RENORMALONS FROM INTEGRABILITY
A RESURGENT INSIGHT INTO QFT'S CONUNDRUM

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Trans-series in QFT

The divergence menace

Many if not most series in QFT are asymptotic, i.e. divergent (Dyson 1953). Typically they are of the form:

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If φ is *Borel summable* and we recover the “true” function $\varphi(z)$ from the Borel sum

$$s(\varphi)(z) = \int_0^\infty \mathrm{e}^{-\zeta}\widehat{\varphi}(z\zeta)\,d\zeta. \tag{1.3}$$

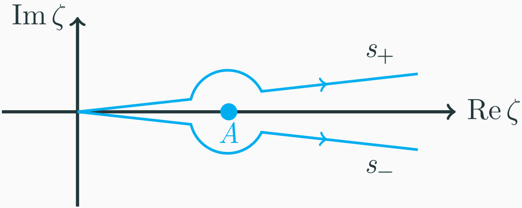
Ambiguity strikes back

If we Borel transform the example from before with $A > 0$

$$F_p(g) \sim \sum_{k\geq 0}^{\infty} (A^{-k} k!) g^k \Rightarrow \widehat{F}(\zeta) = \frac{1}{1-\zeta/A} \tag{1.4}$$

There's a pole on $\mathbb{R}^+$! We can deform the contour to go slightly above or below the real axis. But an ambiguity remains

$$s_+(F)(g) - s_-(F)(g) = 2\pi \mathrm{i} A g^{-1} \mathrm{e}^{-A/g} \tag{1.5}$$



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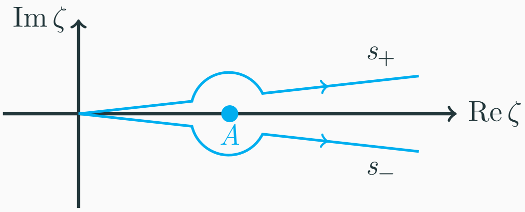

Return of non-perturbative physics

Ambiguities can be cancelled by non-perturbative sectors. The "true" function is then given by a **trans-series**

$$\Phi(z) = \sum_{k\geq 0} c_k z^k + \sum_i \mathcal{C}_i^{\pm} \mathrm{e}^{-A_i/z} z^{b_i} \sum_{k\geq 0} c_k^{(i)} z^k + \cdots \tag{1.6}$$

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We saw an example where

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To cancel we must have

$$\mathcal{C}^+ - \mathcal{C}^- = -2\pi \mathrm{i} A, \quad b = -1, c_0 = 1. \tag{1.8}$$

How I learned to stop worrying and love divergent series

Generally the leading term in the trans-series (top) implies a contribution to the asymptotic behaviour of the perturbative series (bottom)

$$\begin{gathered} \mathcal{C}_1^{\pm} g^{-b_1} \mathrm{e}^{-A_1/g} \left(\psi_{1,0} + \mathcal{O}(g)\right), \quad \mathcal{C}_1^{+} = \mathcal{C}_1^{-} - \mathrm{i}\mathsf{S}_1 \\ \Leftrightarrow \\ c_k \sim \frac{\mathsf{S}_1}{2\pi} A_1^{-k-b_1} \Gamma(k+b_1) \left(\psi_{1,0} + \mathcal{O}(k^{-1})\right), \qquad k \gg 1, \end{gathered} \tag{1.9}$$

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However resurgence helps both **make sense** of what we know and **explore** what we don't know.

Resurgence all the way down?

We can summarize the previous ideas into two levels (Di Pietro et al. 2021)

- **Weak Resurgence**: Observables in QFT are given by unambiguous Borel summations of trans-series. We can learn some information from perturbation theory but we might need non-perturbative methods to find the full trans-series.
- **Strong Resurgence**: From analysis of the large order behaviour of perturbation theory we can specify the full trans-series up to the numerical values of $\mathcal{C}_i^{\pm}$. Doesn't always work.

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Renormalons in a nutshell

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You could hope that all we need in QFT is perturbative theory and then perturbations around instantons. The transseries would follow naturally from expanding the path integral around saddle points. It happens in Quantum Mechanics (Bender and Wu 1969) and very SUSY theories (Nekrasov 2002). This was a common view in the early 70's.

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However...

Meet the renormalons

A series can also diverge because individual Feynman diagrams through their momenta integration become too big. We call this a **renormalon** effect (discovered in renormalizable theories, and baptised in analogy with instantons.).

