+1} = \theta_n + p_{n+1}$$



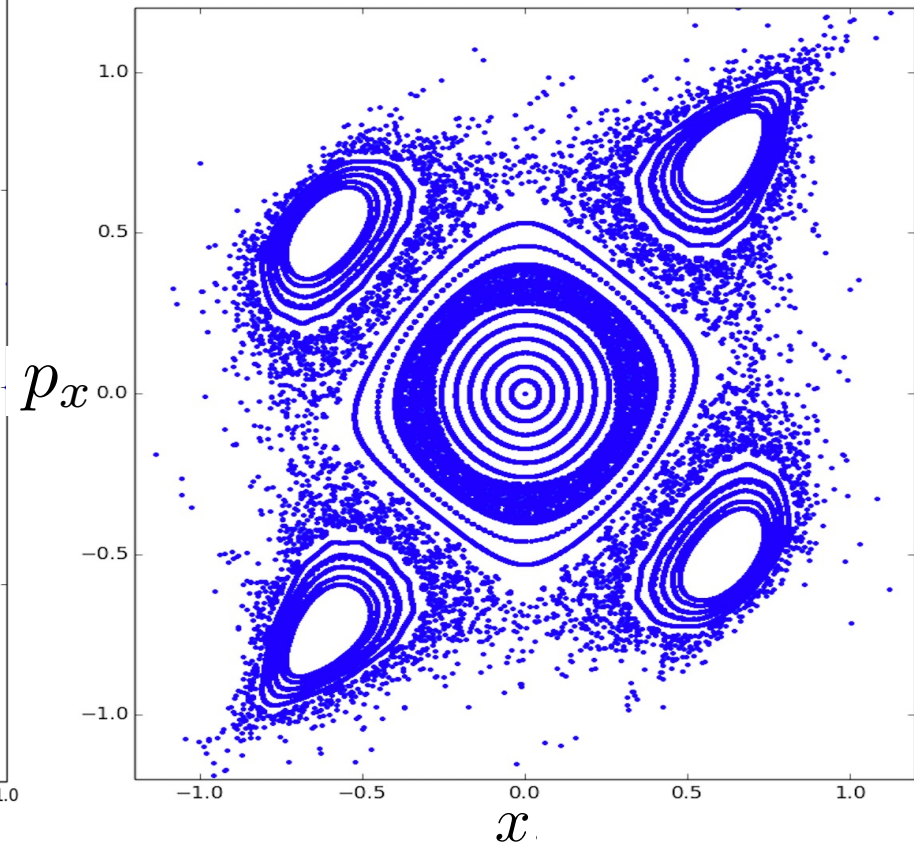
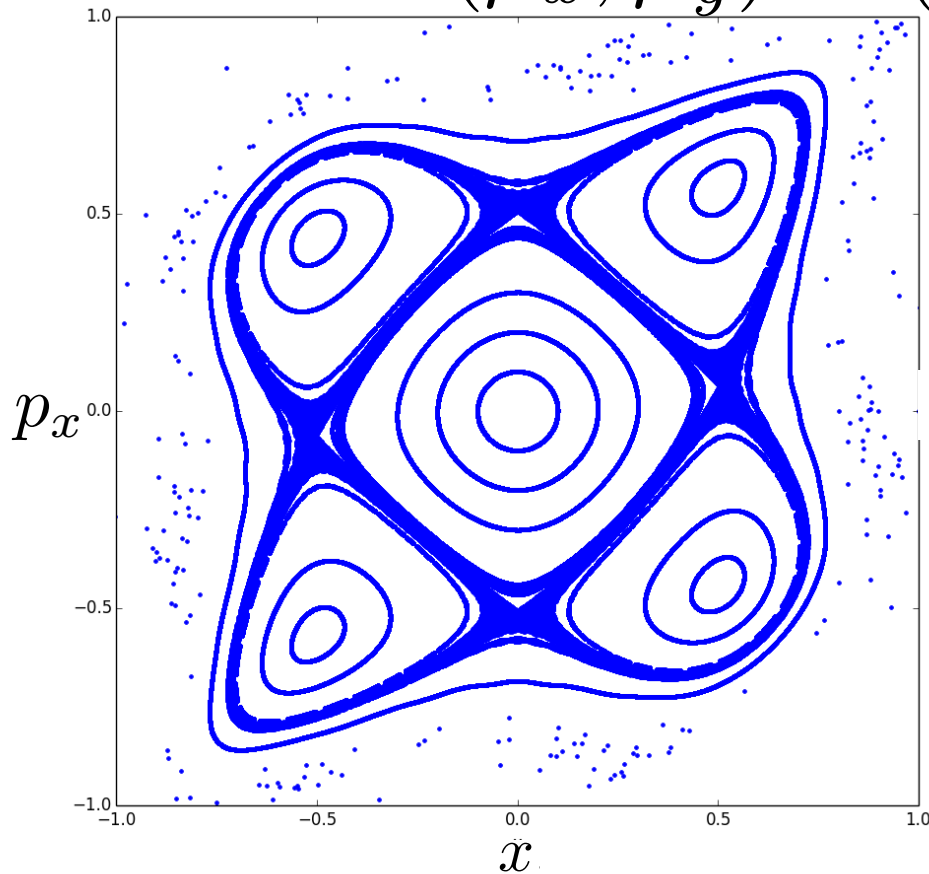
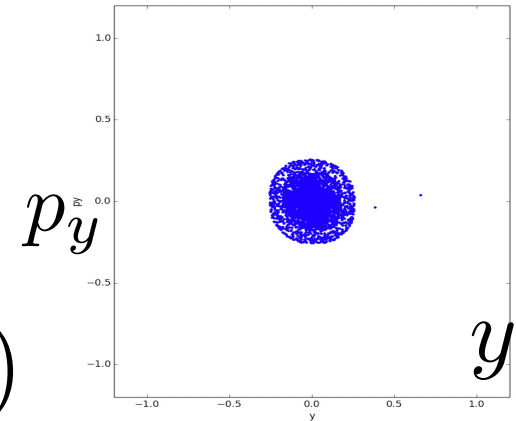


# Increasing dimensions



■ For  $(y_0, p_{y0}) \neq (0, 0)$ , i.e. by adding another degree of freedom **chaotic motion is enhanced**

$$(\mu_x, \mu_y) = (0.22, 0.24)$$





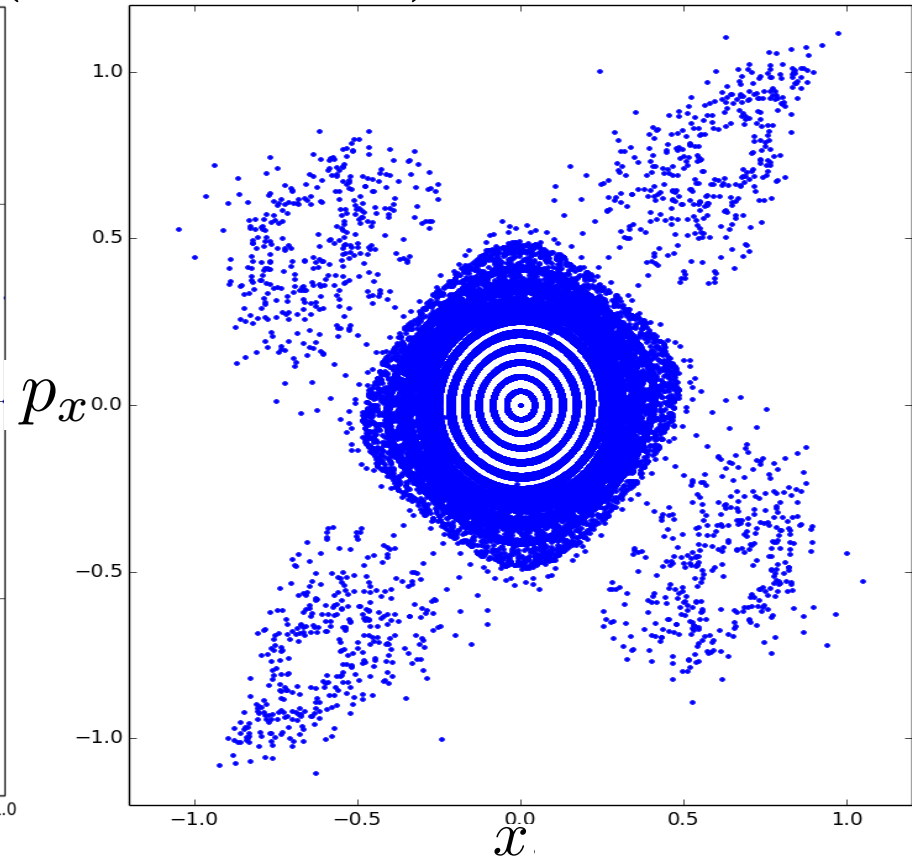
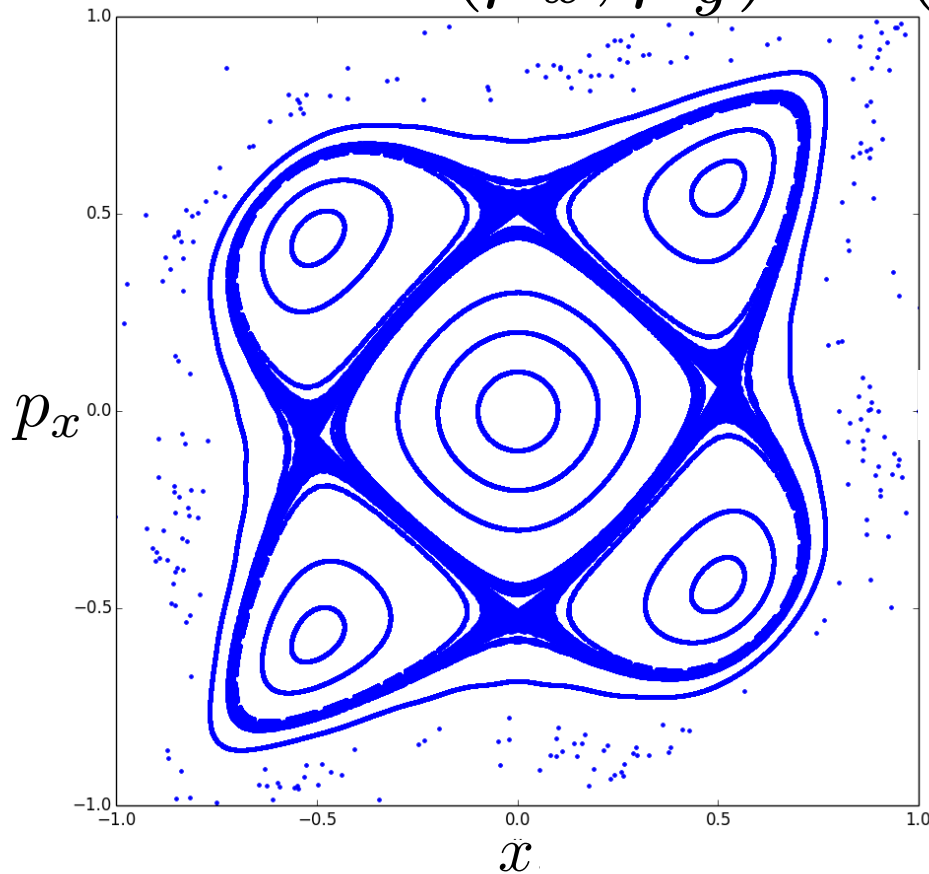
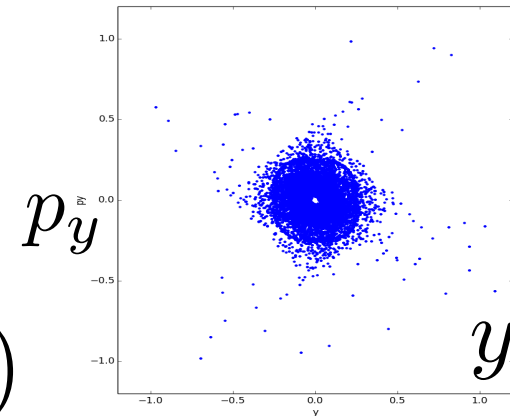
# Increasing dimensions



■ For  $(y_0, p_{y0}) \neq (0, 0)$ , i.e. by adding another degree of freedom **chaotic motion is enhanced**

■ At the same time, **analysis of phase space on surface of section** becomes **difficult** to interpret, as these are projections of 4D objects on a 2D plane

$$(\mu_x, \mu_y) = (0.22, 0.24)$$





- Computing/measuring **dynamic aperture (DA)** or particle survival

A. Chao et al., PRL 61, 24, 2752, 1988;  
F. Willeke, PAC95, 24, 109, 1989.

- Computation of Lyapunov exponents

F. Schmidt, F. Willeke and F. Zimmermann, PA, 35, 249, 1991;  
M. Giovannozzi, W. Scandale and E. Todesco, PA 56, 195, 1997

- Variance of unperturbed action (a la Chirikov)

B. Chirikov, J. Ford and F. Vivaldi, AIP CP-57, 323, 1979  
J. Tennyson, SSC-155, 1988;  
J. Irwin, SSC-233, 1989

- Fokker-Planck diffusion coefficient in actions

T. Sen and J.A. Elisson, PRL 77, 1051, 1996  
YP and F. Zimmerman, PRSTAB, 5 (7), 074001

- Frequency map analysis

# Dynamic aperture

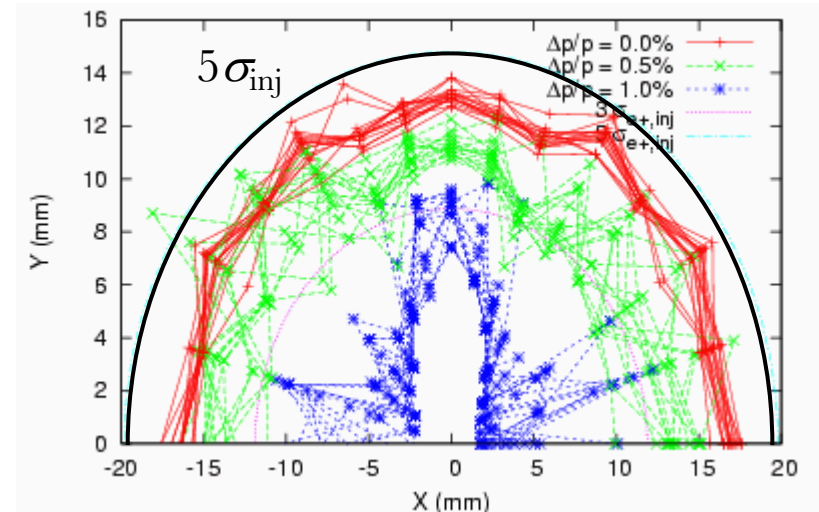
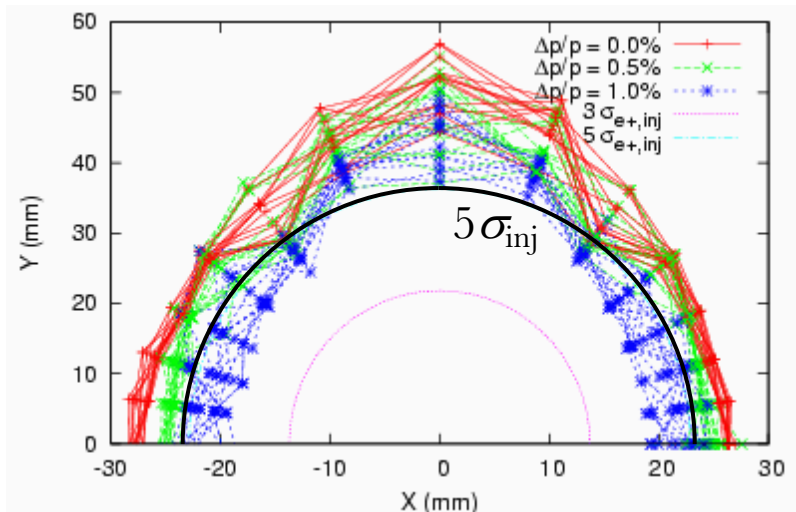


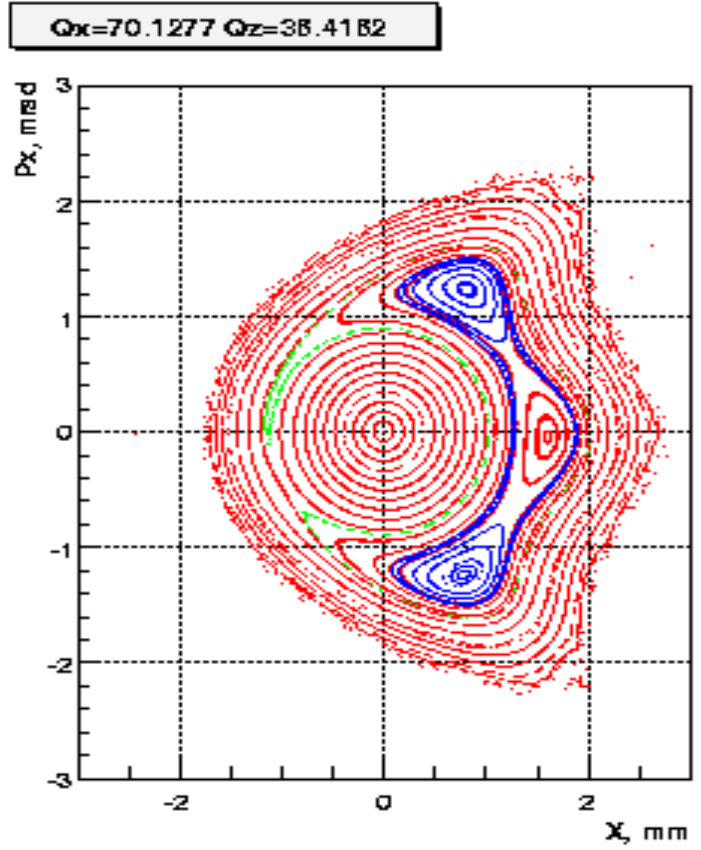
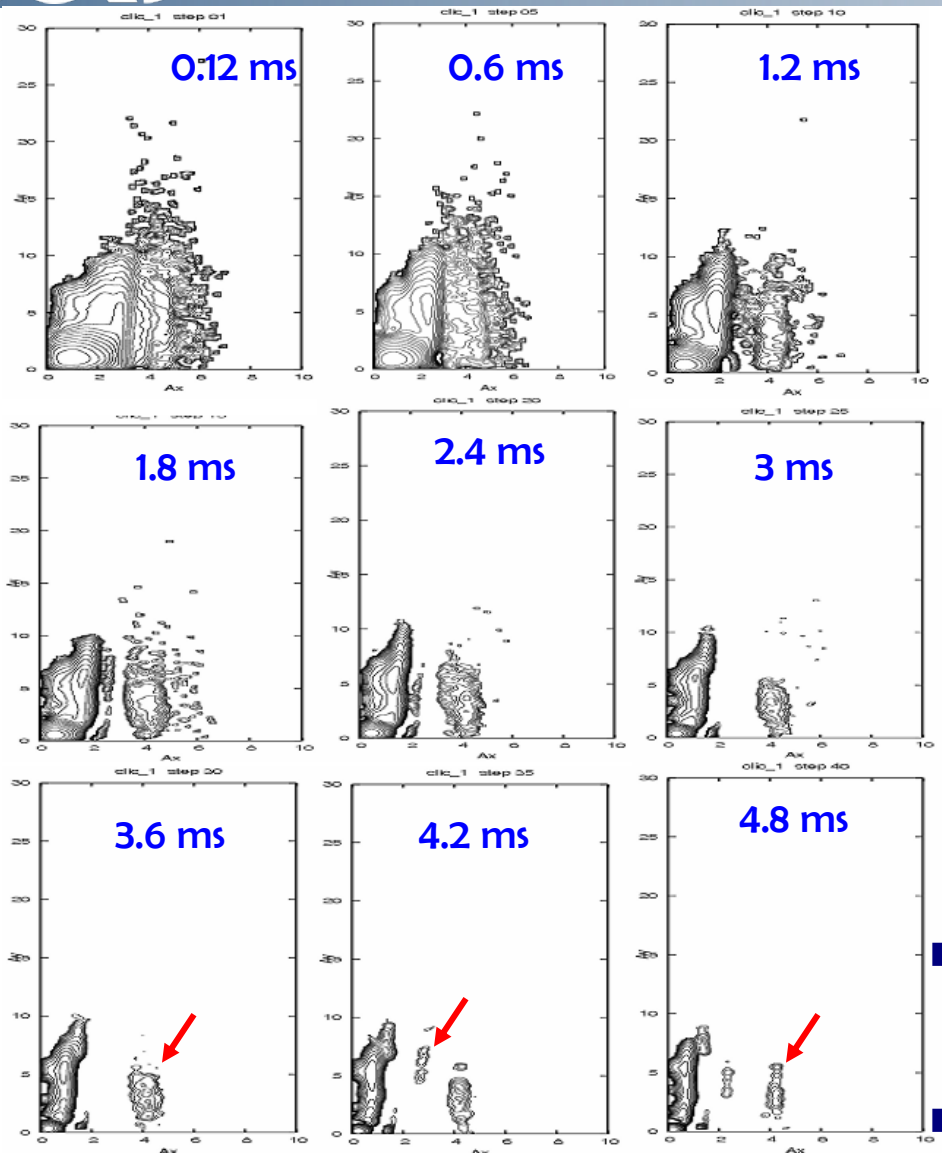


- The most direct way to evaluate the non-linear dynamics performance of a ring is the computation of **Dynamic Aperture**
- Particle motion due to multi-pole errors is generally non-bounded, so chaotic particles can **escape to infinity**
- This is not true for all non-linearities (e.g. the beam-beam force)
- Need a **symplectic** tracking code to follow particle trajectories (a lot of initial conditions) for a **number of turns** (depending on the given problem) until the particles start getting lost. This **boundary** defines the **Dynamic aperture**
- As multi-pole errors may not be completely known, one has to track through **several machine models** built by **random distribution** of these errors
- One could start with **4D** (only transverse) tracking but certainly needs to simulate **5D** (constant energy deviation) and finally **6D** (synchrotron motion included)



- Dynamic aperture plots show the maximum initial values of stable trajectories in x-y coordinate space at a particular point in the lattice, for a range of energy errors.
  - The beam size can be shown on the same plot.
  - Generally, the goal is to allow some significant margin in the design - the measured dynamic aperture is often smaller than the predicted dynamic aperture.



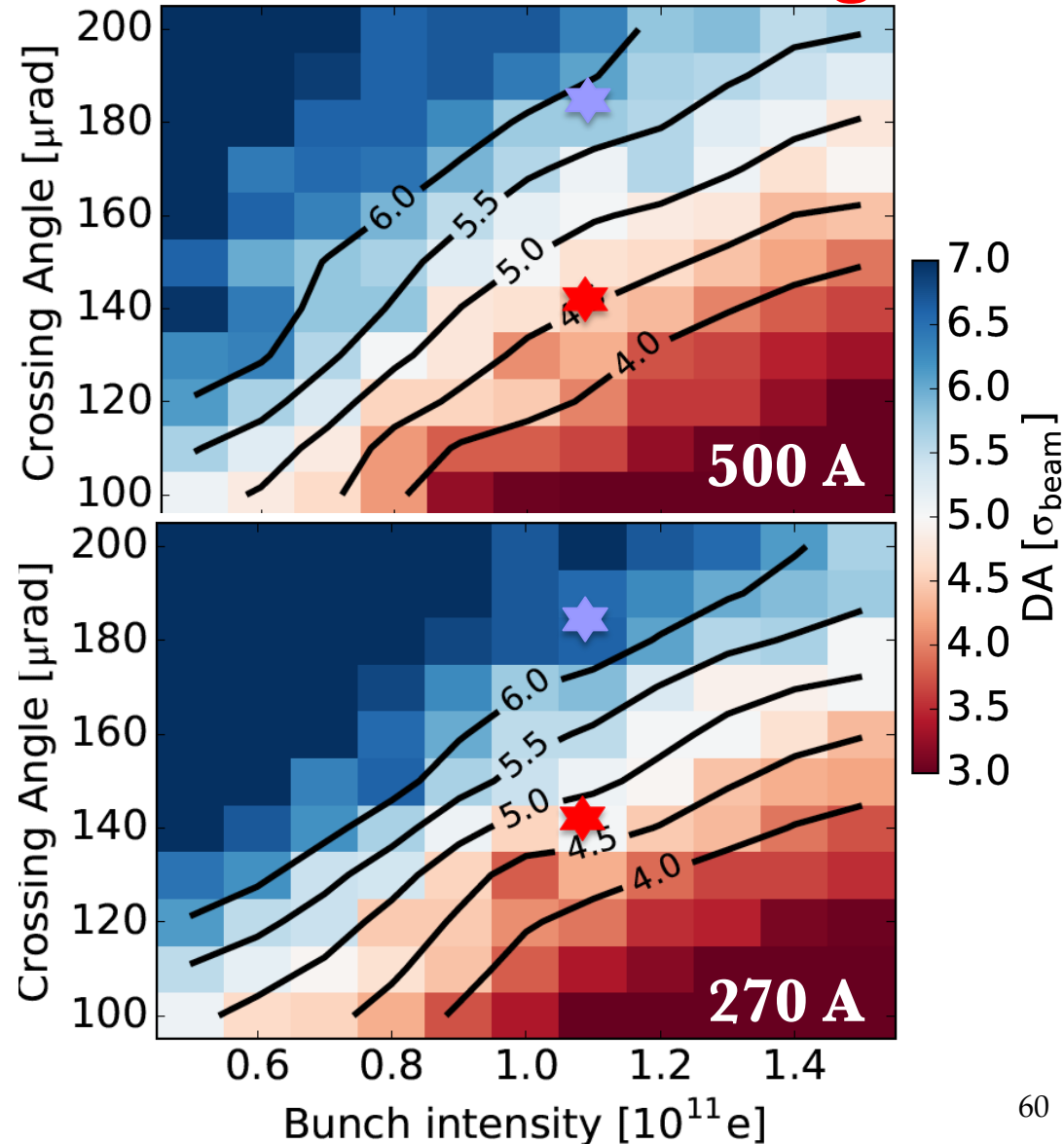


- Including radiation damping and excitation shows that 0.7% of the particles are lost during the damping
- Certain particles seem to damp away from the beam core, on resonance islands



