“PRE-INTRODUCTION” TO RESURGENCE AND RENORMALONS

Or how I learned to stop worrying and love divergent series

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The convergence that wasn't

Often in QM/QFT, observables $f(g)$ computed pertubatively in some variable g result in asymptotic series.

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Typically, for a fixed g, the terms approach the right function up some N^*, after which they diverge. This is the **optimal truncation**.

A typical asymptotic series

Many series expansions in physics are of the form

$$F_N(g) = \sum_{k=1}^{N} a_k g^k, \tag{0.1}$$

where for large k

$$a_k \sim A^{-k} k!\ . \tag{0.2}$$

The radius of convergence

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We can try minimizing $a_k g^k$. We find that

$$N^* = |A/g|\,, \tag{0.4}$$

And thus the optimal error is

$$F(g) - F_{N^*}(g) \sim \mathrm{e}^{-|A/g|}. \tag{0.5}$$

Borel summation

Many series, including some conventionally divergent series, can be resummed through **Borel (re)summation**.

The Borel transform of a series is given by

$$\varphi(z) = \sum_{k \geq 0} b_k z^k \to \widehat{\varphi}(\zeta) = \sum_{k \geq 0} \frac{b_k}{k!} \zeta^k \tag{0.6}$$

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If the Laplace transform converges, φ is *Borel summable* with Borel sum

$$s(\varphi)(z) = \int_0^\infty \mathrm{e}^{-\zeta}\widehat{\varphi}(z\zeta)\,d\zeta \tag{0.7}$$

which recovers the "true" function.

Why Borel?

$$\widehat{\varphi}(\zeta) = \sum_{k \geq 0} \frac{b_k}{k!} \zeta^k \Leftrightarrow s(\varphi)(z) = \int_0^\infty \mathrm{e}^{-\zeta} \widehat{\varphi}(z\zeta) d\zeta \tag{0.8}$$

Term by term, recovers the asymptoptic series

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The Borel transform is $\sim$ the inverse of a Laplace transform.
Natural relation to ODEs.

The ambiguity strikes back

Let us look at the Borel transform of the “typical case”

$$F_p(g) \sim \sum_{k\geq 0}^{\infty}(A^{-k}k!)g^k \Rightarrow \widehat{F}(\zeta) = \frac{1}{1-\zeta/A} \tag{0.10}$$

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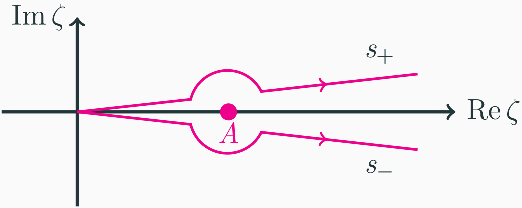

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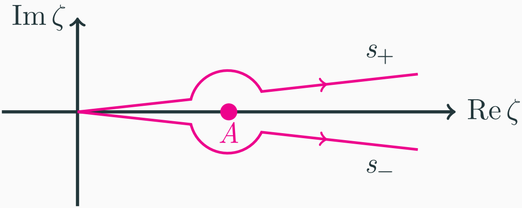

But there is an ambiguity: $s_+(F)(g) - s_-(F)(g) = 2\pi \mathrm{i} e^{-A/g}$

Trans-series

Ambiguities can be cancelled by non-perturbative sectors. The "true" function is then given by a **trans-series**

$$\Phi(z) = \sum_{k\geq 0} c_k z^k + \sum_i \mathcal{C}_i^{\pm} \mathrm{e}^{-A_i/z} z^{b_i} \sum_{k\geq 0} c_k^{(i)} z^k + \cdots \tag{0.11}$$

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In order to cancel ambiguities the **trans-series parameters** $\mathcal{C}_i^{\pm}$ must depend themselves on the ray in the complex plane where we perform the Borel summation.

We can further augment with the monomials $\log(z)$, $\exp(-\exp(A_i/z))$, $\log(\log(z))$, $\exp(-\exp(\exp(A_i/z)))$, ...

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Back to the pole

We saw an example where

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To cancel we must have

$$\mathcal{C}^+ - \mathcal{C}^- = -2\pi \mathrm{i} A, \quad b = -1, c_0 = 1. \tag{0.13}$$

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Why do different $\mathcal{C}^\pm$ make sense? Think of the Airy function (oscillating in one side, decaying in the other).

Non-perturbative physics awakens

An example of non-perturbative physics you might have heard of is **instantons**.

Instantons are non-trivial solutions of the classical EOM with **finite** action

$$\left.\frac{\delta S}{\delta \phi}\right|_{\phi=\phi_{\text{instanton}}} = 0 \quad \wedge \quad \infty > S[\phi_{\text{instanton}}] > 0 \tag{0.14}$$

In the $1/\hbar$ expansion we can see how instanton saddles give rise to a trans-series

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\phi \mathrm{e}^{-\frac{1}{\hbar}S[\phi]} \\ &= (\text{pertubation around vacuum}) + \\ &+ \mathrm{e}^{-\frac{1}{\hbar}S[\phi_{\text{instanton}}]}(\text{pertubation around instanton}) + \cdots \end{aligned} \tag{0.15}$$

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In fact, in many relevant theories, like Yang Mills, the asymptotic behaviour of the perturbative series is at leading order determined by renormalons, not instantons!

Rise of the renormalon

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$$\mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = \beta(g) = -\beta_0 g^2 - \beta_1 g^3 + \mathcal{O}(g^4)$$

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We find the RG invariant scale

$$\mathcal{I} = \mu g^{-\frac{\beta_1}{\beta_0^2}} \mathrm{e}^{-\frac{1}{\beta_0 g}} (1 + \mathcal{O}(g)) \tag{0.16}$$

This is a non-perturbative effect and gives rise to the **renormalon** pole.