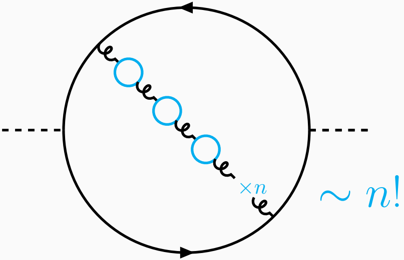

Figure 1: A typical renormalon diagram in particle physics.

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What do we know about renormalons?

What's in a name?

From renormalization and diagrammatic arguments, Parisi (1978) and 't Hooft (1979) argued that in asymptotically free theories, the Borel transform of an observable $F(g)$ should have singularities at

$$\zeta = \frac{\ell}{2|\beta_0|}, \quad \ell \in \mathbb{Z}, \tag{2.10}$$

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New renormalons from integrability

Bethe(r) idea

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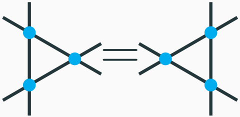
Asymptotically free integrable QFTs are both very rich and exactly solvable, which makes them great toy models. E.g.:



- $O(N)$ Gross-Neveu model
- $O(N)$ non-linear sigma model (NLSM)
- $\mathcal{N} = 1$ SUSY $O(N)$ NLSM
- $SU(N) \times SU(N)$ principal chiral field

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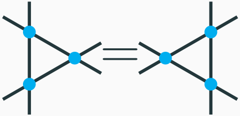

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For example, their mass gaps were computed in the 90's (Forgacs et al. Hasenfratz et al., Evans et al., Balog et al., ...)

In order to use integrability to our advantage, we add a chemical potential $h > m$ coupled to a conserved charge Q such that it excites a single species of particles of the lowest mass m in the ground state

$$\mathsf{H} \to \mathsf{H} - h\mathsf{Q}. \tag{3.11}$$

In this case the ground state, populated by particles, can be described by the **Bethe ansatz integral equation**

$$\epsilon(\theta) - \int_{-B}^{B} K(\theta - \theta')\epsilon(\theta')\mathrm{d}\theta' = h - m\cosh\theta, \quad \epsilon(\pm B) = 0, \tag{3.12}$$

where ϵ is like a Fermi density over rapidities θ. B is a function of h specified by the "Fermi level", and the kernel K is specified by the S-matrix of the excited particles, which is known exactly thanks to integrability.

In a previous episode...

Thanks to a method by Volin, at weak coupling ($B \gg 1$) one can can turn the integral equation into a series of recursive algebraic solution that give the perturbative expansion of the solution and some observables. This can be done exactly (we get 40-50 coefficients) or numerically (Abbott et al. got $\sim$2000 coeffiecients for the $O(4)$ NLSM).

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An interesting observable is the free energy

$$\mathcal{F}(h) = -\frac{m}{2\pi} \int_{-B}^{B} \epsilon(\theta) \cosh\theta \mathrm{d}\theta \,. \tag{3.13}$$

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Using Volin's method we tested that the leading large order behaviour of the perturbative series of $\mathcal{F}(h)$ matched Parisi's prediction with $\ell = 2$. But the integral equation is an **exact** solution, it should know the full trans-series!

A good vintage

Let us focus on the example of the **Gross-Neveu model**. We have $N > 4$ Majorana fermions χ with a 4 point interaction

$$\mathcal{L} = \frac{\mathrm{i}}{2}\bar{\chi} \cdot \partial\!\!\!/ \chi + \frac{g^2}{8}(\bar{\chi} \cdot \chi)^2 \tag{3.14}$$

The theory is asymptotically free, so we the running coupling is evaluated at scale $\mu = h$. The following expansions are equivalent

$$\bar{g}(h) \ll 1 \quad \Leftrightarrow \quad h \gg \Lambda \quad \Leftrightarrow \quad B \gg 1 \tag{3.15}$$

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but they all have different complications. The most simple choice is to write our results as a function of

$$\frac{1}{\alpha} + \Delta \log \alpha = \log \frac{h}{\Lambda}, \quad \alpha \sim 2|\beta_0|\bar{g}(h)^2 \sim 1/B \tag{3.16}$$

We can transform the Bethe Ansatz equation into an equation for $u(\omega)$

$$u(\omega) = \frac{\mathrm{i}}{\omega} + \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{\mathrm{e}^{2\mathrm{i}B\omega'}}{\omega + \omega' + \mathrm{i}0} \rho(\omega') u(\omega') \mathrm{d}\omega' \tag{3.17}$$

To the complex plane and beyond

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Because of $\mathrm{e}^{2\mathrm{i}B\omega'}$ we must deform the contour integral around the positive imaginary axis but…

- $\rho(\omega)$ is discontinuous along the positive imaginary axis
- $\rho(\omega)$ has poles along the imaginary axis, whose residues depend on the branch. The poles are at

$$\omega = \mathrm{i}\xi_k, \quad \xi_k = (2k+1)\frac{N-2}{N-4}, \quad k \in \mathbb{N} \tag{3.18}$$

So we must careful about how we proceed.