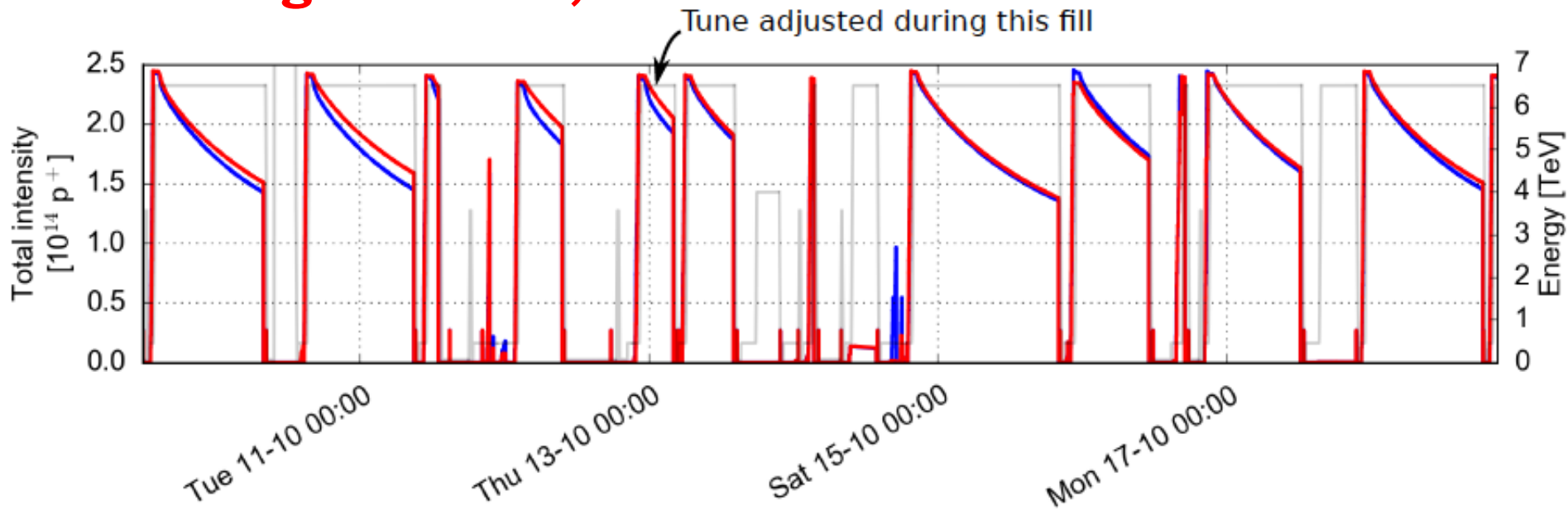
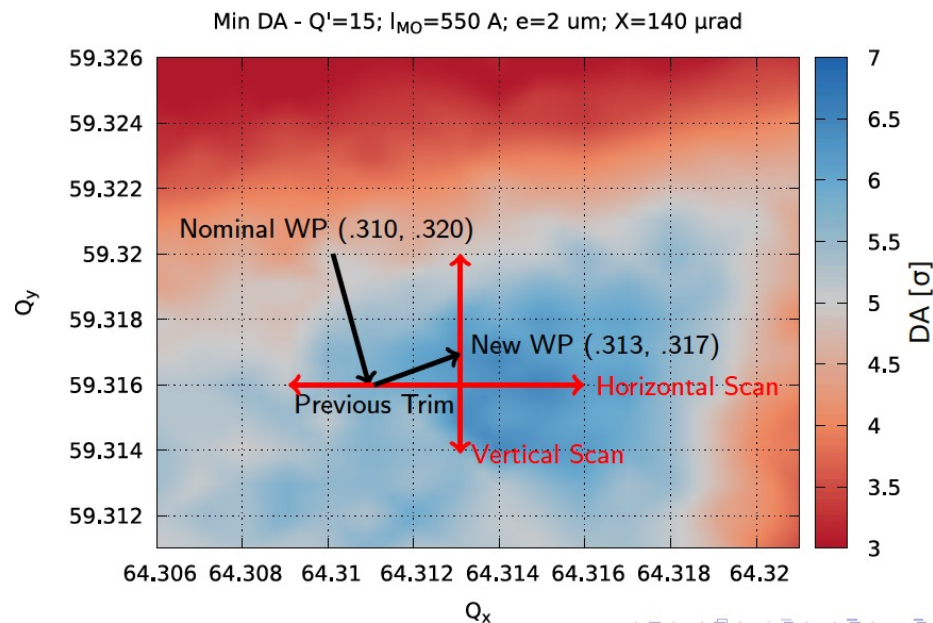
**D. Pellegrini**

- **Min. Dynamic Aperture (DA) with intensity vs crossing angle, for nominal optics ( $\beta^* = 40$  cm) and BCMS beam ( $2.5 \mu\text{m}$  emittance), 15 units of chromaticity**
- For  $1.1 \times 10^{11}$  p
  - At  $\theta_c/2 = 185 \mu\text{rad}$  ( $\sim 12 \sigma$  separation), DA around  $6 \sigma$  (good lifetime observed)
  - At  $\theta_c/2 = 140 \mu\text{rad}$  ( $\sim 9 \sigma$  separation), DA below  $5 \sigma$  (reduced lifetime observed)
  - Improvement for low octupoles, low chromaticity and WP optimisation (observed in operation)



- B1 suffering from lower lifetime in the LHC
- DA simulations predicted the required adjustment
- Fine-tune scan performed and applied in operation, solving B1 lifetime problem

**D. Pellegrini et al., 2016**





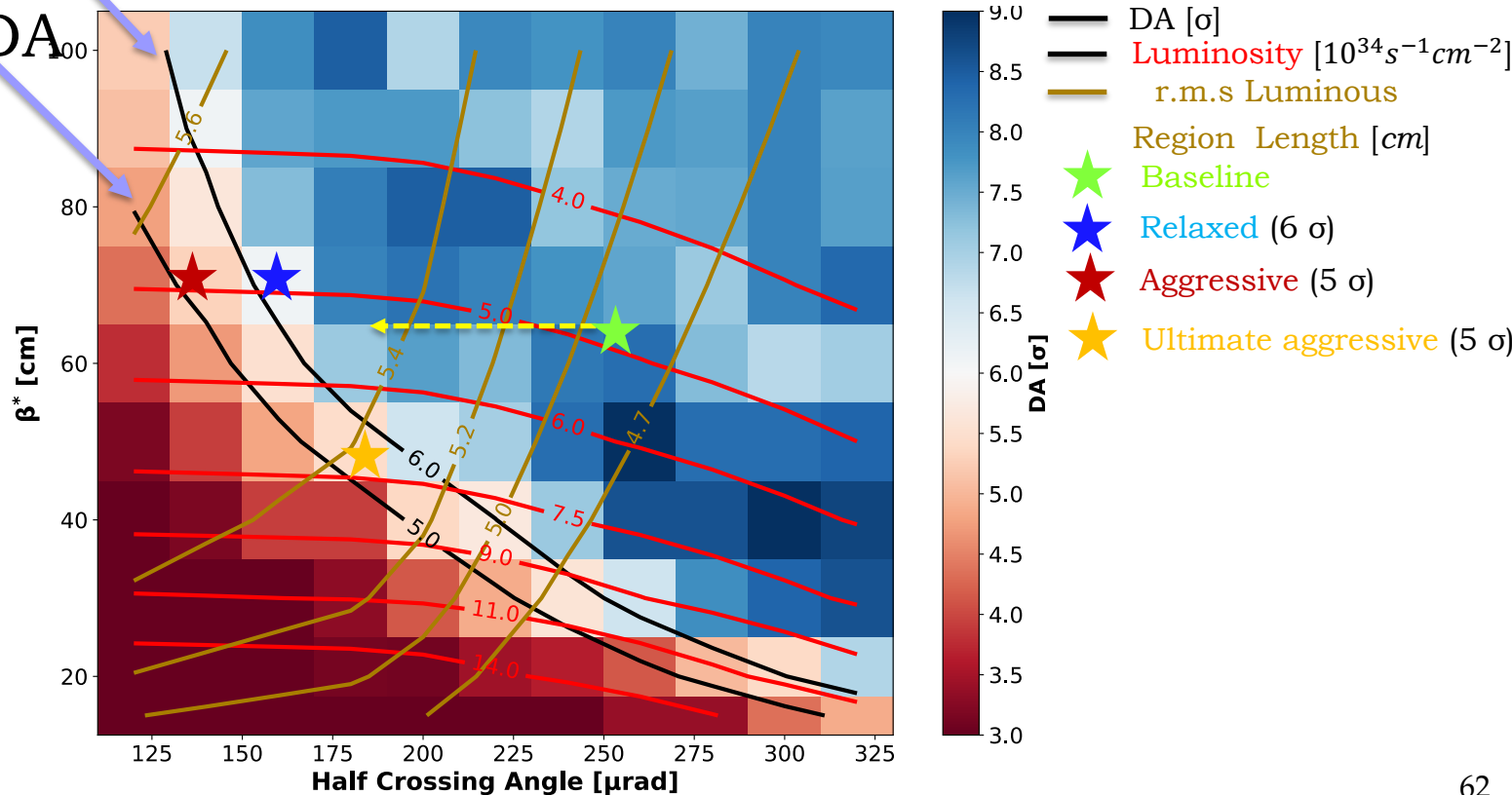
- Reduction of **crossing angle** at constant luminosity, reduces **pileup density** (by elongating the luminous region) and **triplet irradiation**

**YP, N. Karastathis and D. Pellegrini et al., 2018**

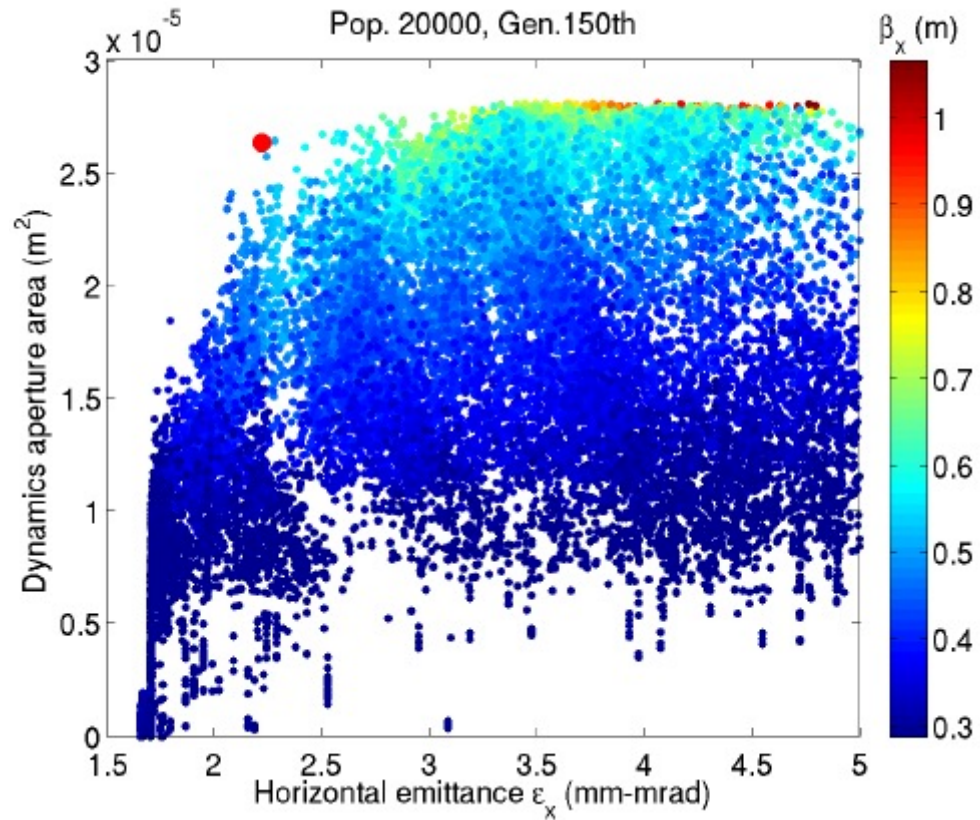
Non-linear dynamics Phenomenology, CERN Accelerator School, Novembre 2022

Relaxed DA

Aggressive DA

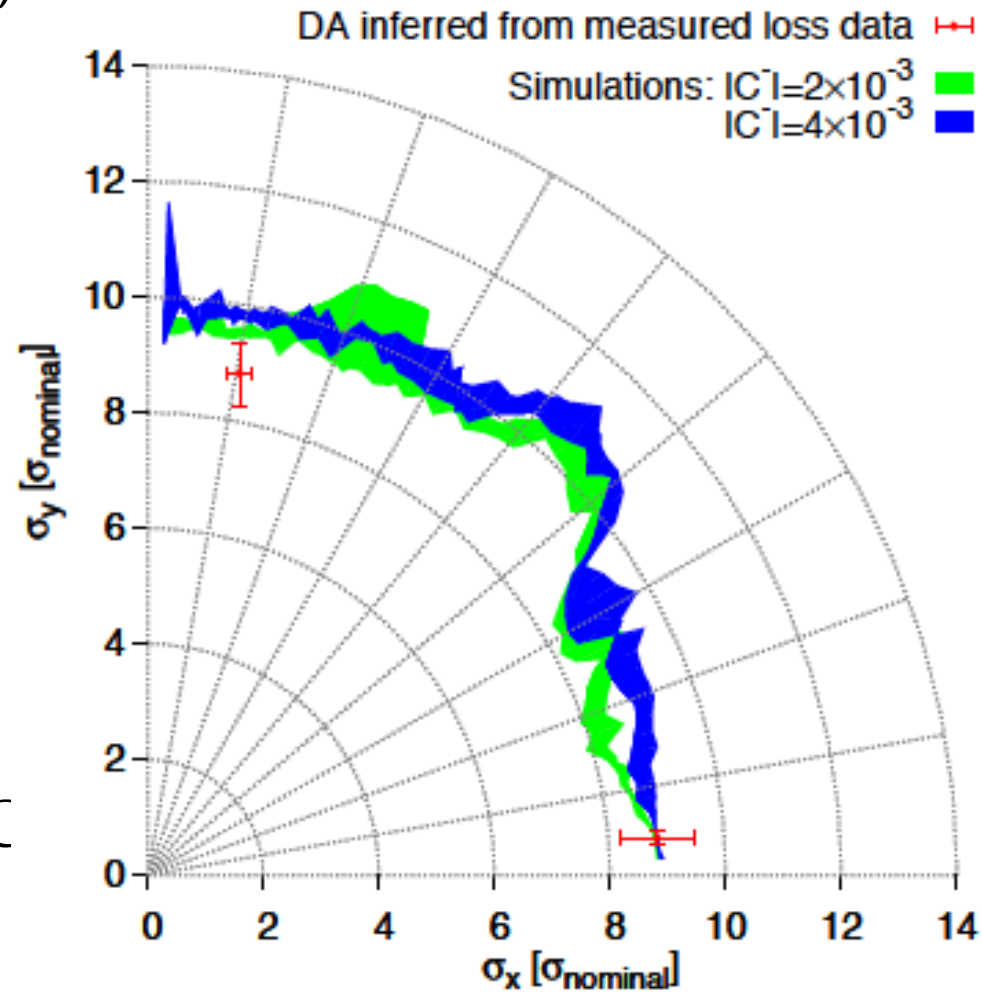


- **MOGA – Multi Objective Genetic Algorithms** are being recently used to optimise linear but also non-linear dynamics of electron low emittance storage rings
- Use **knobs** quadrupole strengths, chromaticity sextupoles and correctors with some constraints
- Target ultra-low horizontal emittance, increased lifetime and high dynamic aperture





- During LHC design phase, DA target was 2x higher than collimator position, due to statistical fluctuation, finite mesh, linear imperfections, short tracking time, multi-pole time dependence, ripple and a 20% safety margin
- Better knowledge of the model led to good agreement between measurements and simulations for actual LHC
- Necessity to build an accurate magnetic model (from beam based measurements)



**E.Mclean, PhD thesis, 2014**



# Lyapunov exponent



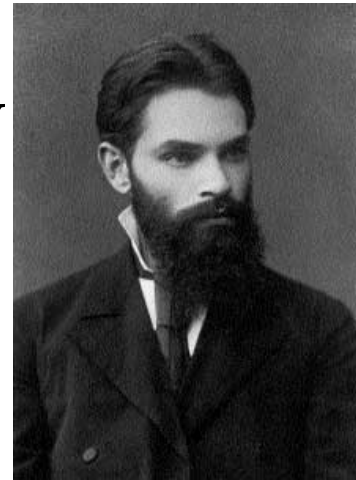
- Chaotic motion implies **sensitivity to initial condition**
- Two infinitesimally close **chaotic trajectories** in phase space with initial difference  $\delta\mathbf{Z}_0$  will end-up diverging with rate

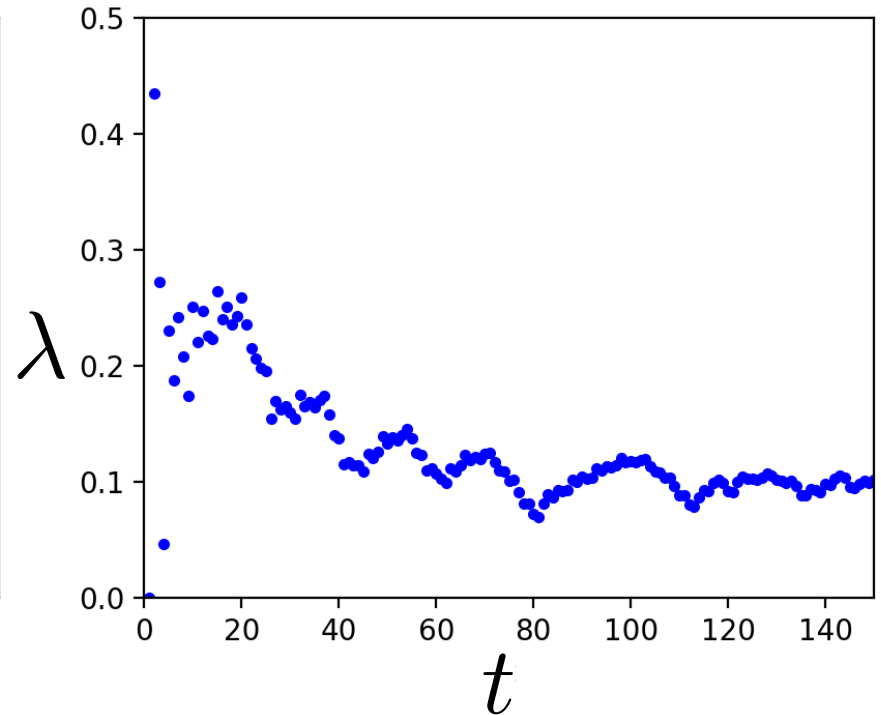
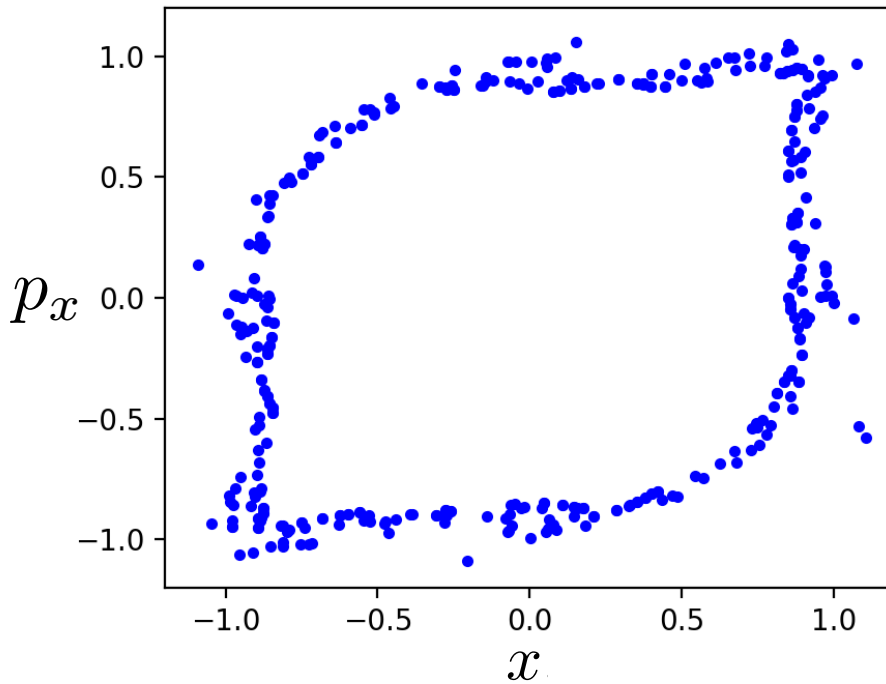
$$|\delta\mathbf{Z}(t)| \approx e^{\lambda t} |\delta\mathbf{Z}_0|$$

with  $\lambda$  the maximum **Lyapunov exponent**

- There is as many exponents as the phase space dimensions (Lyapunov spectrum)
- The largest one is the **Maximal Lyapunov exponent (MLE)** is defined as

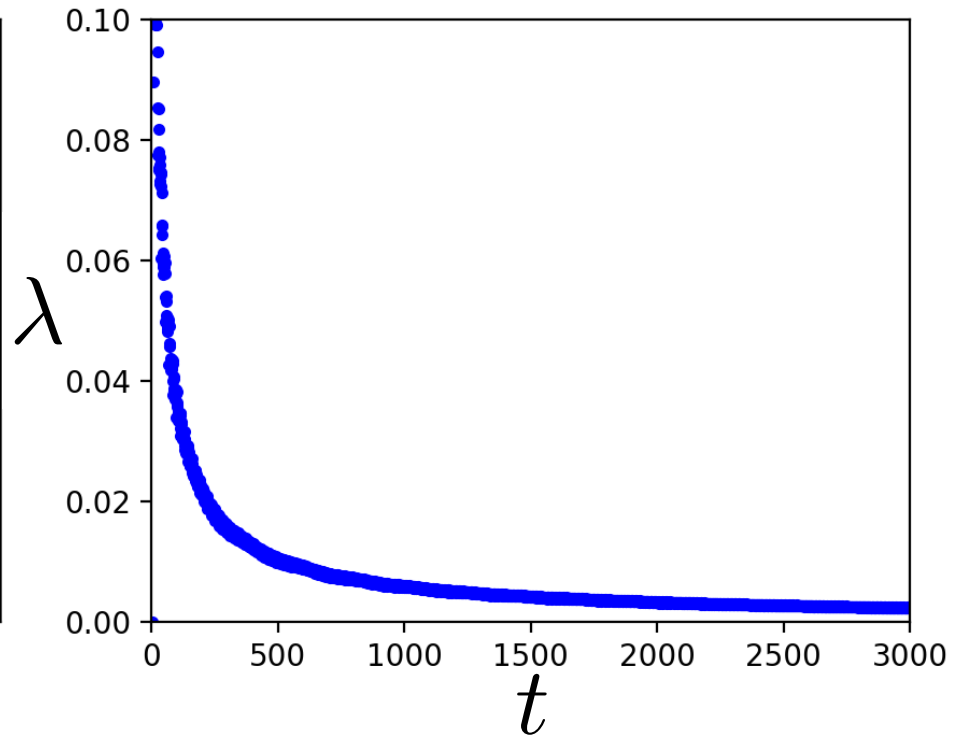
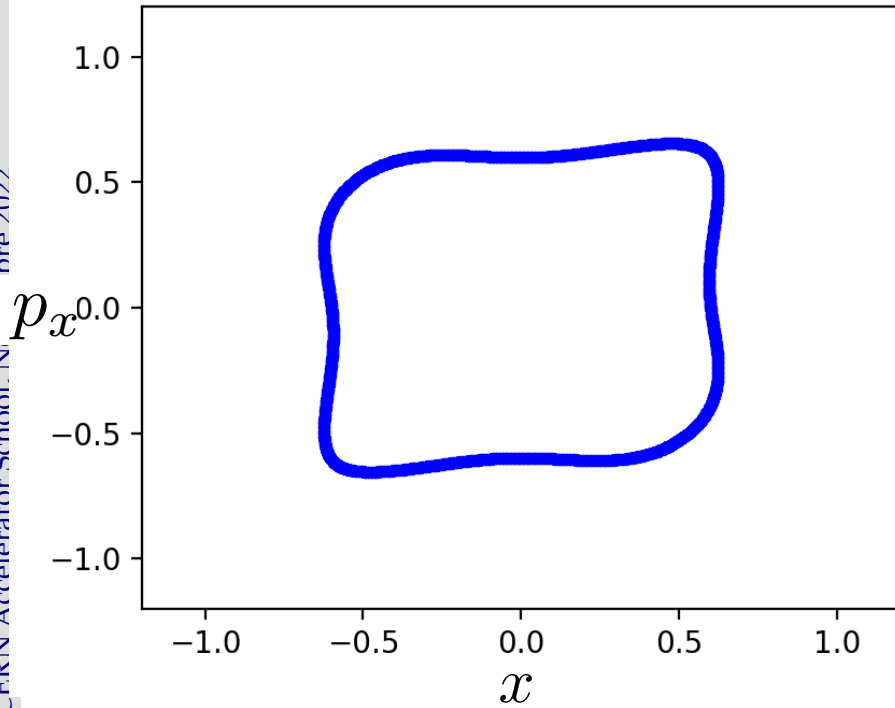
$$\lambda = \lim_{t \rightarrow \infty} \lim_{\delta\mathbf{Z}_0 \rightarrow 0} \frac{1}{t} \ln \frac{|\delta\mathbf{Z}(t)|}{|\delta\mathbf{Z}_0|}$$





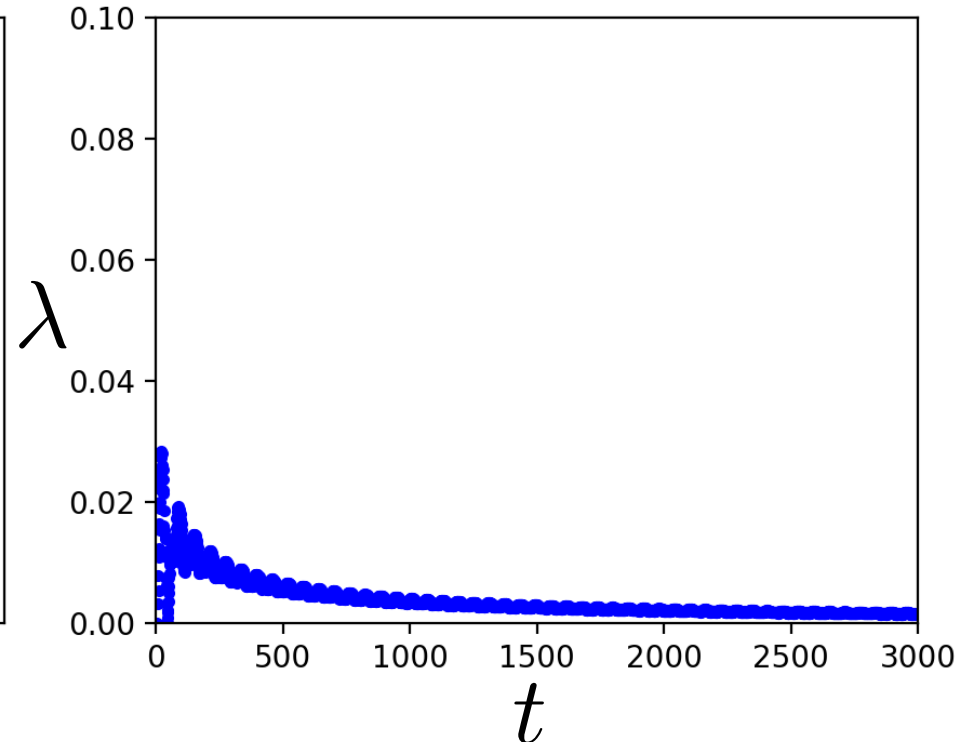
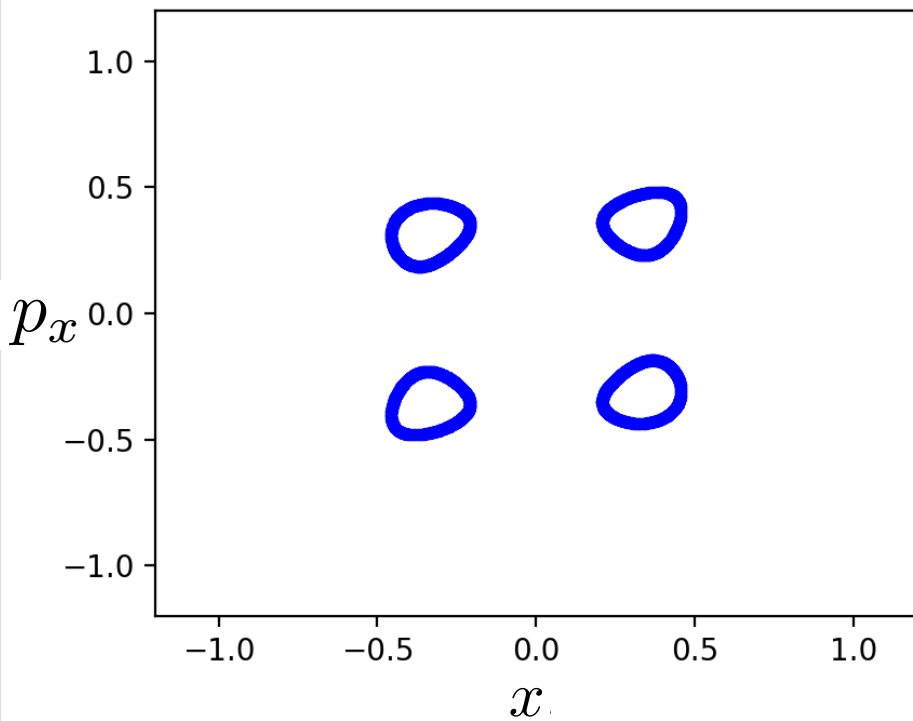
- Maximum Lyapunov exponent converges towards a **positive value** for a chaotic orbit

$$\lambda = \lim_{t \rightarrow \infty} \lim_{\delta \mathbf{Z}_0 \rightarrow 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$



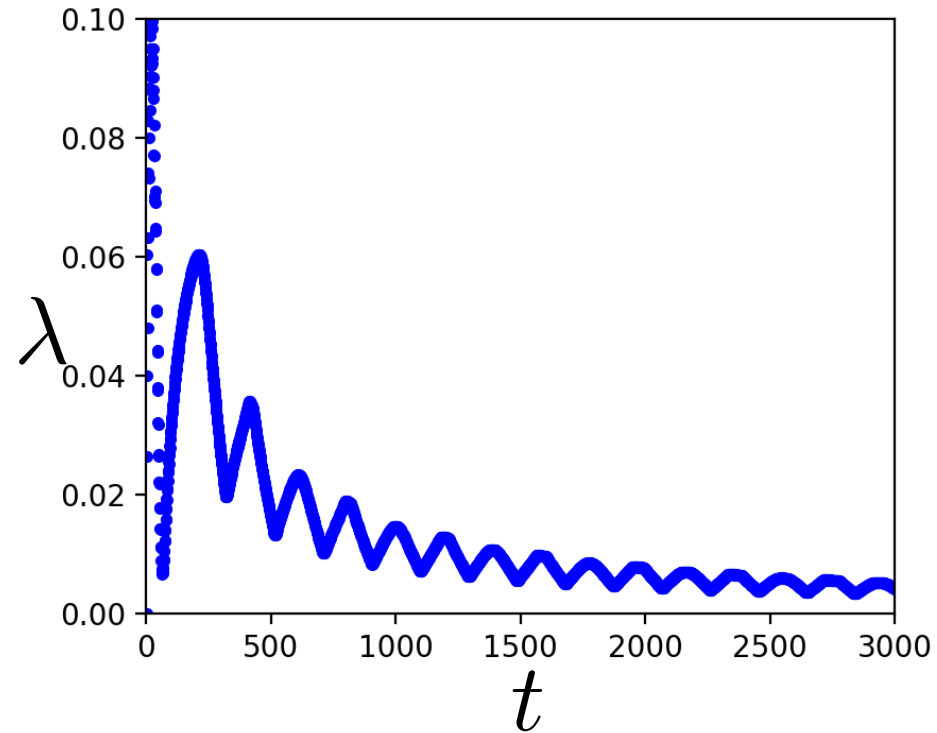
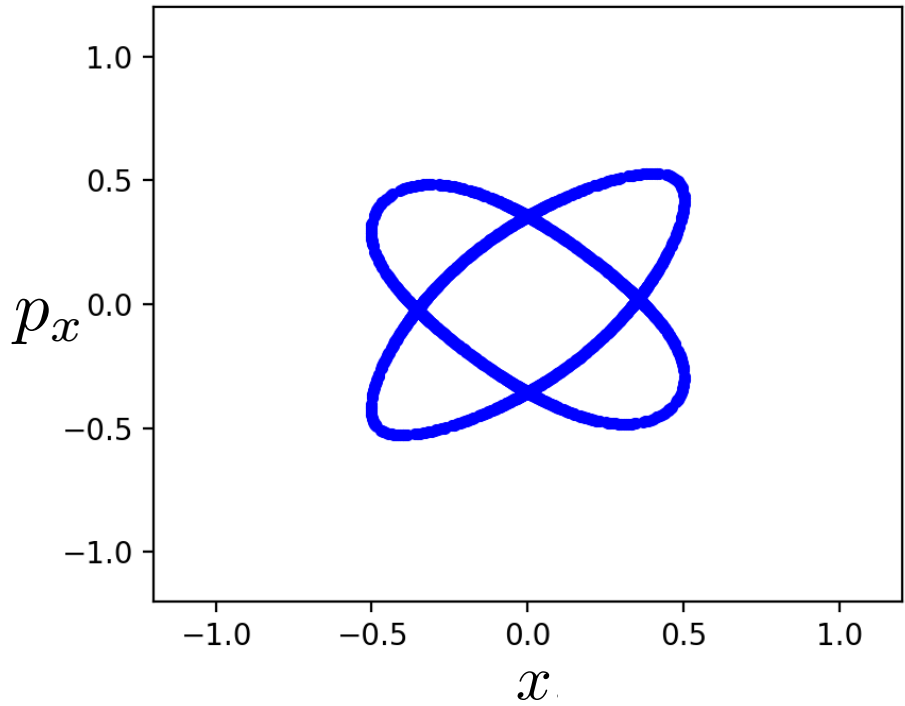
- Maximum Lyapunov exponent converges towards **zero** for a chaotic orbit

$$\lambda = \lim_{t \rightarrow \infty} \lim_{\delta \mathbf{Z}_0 \rightarrow 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$



■ Maximum Lyapounov exponent converges more slowly towards **zero** for a resonant orbit

$$\lambda = \lim_{t \rightarrow \infty} \lim_{\delta \mathbf{Z}_0 \rightarrow 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$



- Maximum Lyapunov exponent converges more slowly towards **zero** for a resonant orbit, in particular close to the separatrix

$$\lambda = \lim_{t \rightarrow \infty} \lim_{\delta \mathbf{Z}_0 \rightarrow 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$

# Frequency Map Analysis



- Frequency Map Analysis (FMA) is a numerical method which springs from the studies of J. Laskar (Paris Observatory) putting in evidence the chaotic motion in the Solar Systems
- FMA was successively applied to several dynamical systems
  - Stability of Earth Obliquity and climate stabilization (Laskar, Robutel, 1993)
  - 4D maps (Laskar 1993)
  - Galactic Dynamics (Y.P and Laskar, 1996 and 1998)
  - Accelerator beam dynamics: lepton and hadron rings (Dumas, Laskar, 1993, Laskar, Robin, 1996, Y.P, 1999, Nadolski and Laskar 2001)



- Consider an integrable Hamiltonian system of the usual form

$$H(\mathbf{J}, \varphi, \theta) = H_0(\mathbf{J})$$

- Hamilton's equations give
 
$$\dot{\phi}_j = \frac{\partial H_0(\mathbf{J})}{\partial J_j} = \omega_j(\mathbf{J}) \Rightarrow \phi_j = \omega_j(\mathbf{J})t + \phi_{j0}$$

$$\dot{J}_j = -\frac{\partial H_0(\mathbf{J})}{\partial \phi_j} = 0 \Rightarrow J_j = \text{const.}$$

- The actions define the surface of an invariant torus

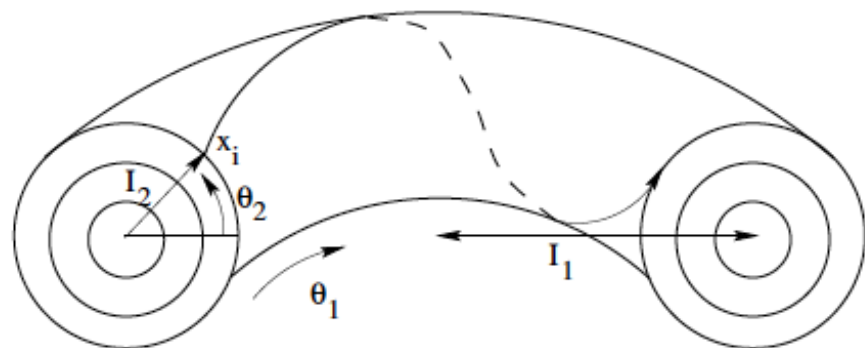
- In complex coordinates the motion is described by

$$\zeta_j(t) = J_j(0)e^{i\omega_j t} = z_{j0}e^{i\omega_j t}$$

- For a **non-degenerate** system  $\det \left| \frac{\partial \omega(J)}{\partial J} \right| = \det \left| \frac{\partial^2 H_0(J)}{\partial J^2} \right| \neq 0$

there is a one-to-one correspondence between the actions and the frequency, a frequency map can be defined parameterizing the tori in the frequency space

$$F : (\mathbf{I}) \longrightarrow (\omega)$$



- If a transformation is made to some new variables

$$\zeta_j = I_j e^{i\theta_j t} = z_j + \epsilon G_j(\mathbf{z}) = z_j + \epsilon \sum_{\mathbf{m}} c_{\mathbf{m}} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}$$

- The system is still integrable but the tori are distorted
- The motion is then described by

$$\zeta_j(t) = z_{j0} e^{i\omega_j t} + \sum_{\mathbf{m}} a_{\mathbf{m}} e^{i(\mathbf{m} \cdot \boldsymbol{\omega}) t}$$

i.e.

a quasi-periodic function of time, with

$$a_{\mathbf{m}} = \epsilon c_{\mathbf{m}} z_{10}^{m_1} z_{20}^{m_2} \dots z_{n0}^{m_n} \text{ and } \mathbf{m} \cdot \boldsymbol{\omega} = m_1 \omega_1 + m_2 \omega_2 + \dots + m_n \omega_n$$

- For a non-integrable Hamiltonian,  $H(\mathbf{I}, \theta) = H_0(\mathbf{I}) + \epsilon H'(\mathbf{I}, \theta)$  and especially if the perturbation is small, most tori persist (**KAM** theory)
- In that case, the motion is still quasi-periodic and a frequency map can be built
- The **regularity** (or not) of the map reveals stable (or chaotic) motion



- When a quasi-periodic function  $f(t) = q(t) + ip(t)$  in the complex domain is given numerically, it is possible to recover a quasi-periodic approximation

$$f'(t) = \sum_{k=1}^N a'_k e^{i\omega'_k t}$$

in a very precise way over a finite time span  $[-T, T]$  several orders of magnitude more precisely than simple Fourier techniques

- This approximation is provided by the Numerical Analysis of Fundamental Frequencies – **NAFF** algorithm
- The frequencies  $\omega'_k$  and complex amplitudes  $a'_k$  are computed through an iterative scheme.



- The first frequency  $\omega'_1$  is found by the location of the maximum of

$$\phi(\sigma) = \langle f(t), e^{i\sigma t} \rangle = \frac{1}{2T} \int_{-T}^T f(t) e^{-i\sigma t} \chi(t) dt$$

where  $\chi(t)$  is a weight function

- In most of the cases the Hanning window filter is used  $\chi_1(t) = 1 + \cos(\pi t/T)$
- Once the first term  $e^{i\omega'_1 t}$  is found, its complex amplitude  $a'_1$  is obtained and the process is restarted on the remaining part of the function

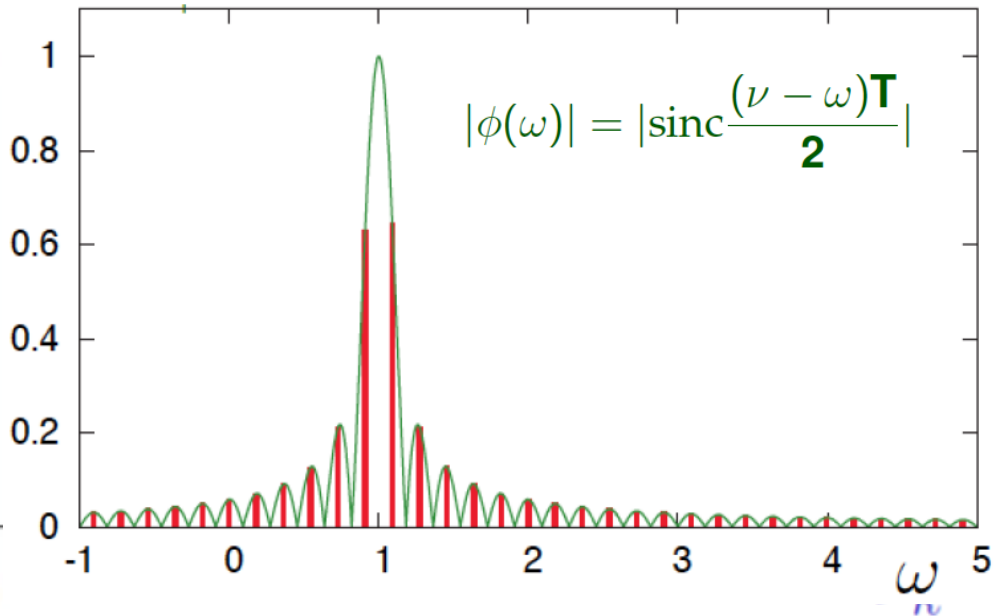
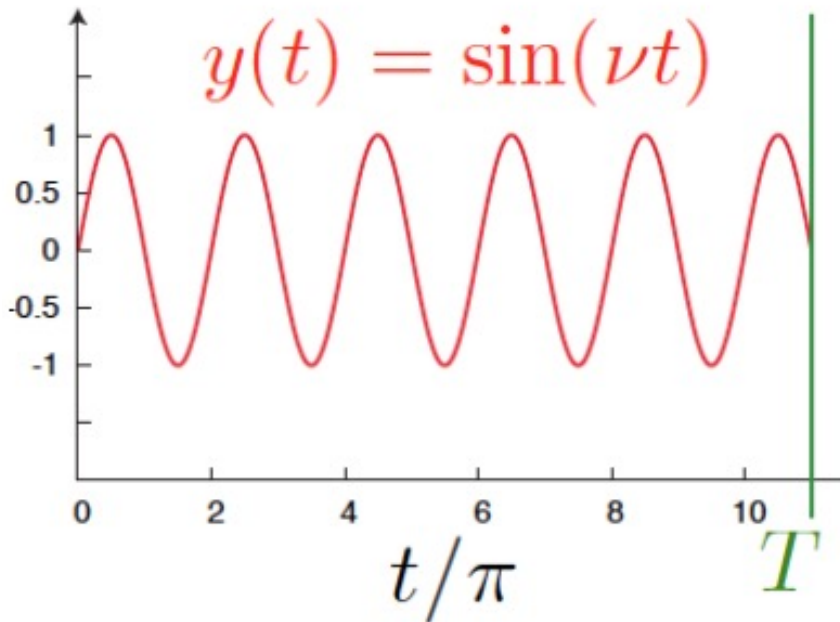
$$f_1(t) = f(t) - a'_1 e^{i\omega'_1 t}$$

- The procedure is continued for the number of desired terms, or until a required precision is reached



- The accuracy of a simple FFT even for a simple sinusoidal signal is not better than  $|\nu - \nu_T| = \frac{1}{T}$
- Calculating the Fourier integral explicitly

$\phi(\omega) = \langle f(t), e^{i\omega t} \rangle = \frac{1}{T} \int_0^T f(t) e^{-i\omega t} dt$  shows that the maximum lies in between the main peaks of the FFT

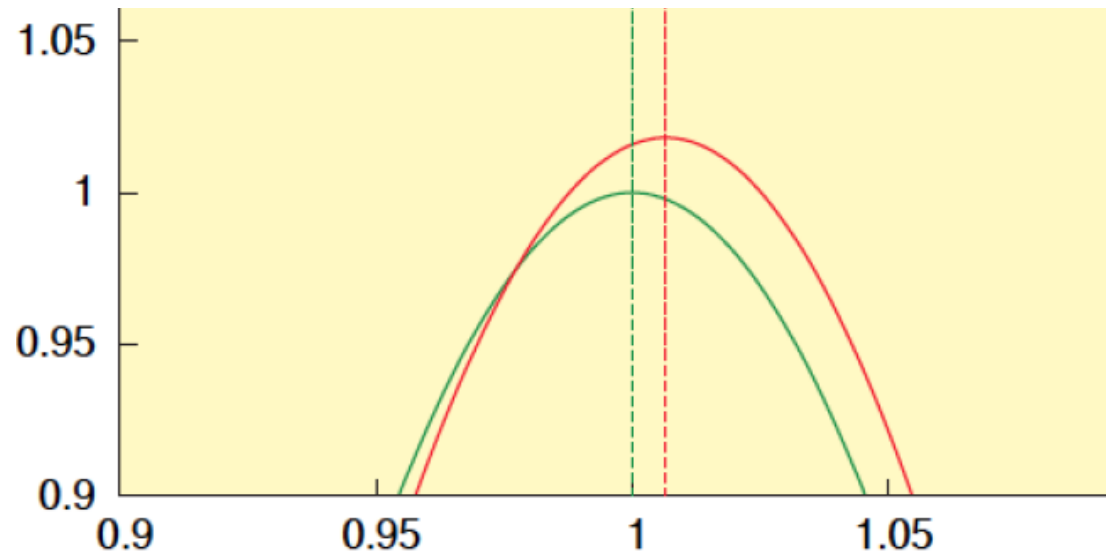
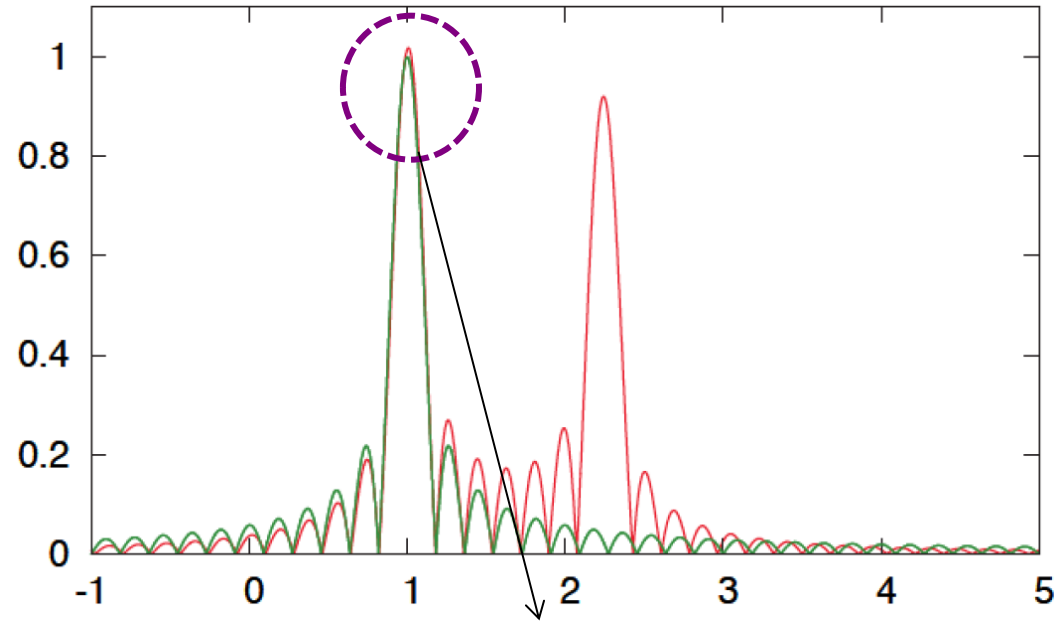




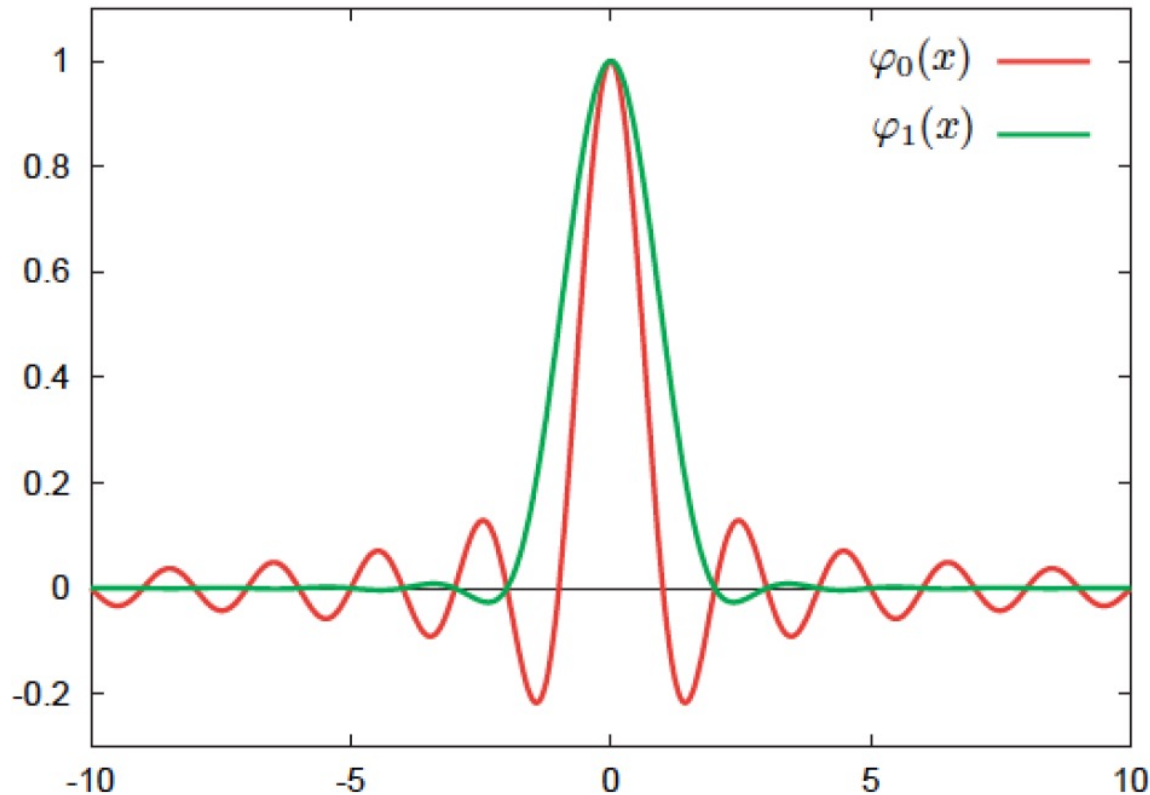
- A more complicated signal with two frequencies

$$f(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}$$

shifts slightly the maximum with respect to its real location



- A window function like the Hanning filter  $\chi_1(t) = 1 + \cos(\pi t/T)$  kills side-lobes and allows a very accurate determination of the frequency



- For a general window function of order  $p$

$$\chi_p(t) = \frac{2^p (p!)^2}{(2p)!} (1 + \cos \pi t)^p$$

Laskar (1996) proved a theorem stating that the solution provided by the NAFF algorithm converges asymptotically towards the real KAM quasi-periodic solution with precision

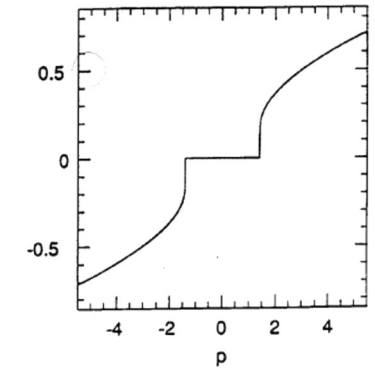
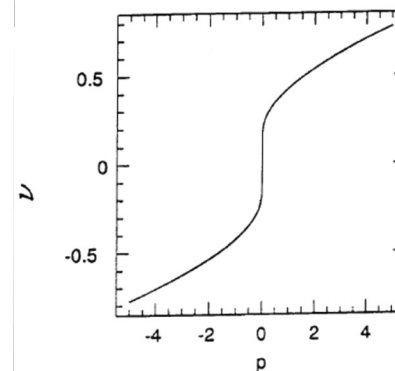
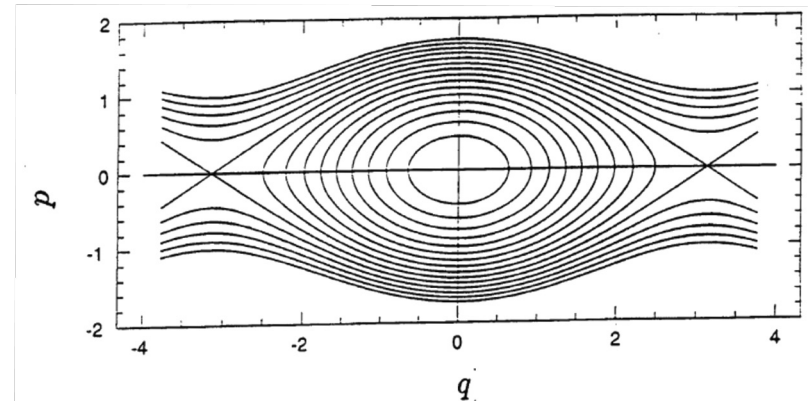
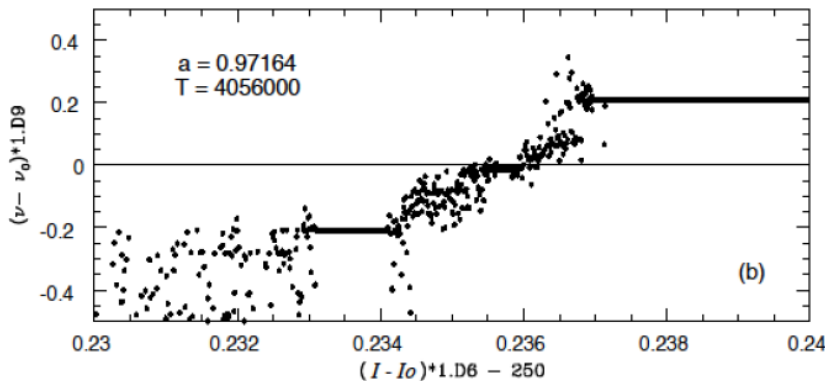
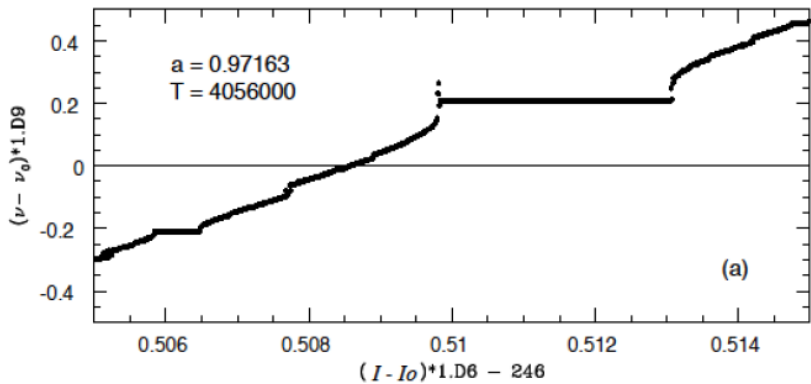
$$\nu_1 - \nu_1^T \propto \frac{1}{T^{2p+2}}$$