A plot is worth more than 10^3 equations

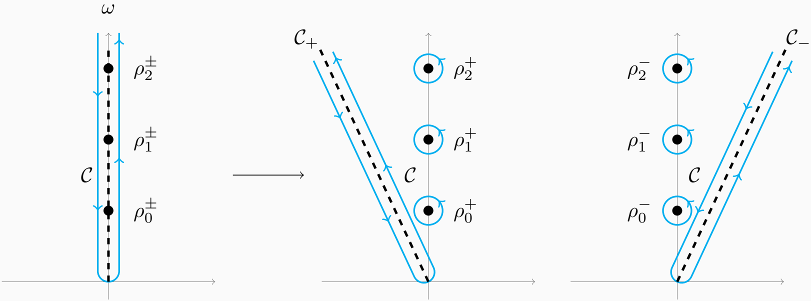

Figure 2: Deforming the contour in the complex plane.

Integrals transfigure into trans-series

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\mathrm{e}^{2\mathrm{i}B\omega'}}{\omega-\mathrm{i}}\rho(\omega')u(\omega')\mathrm{d}\omega' = \int_{\mathcal{C}^{\pm}}\frac{\mathrm{e}^{-2B\xi}u(\xi)}{\xi-1}\frac{\operatorname{disc}\rho(\mathrm{i}\xi)}{2\pi\mathrm{i}}\mathrm{d}\xi + \mathrm{e}^{-2B}\rho(\mathrm{i}\pm 0)u(\mathrm{i}) + \sum_{n\geq 1}\mathrm{e}^{-2B\xi_n}\rho_n^{\pm}\frac{u(\mathrm{i}\xi_n)}{\xi_n-1} \quad (3.19)$$

where $\rho_n^{\pm}$ are the residues of $\rho(\omega)$ in the two branches.

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The sketch of what happens next:

$$\int_{\mathcal{C}^\pm} \frac{\mathrm{e}^{-2B\xi}u(\xi)}{\xi-1}\frac{\operatorname{disc}\rho(\mathrm{i}\xi)}{2\pi\mathrm{i}}\mathrm{d}\xi \to \cdots \to \quad \frac{c_0}{B}\left\{1+\mathcal{O}\left(\frac{1}{B}\right)\right\}$$

$$\mathrm{e}^{-2B\xi_n}\rho_n^\pm\frac{u(\mathrm{i}\xi_n)}{\xi_n-1} \to \cdots \to \quad \mathrm{e}^{-2B\xi_n}\mathcal{C}_n^\pm\left\{1+\mathcal{O}\left(\frac{1}{B}\right)\right\}$$

where the $\mathcal{C}_{0,n}^\pm$ depend on the branch choice through the $\rho_n^\pm$.

Alas, the trans-series

We find a formal series with exponential suppressed terms and its ambiguous coefficients: the trans-series!

$$\mathcal{F}(h) = -\frac{h^2}{2\pi}\Big\{ (1+\mathcal{O}(\alpha)) - \mathrm{e}^{-\frac{2}{\alpha}}\alpha^{\frac{2}{N-2}}\mathcal{C}_0^{\pm} + \sum_{k\geq 1} \mathrm{e}^{-\frac{2k}{\alpha}\frac{N-2}{N-4}}\alpha^{\frac{2k}{N-4}}\mathcal{C}_k^{\pm}\,(1+\mathcal{O}(\alpha))\Big\} \tag{3.20}$$

(remember $\alpha \sim 1/B \sim 2|\beta_0|\bar{g}(h)^2$).

Alas, the trans-series

We find a formal series with exponential suppressed terms and its ambiguous coefficients: the trans-series!

$$\mathcal{F}(h) = -\frac{h^2}{2\pi}\Bigg\{ (1+\mathcal{O}(\alpha)) - \mathrm{e}^{-\frac{2}{\alpha}}\alpha^{\frac{2}{N-2}}\mathcal{C}_0^{\pm} + \sum_{k\geq 1} \mathrm{e}^{-\frac{2k}{\alpha}\frac{N-2}{N-4}}\alpha^{\frac{2k}{N-4}}\mathcal{C}_k^{\pm}\,(1+\mathcal{O}(\alpha))\Bigg\} \tag{3.20}$$

(remember $\alpha \sim 1/B \sim 2|\beta_0|\bar{g}(h)^2$).

What do the $\mathcal{C}^{\pm}_{k\geq 1}$ look like?

$$\mathcal{C}_k^{\pm} = \left\{\cos\left(\frac{k\pi}{N-4}\right) \mp \mathrm{i}\sin\left(\frac{k\pi}{N-4}\right)\right\} \times \text{(real function of } k, \Delta)$$

Extraordinary claims require extraordinary evidence

If this is the trans-series for the exact function of the observable $\mathcal{F}(h)$ we should be able to test it

- By comparing the discontinuity of the $\mathcal{C}_n^{\pm}$ with the large order behaviour of the perturbative series found with Volin's method.
- By comparing the resummation of the perturbative series with the numeric solution of the exact integral equation and see what exponentially suppressed terms are missing.

We have done many such numerical tests with success.

Unexpected developments

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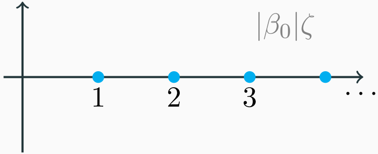
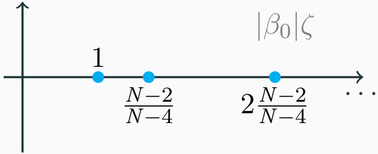
How does our analytic trans-series compare to renormalon predictions? Let's us look at Borel singularities.





Instead of

$$\zeta = \frac{k}{|\beta_0|}, \quad k \in \mathbb{N}$$

we have

$$\zeta = \frac{1}{|\beta_0|}, \frac{k}{|\beta_0|}\frac{N-2}{N-4}.$$

So our trans-series is very different from what the standard lore predicted!

Hidden in plain large N

In the large N limit we match known results but the new renormalons move to the “traditional” renormalon predictions

$$\frac{\ell}{|\beta_0|}\frac{N-2}{N-4} \to \frac{\ell}{|\beta_0|} \tag{3.21}$$

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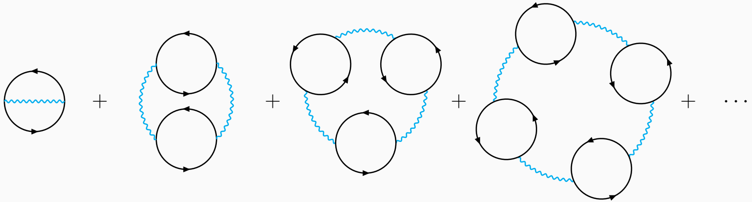

This shows that these effects are “renormalons”, at LO they are associated with the factorial growth of diagrams.

The instanton in the room

We can apply our scheme to the $O(N)$ **non-linear sigma model**, to investigate the Borel singularities of the free energy.

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- $O(N \geq 4)$ **non-linear sigma model**

$$\zeta_{IR} = 1/|\beta_0| \quad \text{and} \quad \zeta_\ell = \ell(N-2)/|\beta_0| \quad \ell \in \mathbb{N} \tag{3.23}$$

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- $O(3)$ **non-linear sigma model**

  $$\zeta_\ell = \ell/|\beta_0| \tag{3.24}$$

  Already at order $\mathrm{e}^{-2/\alpha}$ there is an additional (real, unambiguous) non-trivial series which is invisible to the perturbative series. Related to (stable) instantons?

A little bit of both

Some models have both "renormalon-like" and "instanton-like" singularities.

- $\mathcal{N} = 1$ SUSY $O(N)$ non-linear sigma model

$$\zeta_\ell = \frac{\ell}{|\beta_0|}\frac{N-2}{N-4}, \quad \zeta'_\ell = \frac{\ell(N-2)}{|\beta_0|} \quad \text{and} \quad \zeta_{\ell_1} + \zeta'_{\ell_2} \tag{3.25}$$

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- $SU(N)$ principal chiral field

$$\zeta_{IR} = \frac{1}{|\beta_0|}, \quad \zeta_\ell = \frac{\ell}{|\beta_0|} \frac{N}{N-1}, \quad \zeta'_\ell = \frac{\ell N}{|\beta_0|}, \quad \zeta_{\ell_1} + \zeta'_{\ell_2} \tag{3.26}$$

  A traditional IR renormalon, new renormalons and instanton-like singularities. Similar thing structure is found for $SU(N)/SO(N)$ coset NLSM.