- In particular, for no filter (i.e.  $p = 0$ ) the precision is  $\frac{1}{T^2}$ , whereas for the Hanning filter ( $p = 1$ ), the precision is of the order of  $\frac{1}{T^4}$





- In the vicinity of a resonance the system behaves like a **pendulum**
- Passing through the **elliptic point** for a fixed angle, a **fixed frequency** (or rotation number) is observed
- Passing through the **hyperbolic point**, a **frequency jump** is observed



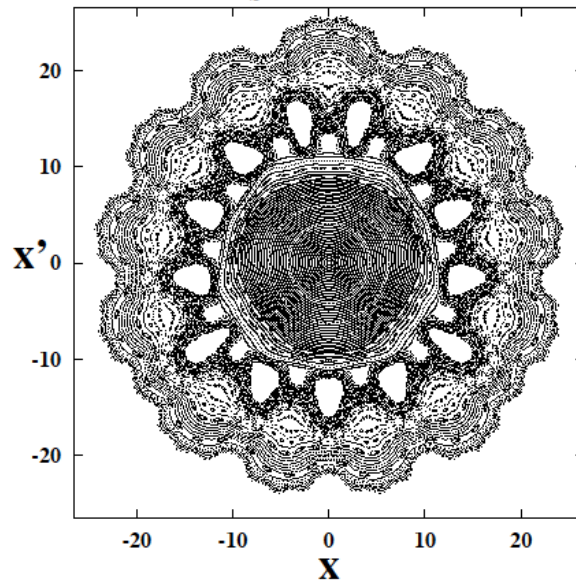
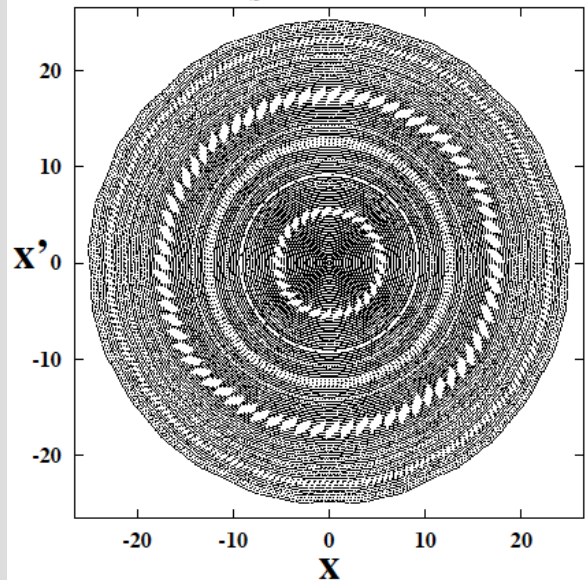


# Example: Frequency map for BBLR

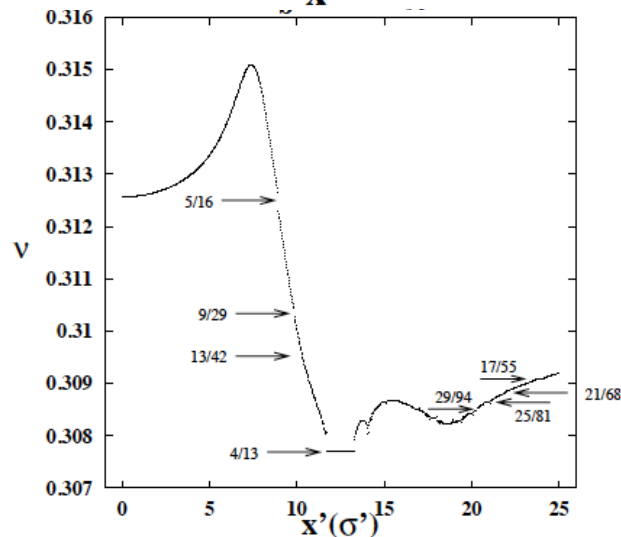
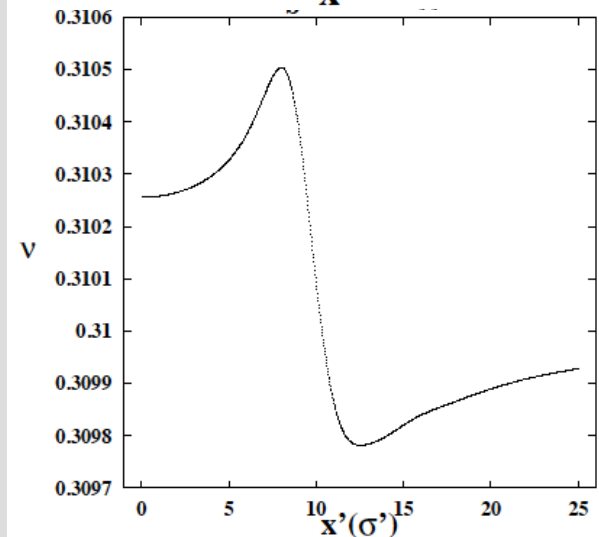


$N_b = 1 \times 10^{10}$

$N_b = 1 \times 10^{11}$



■ Simple Beam-beam long range (BBLR) kick and a rotation



Non-L

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = \begin{pmatrix} \cos \mu & \beta^* \sin \mu \\ -\sin \mu / \beta^* & \cos \mu \end{pmatrix} \begin{pmatrix} x + f(x') \\ x' \end{pmatrix}_0 \quad f(x') = K \left[ \frac{1}{x' + \theta_c} \left( 1 - e^{-\frac{(x' + \theta_c)^2}{2\sigma_x'^2}} \right) - \frac{1}{\theta_c} \left( 1 - e^{-\frac{\theta_c^2}{2\sigma_x'^2}} \right) \right]$$

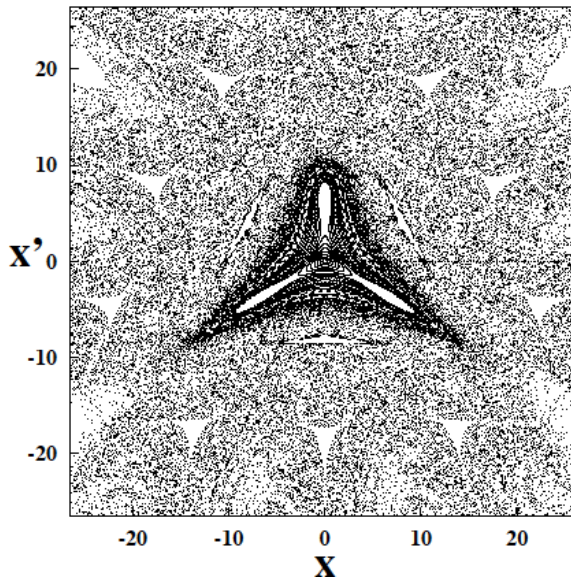
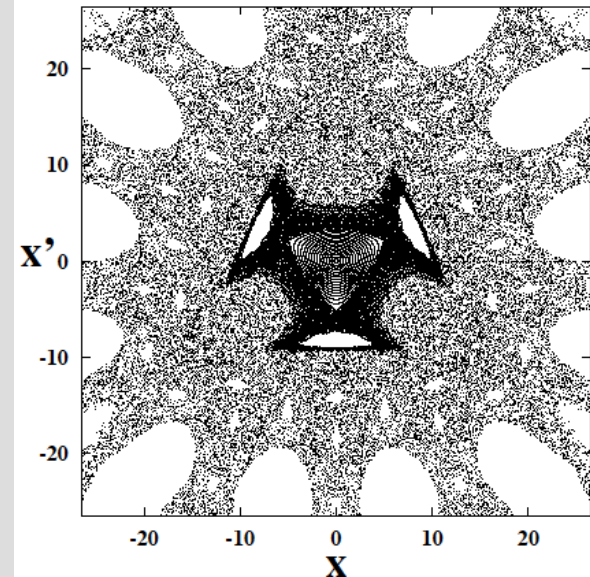


# Example: Frequency map for BBLR



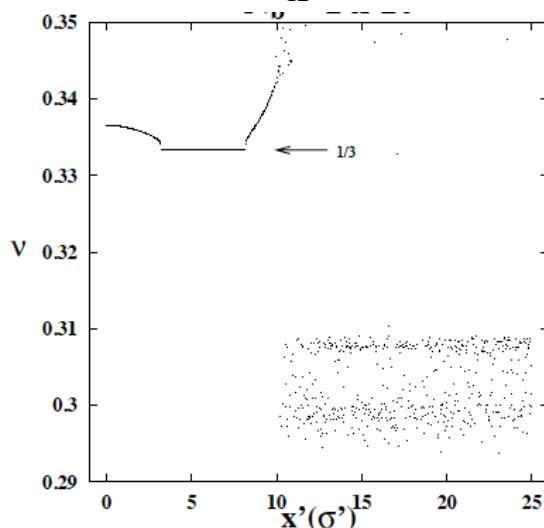
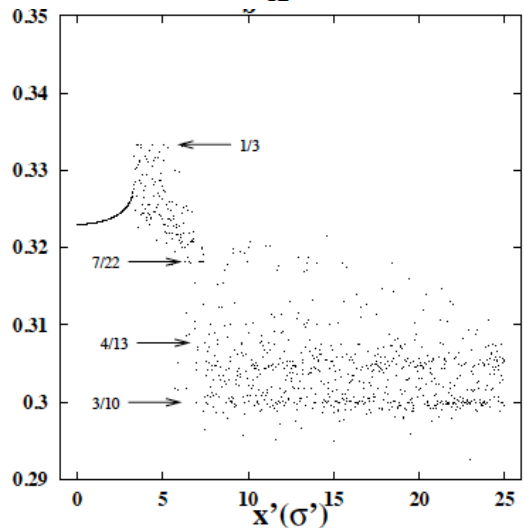
$N_b = 5 \times 10^{11}$

$N_b = 1 \times 10^{12}$



- Simple Beam-beam long range (BBLR) kick and a rotation

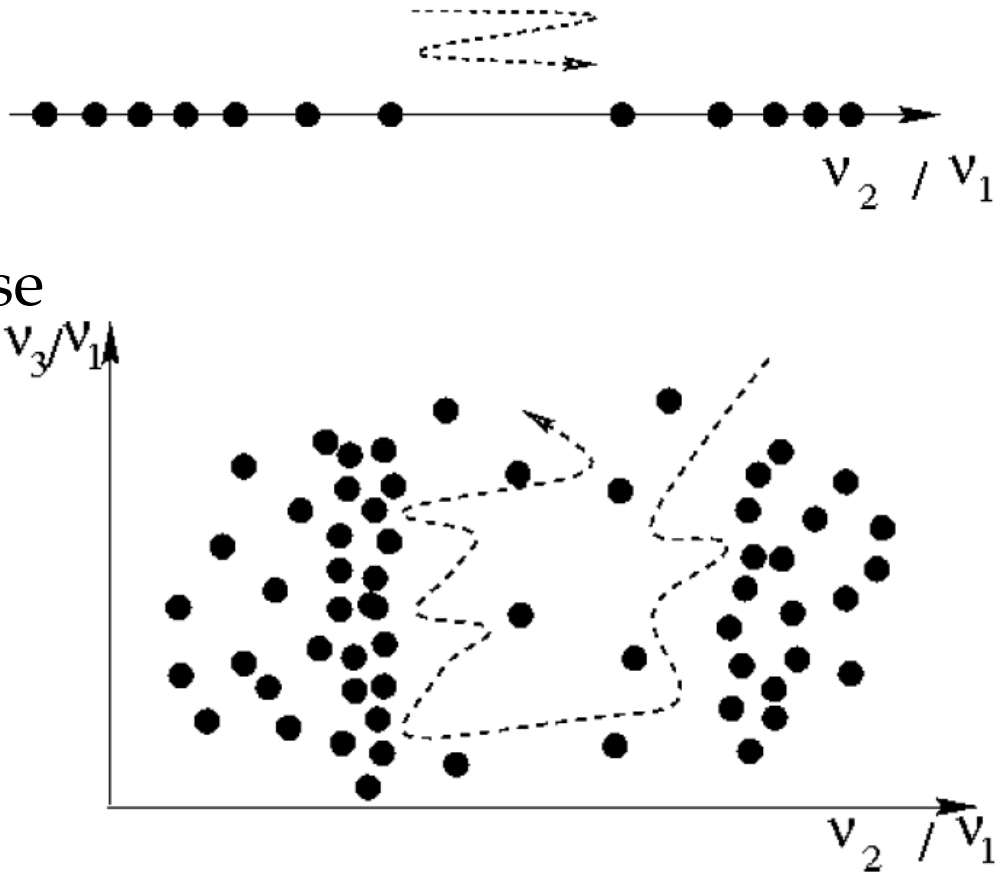
Non-linear dynamics Phenomenology, CERN A



$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = \begin{pmatrix} \cos \mu & \beta^* \sin \mu \\ -\sin \mu / \beta^* & \cos \mu \end{pmatrix} \begin{pmatrix} x + f(x') \\ x' \end{pmatrix}_0 \quad f(x') = K \left[ \frac{1}{x' + \theta_c} \left( 1 - e^{-\frac{(x' + \theta_c)^2}{2\sigma_x'^2}} \right) - \frac{1}{\theta_c} \left( 1 - e^{-\frac{\theta_c^2}{2\sigma_x'^2}} \right) \right]$$



- For a **2 degrees of freedom** Hamiltonian system, the **frequency space** is a **line**, the **tori** are dots on this lines, and the **chaotic zones** are **confined** by the existing KAM tori
- For a system with 3 or more degrees of freedom, KAM tori are still represented by dots but do not prevent chaotic trajectories to diffuse
- This topological possibility of particles diffusing is called **Arnold diffusion**
- This diffusion is supposed to be extremely small in their vicinity, as tori act as effective barriers (**Nechoroshev theory**)

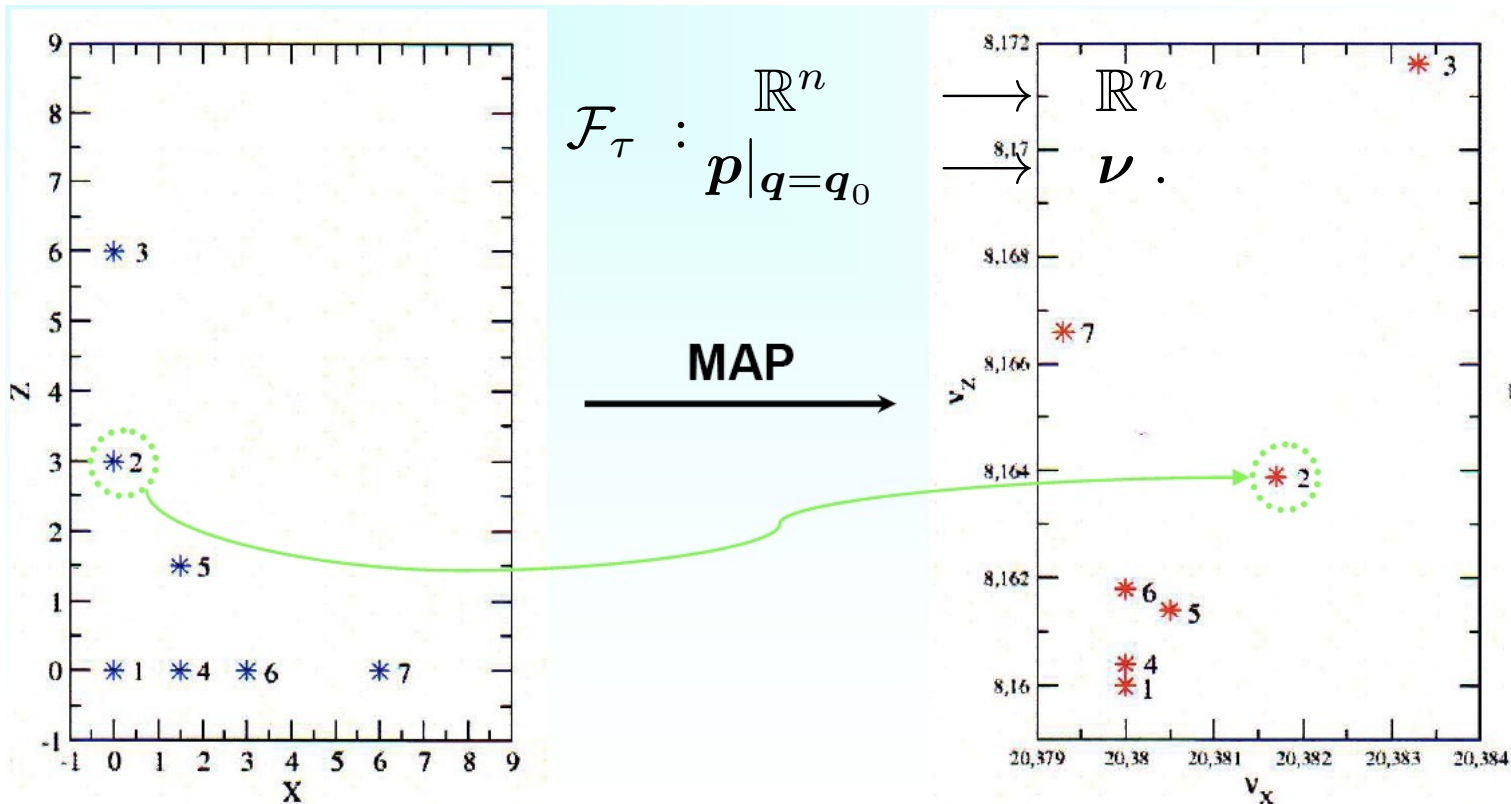


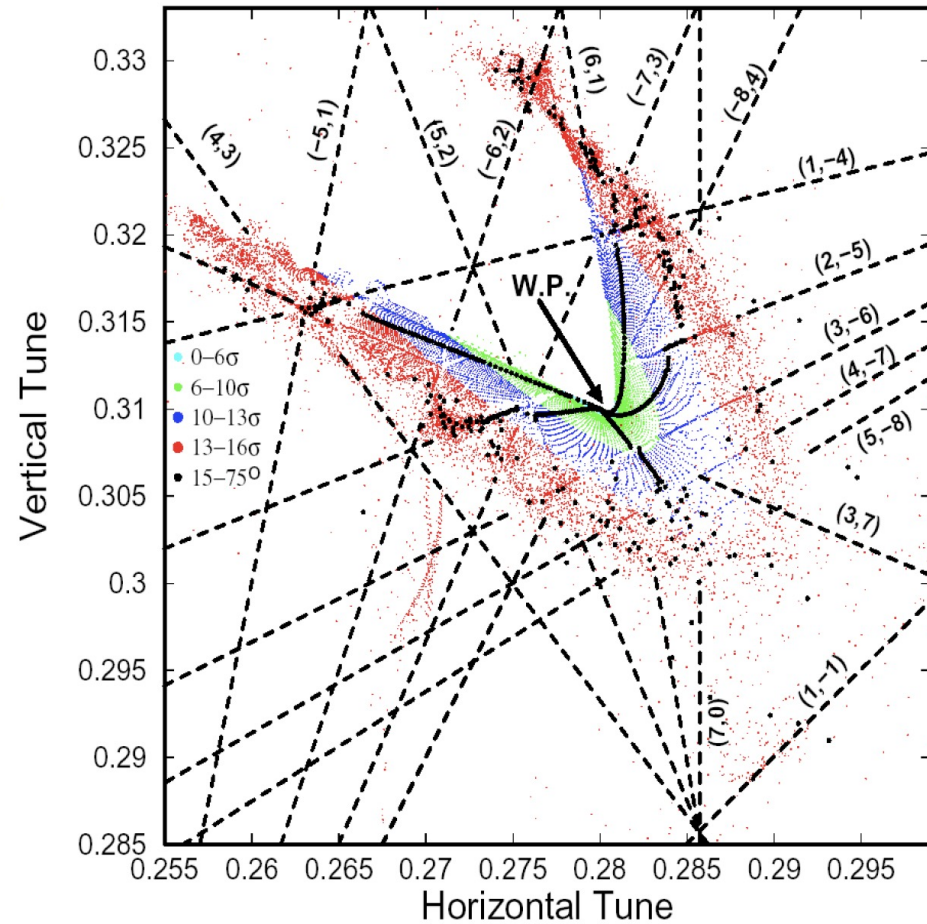
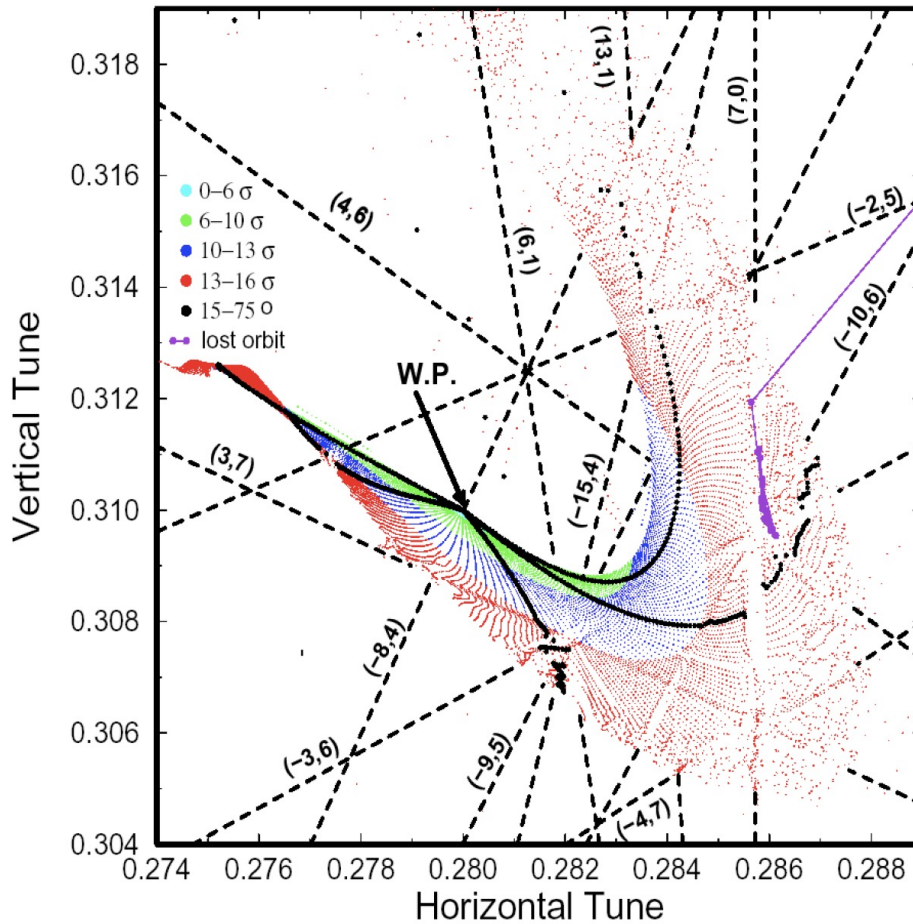


# Building the frequency map



- Choose coordinates  $(x_i, y_i)$  with  $p_x$  and  $p_y=0$
- Numerically integrate the phase trajectories through the lattice for sufficient number of turns
- Compute through NAFF  $Q_x$  and  $Q_y$  after sufficient number of turns
- Plot them in the tune diagram





- Frequency maps for the target error table (left) and an increased random skew octupole error in the superconducting dipoles (right)