Look at it from a (topological) angle

In the $O(3)$ NLSM and some coset NLSM, we can add a topological term with $\vartheta = 0, \pi$ (analogue to the QCD ϑ-angle),

$$q(x) \sim \mathrm{i}\vartheta(\epsilon^{abc}\epsilon_{\mu\nu}S_a\partial_\mu S_b\partial_\nu S_c)/(8\pi), \tag{3.27}$$

keeping the perturbative series and integrability.

Non-perturbative contributions which are directly tied to the perturbative series are not affected, but the real, unfixed, contributions change signs. Schematically,

$$(\mathrm{Re}\,\mathcal{C}_1^+ \pm \mathrm{i}\,\mathrm{Im}\,\mathcal{C}_1^+)\mathrm{e}^{-2\xi_1/\alpha} \to (-\mathrm{Re}\,\mathcal{C}_1^+ \pm \mathrm{i}\,\mathrm{Im}\,\mathcal{C}_1^+)\mathrm{e}^{-2\xi_1/\alpha},$$

for both "instanton-like" and "renormalon-like" contributions.

This is what we would expect from instantons. What does this say about renormalons?

Conclusion

Take-home ideas

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- Large N can be deceiving.

Thank you!

Thank you!

Useful links

- For reviews on resurgence check out Mariño's lecture notes (arXiv:1206.6272) or the review by Aniceto et al. (arXiv:1802.10441).
- The canonical review on renormalons is Beneke's (arXiv:hep-ph/9807443).
- The results in this presentation are mainly from (arXiv:2111.11951), as well as partially from (arXiv:2205.04495) and (arXiv:1909.12134).

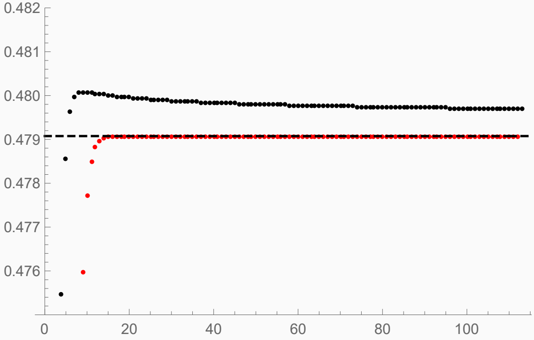
Backup slides

Backup slides

Large order behaviour

One important test is to take long perturbative series from Volin's method and compare with the Stokes constants $\mathcal{C}_0^+ - \mathcal{C}_0^- = -\mathrm{i}\mathsf{S}_0$. With an asymptotic c_k we can construct an auxiliary series s_k such that

$$c_k \sim \frac{\mathsf{S}_0}{2\pi} A_1^{-k-b_1} \Gamma(k+b_1) \Rightarrow s_k \sim \mathsf{S}_0, \quad k \gg 1$$



Intermezzo

Because the $\mathrm{e}^{-\frac{2}{\alpha}}$ term is very simple in

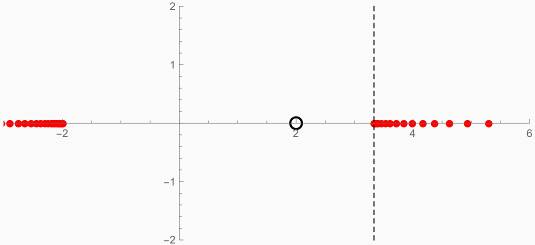
$$\mathcal{F}(h) = -\frac{h^2}{2\pi}\Big\{ (1+\mathcal{O}(\alpha)) - \mathrm{e}^{-\frac{2}{\alpha}}\alpha^{\frac{2}{N-2}}\mathcal{C}_0^{\pm} + \mathrm{e}^{-\frac{2}{\alpha}\frac{N-2}{N-4}}\alpha^{\frac{2}{N-4}}\mathcal{C}_1^{\pm}\,(1+\mathcal{O}(\alpha)+\cdots)\Big\}$$

we can subtract its contribution to the asymptotics and see the effects of the term $\mathrm{e}^{-\frac{2}{\alpha}\frac{N-2}{N-4}}$.

The singularity marks the spot

With $N = 7$, we plot the singularities of and approximation of the Borel transform (they approximate a cut) after subtracting the leading order behaviour.

The removed singularity is at 2 (i.e. $1/|\beta_0|$) and the next predicted singularity is at $10/3$ (i.e. $5/3|\beta_0|$).



Keeping it real

We can calculate the difference between the Borel summation of the perturbative part, with the leading ambiguities cancelled, and the numerical solution of the Bethe ansatz equation and then compare this difference with the non-perturbative predictions from the trans-series.

We plot with $N = 7$, the dashed line includes only the non-perturbative effect $\propto e^{-2/\alpha}$ and the solid line also includes the subleading effect.