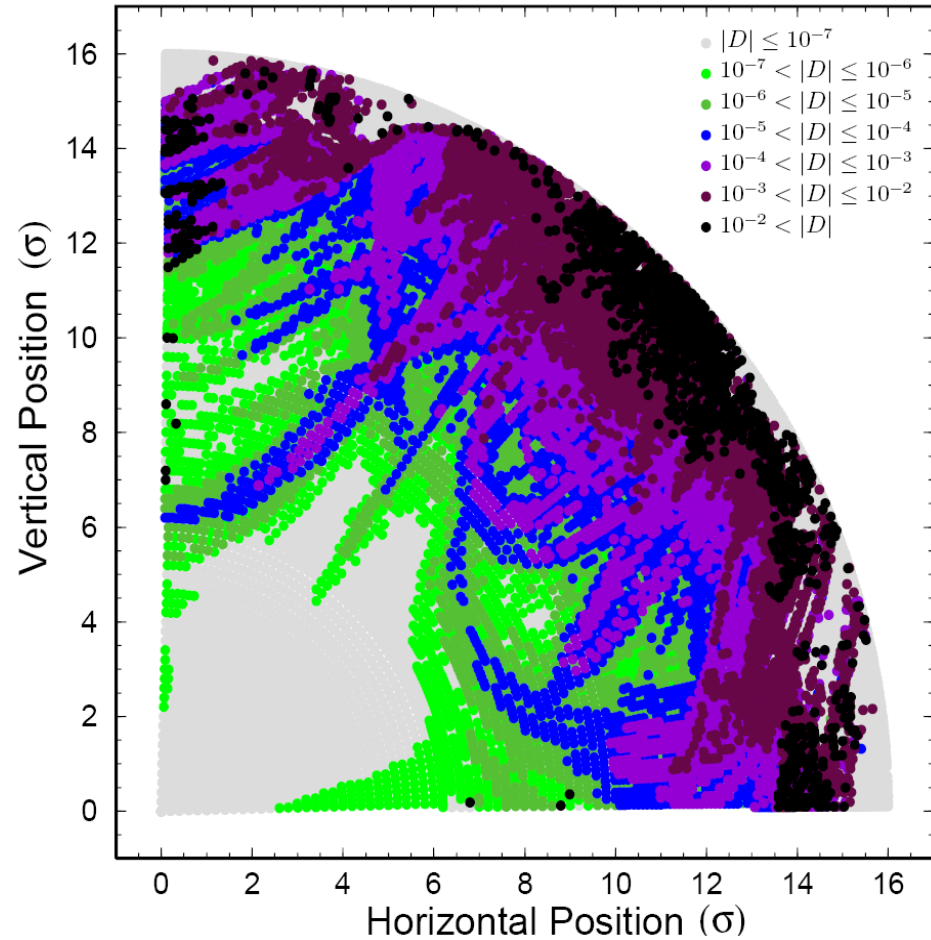
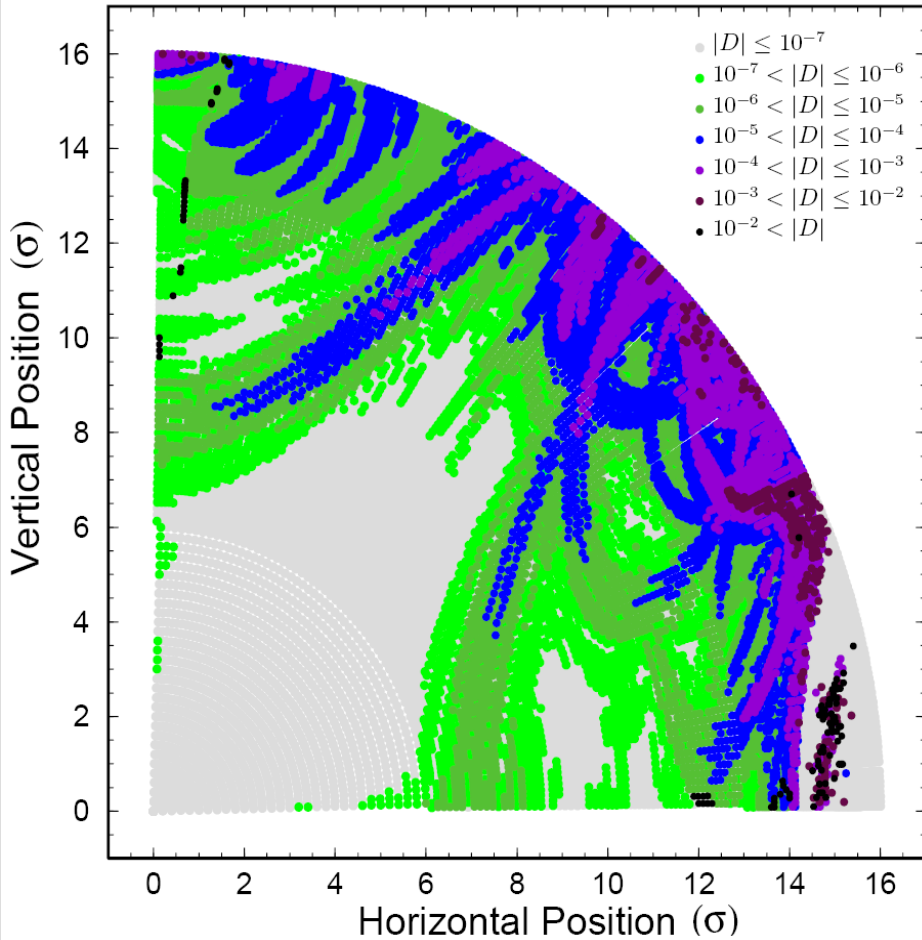


- Calculate frequencies for two equal and successive time spans and compute frequency diffusion vector:

$$\mathbf{D}|_{t=\tau} = \boldsymbol{\nu}|_{t \in (0, \tau/2]} - \boldsymbol{\nu}|_{t \in (\tau/2, \tau]}$$

- Plot the initial condition space color-coded with the norm of the diffusion vector
- Compute a diffusion quality factor by averaging all diffusion coefficients normalized with the initial conditions radius

$$D_{QF} = \left\langle \frac{|\mathbf{D}|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \right\rangle_R$$



Diffusion maps for the target error table (left) and an increased random skew octupole error in the super-conducting dipoles (right)

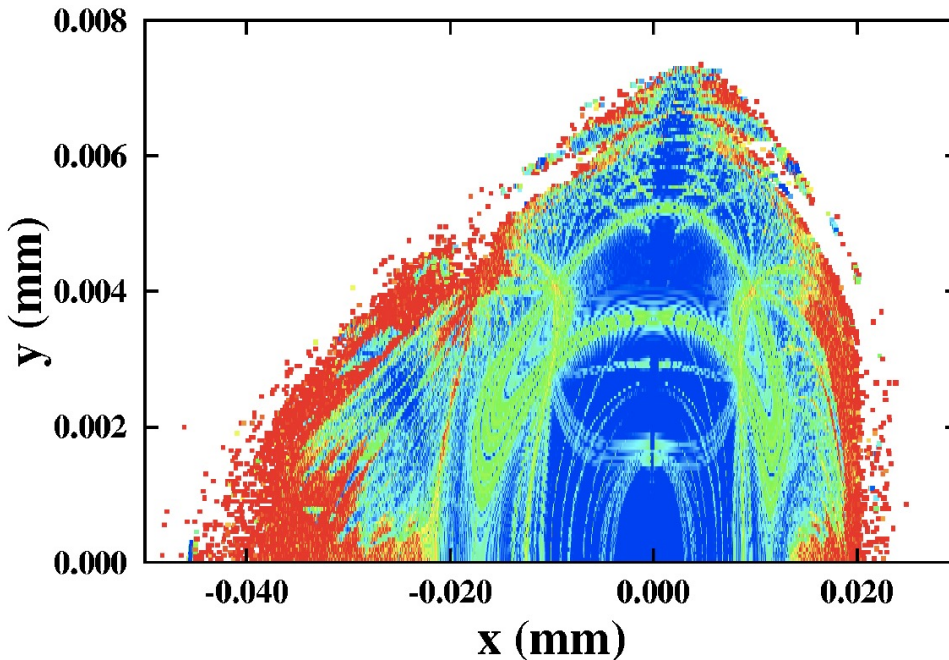
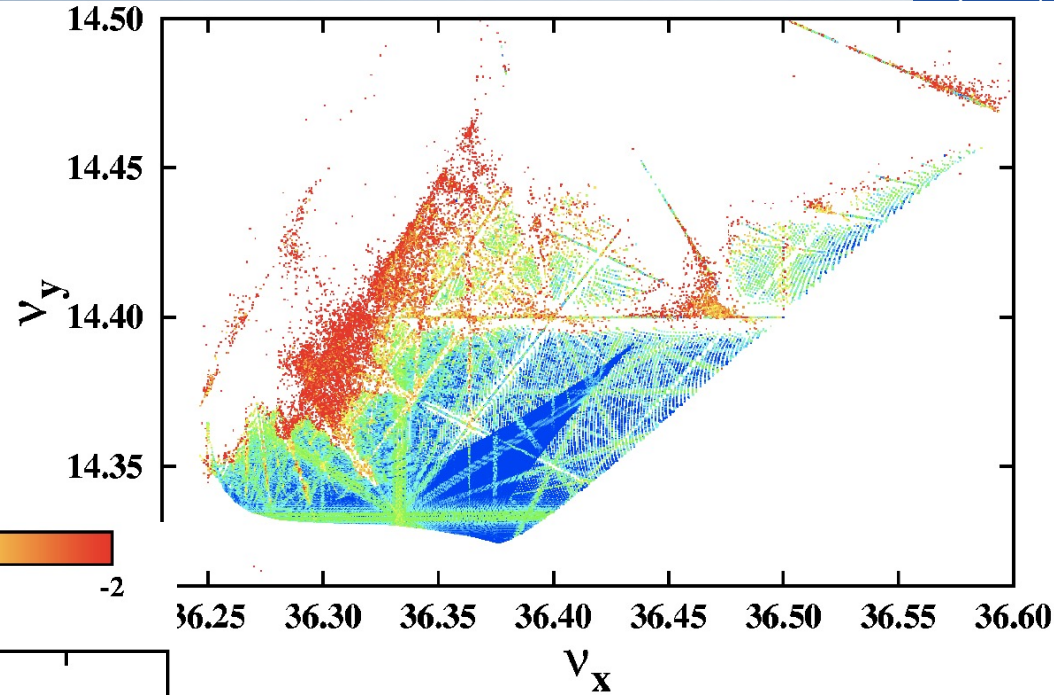




# Example: Frequency Map for the ESRF

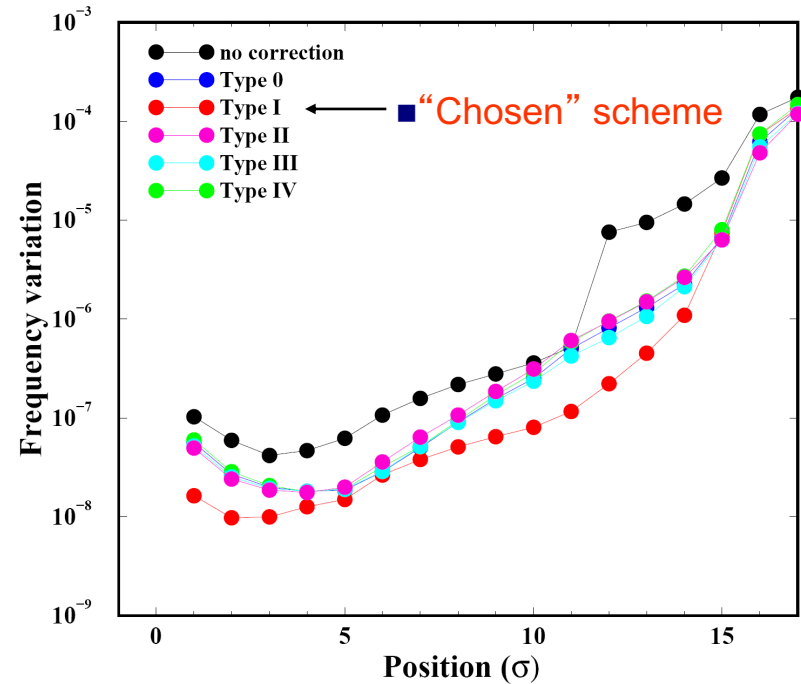
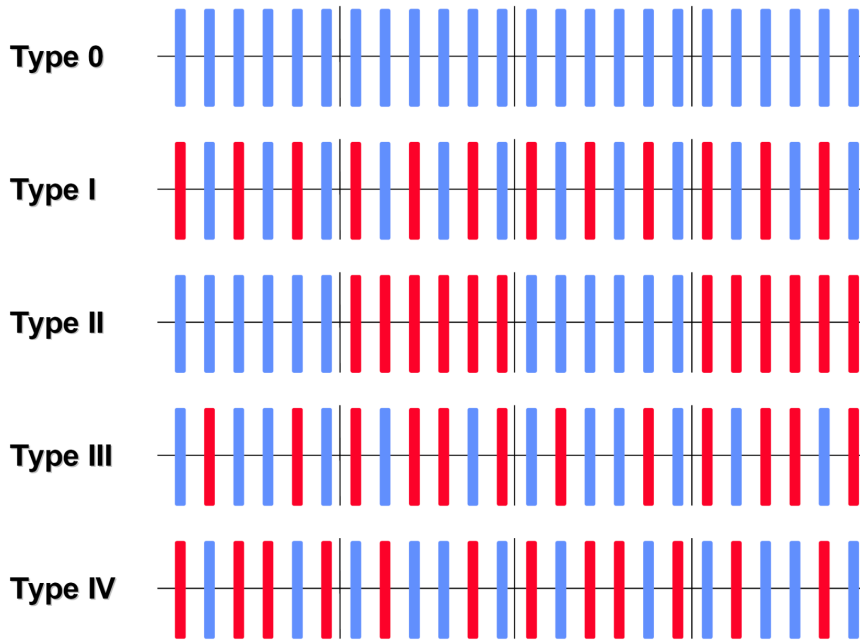


- All dynamics represented in these two plots
- Regular motion represented by blue colors (close to zero amplitude particles or working point)

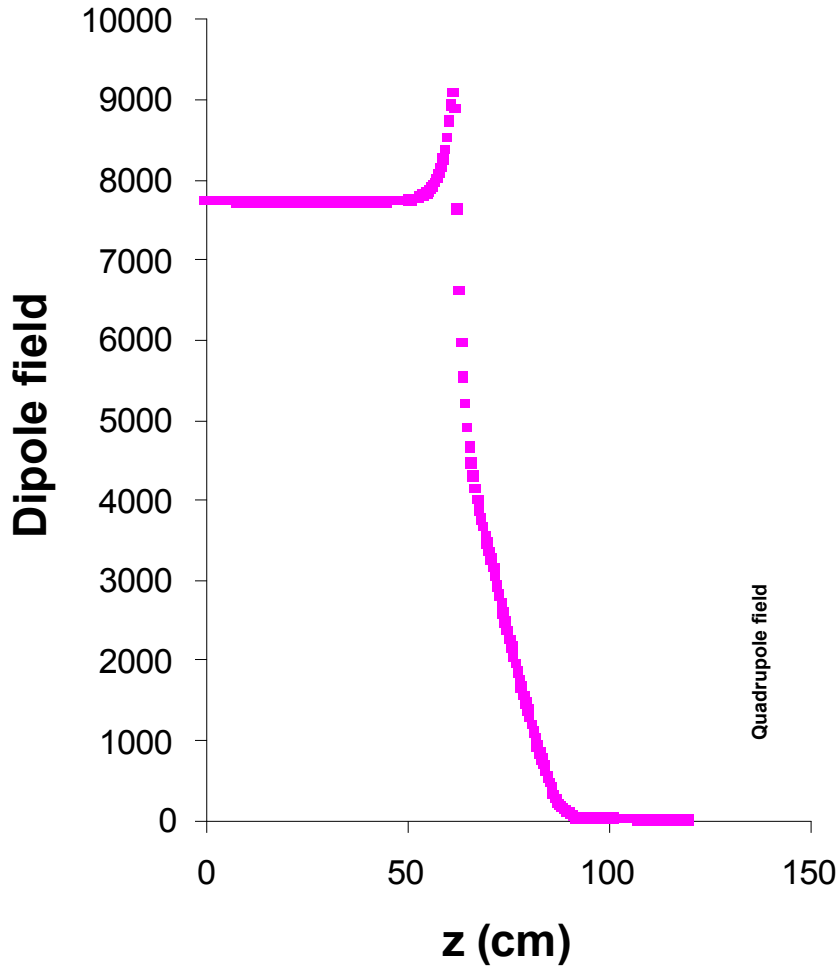


- Resonances appear as distorted lines in frequency space (or curves in initial condition space)
- Chaotic motion is represented by red scattered particles and defines dynamic aperture of the machine

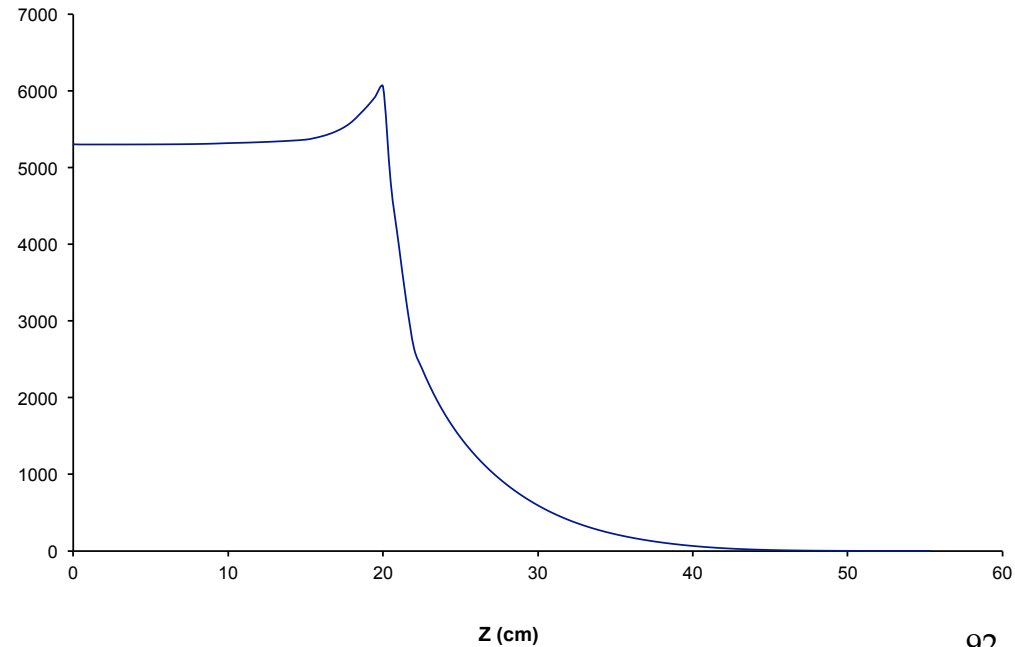
# Numerical Applications



- Comparison of correction schemes for  $b_4$  and  $b_5$  errors in the LHC dipoles
- Frequency maps, resonance analysis, tune diffusion estimates, survival plots and short term tracking, proved that only half of the correctors are needed



- Up to now we considered only transverse fields
- Magnet fringe field is the longitudinal dependence of the field at the magnet edges
- Important when magnet aspect ratios and/or emittances are big





General field expansion for a quadrupole magnet:

$$B_x = \sum_{m,n=0}^{\infty} \sum_{l=0}^m \frac{(-1)^m x^{2n} y^{2m+1}}{(2n)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l]}$$

$$B_y = \sum_{m,n=0}^{\infty} \sum_{l=0}^m \frac{(-1)^m x^{2n+1} y^{2m}}{(2n+1)!(2m)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l]} \quad .$$

$$B_z = \sum_{m,n=0}^{\infty} \sum_{l=0}^m \frac{(-1)^m x^{2n+1} y^{2m+1}}{(2n+1)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l+1]}$$

and to leading order

$$B_x = y \left[ b_1 - \frac{1}{12} (3x^2 + y^2) b_1^{[2]} \right] + O(5)$$

$$B_y = x \left[ b_1 - \frac{1}{12} (3y^2 + x^2) b_1^{[2]} \right] + O(5)$$

$$B_z = xy b_1^{[1]} + O(4)$$

The quadrupole fringe to leading order has an octupole-like effect



## ■ From the hard-edge Hamiltonian

$$H_f = \frac{\pm Q}{12B\rho(1+\frac{\delta p}{p})} (y^3 p_y - x^3 p_x + 3x^2 y p_y - 3y^2 x p_x),$$

the first order shift of the frequencies with amplitude can be computed analytically

$$\begin{pmatrix} \delta\nu_x \\ \delta\nu_y \end{pmatrix} = \begin{pmatrix} a_{hh} & a_{hv} \\ a_{hv} & a_{vv} \end{pmatrix} \begin{pmatrix} 2J_x \\ 2J_y \end{pmatrix},$$

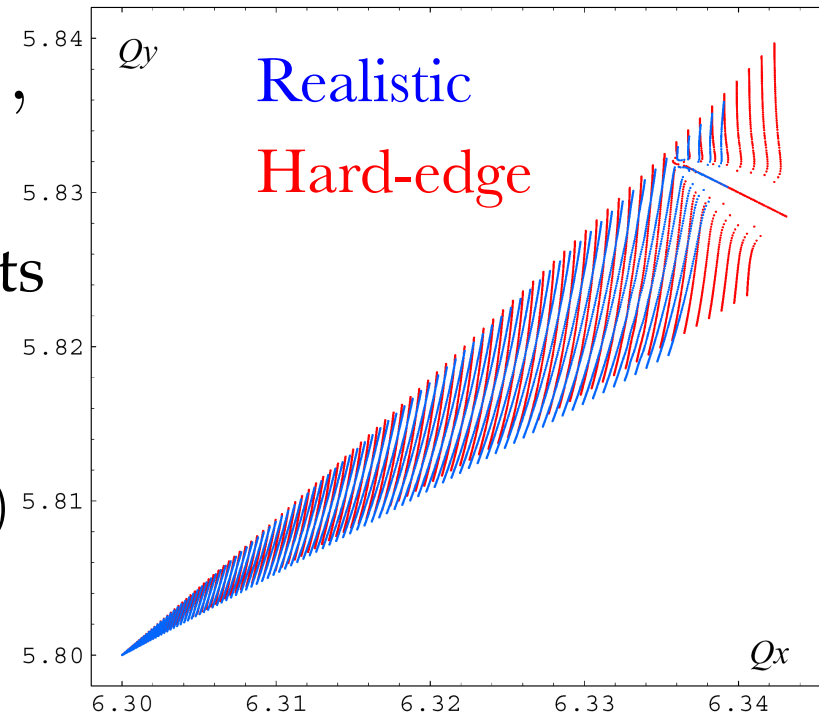
with the "anharmonicity" coefficients (torsion)

$$a_{hh} = \frac{-1}{16\pi B\rho} \sum_i \pm Q_i \beta_{xi} \alpha_{xi}$$

$$a_{hv} = \frac{1}{16\pi B\rho} \sum_i \pm Q_i (\beta_{xi} \alpha_{yi} - \beta_{yi} \alpha_{xi})$$

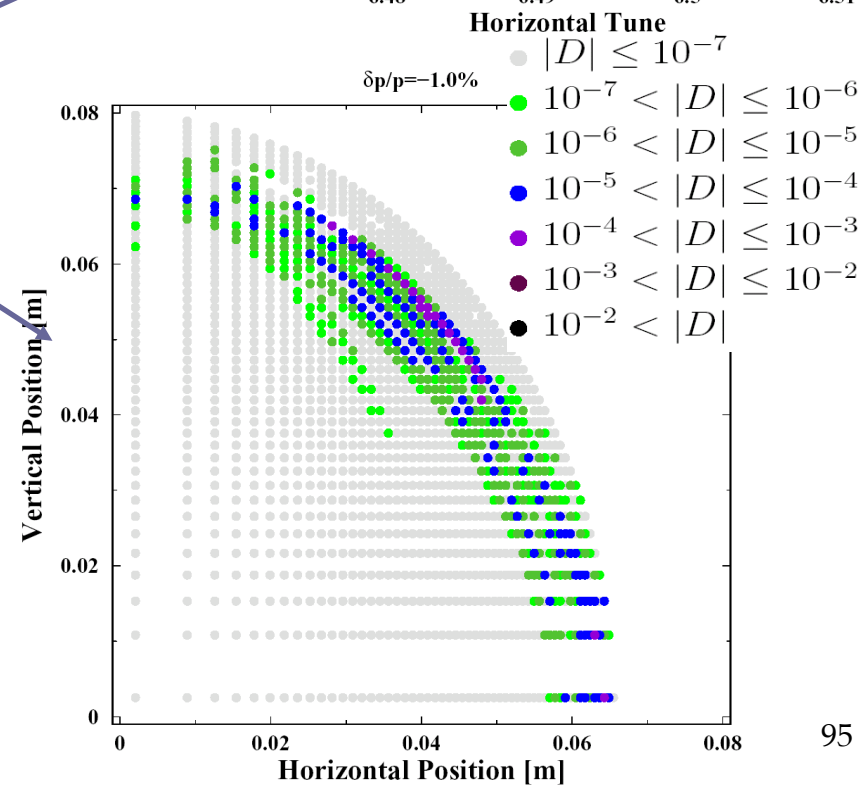
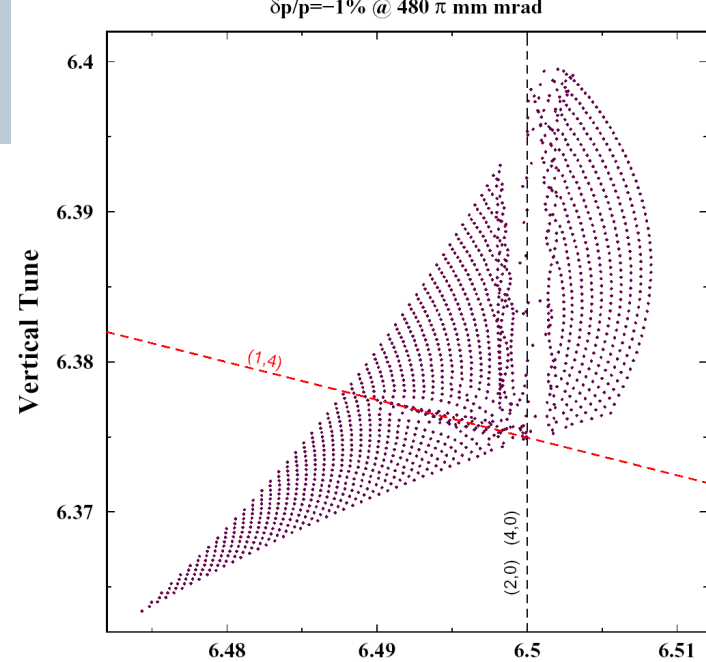
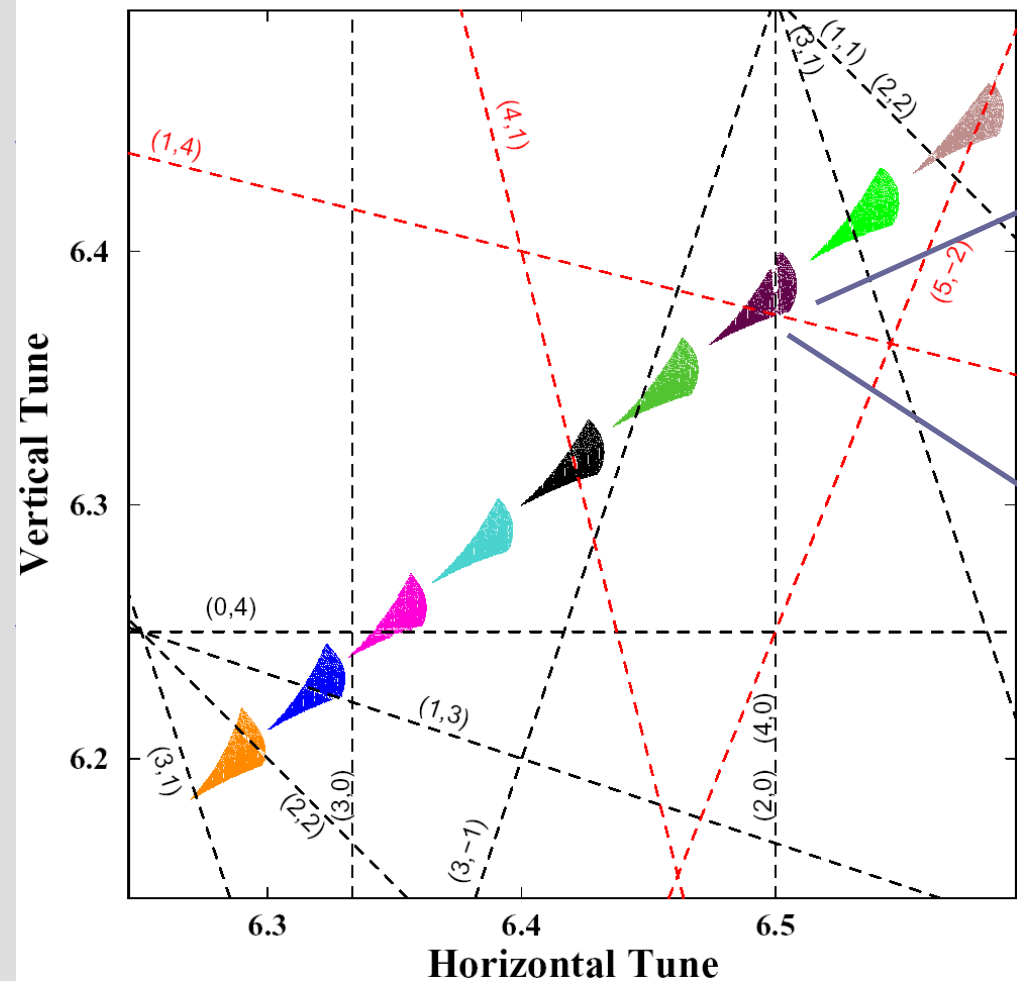
$$a_{vv} = \frac{1}{16\pi B\rho} \sum_i \pm Q_i \beta_{yi} \alpha_{yi}$$

Tune footprint for the SNS based on hard-edge (red) and realistic (blue) quadrupole fringe-field



# SNS Working Point $(Q_x, Q_y) = (6.4, 6.3)$

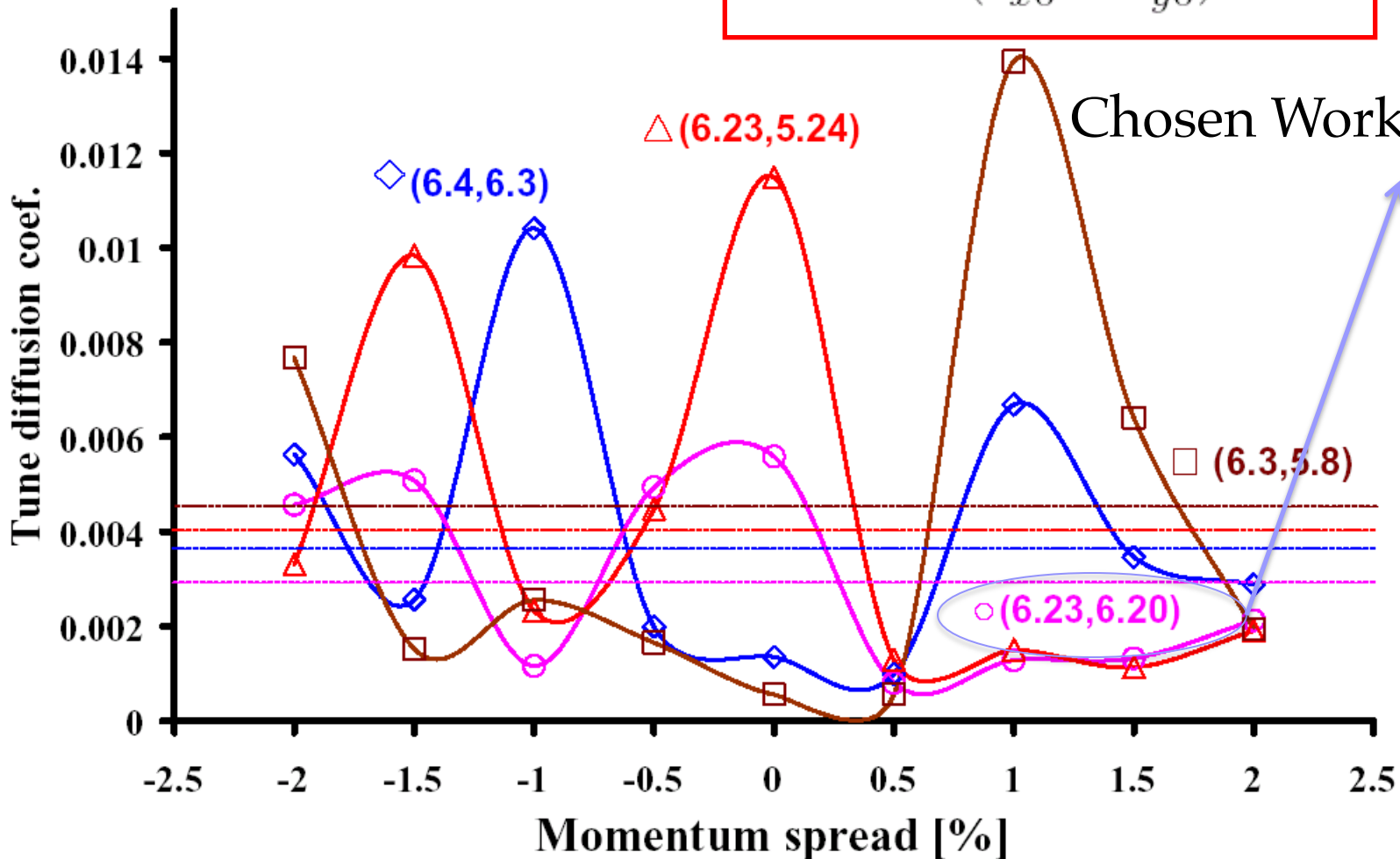
$\delta p/p = [2\%, -2\%]$  @  $480 \pi$  mm mrad





## Tune Diffusion quality factor

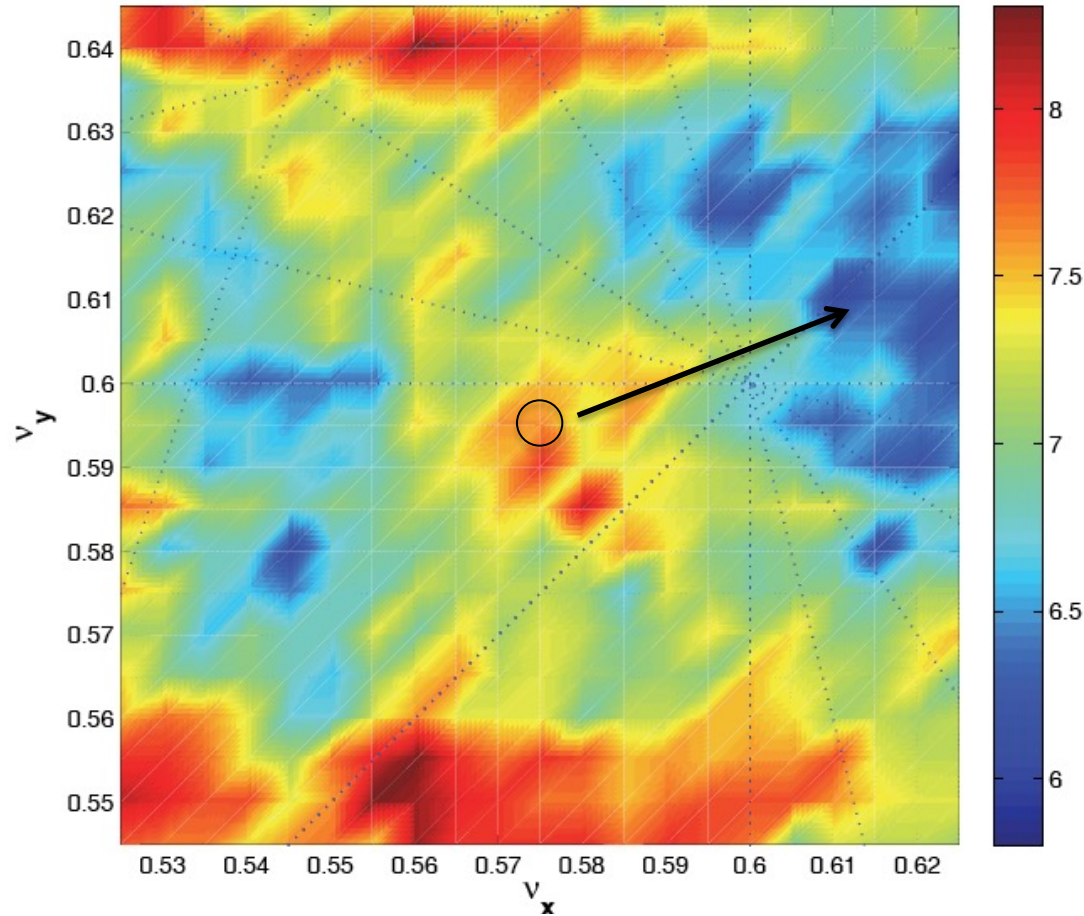
$$D_{QF} = \left\langle \frac{|D|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \right\rangle_R$$





- Figure of merit for choosing best working point is sum of diffusion rates with a constant added for every lost particle
- Each point is produced after tracking 100 particles
- Nominal working point had to be moved towards “blue” area

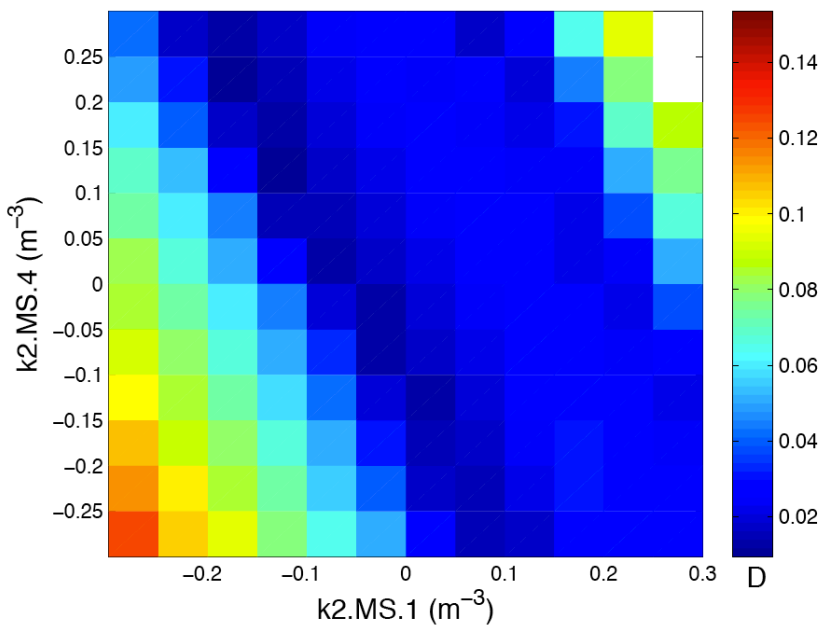
$$e^D = \sqrt{\frac{(\nu_{x,1} - \nu_{x,2})^2 + (\nu_{y,1} - \nu_{y,2})^2}{N/2}}$$



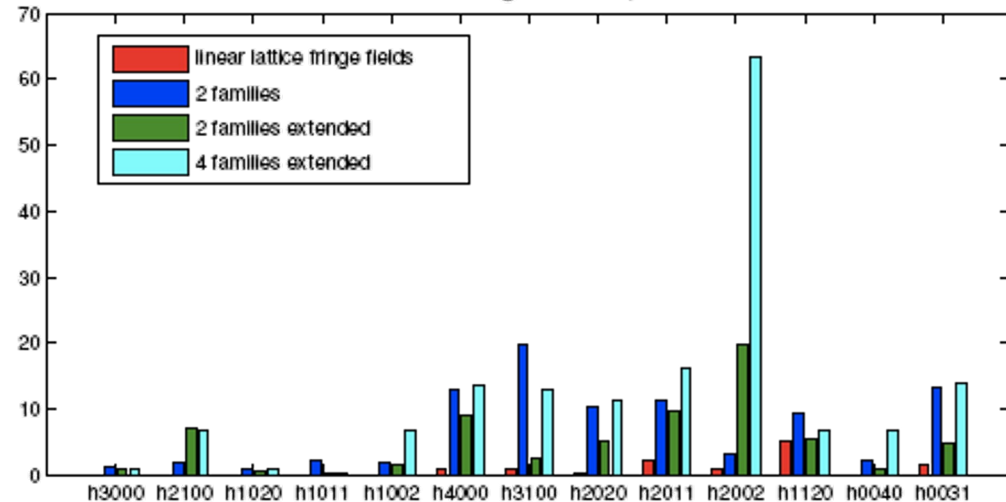
$$WPS = 0.1N_{lost} + \sum e^D$$



Normalized diffusion sum ( $Q_x=11.78, Q_y=6.7$ )



Hamiltonian driving terms up to 4<sup>th</sup> order

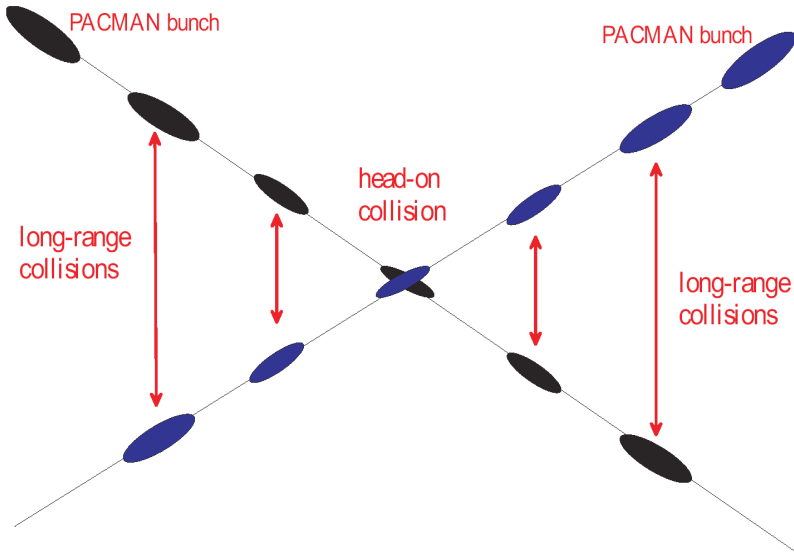


- Comparing different chromaticity sextupole correction schemes and working point optimization using normal form analysis, frequency maps and finally particle tracking
- Finding the adequate sextupole strengths through the tune diffusion coefficient



Variable	Symbol	Value
Beam energy	$E$	7 TeV
Particle species	...	protons
Full crossing angle	$\theta_c$	300 $\mu\text{rad}$
rms beam divergence	$\sigma'_x$	31.7 $\mu\text{rad}$
rms beam size	$\sigma_x$	15.9 $\mu\text{m}$
Normalized transv. rms emittance	$\gamma\varepsilon$	3.75 $\mu\text{m}$
IP beta function	$\beta^*$	0.5 m
Bunch charge	$N_b$	$(1 \times 10^{11} - 2 \times 10^{12})$
Betatron tune	$Q_0$	0.31

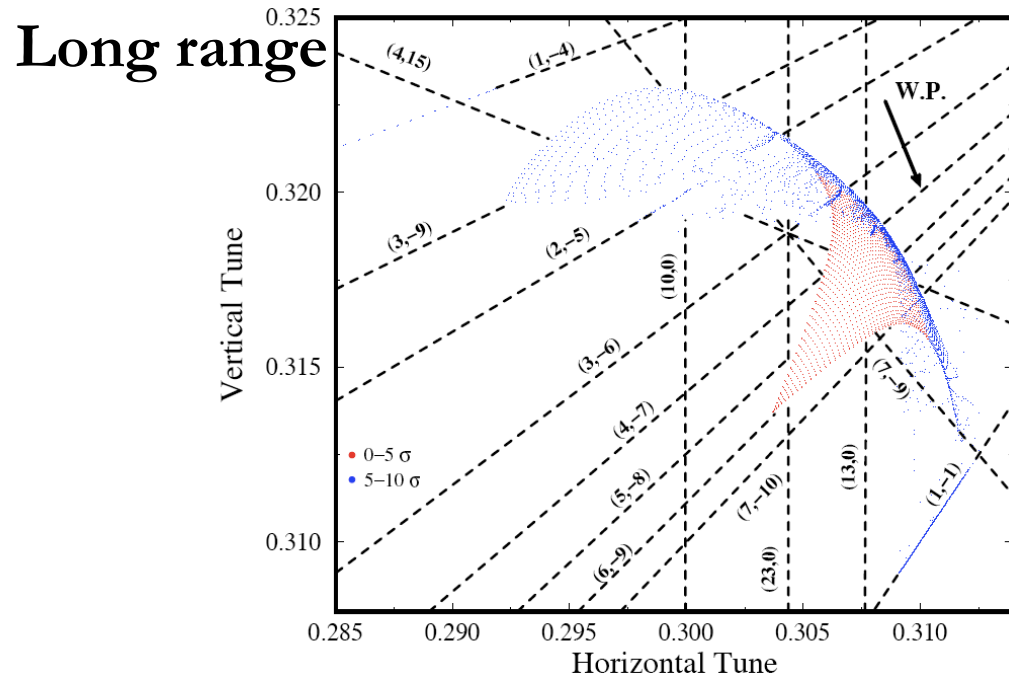
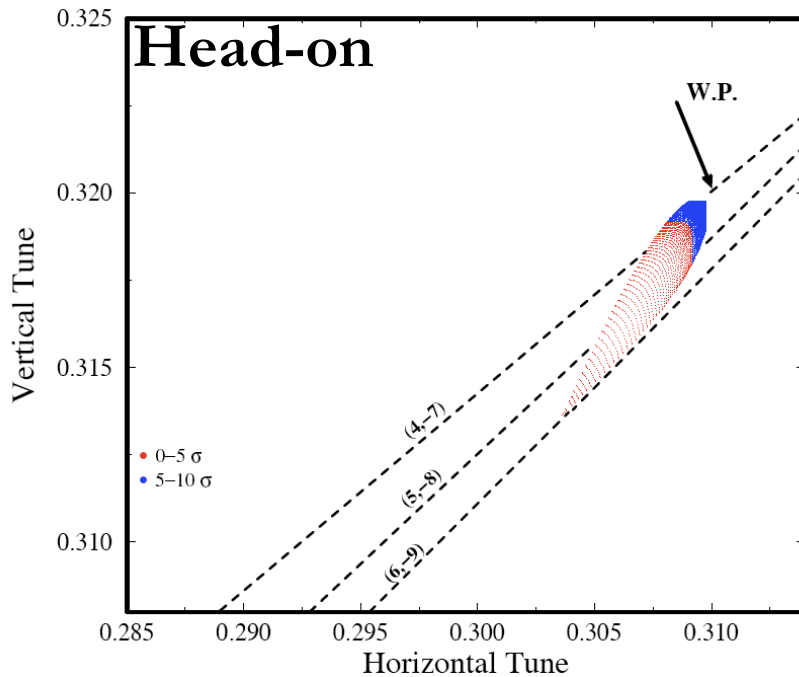
■ Long range beam-beam interaction represented by a 4D kick-map



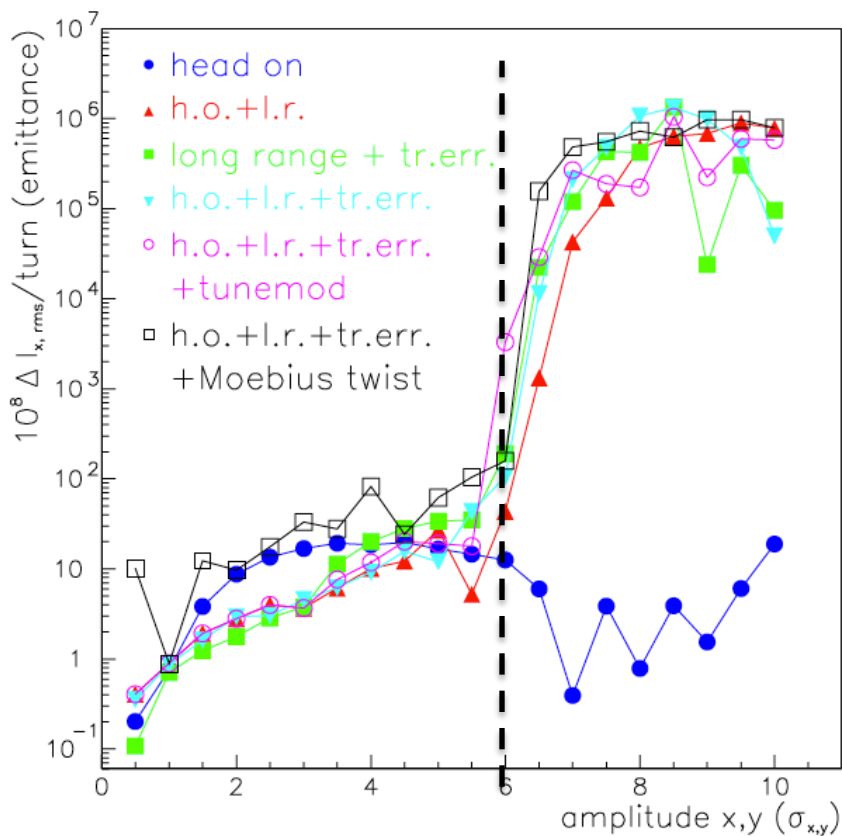
$$\Delta x = - n_{par} \frac{2r_p N_b}{\gamma} \left[ \frac{x' + \theta_c}{\theta_t^2} \left( 1 - e^{-\frac{\theta_t^2}{2\theta_{x,y}^2}} \right) - \frac{1}{\theta_c} \left( 1 - e^{-\frac{\theta_c^2}{2\theta_{x,y}^2}} \right) \right]$$

$$\Delta y = - n_{par} \frac{2r_p N_b}{\gamma} \frac{y'}{\theta_t^2} \left( 1 - e^{-\frac{\theta_t^2}{2\theta_{x,y}^2}} \right)$$

with  $\theta_t \equiv \left( (x' + \theta_c)^2 + y'^2 \right)^{1/2}$

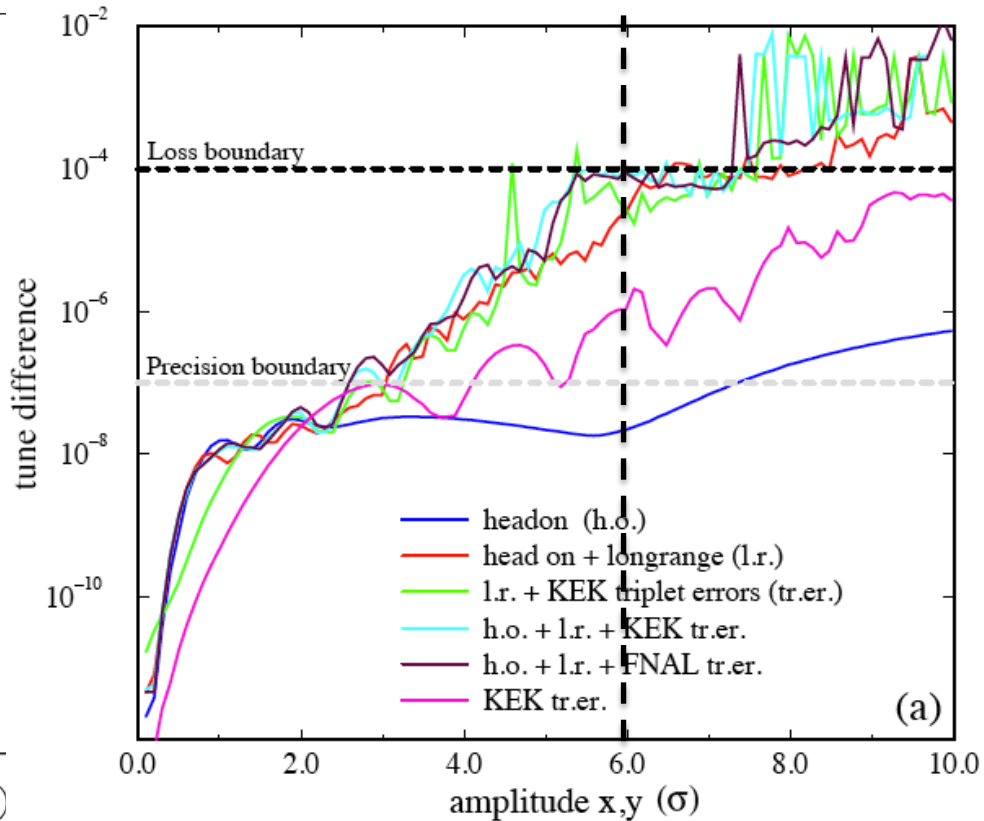
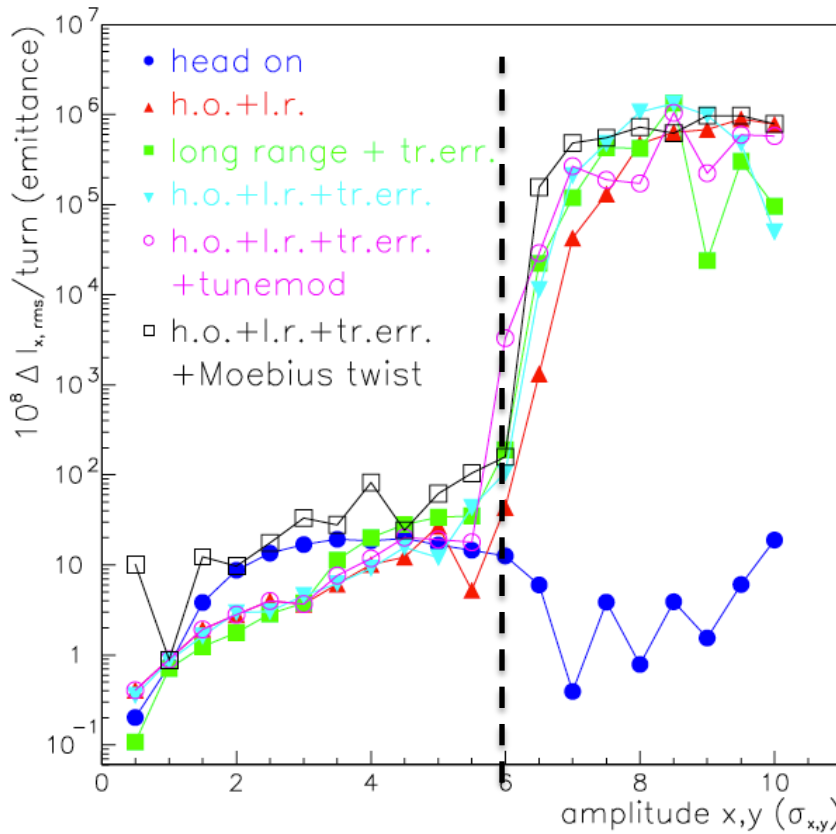


- Proved dominant effect of long range beam-beam effect
- Dynamic Aperture (around  $6\sigma$ ) located at the folding of the map (indefinite torsion)
- Experimental effort to compensate beam-beam long range effect with wires ( $1/r$  part of the force) or octupoles



- In the chaotic region of phase space, the action diffusion coefficient per turn can be estimated by averaging over the quasi-randomly varying betatron phase variable as

$$D(J) = \frac{1}{2\pi} \int_0^{2\pi} d\phi [\Delta J(\phi)]^2$$



- Very good agreement of diffusive aperture boundary (action variance) with frequency variation (loss boundary corresponding to around 1 integer unit change in  $10^7$  turns)

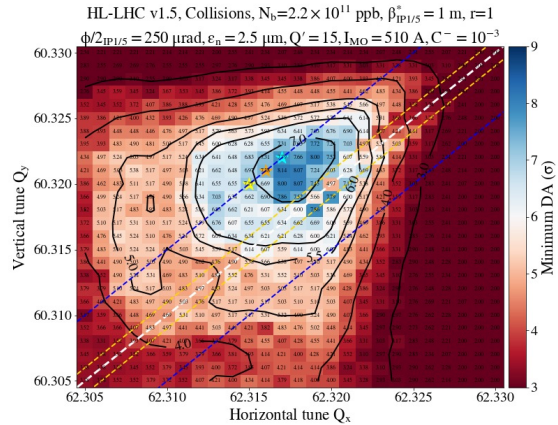


## Optimizing operational scenario

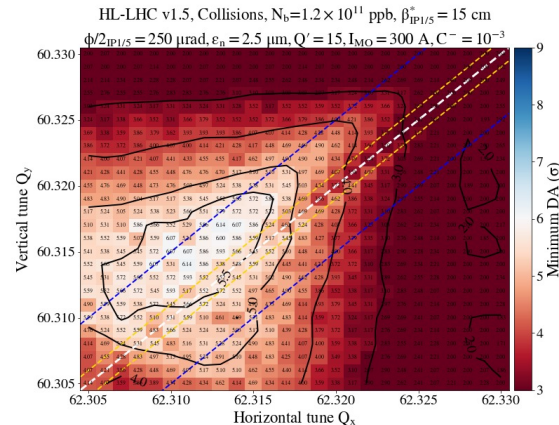
Tune scans color-coded with minimum DA for the selection of the best working point for operation.

S. Kostoglou et al.

### Start of $\beta^*$ -leveling

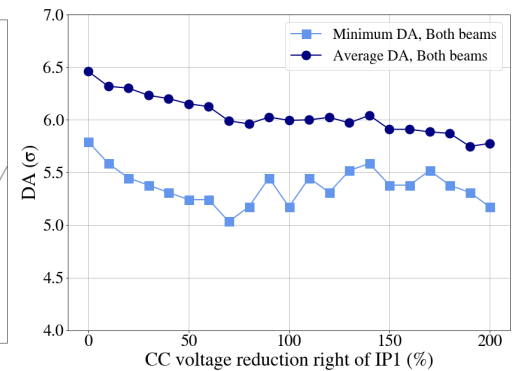
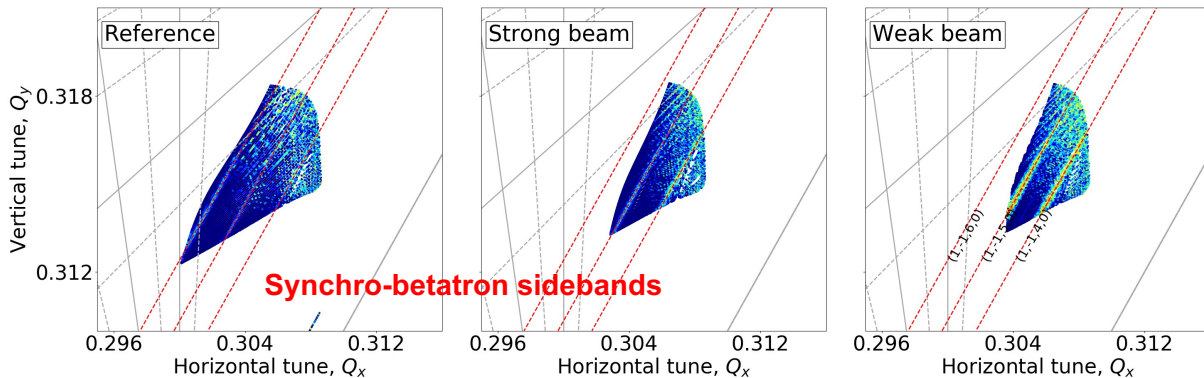


### End of $\beta^*$ -leveling



## CC orbit bump non-closure

Impact of the non-closure of the CC orbit bump due to cavity voltage error:  $V_{err} = V_{nom}(1 - \epsilon)$





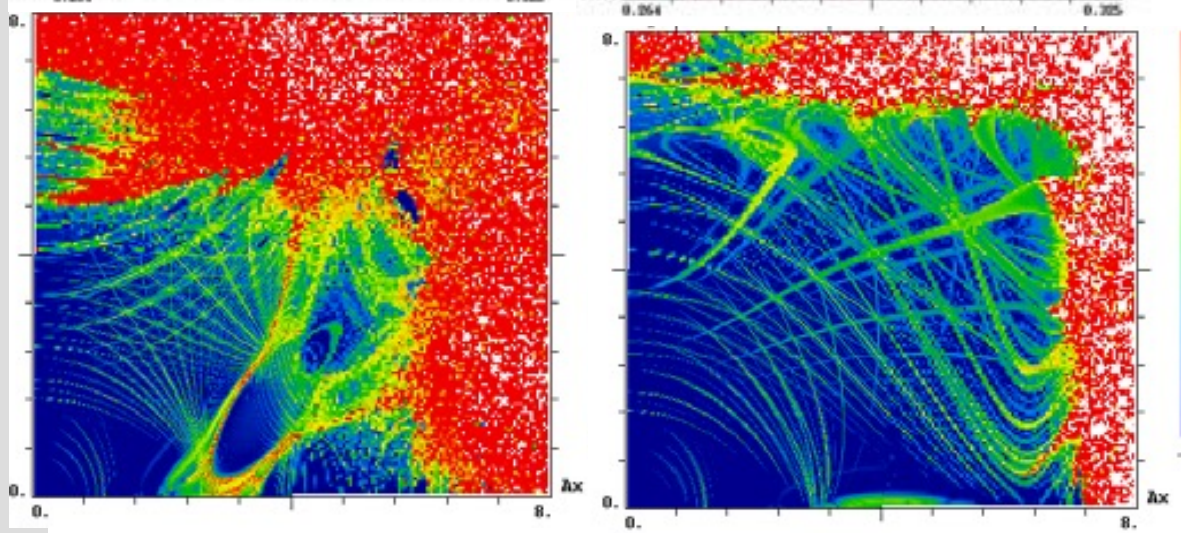
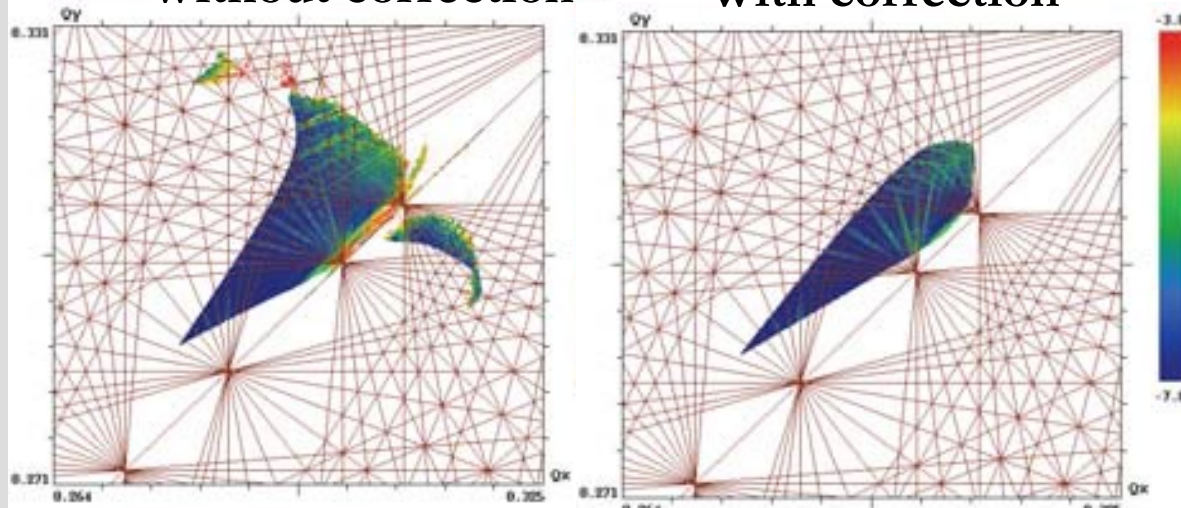
# Wire compensation



2022

Without correction

With correction

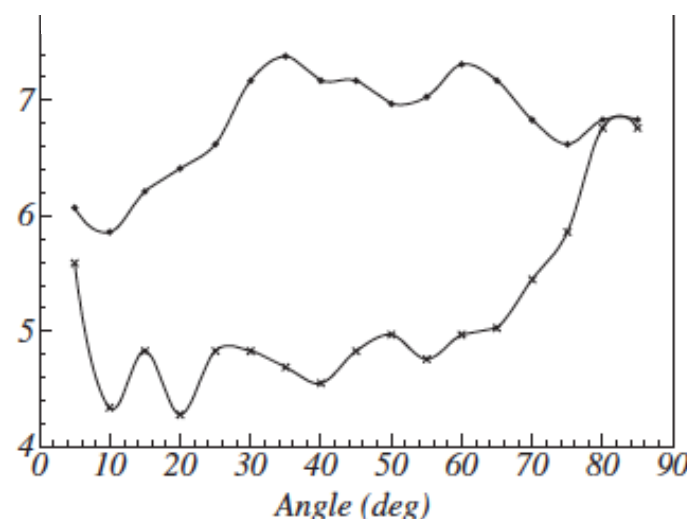


- Current baring wire can improve DA by 1-2  $\sigma$
- Tests in the LHC during 2017-2018

Reduced crossing angle  
of  $450\mu\text{rad}$  @ 15cm

S. Fartoukh, YP et al., PRSTAB, 2015

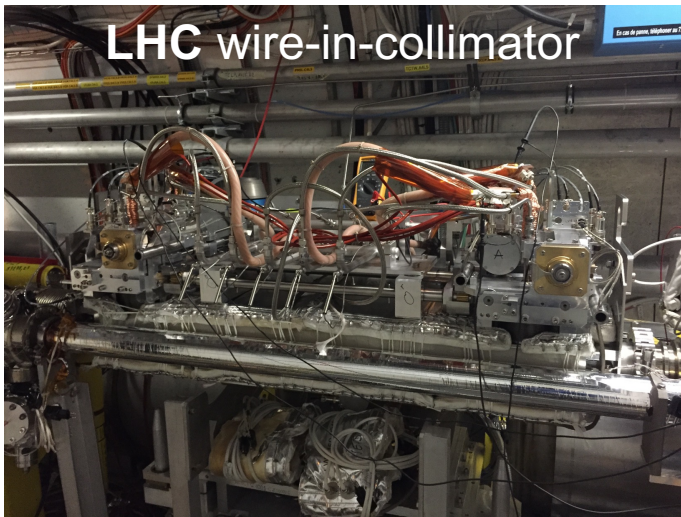
- Nominal bunches with wire correction
- ×— Nominal bunches without wire correction



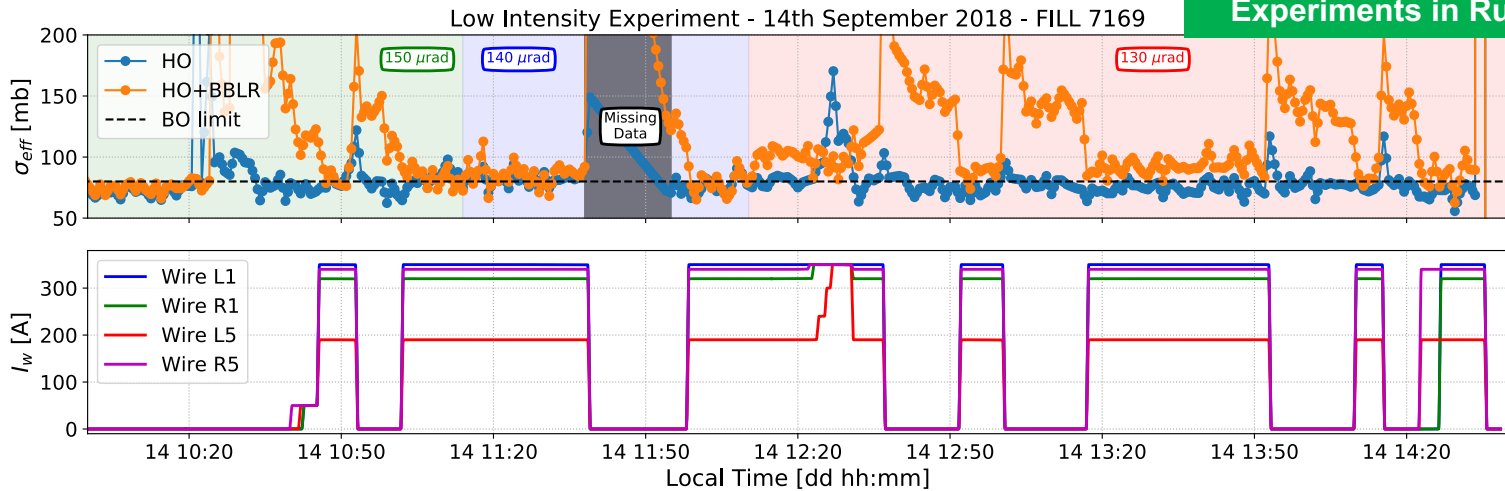
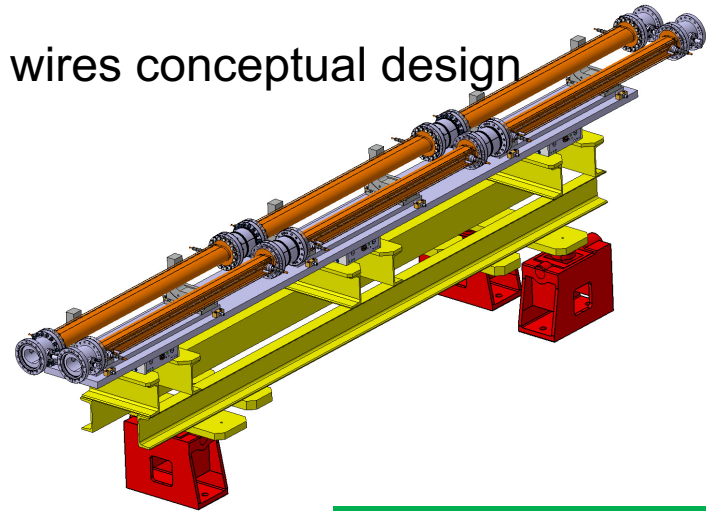


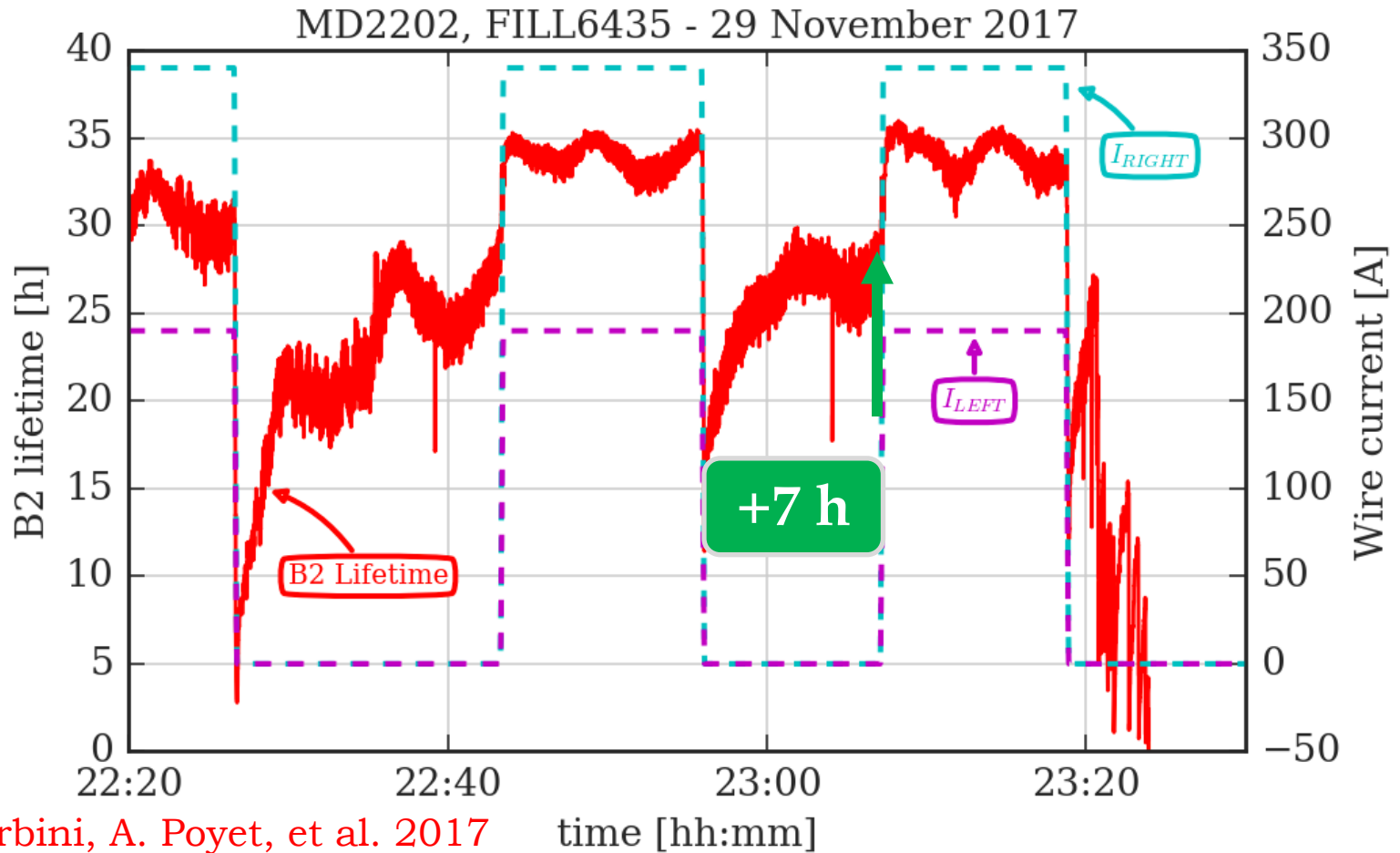
G. Sterbini, A. Poyet, A. Rossi, A. Bertarelli, et al.

CHALLENGE: propose solutions to tackle the Beam Dynamics limitations



HL-LHC wires conceptual design





- Wire current @ 340 / 190 A and collimator jaw at  $5.5 \sigma_{\text{coll}}$
- **Compensating effect** of the wires visible on beam **lifetime**

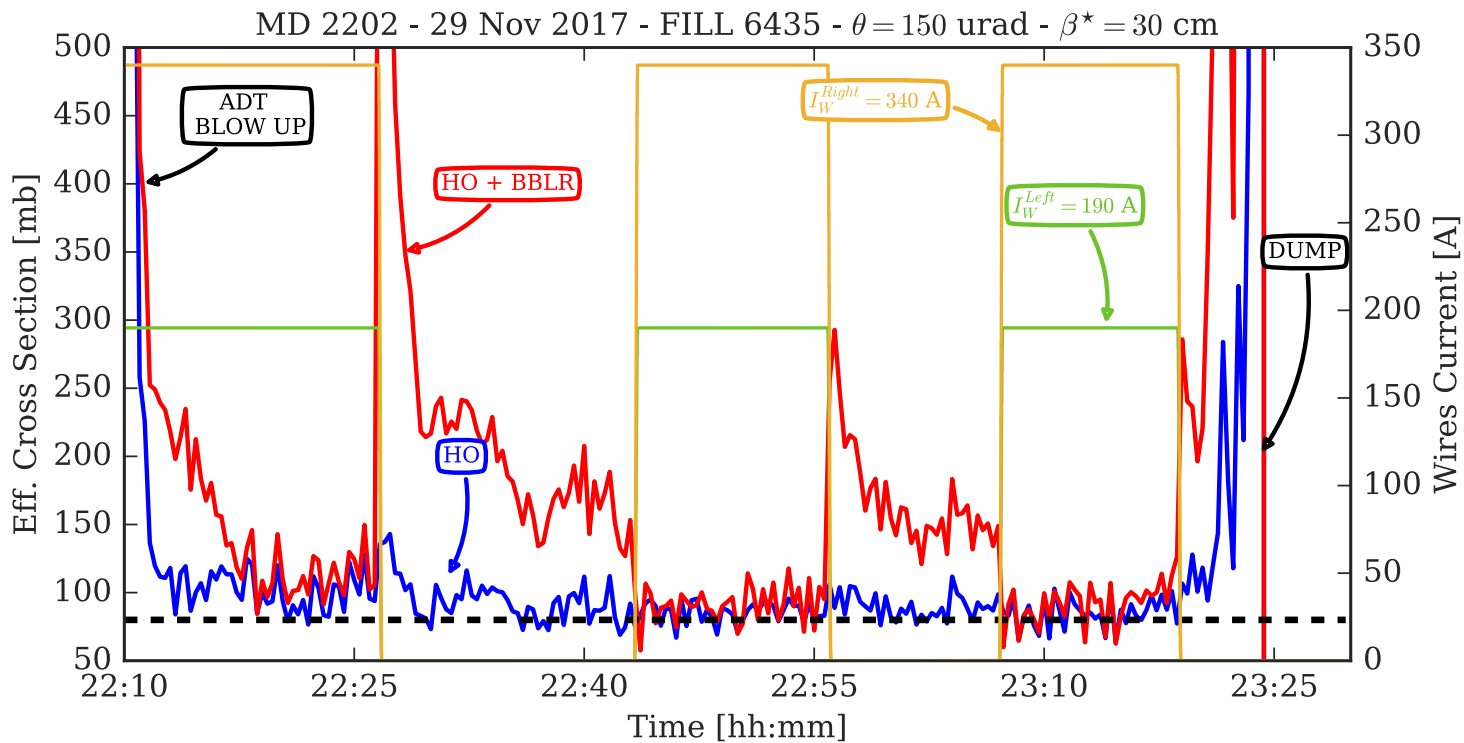


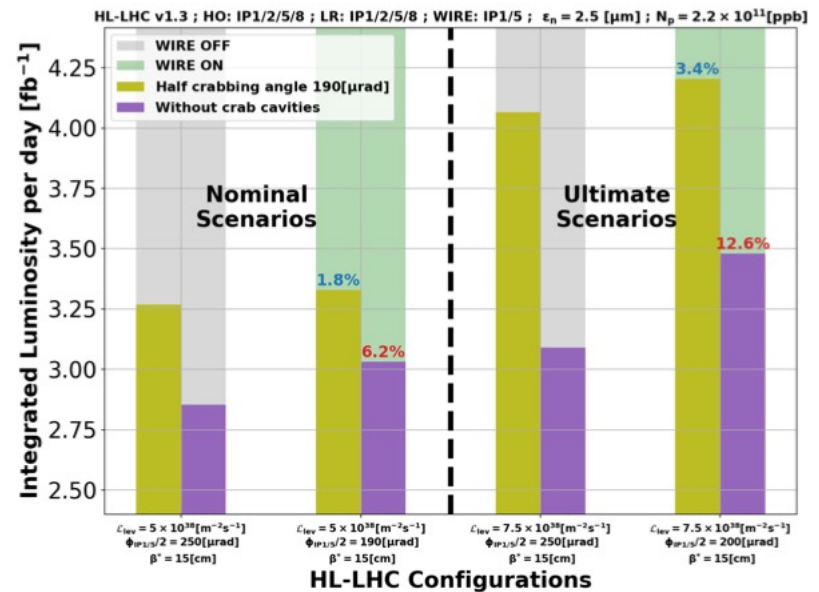
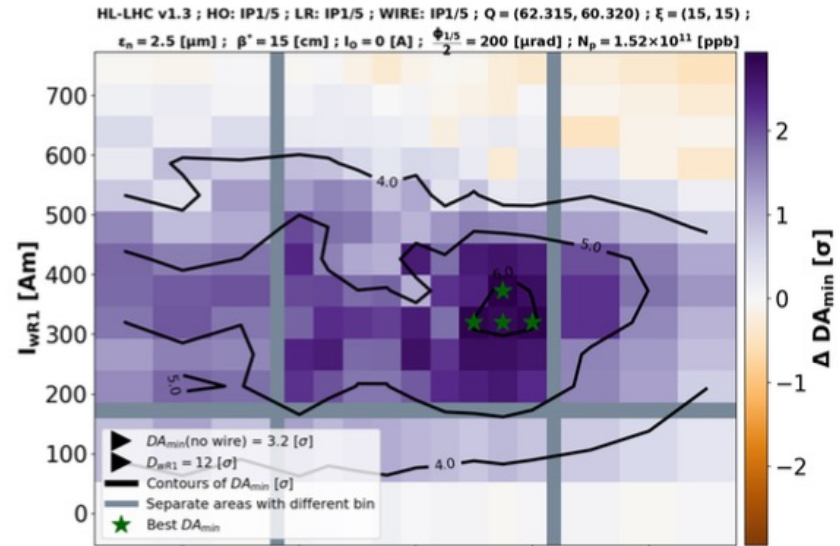
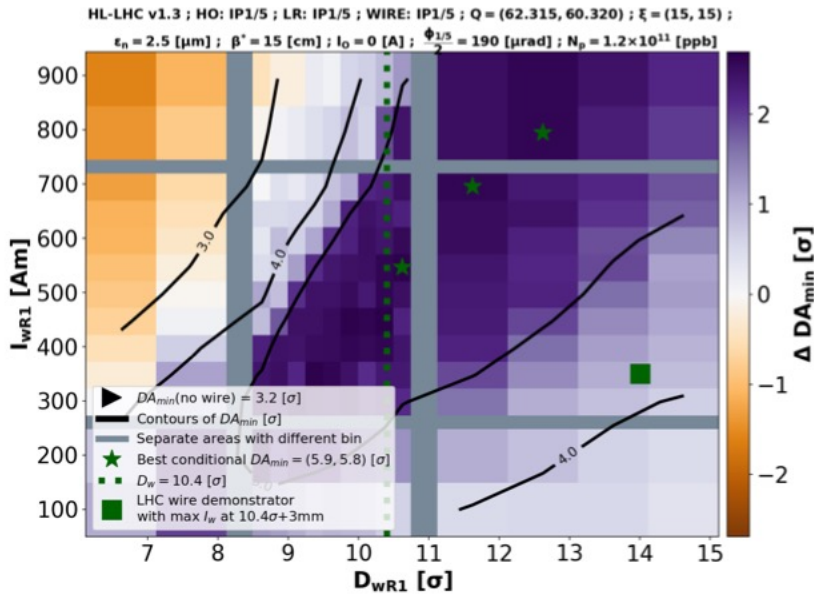
$$\sigma_{EFF} = - \frac{1}{\sum_{IP} L_{IP}} \frac{dN}{dt}$$

Annotations:
 

- Instantaneous luminosity (points to  $\sum_{IP} L_{IP}$ )
- Intensity loss-rate (points to  $\frac{dN}{dt}$ )

- Compensating effect of the wires visible on effective x-section**

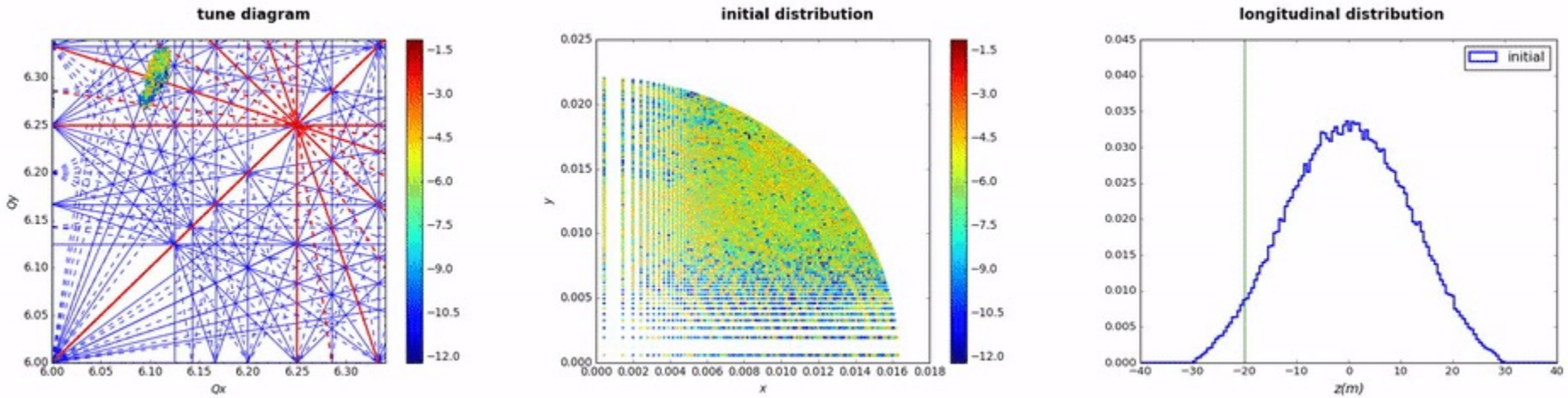




**CHALLENGE:** Predict and optimize the future (HL-LHC) by

- taking into account Machine Protection constraints and
- minimizing complexity, saving commissioning time and maximize machine availability.

# Frequency Map Analysis with modulation



F.Asvesta, et al., 2017

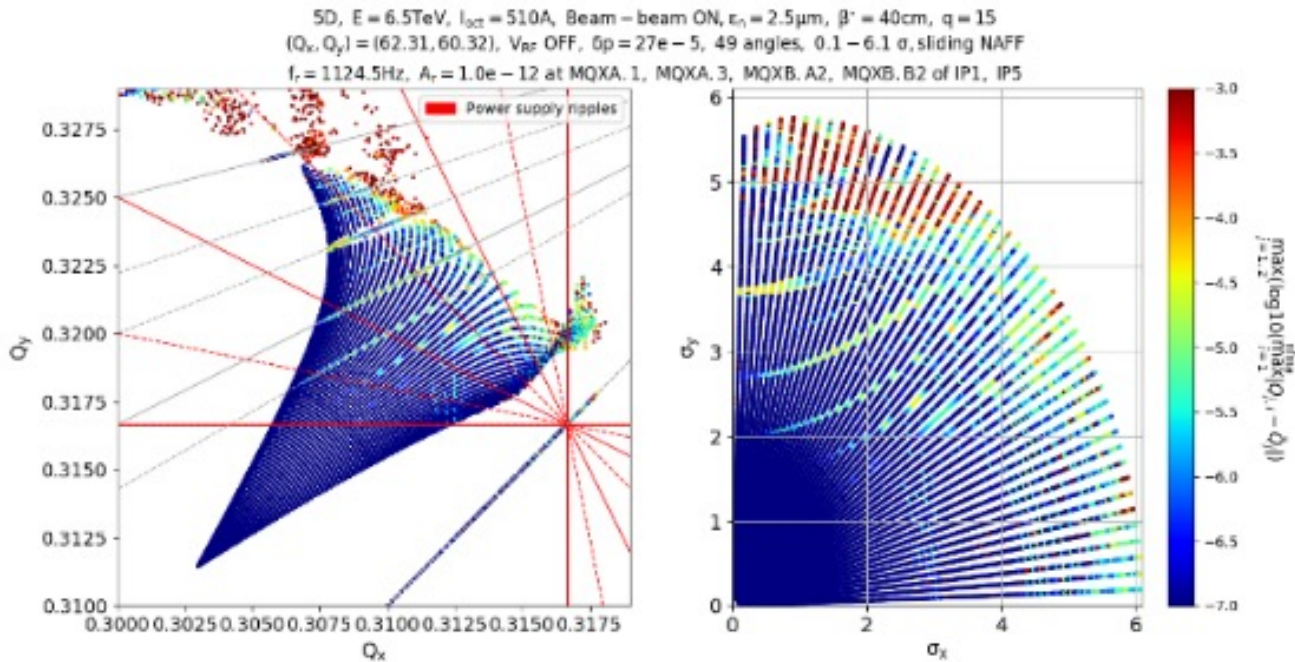
- ❑ Evolution of frequency map over different longitudinal position
- ❑ Tunes acquired over each longitudinal period
- ❑ Particles with similar longitudinal offset but different amplitudes experience the resonance in different manner
- ❑ Particles with different longitudinal offset may experience different resonances



- Quadrupoles of the **inner triplet** right and left of **IP1 and IP5**, **large beta-functions** increase the sensitivity to non-linear effects
- **Resonance conditions:** S. Kostoglou, et al., 2018

$$aQ_x + bQ_y + c \frac{f_{\text{modulation}}}{f_{\text{revolution}}} = k \text{ for } a, b, c, k \text{ integers}$$

**-By increasing the modulation depth, sidebands start to appear in the FMAs**



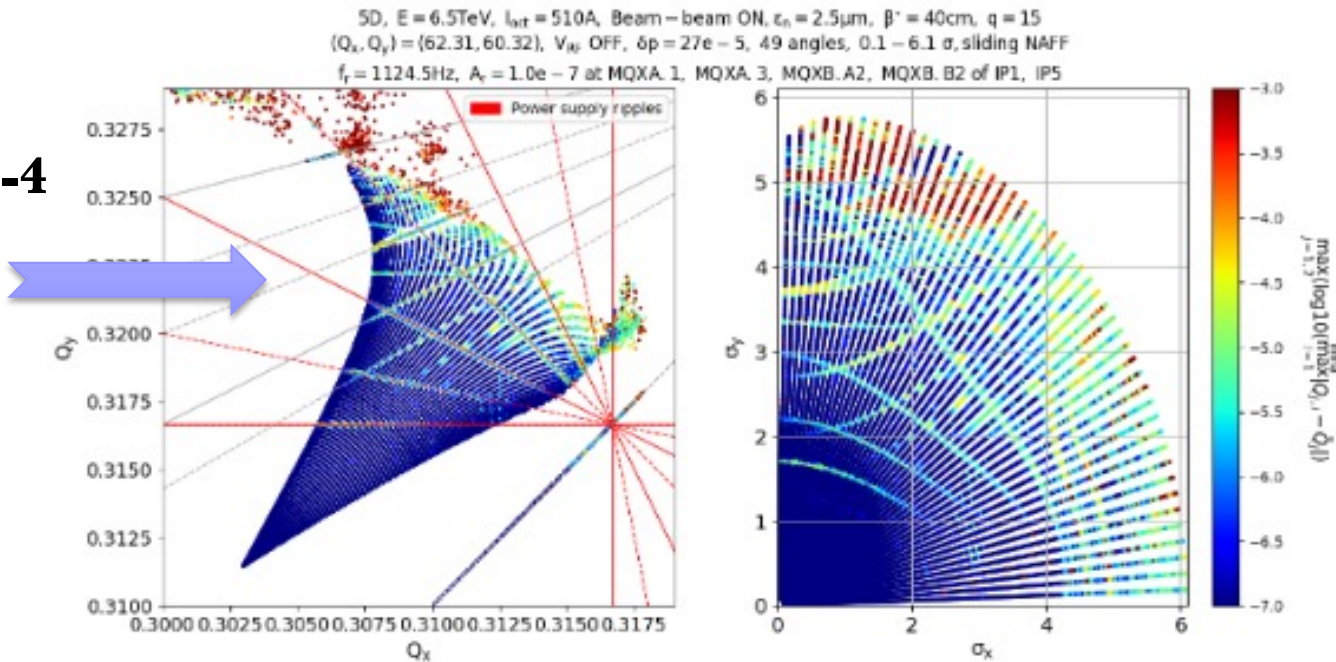


- Quadrupoles of the **inner triplet** right and left of **IP1 and IP5**, **large beta-functions** increase the sensitivity to non-linear effects
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$$aQ_x + bQ_y + c \frac{f_{\text{modulation}}}{f_{\text{revolution}}} = k \text{ for } a, b, c, k \text{ integers}$$

**-By increasing the modulation depth, sidebands start to appear in the FMAs**

$\Delta Q = 1e-4$

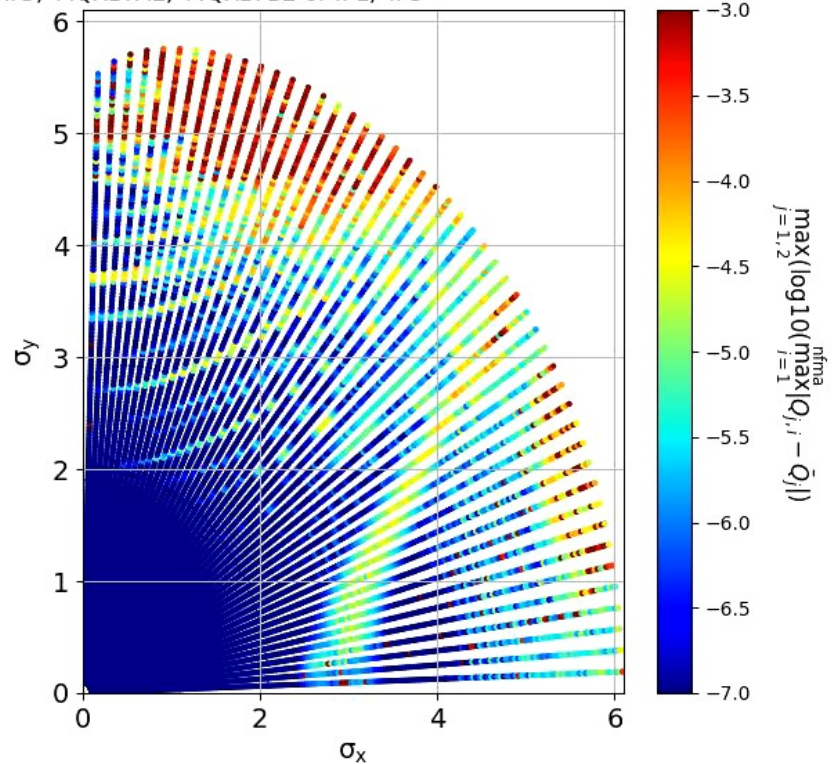
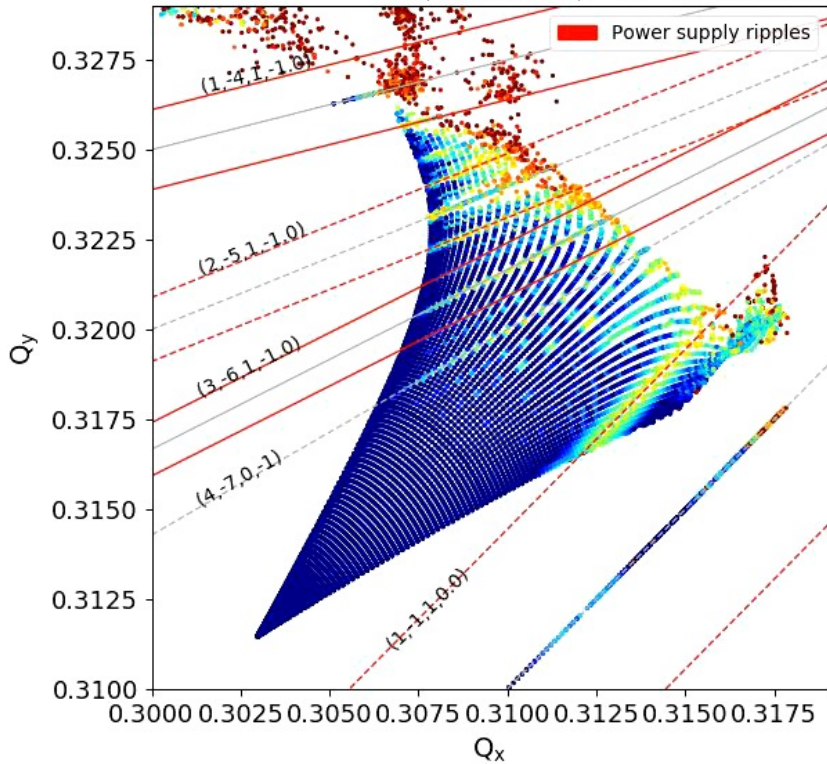






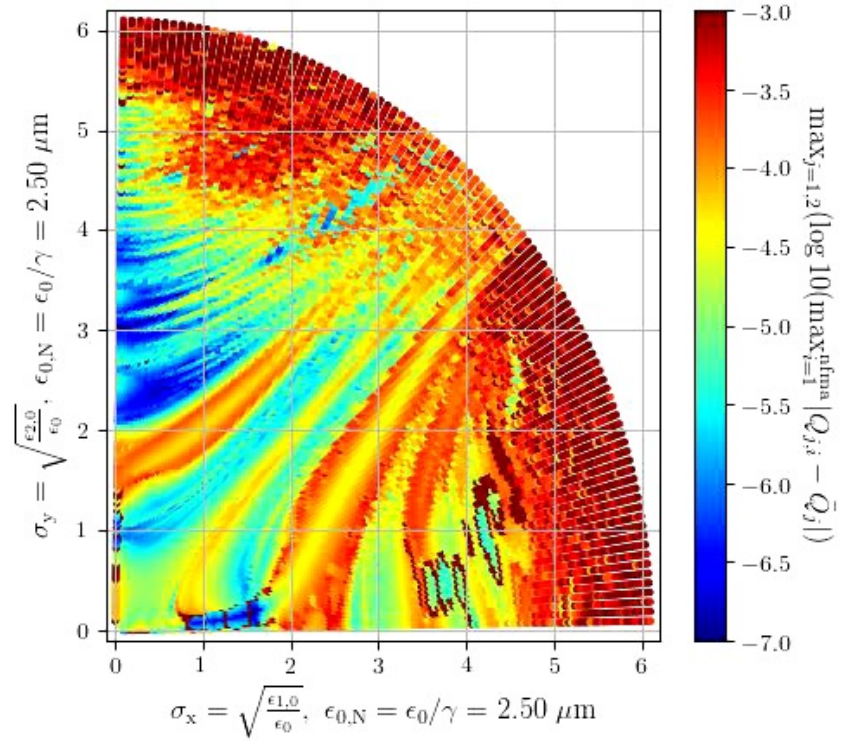
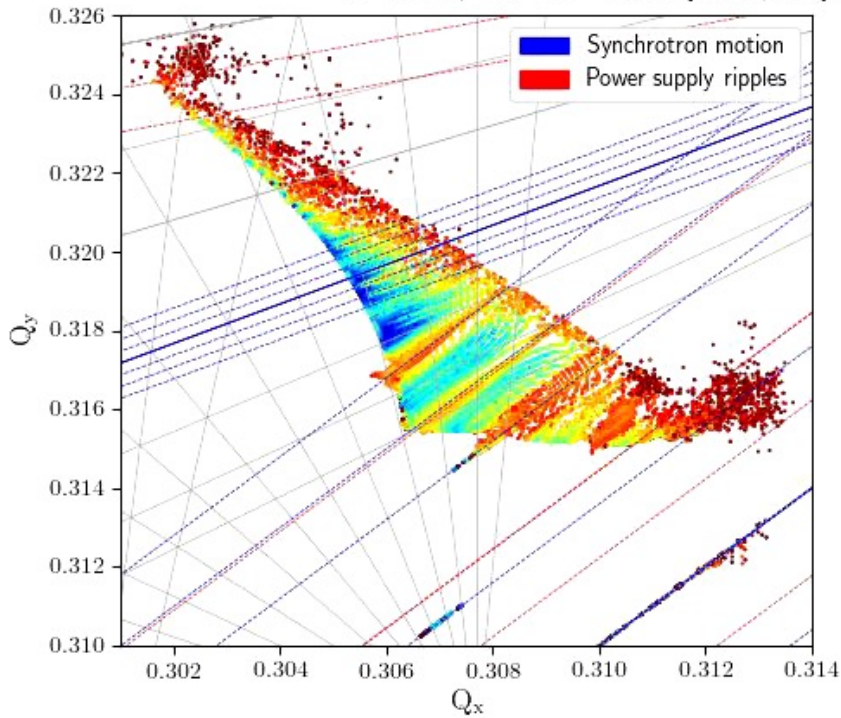
## Scan of different ripple frequencies (50-900 Hz)

5D,  $E = 6.5\text{TeV}$ ,  $I_{\text{oct}} = 510\text{A}$ , Beam - beam ON,  $\epsilon_n = 2.5\mu\text{m}$ ,  $\beta' = 40\text{cm}$ ,  $q = 15$   
 $(Q_x, Q_y) = (62.31, 60.32)$ ,  $V_{\text{RF}}$  OFF,  $\delta p = 27e-5$ , 49 angles,  $0.1 - 6.1 \sigma$ , sliding NAFF  
 $f_r = 50.0\text{Hz}$ ,  $A_r = 10^{-7}$  at MQXA. 1, MQXA. 3, MQXB. A2, MQXB. B2 of IP1, IP5



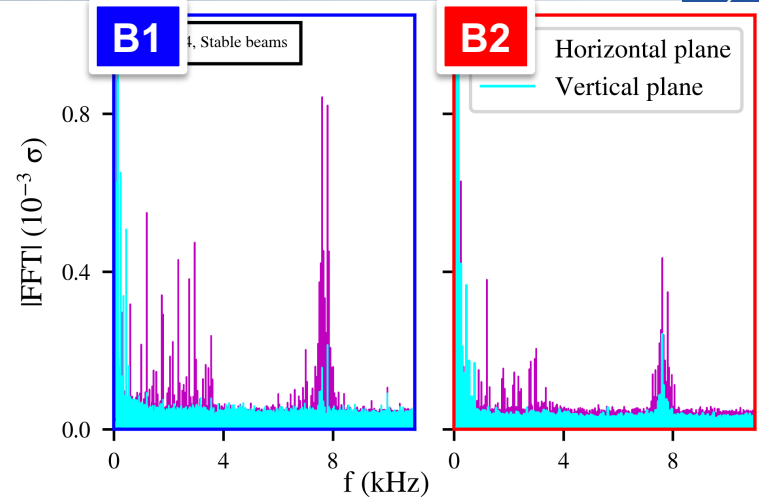


6D,  $E = 6.5\text{TeV}$ ,  $I_{\text{occt}} = 510\text{A}$ , Beam - beam ON,  $\epsilon_n = 2.5\mu\text{m}$ ,  $\beta^* = 40\text{cm}$ ,  $q = 0$   
 $(Q_x, Q_y) = (62.31, 60.32)$ ,  $V_{\text{RF}}$  ON,  $\delta p = 27 \cdot 10^{-5}$ , 99 angles,  $0.1 - 6.1 \sigma$ , sliding NAFF  
 $f_r = 50\text{Hz}$ ,  $A_r = 10^{-7}$  at MQXA.1, MQXA.3, MQXB.A2, MQXB.B2 of IP1, IP5



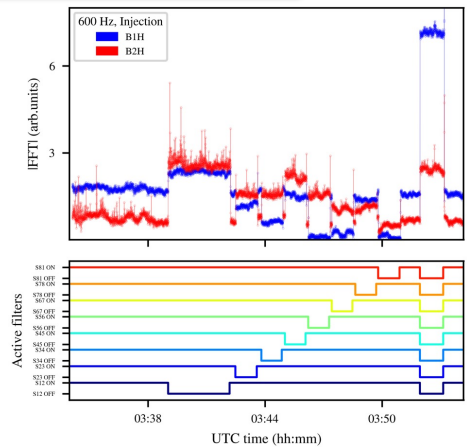
**CHALLENGE:** chasing the source of the  $10^{-3} \sigma$  beam oscillations and its impact on beam lifetime.

- 50 Hz harmonics on transverse beam spectrum.
- Harmonics of **Beam 1** x2 larger amplitudes than **Beam 2**.

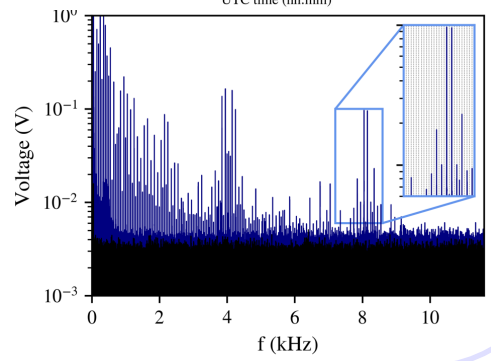


## Perturbation source

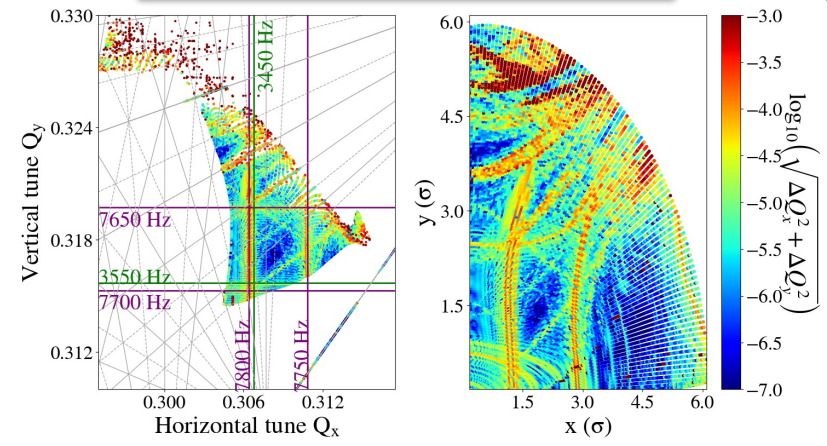
Power converters of main dipoles: tests with active filters



UPS voltage measurements, additional studies in Run 3



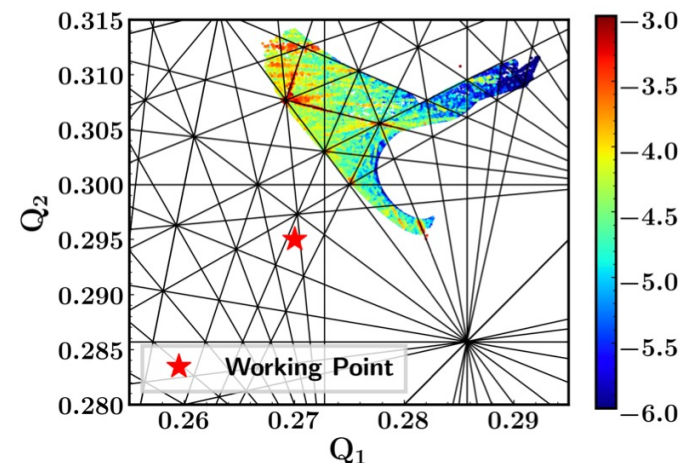
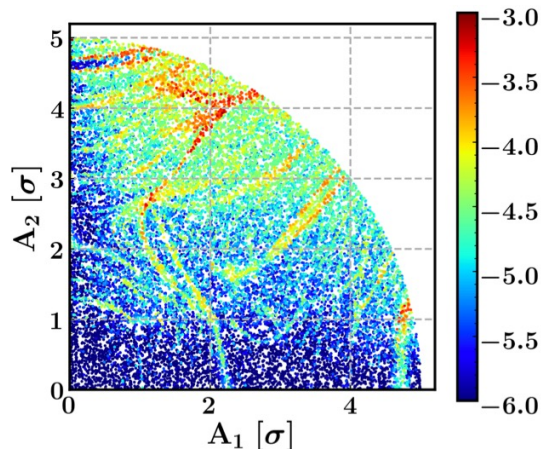
## Impact on beam lifetime



Single-particle tracking simulations:  
**22h** beam lifetime for **Beam 1**  
 and **27h** for **Beam 2**



- Analysis of experimental data from the LHC shows a slow beam degradation due to e-cloud both at injection and in collisions
- A development effort was launched to acquire the ability of simulating the effect of e-cloud forces within a symplectic tracking code over the required long timescales (10M turns). The development included:
  - Theoretical framework
  - Tricubic interpolator in sixtracklib tracking code to apply forces from a recorded pinch in a symplectic way
  - Software infrastructure to simulate and condition the electron pinches and setup the simulation from the MAD-X description of the machine
- Presently capable of simulating 10 M turns (15 minutes of beam time) by exploiting the computational power of GPUs



K. Paraschou, G. Iadarola, et al.



- Appearance of **fixed points** (periodic orbits) determine **topology** of the phase space
- **Perturbation** of unstable (hyperbolic points) opens the path to chaotic motion
- Resonance can overlap enabling the rapid diffusion of orbits
- **Dynamic aperture** by brute force tracking (with symplectic numerical integrators) is the usual quality criterion for evaluating non-linear dynamics performance of a machine
- **Frequency Map Analysis** is a numerical tool that enables to study in a global way the dynamics, by identifying the excited **resonances** and the extent of **chaotic** regions
- It can be directly applied to **tracking** and **experimental** data
- A combination of these modern methods enable a thorough analysis of non-linear dynamics and lead to a robust design

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C. Skokos, K. Skoufaris, C. Steier,  
F. Schmidt, G. Sterbini, A. Wolski,  
F. Zimmermann